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Xi CHEN & Andrew HASSELL

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RESOLVENT AND SPECTRAL MEASURE ON NON-TRAPPING ASYMPTOTICALLY HYPERBOLIC MANIFOLDS II: SPECTRAL MEASURE, RESTRICTION THEOREM, SPECTRAL MULTIPLIERS

by Xi CHEN & Andrew HASSELL (*)

Abstract. — We consider the Laplacian Δ on an asymptotically hyperbolic manifold X, as defined by Mazzeo and Melrose. We give pointwise bounds on the Schwartz kernel of the spectral measure for the operator $(\Delta - n^2/4)_+^{1/2}$ on such manifolds, under the assumptions that X is nontrapping and there is no resonance at the bottom of the spectrum. This uses the construction of the resolvent given by Mazzeo and Melrose, Melrose, Sá Barreto and Vasy, the present authors, and Wang.

We give two applications of the spectral measure estimates. The first, following work due to Guillarmou and Sikora with the second author in the asymptotically conic case, is a restriction theorem, that is, a $L^p(X) \to L^{p'}(X)$ operator norm bound on the spectral measure. The second is a spectral multiplier result under the additional assumption that X has negative curvature everywhere, that is, a bound on functions of the Laplacian of the form $F((\Delta - n^2/4)_+^{1/2})$, in terms of norms of the function F. Compared to the asymptotically conic case, our spectral multiplier result is weaker, but the restriction estimate is stronger.

Résumé. — Nous considérons le Laplacien Δ sur une variété asymptotiquement hyperbolique X au sens de Mazzeo et Melrose. Nous donnons des estimations ponctuelles sur le noyau de Schwartz de la mesure spectrale pour l'opérateur $(\Delta - n^2/4)_+^{1/2}$ sur ces variétés, sous l'hypothèse qu'il n'y ni trajectoires captées dans X, ni résonance au bas du spectre. Nous utilisons la construction de la résolvante par Mazzeo et Melrose, Sá Barreto et Vasy, Wang, et nous-mêmes.

Nous donnons deux applications des estimations de la mesure spectrale. La première, qui prolonge l'étude de Guillarmou et Sikora avec le deuxième auteur dans

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le cas asymptotiquement conique, est un théorème de restriction: c'est-à-dire une borne sur la norme d'opérateur $L^p(X) \to L^{p'}(X)$ de la mesure spectrale. La seconde est un résultat de type multiplicateur spectral sous l'hypothèse additionnelle que X est à courbure strictement négative partout. Plus précisément, nous donnons une estimation sur les fonctions du laplacien de la forme $F((\Delta - n^2/4)_+^{1/2})$ en termes de normes de la fonction F. Par rapport au cas asymptotiquement conique, notre résultat de multiplicateur spectral est plus faible, mais l'estimation de restriction est plus forte.

1. Introduction

This paper, following [10], is the second in a series of three devoted to the analysis of the resolvent family and spectral measure for the Laplacian on an asymptotically hyperbolic, nontrapping manifold. The third paper, by the first author alone, will establish global-in-time Strichartz estimates on such a manifold.

Let (X°, g) be an asymptotically hyperbolic manifold of dimension $n+1$ (see Section 1.5 for the precise definition of "asymptotically hyperbolic"). Let Δ be the positive Laplacian on (X°, g), which is essentially self-adjoint on $C_c^\infty(X^\circ)$. It is well known that the spectrum of Δ is absolutely continuous on $[n^2/4, \infty)$ [26] with possibly finitely many eigenvalues (of finite multiplicity) in $(0, n^2/4)$. We write P for the operator

$$P = (\Delta - n^2/4)_+^{1/2}, \tag{1.1}$$

where the subscript $+$ indicates positive part. Thus, P vanishes on the pure point eigenspaces. In this paper, we analyze the spectral measure $\mathrm{d}E_P(\boldsymbol{\lambda})$ of the operator P, under the assumption that (X°, g) is nontrapping (that is, every geodesic reaches infinity both forward and backward) and that there is no resonance at the bottom of the continuous spectrum, $n^2/4$. To do this, we express the spectral measure $\mathrm{d}E_P(\boldsymbol{\lambda})$ in terms of the boundary values of the resolvent $(\Delta - n^2/4 - (\boldsymbol{\lambda} \pm \imath 0)^2)^{-1}$ just "above" and "below" the spectrum in $\mathbb{C}$. We then use the construction of the resolvent given by Mazzeo and Melrose [27] (valid when the spectral parameter lies in a compact set), Melrose, Sá Barreto and Vasy [29] (high energy estimates for a perturbation of the hyperbolic metric) and the present authors [10] (and, independently, [38]) in the general high-energy case to get precise information about the Schwartz kernel of the spectral measure. In particular, following the work of the second author with Guillarmou and Sikora [17] in the asymptotically conic setting, this will allow us to obtain precise pointwise bounds on the Schwartz kernel, when (micro)localized near the diagonal in a certain sense.

We then apply these pointwise kernel bounds to prove operator norm estimates on the spectral measure $\mathrm{d}E_P(\boldsymbol{\lambda})$, and on general functions $F(P)$ of the operator P, again following the general strategy of [17]. However, there are key differences in the results we prove here compared to the asymptotically conic case, which can be traced to the exponential, as opposed to polynomial, growth of the volume of large balls in the present setting. In the case of the restriction theorem, that is, an $L^p \to L^{p'}$ bound on the spectral measure, we prove more: we obtain an estimate for all $p \in [1,2)$, while in the asymptotically conic case, it is well known that such an estimate fails for $p > 2(d+1)/(d+3)$, where d is the dimension. In the case of the spectral multiplier result, that is, boundedness of $F(P)$, where we assume only a finite amount of Sobolev regularity on F, boundedness on $L^p(X)$ spaces fails for $p \neq 2$ due to results of Clerc–Stein [11] and Taylor [35]. Instead, we obtain boundedness on $L^p(X) + L^2(X)$ for $p \in [1,2)$, provided X is negatively curved.

1.1. The spectral measure

Consider functions of an abstract (unbounded) self-adjoint operator L on a Hilbert space H. These are defined by the spectral theorem for unbounded self-adjoint operators (for example, see [31, p. 263]). One standard version of this theorem says that there is a one-to-one correspondence between self-adjoint operators L and increasing, right-continuous families of projections $E(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathbb{R}$, having the property that the strong limit of $E(\boldsymbol{\lambda})$ as $\boldsymbol{\lambda} \to -\infty$ is the zero operator and as $\boldsymbol{\lambda} \to +\infty$ is the identity. The correspondence is given by

$$L = \int_{-\infty}^{\infty} \boldsymbol{\lambda} \, \mathrm{d}E(\boldsymbol{\lambda});$$

if $g(\,\cdot\,)$ is a real-valued Borel function on $\mathbb{R}$, then

$$g(L) = \int_{-\infty}^{\infty} g(\boldsymbol{\lambda}) \, \mathrm{d}E(\boldsymbol{\lambda})$$

with domain

$$\left\{ \psi : \int_{-\infty}^{\infty} |g(\boldsymbol{\lambda})|^2 \, \mathrm{d}\langle \psi, E(\boldsymbol{\lambda})\psi \rangle < \infty \right\}$$

is self-adjoint. Here the formula means

$$\langle g(L)\psi, \psi \rangle = \int_{-\infty}^{\infty} g(\boldsymbol{\lambda}) \, \mathrm{d}\langle E(\boldsymbol{\lambda})\psi, \psi \rangle,$$

which can be interpreted as a Stieltjes integral since $\langle E(\boldsymbol{\lambda})\psi, \psi\rangle$ is a non-decreasing function of $\boldsymbol{\lambda}$. We call $\mathrm{d}E(\boldsymbol{\lambda})$ the spectral measure associated with the operator L.

In particular we can apply this when $L = P$ and $H = L^2(X, g)$. We then write $\mathrm{d}E_P(\boldsymbol{\lambda})$ for the spectral measure of P. Since P is a positive operator, we only need to integrate over $\boldsymbol{\lambda} \in [0, \infty)$ in this case.

Returning to the abstract operator L, the resolvent family $(L - \boldsymbol{\lambda})^{-1}$ is a holomorphic family of bounded operators on H for $\operatorname{Im} \boldsymbol{\lambda} \neq 0$. In many cases, including in the present setting, the resolvent family extends continuously to the real axis as a bounded operator in a weaker sense, e.g. between weighted L^2 spaces, and is then differentiable in $\boldsymbol{\lambda}$ up to the real axis. In that case, we find that $E(\boldsymbol{\lambda})$ is differentiable in $\boldsymbol{\lambda}$ and we have Stone's formula

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}} E(\boldsymbol{\lambda}) = \frac{1}{2\pi\imath}\Big((L - (\boldsymbol{\lambda} + \imath 0))^{-1} - (L - (\boldsymbol{\lambda} - \imath 0))^{-1}\Big). \tag{1.2}$$

In this case we write (abusing notation somewhat) $\mathrm{d}E(\boldsymbol{\lambda})$ for the derivative of $E(\boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$. Stone's formula gives a mechanism for analyzing the spectral measure, namely we need to analyze the limit of the resolvent $(L - \boldsymbol{\lambda})^{-1}$ on the real axis. In the case of P, we notice that the spectral measure $\mathrm{d}E_P(\boldsymbol{\lambda})$ for P is $2\boldsymbol{\lambda}$ times the spectral measure at $n^2/4 + \boldsymbol{\lambda}^2$ for Δ. This gives us the distributional formula

$$\mathrm{d}E_P(\boldsymbol{\lambda}) = \frac{\boldsymbol{\lambda}}{\pi\imath}\Big((\Delta - (n^2/4 + \boldsymbol{\lambda}^2 + \imath 0))^{-1} - (\Delta - (n^2/4 + \boldsymbol{\lambda}^2 - \imath 0))^{-1}\Big). \tag{1.3}$$

1.2. Restriction theorem via spectral measure

Stein [34] and Tomas [37] proved estimates for the restriction of the Fourier transform of an L^p function to the sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$:

$$\int_{\mathbb{S}^{d-1}} |\hat{f}|^2 \,\mathrm{d}\sigma \leqslant C\|f\|^2_{L^p(\mathbb{R}^d)}, \quad p \in [1, 2(d+1)/(d+3)].$$

Alternatively, we may formulate the estimate in terms of the restriction operator R to the hypersphere,

$$R(f)(\xi) = \int_{\mathbb{R}^d} e^{-\imath x\cdot\xi} f(x)\,\mathrm{d}x, \quad |\xi| = 1.$$

The Stein–Tomas theorem is equivalent to the boundedness of

$$R : L^p(\mathbb{R}^d) \longrightarrow L^2(\mathbb{S}^{d-1}),$$

which in turn is equivalent to the boundedness of

$$R^*R : L^p(\mathbb{R}^d) \longrightarrow L^{p'}(\mathbb{R}^d).$$

The Schwartz kernel of R^*R,

$$\int_{|\xi|=1} e^{\imath(x-y)\cdot\xi}\,\mathrm{d}\xi\,,$$

is $(2\pi)^d$ times the spectral measure $\mathrm{d}E_{\sqrt{\Delta}}(1)$ for the square root of the flat Laplacian on $\mathbb{R}^d$, since the spectral projection $E_{\sqrt{\Delta}}(\boldsymbol{\lambda})$ of $\sqrt{\Delta}$ can be written as $\mathcal{F}^{-1}(\chi_{B(0,\boldsymbol{\lambda})})\mathcal{F}$. Therefore, one may rewrite the restriction theorem as the following estimate:

$$\begin{aligned}(1.4)\quad \|\mathrm{d}E_{\sqrt{\Delta}}(\boldsymbol{\lambda})\|_{L^p\to L^{p'}}\ \big(=\boldsymbol{\lambda}^{d(2/p-1)-1}\|\mathrm{d}E_{\sqrt{\Delta}}(1)\|_{L^p\to L^{p'}}\ \big)&\\ \leqslant C\boldsymbol{\lambda}^{d(2/p-1)-1}&,\end{aligned}$$

provided $p\in[1,2(d+1)/(d+3)]$. This naturally leads to the question: for which Riemannian manifolds (N,g) does the spectral measure for $\sqrt{\Delta_{N,g}}$ map $L^p(N,g)$ to $L^{p'}(N,g)$ for some $p\in[1,2)$, and how does the norm depend in the spectral parameter? We refer to such an estimate as a "restriction estimate" or a "restriction theorem". Such a result is a continuous spectral analogue of the well-known discrete restriction theorem of Sogge [32, Chapter 5].

1.3. Results on asymptotically conic spaces

As the present paper is inspired by work by the second author with Guillarmou and Sikora [17] on asymptotically conic spaces, we review the results of [17] here.

Asymptotically conic spaces M, of dimension m, are modelled on spaces that at infinity look like the "large end of a cone"; that is, have one end diffeomorphic to $(r_0,\infty)\times Y$, where Y is a closed manifold of dimension $m-1$, with a metric of the form

$$\mathrm{d}r^2+r^2g_0(y,\mathrm{d}y)+O\left(\frac{1}{r}\right),\quad r\to\infty\,,$$

where g_0 a metric on Y. Such spaces are Euclidean-like at infinity, in the sense that the volume of balls of radius ρ are uniformly bounded above and below by multiples of ρ^m, and in the sense that the curvature tends to zero, and the local injectivity radius tends to infinity, at infinity. If we add the condition that the manifold be nontrapping, then such spaces are also dynamically similar to Euclidean space (although they may have conjugate points). Consequently, the spectral analysis of such spaces behaves in many ways like Euclidean space. This is illustrated by the results from [17]. On $\mathbb{R}^m$, the spectral measure satisfies pointwise kernel bounds of the form

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\right)^j \mathrm{d}E_{\sqrt{\Delta}}(\boldsymbol{\lambda})(x,y)\right| \leqslant C\boldsymbol{\lambda}^{m-1-j}(1+\boldsymbol{\lambda}|x-y|)^{-(m-1)/2+j}, \tag{1.5}$$

with $j \in \mathbb{N}$, and this estimate is essentially optimal, in the sense that neither exponent can be improved. In [17] it was shown that, if M is an asymptotically conic nontrapping manifold, and Δ its Laplacian, then there is a partition of unity $\mathrm{Id} = \sum_{j=0}^{N} Q_i(\boldsymbol{\lambda})$, depending on $\boldsymbol{\lambda}$, and $\delta > 0$ such that

$$\begin{aligned}\left|Q_i(\boldsymbol{\sigma})\left(\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\right)^j \mathrm{d}E_{\sqrt{\Delta}}(\boldsymbol{\lambda})\right)Q_i^*(\boldsymbol{\sigma})(x,y)\right| \\ \leqslant C\boldsymbol{\lambda}^{m-1-j}(1+\boldsymbol{\lambda}d(x,y))^{-(m-1)/2+j},\end{aligned} \tag{1.6}$$

with $j \in \mathbb{N}$, for $\boldsymbol{\sigma} \in [(1-\delta)\boldsymbol{\lambda}, (1+\delta)\boldsymbol{\lambda}]$, where $d(x,y)$ is the Riemannian distance[(1)]. The $Q_i(\boldsymbol{\lambda})$ are semiclassical pseudodifferential operators (with semiclassical parameter $h = \boldsymbol{\lambda}^{-1}$) with small microsupport. Therefore, the operators $Q_i(\boldsymbol{\sigma})\mathrm{d}E_{\sqrt{\Delta}}(\boldsymbol{\lambda})Q_i^*(\boldsymbol{\sigma})$ can be considered to be the kernel of the spectral measure (micro)localized near the diagonal. Moreover, in the case where there are no conjugate points, then the estimate above is valid without the partition of unity.

This estimate (1.6) was shown to imply a global restriction estimate, that is, an $L^p(M) \to L^{p'}(M)$ operator norm bound on $\mathrm{d}E_{\sqrt{\Delta}}(\boldsymbol{\lambda})$. In fact, this was proved at an abstract level:

Theorem 1.1 ([17, 9][(2)]). — *Let (X,d,μ) be a metric measure space, and L an abstract positive self-adjoint operator on $L^2(X,\mu)$. Suppose that the spectral measure $\mathrm{d}E_{\sqrt{L}}(\boldsymbol{\lambda})$ has a Schwartz kernel satisfying* (1.5) *(with $|x-y|$ replaced by $d(x,y)$) for $j=0$, as well as for $j=m/2-1$ and $j=m/2$ if m is even, or $j=m/2-3/2$ and $j=m/2+1/2$ if m is odd. Then the operator norm estimate*

$$\left\|\mathrm{d}E_{\sqrt{L}}(\boldsymbol{\lambda})\right\|_{L^p(M)\to L^{p'}(M)} \leqslant C\boldsymbol{\lambda}^{m(1/p-1/p')-1}, \quad 1 \leqslant p \leqslant \frac{2(m+1)}{m+3}, \tag{1.7}$$

holds for all $\boldsymbol{\lambda} > 0$. Moreover, if the kernel estimates above hold for some range of $\boldsymbol{\lambda}$, then (1.7) *holds for $\boldsymbol{\lambda}$ in the same range.*

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(1) This was only claimed for $\boldsymbol{\lambda} = \boldsymbol{\sigma}$ in [17], but in [16] it was observed that the same construction gives the more general estimates in (1.6).

(2) This theorem was formulated and partially proved in [17]. See [9] for a complete proof.

Finally, it was shown in [17] that, at an abstract level, such a restriction estimate implies spectral multiplier estimates:

THEOREM 1.2 ([17]). — *Let (X, d, μ) be a metric measure space, such that the volume of each ball of radius ρ is comparable to ρ^m. Suppose L is a positive self-adjoint operator such that $\cos t\sqrt{L}$ satisfies finite propagation speed on $L^2(X)$, and the restriction theorem*

$$\|\mathrm{d}E_{\sqrt{L}}(\boldsymbol{\lambda})\|_{L^p\to L^{p'}} \leqslant C\boldsymbol{\lambda}^{m(1/p-1/p')-1}$$

holds uniformly with respect to $\boldsymbol{\lambda} > 0$ for $1 \leqslant p \leqslant 2(m+1)/(m+3)$. Then there is a uniform operator norm bound on spectral multipliers on $L^p(X)$ of the form

$$\sup_{\alpha>0} \|F(\alpha\sqrt{L})\|_{L^p\to L^p} \leqslant C\|F\|_{H^s}\,, \tag{1.8}$$

where $F \in H^s(\mathbb{R})$ is an even function supported in $[-1, 1]$, and $s > m(1/p-1/2)$.

In particular, one concludes (1.7) and (1.8) when X is an asymptotically conic nontrapping manifold of dimension d.

1.4. Hyperbolic space

We next consider existing results on hyperbolic space. We return to our convention where the dimension is $n + 1$. Using explicit formulae for the Schwartz kernel of functions of the operator $P = (\Delta - n^2/4)_+^{1/2}$, we deduce pointwise bounds

$$|\mathrm{d}E_P(\boldsymbol{\lambda})(z, z')| \leqslant \begin{cases} C\boldsymbol{\lambda}^2, & \text{for } d(z, z') \leqslant 1 \\ C\boldsymbol{\lambda}^2 d(z, z')(1 + \boldsymbol{\lambda} d(z, z'))^{-1} e^{-nd(z,z')/2}, & \text{for } d(z, z') \geqslant 1 \end{cases} \tag{1.9}$$

for $\boldsymbol{\lambda} \leqslant 1$, and derivative estimates[(3)]

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\right)^j \mathrm{d}E_P(\boldsymbol{\lambda})(z, z')\right| \leqslant \begin{cases} C\boldsymbol{\lambda}^{n-j}(1 + d(z, z')\boldsymbol{\lambda})^{-n/2+j}, & \text{for } d(z, z') \leqslant 1 \\ C\boldsymbol{\lambda}^{n/2} d(z, z')^j e^{-nd(z,z')/2}, & \text{for } d(z, z') \geqslant 1\,, \end{cases} \tag{1.10}$$

(3) We can obtain derivative estimates for $\boldsymbol{\lambda} \leqslant 1$ also, but we do not need such estimates in the low energy case.

when $\boldsymbol{\lambda} \geqslant 1$. Closely related pointwise bounds for the wave kernels $\cos tP$ and $P^{-1}\sin tP$, the heat kernel e^{-tP^2} and the Schrödinger propagator $e^{\imath tP^2}$ on hyperbolic space have been exploited in various works; see for example [1, 5, 6, 12].

To the authors' knowledge, the recent paper [21] by Huang and Sogge is the only previous paper in which restriction estimates for hyperbolic space have been considered. Huang and Sogge proved restriction estimates for p in the same range $[1, 2(d+1)/(d+3)]$ as for Euclidean space, using the exact expression for the hyperbolic resolvent, and complex interpolation, in the manner of Stein's original proof of the Stein–Tomas restriction theorem [37] (this argument was presented in an abstract formulation in [17]). In fact, on hyperbolic space (and, as we shall show, asymptotically hyperbolic nontrapping spaces), restriction estimates are valid for all $p \in [1, 2)$ (see Section 2 for a very simple proof on $\mathbb{H}^3$.)

Spectral multiplier estimates on hyperbolic and asymptotically hyperbolic spaces on L^p spaces (much more general than those considered here) have been well studied. It was pointed out by Clerc and Stein [11] for symmetric spaces and Taylor [35] for spaces with exponential volume growth and C^∞ bounded geometry that a *necessary* condition for $F(P)$ to be bounded is that F admit an analytic continuation to a strip in the complex plane. Cheeger, Gromov and Taylor [7], and Taylor [35] showed that if M has C^∞ bounded geometry and injectivity radius bounded from below, then $F(\sqrt{P})$ maps $L^p(M)$ into itself for $1 < p < \infty$, provided that F is holomorphic and even on the strip $\{z \in \mathbb{C} : |\mathrm{Im} z| < W\}$ for some W and satisfies symbol estimates $|F^{(j)}(z)| \leqslant C_j \langle z \rangle^{k-j}$ on the strip.

By constrast, we want to consider the mapping properties of $F(P)$ where F has only finite Sobolev regularity. This is motivated by typical applications of spectral multipliers in harmonic analysis, such as Riesz means, and in PDE, in which one often wants to restrict to a dyadic frequency interval, that is, to the range of a spectral projector of the form $1_{[2^j, 2^{j+1}]}(P)$, or a smoothed version of this. Clearly, such a spectral multiplier cannot have an analytic continuation to a strip. On the other hand, the work of Clerc–Stein and Taylor shows that boundedness on L^p, $p \neq 2$, cannot be expected. This motivates us to search for replacements for L^p spaces, on which spectral multipliers are bounded.

1.5. Asymptotically hyperbolic manifolds

The geometric setting in the present paper is that of asymptotically hyperbolic manifolds. An asymptotically hyperbolic manifold (X°, g) is the interior of a compact manifold X with boundary, such that the Riemannian metric g takes a specific degenerate form near the boundary of X. Specifically, near each boundary point, there are local coordinates (x, y), where x is a boundary defining function and y restrict to local coordinates on ∂X, such that g takes the form

$$g = \frac{\mathrm{d}x^2 + g_0(x, y, \mathrm{d}y)}{x^2}, \tag{1.11}$$

where $g_0(x, y, dy)$ is a family of metrics on ∂X, smoothly parametrized by x. Under the metric g, the interior X° of X is a complete Riemannian manifold.

As is well known, $n + 1$-dimensional hyperbolic space takes this form in the Poincaré ball model. Indeed, $\mathbb{H}^{n+1}$ is given by the interior of the unit ball in $\mathbb{R}^{n+1}$, with the metric

$$g = \frac{4\mathrm{d}z^2}{(1 - |z|^2)^2}, \tag{1.12}$$

where $z = (z_1, \ldots, z_{n+1})$ are the standard coordinates on $\mathbb{R}^{n+1}$. Other examples include all convex co-compact hyperbolic manifolds, and compactly supported metric perturbations of these.

Such spaces are termed asymptotically hyperbolic spaces as the sectional curvatures tend to -1 at infinity [27]. Analytically, they have many similarities to hyperbolic spaces. Consider the resolvent $R(\zeta) := (\Delta - \boldsymbol{\zeta}(n - \boldsymbol{\zeta}))^{-1}$ on $\mathbb{H}^{n+1}$, which is well-defined as a bounded operator on $L^2(\mathbb{H}^{n+1})$ for $\operatorname{Re} \boldsymbol{\zeta} > n/2$. Notice that the axis $\operatorname{Re} \boldsymbol{\zeta} = n/2$ corresponds to the spectrum of Δ, and the point $\boldsymbol{\zeta} = n/2 \pm \imath \boldsymbol{\lambda}$ corresponds to the point $|\boldsymbol{\lambda}|$ in the spectrum of $P = (\Delta - n^2/4)_+^{1/2}$. On $\mathbb{H}^{n+1}$, the resolvent $R(\boldsymbol{\zeta})$ extends to a holomorphic function of $\boldsymbol{\zeta} \in \mathbb{C}$ when n is even, and a meromorphic function with poles at $\{0, -1, -2, \ldots\}$ when n is odd.

For asymptotically hyperbolic spaces, it is known from works of Mazzeo–Melrose [27] and Guillarmou [14] that the resolvent $(\Delta - \boldsymbol{\zeta}(n - \boldsymbol{\zeta}))^{-1}$ extends to be a meromorphic function of $\boldsymbol{\zeta}$ on $\mathbb{C} \setminus \{(n-1)/2 - k \mid k = 1, 2, 3, \ldots\}$, and extends to be meromorphic on the whole of $\mathbb{C}$ provided that g is even in x, that is, a smooth function of x^2. In addition, it is holomorphic in a neighbourhood of the spectral axis $\operatorname{Re} \boldsymbol{\zeta} = n/2$ except possibly at the point $n/2$ itself, corresponding to the bottom of the continuous spectrum, which could be a simple pole [4]. In the present article, we shall *assume* that the resolvent is holomorphic at $\boldsymbol{\zeta} = n/2$ as well. We point out that our estimates will certainly fail in the case of a resonance at the bottom of the spectrum, but weaker estimates will remain valid; see [15, 23] for an analysis of zero-resonances in the asymptotically Euclidean case.

1.6. Main results

1.6.1. Pointwise estimates on the spectral measure

Our first main result, analogous to (1.6), is that there is a partition of the identity, $\mathrm{Id} = \sum_{j=0}^{N} Q_i(\boldsymbol{\lambda})$ on $L^2(X)$ such that the diagonal terms in the two-sided decomposition of $\mathrm{d}E_P(\boldsymbol{\lambda})$ satisfy the same type of pointwise bounds as are valid on hyperbolic space. In fact, following [19], we prove a slightly stronger result, in which we retain information about the oscillatory nature of the kernel as $\boldsymbol{\lambda} \to \infty$.

Before stating the result, we refer to Section 3 for the definition of the double space X_0^2, the blow-up of X^2 at the boundary of the diagonal; see Figure 3.1. This space has 3 boundary hypersurfaces: the lift to X_0^2 of the left and right boundaries in X^2, denoted FL and FR, respectively, and the "front face" FF created by blowup. We denote boundary defining functions for these boundary hypersurfaces by ρ_L, ρ_R and ρ_F respectively.

Theorem 1.3. — *Let (X°, g) be an asymptotically hyperbolic nontrapping manifold with no resonance at the bottom of the spectrum, and let P be given by* (1.1)*. Then for low energies, $\boldsymbol{\lambda} \leqslant 1$, the Schwartz kernel of the spectral measure $\mathrm{d}E_P(\boldsymbol{\lambda})$ takes the form*

$$(1.13)\quad \mathrm{d}E_P(\boldsymbol{\lambda})(z,z') = \boldsymbol{\lambda}\Big((\rho_L\rho_R)^{n/2+\imath\boldsymbol{\lambda}} a(\boldsymbol{\lambda}, z, z') - (\rho_L\rho_R)^{n/2-\imath\boldsymbol{\lambda}} a(-\boldsymbol{\lambda}, z, z')\Big),$$

where $a \in C^\infty([-1,1]_{\boldsymbol{\lambda}} \times X_0^2)$.

For high energies, $\boldsymbol{\lambda} \geqslant 1$, one can choose a finite pseudodifferential operator partition of the identity operator,

$$Id = \sum_{k=0}^{N} Q_k(\boldsymbol{\lambda}),$$

such that the Q_j are bounded on L^p, uniformly in λ, for each $p \in (1,\infty)$, and such that the microlocalized spectral measure, that is, any of the compositions $Q_k(\boldsymbol{\lambda})\mathrm{d}E_P(\boldsymbol{\lambda})Q_k^*(\boldsymbol{\lambda})$, $0 \leqslant k \leqslant N$, takes the form

$$(1.14)\quad Q_k(\boldsymbol{\lambda})\mathrm{d}E_P(\boldsymbol{\lambda})Q_k^*(\boldsymbol{\lambda})(z,z') = \boldsymbol{\lambda}^n\Big(\sum_{\pm} e^{\pm\imath\boldsymbol{\lambda} d(z,z')} b_\pm(\boldsymbol{\lambda}, z, z')\Big) + (\rho_L\rho_R)^{n/2+\imath\boldsymbol{\lambda}}\, a_+ + (\rho_L\rho_R)^{n/2-\imath\boldsymbol{\lambda}} a_- + (xx')^{n/2+\imath\boldsymbol{\lambda}}\, \tilde{a}_+ + (xx')^{n/2-\imath\boldsymbol{\lambda}}\, \tilde{a}_-$$

where $a_\pm$ is in

$$\boldsymbol{\lambda}^{-\infty}C^\infty([0,1]_{\boldsymbol{\lambda}^{-1}} \times X_0^2)$$

and $\tilde{a}_\pm$ is in

$$\boldsymbol{\lambda}^{-\infty}C^\infty([0,1]_{\boldsymbol{\lambda}^{-1}} \times X^2)\,,$$

and the functions $b_\pm$ satisfy the following. For small distance, $d(z,z') \leqslant 1$, we have

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}\boldsymbol{\lambda}^j} b_\pm(\boldsymbol{\lambda}, z, z')\right| \leqslant C\boldsymbol{\lambda}^{-j}\big(1+\boldsymbol{\lambda}d(z,z')\big)^{-n/2}. \tag{1.15}$$

For $d(z,z') \geqslant 1$, $b_\pm$ is $\boldsymbol{\lambda}^{-n/2}$ times a smooth function of $\boldsymbol{\lambda}^{-1}$, decaying to order $n/2$ at FL *and* FR*:*

$$b_\pm(\boldsymbol{\lambda}, z, z') \in \boldsymbol{\lambda}^{-n/2}(\rho_L\rho_R)^{n/2}C^\infty([0,1]_{\boldsymbol{\lambda}^{-1}} \times X_0^2)\,. \tag{1.16}$$

Moreover, if (X°, g) is in addition simply connected with nonpositive sectional curvatures, then the estimates above are true for the spectral meaure without microlocalization, i.e. in this case we can take $\{Q_i(\boldsymbol{\lambda})\}$ to be the trivial partition of unity.

Remark 1.4. — We can split the continuous spectrum of P at any point $\boldsymbol{\lambda} \in (0,\infty)$ to differentiate high and low energies.

Using this structure theorem, we prove pointwise bounds on the microlocalized spectral measure:

Theorem 1.5. — *Let (X°, g) be as above. Then for low energies, $\boldsymbol{\lambda} \leqslant 1$, we have pointwise estimates on the spectral measure of the form*

$$\begin{aligned} &|\mathrm{d}E_P(\boldsymbol{\lambda})(z,z')| \\ &\qquad \leqslant \begin{cases} C\boldsymbol{\lambda}^2, & \text{for } d(z,z') \leqslant 1 \\ C\boldsymbol{\lambda}^2 d(z,z')(1+\boldsymbol{\lambda}d(z,z'))^{-1}e^{-nd(z,z')/2}, & \text{for } d(z,z') \geqslant 1. \end{cases} \end{aligned} \tag{1.17}$$

For high energies, $\boldsymbol{\lambda} \geqslant 1$, one has, for sufficiently small $\delta > 0$ and $\boldsymbol{\sigma} \in [(1-\delta)\boldsymbol{\lambda}, (1+\delta)\boldsymbol{\lambda}]$

$$\begin{aligned} &\left|Q_k(\boldsymbol{\sigma})\left(\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\right)^j \mathrm{d}E_P(\boldsymbol{\lambda})\right)Q_k^*(\boldsymbol{\sigma})(z,z')\right| \\ &\qquad\qquad \leqslant \begin{cases} C\boldsymbol{\lambda}^{n-j}(1+d(z,z')\boldsymbol{\lambda})^{-n/2+j}, & \text{for } d(z,z') \leqslant 1 \\ C\boldsymbol{\lambda}^{n/2}d(z,z')^j e^{-nd(z,z')/2}, & \text{for } d(z,z') \geqslant 1. \end{cases} \end{aligned} \tag{1.18}$$

As before, if (X°, g) is in addition simply connected with nonpositive sectional curvatures, then the estimates above are true for the spectral meaure without microlocalization, i.e. in this case we can take $\{Q_i(\boldsymbol{\lambda})\}$ to be the trivial partition of unity.

1.6.2. Restriction theorem

Using Theorem 1.5, we prove

Theorem 1.6. — *Suppose (X, g) is an $n+1$-dimensional non-trapping asymptotically hyperbolic manifold with no resonance at the bottom of the continuous spectrum. Then for some constant $C = C(p)$ we have the following estimate for $\boldsymbol{\lambda} \leqslant 1$:*

$$\|\mathrm{d}E_P(\boldsymbol{\lambda})\|_{L^p \to L^{p'}} \leqslant C\boldsymbol{\lambda}^2, \quad 1 \leqslant p < 2. \tag{1.19}$$

For $\boldsymbol{\lambda} \geqslant 1$, we have the estimate

$$\|\mathrm{d}E_P(\boldsymbol{\lambda})\|_{L^p \to L^{p'}} \leqslant \begin{cases} C\boldsymbol{\lambda}^{(n+1)(1/p-1/p')-1}, & 1 \leqslant p \leqslant \frac{2(n+2)}{n+4}, \\ C\boldsymbol{\lambda}^{n(1/p-1/2)}, & \frac{2(n+2)}{n+4} \leqslant p < 2. \end{cases} \tag{1.20}$$

Remark 1.7. — The range of exponents p is greater for a hyperbolic space than for a conic (Euclidean) space. Indeed, it includes all $p < 2$, while on Euclidean space $\mathbb{R}^d$, the well-known Knapp example shows that the restriction estimate cannot hold for $p > 2(d+1)/(d+3)$. (The Knapp example does not apply to hyperbolic space as it relies on the dilation symmetry of $\mathbb{R}^d$.) For high energies, $\boldsymbol{\lambda} \geqslant 1$, the exponent is the same as on $\mathbb{R}^d$ for the range $1 \leqslant p \leqslant 2(d+1)/(d+3)$ but again we get the full range of p up to $p = 2$. Naturally, the constant C blows up as $p \to 2$.

This surprising result is closely tied to a non-Euclidean feature of hyperbolic space related to the Kunze–Stein phenomenon [25]. The Kunze–Stein phenomenon for semisimple Lie groups is that there is a much larger set of exponents p, q, r for which one has

$$L^p * L^q \subset L^r,$$

compared to Euclidean space. Since $\mathbb{H}^{n+1}$ can be viewed as

$$SO(n+1,1)/SO(n+1),$$

this has consequences for convolution on $\mathbb{H}^{n+1}$. Anker and Pierfelice [1], [2, Section 4] showed that convolution with a radial kernel $\kappa(r)$ satisfies

$$\begin{aligned} &\|f * \kappa\|_{L^q(\mathbb{H}^{n+1})} \\ &\quad \leqslant C_q \|f\|_{L^{q'}(\mathbb{H}^{n+1})} \left(\int_0^\infty (\sinh r)^n (1+r) e^{-nr/2} |\kappa(r)|^{q/2} \,\mathrm{d}r \right)^{2/q}, \end{aligned} \tag{1.21}$$

with $q \geqslant 2$. From this we see that if $\kappa(r)$ is smooth and decays as $e^{-nr/2}$, then convolution with κ maps L^p to $L^{p'}$ for all $p \in [1, 2)$. Additionally, this non-Euclidean feature also affects the range of valid Strichartz estimates on (asymptotically) hyperbolic manifolds (see [1, 8, 22]).

1.6.3. Spectral multipliers

Our result for spectral multipliers is restricted to the case where the manifold is, in addition, a Cartan–Hadamard manifold, i.e. simply connected with nonpositive sectional curvatures.

Theorem 1.8. — *Suppose (X, g) is an $n+1$-dimensional non-trapping asymptotically hyperbolic manifold with no resonance at the bottom of spectrum. Suppose in addition that X is simply connected with nonpositive sectional curvatures. Then for any $F \in H^s(\mathbb{R})$ supported in $[-1, 1]$ with $s > (n+1)/2$, and for all $p \in [1, 2)$, $F(\alpha P)$ is a bounded operator on $L^p + L^2$ uniformly with respect to parameter α for $0 < \alpha < 1$, in the sense*

$$\sup_{\alpha \in (0,1]} \left\| F(\alpha P) \right\|_{L^p(X)+L^2(X) \longrightarrow L^p(X)+L^2(X)} < \infty .$$

This is weaker than Theorem 1.2, both because the function space is $L^p + L^2$ rather than L^p, but also because we have strengthened the Sobolev condition to $s > (n+1)/2$ for all p. From the perspective of harmonic analysis, it would be interesting to find a "better" function space, that is, more closely associated to the Laplacian, to accommodate the boundedness of the spectral multiplier. Modern harmonic analysis (Calderón–Zygmund theory) is generally built on spaces with a doubling measure, which activates some kind of covering lemma and gives a simple structure of cube nets. Though some authors have investigated non-doubling spaces, the advances are mainly restricted to spaces of polynomial growth, which are "semi-doubling". In any case, the harmonic analysis on space of exponential growth is barely explored. One recent work along these lines is due to Bouclet [3], where it is shown that semiclassical spectral multipliers are bounded on appropriate *weighted* L^p spaces in a setting with exponential volume growth. The authors plan to pursue this question in future publications.

1.7. Strichartz estimates on asymptotically hyperbolic manifolds

In the third paper in this series, [8], the first author will prove global-in-time Strichartz type estimates without loss on non-trapping asymptotically hyperbolic manifolds. Namely, for solutions of the inhomogeneous Schrödinger equation,

$$\begin{cases} \imath \frac{\partial}{\partial t} u + \Delta u = F(t, z) \\ u(0, z) = f(z) \end{cases}$$

with f and F orthogonal to eigenfunctions of Δ on an $n+1$-dimensional asymptotically hyperbolic manifold X, one has the estimate

$$\|u\|_{L^p(\mathbb{R},L^q(X))} \leqslant C\|f\|_{L^2(X)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(X))}$$

provided the pairs (q,r) and $(\tilde{q},\tilde{r})$ are hyperbolic Schrödinger admissible pairs of exponents.

1.8. Outline of the paper

The paper is organized as follows. In Section 2, we show how the main results in Section 1.6 follow in the simple case of hyperbolic 3-space $\mathbb{H}^3$. In Section 3, we review the geometry and analysis of asymptotically hyperbolic manifolds, recalling the main results of [27] and [10]. In Section 4 we prove the restriction estimate, Theorem 1.6, for low energy, which exploits, in some sense, the Kunze–Stein phenomenon on $\mathbb{H}^{n+1}$.

In Section 5, in preparation for the high-energy estimates, we show how the microlocal support of the spectral measure may be localized by pre- and post-composition by pseudodifferential operators. In Section 6 we prove Theorem 1.3. This uses, in a crucial way, the semiclassical Lagrangian structure of the high-energy spectral measure proved in [10] and [38]. Next we establish a factorization of spectral measure in Section 7. It is used in Section 8 for the proof of Theorem 1.6 at high energies. Finally, in Section 9, we prove the spectral multiplier result, Theorem 1.8.

2. The model space $\mathbb{H}^3$

In this section we illustrate the results of Theorems 1.5, 1.6 and 1.8 in the simple case of hyperbolic space. We focus on the case of $\mathbb{H}^3$, in which the formulae are particularly simple.

Hyperbolic space can be defined in terms of the half space model

$$\mathbb{H}^{n+1} = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid x > 0\},$$

equipped with the metric

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{x^2},$$

or in terms of the Poincaré disc model, as in (1.12). For odd dimensions, that is, when $n = 2k$ is even, the Schwartz kernel of $g(P)$ is given by the explicit formula

$$\frac{1}{\sqrt{2\pi}} \left(-\frac{1}{2\pi} \frac{1}{\sinh(r)} \frac{\partial}{\partial r} \right)^k \hat{g}(r), \tag{2.1}$$

where $P = (\Delta - n^2/4)^{1/2}$ as before, and r is geodesic distance on $\mathbb{H}^{n+1}$. See [36, p. 105] for proof.

2.1. Kernel bounds for the spectral measure

In particular, $(\Delta - n^2/4 - \boldsymbol{\lambda}^2)^{-1} = (P^2 - \boldsymbol{\lambda}^2)^{-1}$ for $\operatorname{Im}\boldsymbol{\lambda} > 0$ is

$$\begin{aligned} &-\frac{1}{2\imath\boldsymbol{\lambda}}\left(-\frac{1}{2\pi}\frac{1}{\sinh(r)}\frac{\partial}{\partial r}\right)^k e^{\imath\boldsymbol{\lambda}r}, \quad \operatorname{Im}\boldsymbol{\lambda} > 0,\\ &-\frac{1}{2\imath\boldsymbol{\lambda}}\left(-\frac{1}{2\pi}\frac{1}{\sinh(r)}\frac{\partial}{\partial r}\right)^k e^{-\imath\boldsymbol{\lambda}r}, \quad \operatorname{Im}\boldsymbol{\lambda} < 0. \end{aligned} \tag{2.2}$$

Setting now $k = 1$, and applying Stone's formula (1.3), we find that on $\mathbb{H}^3$,

$$\mathrm{d}E_P(\boldsymbol{\lambda}) = \frac{\boldsymbol{\lambda}}{2\pi}\frac{\sin(\boldsymbol{\lambda}r)}{\sinh r}. \tag{2.3}$$

2.2. Restriction estimate

Next, we deduce Theorem 1.6 for $\mathbb{H}^3$. The estimate for low energy follows immediately from (2.3) and (1.21). The estimate for high energy and $p \in [1, 4/3]$ can be deduced from Theorem 1.1:

Proposition 2.1. — *$\mathrm{d}E_P(\boldsymbol{\lambda})$ maps $L^p(\mathbb{H}^3)$ to $L^{p'}(\mathbb{H}^3)$ with a bound $C\boldsymbol{\lambda}^{3(1/p-1/p')-1}$ for all $\boldsymbol{\lambda} > 0$, provided $1 \leqslant p \leqslant 4/3$.*

Proof. — We assert the kernel estimates of Theorem 1.1 hold for this spectral measure, that is,

$$\left|\mathrm{d}E_P(\boldsymbol{\lambda})\right| \leqslant C\frac{\boldsymbol{\lambda}^2}{1+\boldsymbol{\lambda}d(z,z')} \quad \text{and} \quad \left|\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\right)^2 \mathrm{d}E_P(\boldsymbol{\lambda})\right| \leqslant C\big(1+\boldsymbol{\lambda}d(z,z')\big).$$

In fact, one may see

$$\left|\mathrm{d}E_P(\boldsymbol{\lambda})\right| = \left|\frac{\boldsymbol{\lambda}\sin\left(\boldsymbol{\lambda}d(z,z')\right)}{\sinh\left(d(z,z')\right)}\right| \leqslant C\frac{\boldsymbol{\lambda}}{d(z,z')} \leqslant C\frac{\boldsymbol{\lambda}^2}{1+\boldsymbol{\lambda}d(z,z')},$$

when $\boldsymbol{\lambda}d(z,z') > 1$;

$$\left|\mathrm{d}E_P(\boldsymbol{\lambda})\right| = \left|\frac{\boldsymbol{\lambda}\sin\left(\boldsymbol{\lambda}d(z,z')\right)}{\sinh\left(d(z,z')\right)}\right| \leqslant C\boldsymbol{\lambda}^2 \leqslant C\frac{\boldsymbol{\lambda}^2}{1+\boldsymbol{\lambda}d(z,z')},$$

when $\boldsymbol{\lambda} d(z,z') < 1$. On the other hand, it is clear that

$$\begin{aligned}\left|\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\right)^2 \mathrm{d}E_P(\boldsymbol{\lambda})\right| &= \left|\frac{2d(z,z')\cos\big(\boldsymbol{\lambda} d(z,z')\big)}{\sinh\big(d(z,z')\big)} - \frac{\boldsymbol{\lambda} d(z,z')^2\sin\big(\boldsymbol{\lambda} d(z,z')\big)}{\sinh\big(d(z,z')\big)}\right| \\ &\leqslant C\big(1+\boldsymbol{\lambda} d(z,z')\big).\end{aligned}$$

Then applying Theorem 1.1 proves the proposition. □

In the range $p \in [4/3, 2)$ and for high energy, we again use complex interpolation, but rather than applying Theorem 1.1 as a black box, we need to modify the proof slightly. We observe that the spectral measure on $\mathbb{H}^3$ satisfies

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\right)^j \mathrm{d}E_P(\boldsymbol{\lambda})\right| \leqslant \boldsymbol{\lambda} \quad \text{for all } j \geqslant 1. \tag{2.4}$$

We substitute this estimate in place of the kernel bounds of Theorem 1.1, and run the proof of [17, Section 3]. As in that proof, we consider the analytic family of operators $\chi_+^a(\boldsymbol{\lambda} - P)$. The proof works just the same;[(4)] in place of equation (3-7) of [17, Section 3] and the previous equation, we obtain

$$\left\|\chi_+^{\imath s}(\boldsymbol{\lambda} - P)\right\|_{L^2\to L^2} \leqslant Ce^{\pi|s|/2} \tag{2.5}$$

on the line $\operatorname{Re} a = 0$, and

$$\left\|\chi_+^{-b+\imath s}(\boldsymbol{\lambda} - P)\right\|_{L^1\to L^\infty} \leqslant C(1+|s|)e^{\pi|s|/2}\boldsymbol{\lambda} \tag{2.6}$$

on the line $\operatorname{Re} a = -b$, for any $b > 1$. Let $p \in (4/3, 2)$, and choose $b = p/(2-p)$. Using the fact that the spectral measure is $\chi_+^{-1}(\boldsymbol{\lambda} - P)$, and applying complex interpolation, we find that

$$\left\|\mathrm{d}E_P(\boldsymbol{\lambda})\right\|_{L^p\to L^{p'}} \leqslant C\boldsymbol{\lambda}^{(2-p)/p}. \tag{2.7}$$

2.3. Spectral multiplier estimate

The hyperbolic space $\mathbb{H}^3$ is a non-doubling space but rather has exponential volume growth, i.e. the volume of a ball with radius r satisfies $|B(r)| \sim (\sinh r)^2$. The lack of doubling means that we cannot apply Theorem 1.2 directly. Nevertheless, we can decompose the kernel of a spectral multiplier $F(P)$ into two parts, using a cutoff function $\chi_{d(z,z')\leqslant 1}$, say, the characteristic function of $\{(z,z') \in \mathbb{H}^3 \times \mathbb{H}^3 \mid d(z,z') \leqslant 1\}$. We write the

(4) We refer the reader to Section 8 for more details.

operator whose kernel is the kernel of $F(P)$ multiplied by $\chi_{d(z,z')\leqslant 1}$ by $F(P)\chi_{d(z,z')\leqslant 1}$.

Then the proof of Theorem 1.2 applies to $F(P)\chi_{d(z,z')\leqslant 1}$, since all that is required for this proof to work is that doubling is valid for all balls of radius $\leqslant 1$, which is certainly true. We obtain

Lemma 2.2. — *For every even function $F \in H^s(\mathbb{R})$ supported in $[-1,1]$ with $s > 3(1/p - 1/2)$, $F(\alpha P)\chi_{d(z,z')\leqslant 1}$ maps $L^p(\mathbb{H}^3)$ to itself with a uniform bound*

$$\sup_{\alpha>0} \|F(\alpha P)\chi_{d(z,z')\leqslant 1}\|_{L^p(\mathbb{H}^3)\to L^p(\mathbb{H}^3)} \leqslant C\|F\|_{H^s},$$

provided $1 \leqslant p \leqslant 4/3$, where $\chi_{d(z,z')\leqslant 1}$ is the characteristic function of the set $\{(z,z') : d(z,z') \leqslant 1\}$.

In particular, if $s > 3/2$, then this is valid for $p = 1$, and thus by interpolation and duality for all $p \in [1,\infty]$.

For the other part, supported where $d(z,z') \geqslant 1$, we show boundedness from $L^p(\mathbb{H}^3) \to L^2(\mathbb{H}^3)$. By interpolation, it is enough to treat the case $p = 1$, since boundedness $L^2 \to L^2$ follows immediately from the boundedness of F.

The $L^1 \to L^2$ operator norm of an integral operator $K(z_1,z_2)$ is bounded by

$$\sup_{z_2} \left(\int \left|K(z_1,z_2)\right|^2 \mathrm{d}\mu_1 \right)^{1/2}.$$

We express the kernel of $F(P)\chi_{d(z,z')>1}$ using (2.2). So we need to estimate

$$\int_{\mathbb{S}^2\times[1,\infty)} \left| \frac{1}{(2\pi)^{3/2}} \frac{1}{\sinh(r)} \frac{\partial}{\partial r}\hat{F}(r) \right|^2 \sinh^2(r)\,\mathrm{d}r\mathrm{d}\omega \leqslant C \int_1^\infty \left| \frac{\partial}{\partial r}\hat{F}(r)\right|^2 \mathrm{d}r\,.$$

Write $F_\alpha(\boldsymbol{\lambda}) = F(\alpha\boldsymbol{\lambda})$. For any $\alpha > 0$, we get the estimate for F_α:

$$\begin{aligned}\int_{\mathbb{S}^2\times[1,\infty)} \left| \frac{\partial}{\partial r}\hat{F}_\alpha(r)\right|^2 \mathrm{d}r\mathrm{d}\omega &= C\int_1^\infty \left| \frac{\partial}{\partial r}\frac{\hat{F}(r/\alpha)}{\alpha}\right|^2 \mathrm{d}r \\ &\leqslant C\frac{1}{\alpha^3}\int_{1/\alpha}^\infty \left|\frac{\partial}{\partial r}\hat{F}(r)\right|^2 \mathrm{d}r \\ &\leqslant C\int_{1/\alpha}^\infty r^3 \left|\frac{\partial}{\partial r}\hat{F}(r)\right|^2 \mathrm{d}r \\ &\leqslant C\left\|\boldsymbol{\lambda}F(\boldsymbol{\lambda})\right\|^2_{H^{3/2}} \leqslant C\left\|F\right\|^2_{H^{3/2}}\end{aligned}$$

using the compact support of F. Combining this estimate with Lemma 2.2, we have proved Theorem 1.8 in the case of $\mathbb{H}^3$.

3. The geometry and analysis of asymptotically hyperbolic manifolds

3.1. 0-structure

Suppose (X°, g) is an $(n+1)$-dimensional asymptotically hyperbolic manifold. Let X be the compactification. We write x for a boundary defining function, and use local coordinates $(x, y_1, \dots, y_n)$ near a boundary point of X, where $y = (y_1, \dots, y_n)$ restrict to coordinates on ∂X, or $z = (z_1, \dots, z_{n+1})$ in the interior of X.

Consider the space of smooth vector fields on the compactification, X, that are of uniformly finite length. Due to the factor x^{-2} in the metric, such vector fields take the form xV, where V is a smooth vector field on X. Such vector fields are called 0-vector fields, spanned over $C^\infty(X)$ near the boundary by the vector fields $x\partial_x$ and $x\partial_{y_i}$, $1 \leqslant i \leqslant n$. As observed by Mazzeo–Melrose, they are the space of sections of a vector bundle, known as the 0-tangent bundle, ${}^0TX$.

The dual bundle, known as the 0-cotangent bundle and denoted ${}^0T^*X$, is spanned by local sections $\mathrm{d}x/x$ and $\mathrm{d}y_i/x$ near the boundary. It follows that, near the boundary of X, we can write points $q \in {}^0T^*X$ in the form

$$q = \lambda \frac{\mathrm{d}x}{x} + \sum_{j=1}^{n} \mu_j \frac{\mathrm{d}y_j}{x}; \tag{3.1}$$

this defines linear coordinates (λ, μ) on each fibre of ${}^0T^*X$ (near the boundary), depending on the coordinate system (x, y).

The Laplacian Δ on X is built out of an elliptic combination of 0-vector fields. In fact, in local coordinates (x, y) near the boundary of X, with g taking the form (1.11), it takes the form

$$(xD_x)^2 + \imath n x D_x + (xD_{y_i})h^{ij}(xD_{y_j}) \text{ modulo } x\ {}^0\mathrm{Diff}^1(X),$$

where we use ${}^0\mathrm{Diff}^k(X)$ to denote differential operators of order k generated over $C^\infty(X)$ by 0-vector fields.

3.2. The 0-double space

We would like to understand the nature of the Schwartz kernel of the resolvent $(\Delta - \zeta(n-\zeta))^{-1}$, on $X^\circ \times X^\circ$. Following Mazzeo–Melrose, we use a compactification of the double space $X^\circ \times X^\circ$ that reflects the geometry of (X°, g), particularly near the diagonal. This is important as we want

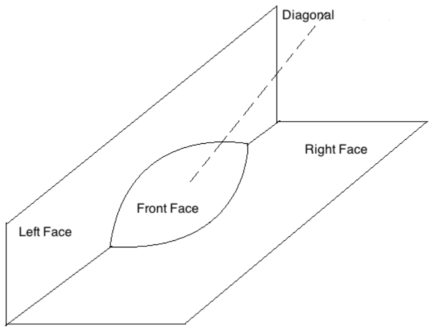

Figure 3.1. The 0*-blown-up double space* $X \times_0 X$

to view the resolvent as some sort of pseudodifferential operator, which means that we need a precise notion of what it means for a distribution to be conormal to the diagonal, uniformly out to infinity.

Compactifying X° to X, we can initially view the resolvent kernel on X^2. However, on this space, the diagonal is not a p-submanifold where it meets the boundary. That is, near the boundary of the diagonal in X^2, there are no local coordinates of the form (x, x', w) where x, resp. x' is a boundary defining function for the left, resp. right, copy of X and w are the remaining coordinates, such that the diagonal is given by the vanishing of a subset of these coordinates. To give a workable definition of conormality to a submanifold, we require it to be a p-submanifold. To remedy this, we blow up (in the real sense) the boundary of the diagonal. This creates a manifold with corners, denoted X_0^2, the "0-double space", with three boundary hypersurfaces: the two original ones, FL "left face" and FR "right face", corresponding to $\{x = 0\}$ and $\{x' = 0\}$ in X^2, and the new face FF, the "front face", created by blowup (see Figure 3.1). We denote a generic boundary defining function for FL, FR or FF by $\rho_{\mathrm{FL}}, \rho_{\mathrm{FR}}$ and ρ_{FF}, respectively.

As in [10], we write down coordinate systems in various regions of X_0^2, in terms of coordinates $(x, y) = (x, y_1, \ldots, y_n)$ near the boundary of X, or $z = (z_1, \ldots, z_{n+1})$ in the interior of X. The unprimed coordinates always indicate those lifted from the left factor of X, while primed coordinates indicate those lifted from the right factor. We label these different regions as follows:

- Region 1: In the interior of X_0^2. Here we use coordinates
$$(z, z') = (z_1, \ldots, z_{n+1}, z'_1, \ldots, z'_{n+1}).$$

- Region 2a: Near FL and away from FF and FR. In this region, we use (x, y, z').
- Region 2b: Near FR and away from FF and FL. Symmetrically, we use (z, x', y').
- Region 3: Near FL $\cap$ FR and away from FF. Here we use (x, y, x', y').
- Region 4a: Near FF and away from FR. This is near the blowup. In this region we can use $s = x/x'$ for a boundary defining function for FF. We use coordinate system
$$s = \frac{x}{x'}\,,\ x',\ y,\ Y = \frac{y'-y}{x'}\,.$$
- Region 4b: Near FF and away from FL. Symmetrically, we use
$$s' = \frac{x'}{x}\,,\ x,\ y',\ Y' = \frac{y-y'}{x}\,.$$
- Region 5: Near the triple corner FL $\cap$ FF $\cap$ FR. In this case, a boundary defining function for FF is $|y'-y|$. By rotating the y coordinates, we can assume that $|y_1' - y_1| \geqslant c|y'-y|$ in a neighbourhood of any given point in the triple corner. Assuming this, we use coordinates
$$s_1 = \frac{x}{y_1'-y_1}\,,\ s_2 = \frac{x'}{y_1'-y_1}\,,\ t = y_1' - y_1\,,\ Z_j = \frac{y_j'-y_j}{y_1'-y_1}\,(j > 1)\,.$$

On X_0^2, the lift of the diagonal, denoted diag_0, meets the boundary in the interior of the front face FF. It has several good geometric properties:

- $\mathrm{diag}_0 \subset X_0^2$ is a p-submanifold disjoint from FL and FR;
- the 0-vector fields $x\partial_x$, $x\partial_{y_i}$ lift from the left and right factors of X to be vector fields on X_0^2 that are non-tangential to diag_0, uniformly down to the boundary of diag_0. Moreover, these vector fields span the normal bundle of diag_0, again uniformly down to the boundary.
- The distance function $d(z, z')$ is smooth in a deleted neighbourhood of diag_0, and its square is a quadratic defining function for the lifted diagonal, i.e. it is smooth and vanishes to precisely second order at diag_0.

3.3. Resolvent kernel

Taking advantage of the first and second geometric properties listed above, Mazzeo and Melrose "microlocalized" the space of 0-differential operators to a calculus of 0-pseudodifferential operators on X. The set of pseudodifferential operators of order m on X, denoted $\Psi_0^m(X)$, is, by definition, the set of operators on half-densities, whose Schwartz kernels are conormal of order m to diag_0, and vanish to infinite order at FL and FR.

Mazzeo and Melrose [27] showed that the resolvent

$$R(\boldsymbol{\lambda}) = \left(\Delta - n^2/4 - \boldsymbol{\lambda}^2\right)^{-1}, \quad \operatorname{Im}\boldsymbol{\lambda} < 0,$$

takes the form

$$R(\boldsymbol{\lambda}) \in \Psi_0^{-2}(X) + \rho_L^{n/2+\imath\boldsymbol{\lambda}} \rho_R^{n/2+\imath\boldsymbol{\lambda}} C^\infty(X \times_0 X). \tag{3.2}$$

For low energy, this description is precise enough to deduce kernel estimates for the spectral measure, restriction estimates, and spectral multiplier theorems. However, as $\boldsymbol{\lambda} \to \infty$, we need a uniform description of the resolvent, and in particular we need to understand its oscillatory nature. For this, we use the description by Melrose–Sá Barreto–Vasy [29] [38] and the present authors [10] (in the first paper of this series) of the high-energy resolvent as a semiclassical Lagrangian distribution. This is associated to the bicharacteristic relation on $X^\circ \times X^\circ$, that is, the submanifold of $T^*X^\circ \times T^*X^\circ$ given by

$$\mathrm{BR} = \left\{ (z, \zeta; z', -\zeta') \;\middle|\; \begin{array}{l} |\zeta|_g = |\zeta'|_g = 1, \\ (z,\zeta) \text{ and } (z',\zeta') \text{ lie on the same bicharacteristic} \end{array} \right\},$$

which is a smooth Lagrangian submanifold provided that X is nontrapping. By "bicharacteristic" we mean here the integral curves of the symbol of Δ on the set where $\sigma(\Delta) = 1$. In this case these are precisely geodesics, viewed as living in the cotangent bundle.

The bicharacteristic relation splits into the forward and backward bicharacteristic relations, BR_+ and BR_-, which[5] consist of those points

$$(z, \zeta; z', -\zeta') \in \mathrm{BR}$$

for which (z, ζ) is on the forward/backward half of the bicharacteristic relative to (z', ζ'). These two halves meet at $\mathrm{BR} \cap N^*\mathrm{diag}$, where $N^*\mathrm{diag}$ denotes the conormal bundle of the diagonal,

$$N^*\mathrm{diag} = \{(z, \zeta, z', -\zeta)\}.$$

We wish to understand the way in which BR compactifies when viewed as living over the double space X_0^2. We consider the bundle ${}^\Phi T^* X_0^2$, obtained by pulling back the bundle $({}^0T^*X)^2$ to X_0^2 by the blowdown map $\beta : X_0^2 \to X^2$. We denote the bundle projection maps by ${}^\Phi\pi : {}^\Phi T^* X_0^2 \to X_0^2$. Then,

[5] The forward bicharacteristic relation BR_+ was denoted FBR in [10].

as explained in [10, Section 3], it is convenient to "shift" BR by the map $T_\pm$ defined by

$$T_\pm(q) = q \mp d(\log \rho_L) \mp d(\log \rho_R), \quad q \in {}^\Phi T^* X_0^2, \tag{3.3}$$

for some choice of boundary defining functions ρ_L for FL and ρ_R for FR; that is, we consider $T_-^{-1}(\mathrm{BR}_-) \cup T_+^{-1}(\mathrm{BR}_+)$. It is convenient here to assume that ρ_L and ρ_R are both constant near diag_0, so that these two shifted Lagrangians join smoothly at $N^* \mathrm{diag}_0$.

In [10] we showed[(6)]

Proposition 3.1. — *The bicharacteristic relation* BR *can be expressed as the union of two relatively open subsets* $\mathrm{BR}^{nd} \cup \mathrm{BR}^*$, *having the following properties.*

- BR^{nd} *contains a neighbourhood of the intersection* $\mathrm{BR} \cap N^* \mathrm{diag}$ *in* BR, *that is, the points* $(z, \zeta, z, -\zeta) \in \mathrm{BR}$.
- *Let* Λ^{nd} *denote the lift of* BR^{nd} *to* ${}^\Phi T^* X_0^2$, *together with its limit points lying over* FF, FL *and* FR. *Let* $\Lambda^{nd}_\pm = \Lambda^{nd} \cap \mathrm{BR}_\pm$ *denote the two halves of this submanifold, meeting at* $N^* \mathrm{diag}_0$. *Then* $\Lambda^{nd}_\pm$ *are manifolds with codimension three corners, with the property that the interior of* Λ^{nd}_+ *is the graph of the differential of the distance function on some deleted neighbourhood* V *of* $(\mathrm{diag}_0 \cup \mathrm{FF}) \subset X_0^2$, *and the interior of* Λ^{nd}_- *is the graph of minus the differential of the distance function on* V. *Thus the projection* ${}^\Phi\pi : \Lambda^{nd} \to X_0^2$ *has full rank restricted to* Λ^{nd}, *except at* $\Lambda^{nd} \cap N^* \mathrm{diag}_0 = \Lambda^{nd}_+ \cap \Lambda^{nd}_-$, *where the rank of the projection* ${}^\Phi\pi : \Lambda^{nd} \to X_0^2$ *drops by* n. *The boundary hypersurfaces of* Λ^{nd} *are* $\partial_{\mathrm{FF}}\Lambda^{nd}$, *lying over* FF, $\partial_{\mathrm{FL}}\Lambda^{nd}$, *lying over* FL *and* $\partial_{\mathrm{FR}}\Lambda^{nd}$, *lying over* FR.
- *The image* $\tilde\Lambda^{nd}$ *of* Λ^{nd} *under the shift* (3.3) *is a smooth Lagrangian submanifold of* $T^* X_0^2$ *(NB: the standard cotangent bundle, not* ${}^\Phi T^* X_0^2$*) with codimension three corners. The projection*

$$\pi : T^* X_0^2 \to X_0^2$$

restricts to a map $\tilde\Lambda^{nd} \to X_0^2$ *with full rank, except at* $\tilde\Lambda^{nd} \cap N^* \mathrm{diag}_0 = \tilde\Lambda^{nd}_+ \cap \tilde\Lambda^{nd}_-$, *where the rank of the projection* ${}^\Phi\pi : \tilde\Lambda^{nd} \to X_0^2$ *drops by* n.
- *Let* $\widetilde{\mathrm{BR}}^*$ *denote the image of* BR^* *under the shift* (3.3), *and let* $\tilde\Lambda^*$ *denote the closure of* $\widetilde{\mathrm{BR}}^*$ *in* $T^* X^2$. *Then* $\tilde\Lambda^*$ *is a smooth Lagrangian*

(6) This was shown for the forward bicharacteristic relation in [10], but the statements in Proposition 3.1 follow immediately.

submanifold of T^*X^2 *(NB: the standard cotangent bundle, not the 0-cotangent bundle) with codimension two corners.*

In terms of these Lagrangian submanifolds we determined the semiclassical nature of the resolvent kernel in [10, Theorem 38]. In view of Stone's formula, (1.2), this has immediate consequences for the spectral measure. We first define (semiclassical) Lagrangian distributions associated to Λ^{nd} and Λ^*, respectively, following [10]. These are given in terms of Lagrangian distributions associated to $\tilde{\Lambda}^{nd}$ and $\tilde{\Lambda}^*$:

DEFINITION 3.2. — *We define the space* $I^k(X_0^2, \Lambda^{nd}; {}^0\Omega^{1/2})$ *as follows: we have* $u \in I^k(X_0^2, \Lambda^{nd}; {}^0\Omega^{1/2})$ *if and only if* $u = u_0 + u_- + u_+$*, where*

- u_0 *is in* $\rho_F^{-(n+1)/2} I^k(X_0^2, \tilde{\Lambda}^{nd}; \Omega^{1/2})$ *and microsupported close to* $N^* \operatorname{diag}_0$*, where "close" means that the shift operation* T *is the identity on the microsupport of* u_0 *(due to our assumption that* $\rho_L = \rho_R = 1$ *in a neighbourhood of* $\partial \operatorname{diag}_0$*);*
- $u_\pm$ *are in* $\rho_F^{-(n+1)/2} (\rho_L \rho_R)^{-(n+1)/2 \mp \imath/h} I^k(X_0^2, \tilde{\Lambda}^{nd}_\pm; \Omega^{1/2})$*, and are microsupported away from* $N^* \operatorname{diag}_0$*.*

We also define $I^k(X^2, \Lambda^*; {}^0\Omega^{1/2})$ *as follows: we say* $u \in I^k(X^2, \Lambda^*; {}^0\Omega^{1/2})$ *if and only if* $u = u_+ + u_-$*, where* $u_\pm \in (xx')^{-(n+1)/2 \mp \imath/h} I^k(X^2, \tilde{\Lambda}^*_\pm; \Omega^{1/2})$*. Here* $\tilde{\Lambda}^*_\pm$ *are the connected components of* $\tilde{\Lambda}^*$*, arising from points in* $\mathrm{BR}_\pm$*.*

In terms of these spaces, we have

THEOREM 3.3. — *Let* (X°, g) *be an asymptotically hyperbolic nontrapping manifold, with no resonance at the bottom of the continuous spectrum. Then the spectral measure* $\mathrm{d}E_P(\boldsymbol{\lambda})$ *with* $\boldsymbol{\lambda} = 1/h$ *can be expressed as a sum of the following terms:*

(1) *A semiclassical Lagrangian distribution in*

$$(\rho_L \rho_R)^{n/2} I^{-1/2}(X_0^2, \Lambda^{nd}; {}^0\Omega^{1/2});$$

(2) *a semiclassical Lagrangian distribution in*

$$(xx')^{n/2} I^{-1/2}(X^2, \Lambda^*; {}^0\Omega^{1/2});$$

(3) *an element of*

$$(\rho_L \rho_R)^{n/2 - \imath/h} h^\infty C^\infty(X_0^2 \times [0, h_0]; {}^0\Omega^{1/2}) \\ + (\rho_L \rho_R)^{n/2 + \imath/h} h^\infty C^\infty(X_0^2 \times [0, h_0]; {}^0\Omega^{1/2}),$$

which can be regarded as an element of type (1) *of order* $-\infty$*;*

(4) *an element of*

$$(xx')^{n/2-\imath/h}h^\infty C^\infty(X^2\times[0,h_0];{}^0\Omega^{1/2}) \\ +(xx')^{n/2+\imath/h}h^\infty C^\infty(X^2\times[0,h_0];{}^0\Omega^{1/2}),$$

which can be regarded as an element of type (2) *of order* $-\infty$.

Proof. — We first remark that the change in order from $+1/2$ for the resolvent in [10, Theorem 38] to $-1/2$ for the spectral measure is simply due to the fact that the semiclassical resolvent in [10] is h^{-2} times the resolvent in (1.3), together with the factor of $\boldsymbol{\lambda}=h^{-1}$ in (1.3).

In [10, Theorem 38] it was shown that the resolvent kernel has a similar, but slightly more complicated structure: in place of the first term above, it consists of a semiclassical pseudodifferential operator, together with a semiclassical intersecting Lagrangian distribution associated to $N^*\operatorname{diag}_0$ together with the forward/backward half of the bicharacteristic relation (for the outgoing/incoming resolvent). We claim that when the incoming resolvent is subtracted from the outgoing, the pseudodifferential part cancels, and what is left is a Lagrangian distribution associated to the full bicharacteristic relation. This follows since the spectral measure satisfies an elliptic equation

$$(h^2\Delta-h^2n^2/4-1)\mathrm{d}E_P(\boldsymbol{\lambda})=0,\quad h=\boldsymbol{\lambda}^{-1}.$$

Therefore, the spectral measure can have no semiclassical wavefront set outside the zero set of the symbol of $h^2\Delta-1$. This excludes all of $N^*\operatorname{diag}_0$ except for that part contained in BR. In addition, propagation of Lagrangian regularity[(7)] shows that the spectral measure is a Lagrangian distribution across $N^*\operatorname{diag}_0$ (given that we already know that it is Lagrangian on both sides of $N^*\operatorname{diag}_0$ corresponding to forward and backward flowout, and given that the Hamilton vector field of the symbol does not vanish at $\mathrm{BR}\cap N^*\operatorname{diag}_0$). This concludes the proof. □

3.4. The distance function on X_0^2

The distance function on X_0^2 satisfies

(7) Propagation of Lagrangian regularity is the statement that, if P is an operator of real principal type, $Pu=O(h^\infty)$, and u is a Lagrangian distribution microlocally in some region V of phase space, then u is also Lagrangian along the bicharacteristics passing through V. It follows from the parametrix construction for Lagrangian solutions of operators of real principal type, and the propagation of singularities theorem.

Proposition 3.4. — *On X_0^2, the Riemannian distance function $d(z,z')$ is given by*

$$d(z,z') = -\log(\rho_L\rho_R) + b(z,z')\,,$$

where $b(z,z')$ is uniformly bounded on X_0^2.

Remark 3.5. — The result in the case that (X°, g) is a small perturbation of $(\mathbb{H}^{n+1}, g_{hyp})$ was shown by Melrose, Sá Barreto and Vasy [29, Section 2].

Proof. — Consider two points $p, p' \in X^\circ$. When (p,p') are in a sufficiently small neighbourhood U of the front face FF, say $p = (x,y)$, $p' = (x',y')$ with $x, x' < \epsilon$ and $d(y,y') < 4\epsilon$ (taken with respect to the metric $h(0)$ at the boundary), then the distance function parametrizes the Lagrangian Λ^{nd}, and it follows from [10, Proposition 20] that this takes the form $-\log(\rho_L\rho_R) + C^\infty(X_0^2)$ in a neighbourhood of FF.

Define $K \subset X^\circ$ to be the compact set $\{x \geqslant \epsilon\}$. Let M be the diameter of K, that is, the maximum distance between two points of K.

Now suppose that $(p,p') \notin U$. In the complement of U, we can take $\rho_L = x$ and $\rho_R = x'$.

If both p and p' lie in K, then the distance between p and p' is at most M, hence $|d(p,p') + \log(\rho_L\rho_R)| \leqslant M + 2\max_K |\log x| = O(1)$.

If one point, say p, lies in K and p' is not in K, then a lower bound on $d(p,p')$ is the distance from p' to the boundary of K, which is exactly $\log\epsilon - \log x = -\log(\rho_L\rho_R) + O(1)$. On the other hand, an upper bound is the length of the path from p' to the closest point p'' on ∂K, plus the distance from p'' to p. This is at most $\log\epsilon - \log x + M = -\log(\rho_L\rho_R) + O(1)$.

If neither point lies in K, then write $p = (x,y)$ and $p' = (x',y')$. Due to the definition of U, we must have $d(y,y') \geqslant 4\epsilon$. We claim that any geodesic between p and p' must enter K. It follows from this claim that a lower bound on the distance between p and p' is the distance from p to ∂K plus the distance between p' to ∂K, which is $-\log x - \log x' + 2\log\epsilon$, that is, $-\log(\rho_L\rho_R) + O(1)$. Also, an upper bound on the distance is clearly $-\log x - \log x' + 2\log\epsilon + M$ which is also $-\log(\rho_L\rho_R) + O(1)$. Thus, to complete the proof, it remains to establish the claim above.

Consider any geodesic that lies wholly within the region $x \leqslant \epsilon$. Parametrize the geodesic with arc length, such that the value of x is maximal at $t = 0$, say, equal to $x_{max} \leqslant \epsilon$. We recall the geodesic equations for (x, y, λ, μ) where these are the 0-cotangent variables as described in [10, Section 2]:

$$
\begin{cases}
\dot{x} = x\lambda \\
\dot{y_i} = xh^{ij}\mu_j \\
\dot{\lambda} = -\left(h^{ij} + \frac{1}{2}x\partial_x h^{ij}\right)\mu_i\mu_j \\
\dot{\mu_i} = \left(\lambda\mu_i - \frac{1}{2}(x\partial_{y_i}h^{jk})\mu_j\mu_k\right).
\end{cases} \tag{3.4}
$$

We also recall that $\lambda^2 + |\mu|^2 = 1$ along the geodesic, where

$$|\mu|^2 = h^{ij}(x,y)\mu_i\mu_j\,.$$

We see that

$$\dot{\lambda} = -|\mu|^2(1+O(x)) = -(1-\lambda^2)(1+O(x))\,.$$

Thus, we have

$$\dot{\lambda} \leqslant -\alpha(1-\lambda^2)\,, \quad \lambda(0) = 0\,. \tag{3.5}$$

for some $\alpha \sim 1 + O(\epsilon)$ slightly less than 1, which can be taken as close as desired to 1 by choosing ϵ sufficiently small. The initial condition $\lambda(0) = 0$ arises as $\dot{x} = 0$ at $t = 0$.

We can integrate the differential inequality (3.5) to obtain

$$\frac{1}{2}\int\left(\frac{1}{1+\lambda} + \frac{1}{1-\lambda}\right)\mathrm{d}\lambda \leqslant -\alpha\int \mathrm{d}t\,,$$

which yields

$$\lambda(t) \leqslant -\frac{1-e^{-2\alpha t}}{1+e^{-2\alpha t}}\,.$$

Plugging this into the equation for x, we find that

$$\dot{x} \leqslant -x\frac{1-e^{-2\alpha t}}{1+e^{-2\alpha t}}\,.$$

Integrating this, we find that

$$\log x \leqslant -\int \frac{1-e^{-2\alpha t}}{1+e^{2\alpha t}}e^{2\alpha t}\,\mathrm{d}t\,,$$

and with the help of the substitution $v = e^{2\alpha t}$, we obtain

$$x \leqslant x_{max}\left(\frac{2e^{\alpha t}}{1+e^{2\alpha t}}\right)^{1/\alpha}\,.$$

Finally we turn to the equation for y. We have

$$|\dot{y}| = x|\mu| = x\sqrt{1-\lambda^2} \leqslant x_{max}\left(\frac{2e^{\alpha t}}{1+e^{2\alpha t}}\right)^{1+1/\alpha}\,.$$

Integrating the RHS from 0 to ∞ at the value $\alpha = 1$ gives x_{max}. We get the same result for negative time, so that means that, along this geodesic, the maximum distance that y can travel, with respect to the $h(0)$ metric, is

$$2x_{max} \int_0^\infty \left(\frac{2e^{\alpha t}}{1 + e^{2\alpha t}} \right)^{1+1/\alpha} \mathrm{d}t\,.$$

This is equal to $2x_{max}$ when $\alpha = 1$, and depends continuously on α, hence is close to $2x_{max}$ for α close to 1, that is, when ϵ is sufficiently small[(8)]. It follows that if $d(y, y') \geqslant 4\epsilon$, the geodesic between p and p' must enter the region $\{x \geqslant \epsilon\}$ (provided ϵ is sufficiently small). This completes the proof of the proposition. □

4. Low energy behaviour of the spectral measure

Pointwise bounds on the spectral measure, and restriction estimates, are readily deduced from the regularity statement (3.2) for the low energy resolvent.

4.1. Pointwise bounds on the spectral measure

The regularity statement (3.2) for the resolvent, together with Stones's formula (1.3), implies that the Schwartz kernel of the low energy spectral measure $\mathrm{d}E_P(\boldsymbol{\lambda})$ takes the form

$$\boldsymbol{\lambda}\Big((\rho_L\rho_R)^{n/2+\imath\boldsymbol{\lambda}} a(\boldsymbol{\lambda}) - (\rho_L\rho_R)^{n/2-\imath\boldsymbol{\lambda}} a(-\boldsymbol{\lambda}) \Big)\,, \tag{4.1}$$

where $a(\boldsymbol{\lambda})$ is a C^∞ function on X_0^2 depending holomorphically on $\boldsymbol{\lambda}$ for small $\boldsymbol{\lambda}$. Here we use our assumption that the resolvent is holomorphic in a neighbourhood of $n^2/4$, the bottom of the essential spectrum; on the other hand, the nontrapping assumption is irrelevant here.

We write the RHS as

$$\begin{aligned} \boldsymbol{\lambda}\Big(&(\rho_L\rho_R)^{n/2+\imath\boldsymbol{\lambda}} - (\rho_L\rho_R)^{n/2-\imath\boldsymbol{\lambda}} \Big) a(0) \\ &+ \boldsymbol{\lambda}\Big((\rho_L\rho_R)^{n/2+\imath\boldsymbol{\lambda}} \big(a(\boldsymbol{\lambda}) - a(0)\big) - (\rho_L\rho_R)^{n/2-\imath\boldsymbol{\lambda}} \big(a(-\boldsymbol{\lambda}) - a(0)\big) \Big)\,, \end{aligned} \tag{4.2}$$

(8) As a check, we note that for the hyperbolic metric on the upper half space, where the geodesics are great circles on planes perpendicular to the boundary and centred on the boundary, the maximum distance is indeed $2x_{max}$.

which implies that the kernel is bounded pointwise by

$$C\boldsymbol{\lambda}(\rho_L\rho_R)^{n/2}\big|\sin(\boldsymbol{\lambda}\log(\rho_L\rho_R))\big| + C'\boldsymbol{\lambda}^2(\rho_L\rho_R)^{n/2}.$$

Using Proposition 3.4 we may write $|\log(\rho_L\rho_R)| = d(z,z') + O(1)$. Then estimating the sine factor by $|\sin s| \leqslant |s|(1+|s|)^{-1}$, we obtain the low energy ($\boldsymbol{\lambda} \leqslant 1$) estimate in Theorem 1.5.

4.2. Restriction estimate

We have just seen that the spectral measure for low energy, $\boldsymbol{\lambda} \leqslant 1$, is bounded pointwise by $\boldsymbol{\lambda}^2$ times $-\log(\rho_L\rho_R)(\rho_L\rho_R)^{n/2}$. Thus, to prove the low energy restriction estimate, it suffices to show that an integral operator, say $A(z,z')$, with kernel bounded pointwise by $-\log(\rho_L\rho_R)(\rho_L\rho_R)^{n/2}$ maps $L^p(X)$ to $L^{p'}(X)$ for all $p \in [1,2)$.

To do this, we break up the kernel $A(z,z')$ into pieces. Let U be a neighbourhood of the front face FF in X_0^2. We consider $A(z,z')1_U$ and $A(z,z')1_{X_0^2\setminus U}$ separately.

First consider $A(z,z')1_{X_0^2\setminus U}$. In this region, we may take $\rho_L = x$ and $\rho_R = x'$. This part of the kernel is therefore bounded by

$$C(-\log x)x^{n/2}(-\log x')x'^{n/2}.$$

Thus, it is easy to check that $A(z,z')1_{X_0^2\setminus U}$ is in $L^{p'}(X\times X)$, for any $p' > 2$. It therefore maps $L^p(X)$ to $L^{p'}(X)$ for all $p \in [1,2)$.

Now consider the remainder of the kernel, $A(z,z')1_U$. We may further decompose the set U into subsets U_i, where on each U_i, we have $x \leqslant \epsilon, x' \leqslant \epsilon$ and $d(y,y_i), d(y',y_i) \leqslant \epsilon$ for some $y_i \in \partial X$ (where the distance is measured with respect to the metric $h(0)$ on ∂X). Choose local coordinates (x,y) on X, centred at $(0,y_i) \in \partial X$, covering the set $V_i = \{x \leqslant \epsilon, d(y,y_i) \leqslant \epsilon\}$, and use these local coordinates to define a map ϕ_i from V_i to a neighbourhood V_i' of $(0,0)$ in hyperbolic space $\mathbb{H}^{n+1}$ using the upper half-space model (such that the map is the identity in the given coordinates).

The map ϕ_i induces a diffeomorphism Φ_i from $U_i \subset X_0^2$ to a subset of $(B^{n+1})^2_0$, the double space for $\mathbb{H}^{n+1}$, covering the set $x \leqslant \epsilon$, $x' \leqslant \epsilon$, $|y|, |y'| \leqslant \epsilon$ in this space. Clearly, this map identifies ρ_L and ρ_R on U_i with corresponding boundary defining functions for the left face and right face on $(B^{n+1})^2_0$. We now consider the kernel

$$\phi_i \circ A1_{U_i} \circ \phi_i^{-1} \tag{4.3}$$

as an integral operator on $(B^{n+1})^2_0$. This kernel is bounded by $(1+r)e^{-nr/2}$, where r is the geodesic distance on $\mathbb{H}^{n+1}$, since $(1+r)$ is comparable to

$-\log(\rho_L\rho_R)$ on $(B^{n+1})^2_0$. Therefore, using (1.21), (4.3) is bounded from $L^p(\mathbb{H}^{n+1})$ to $L^{p'}(\mathbb{H}^{n+1})$ for every $p \in [1,2)$. It is clear that ϕ_i are bounded, invertible maps from $L^p(V_i)$ to $L^p(V_i')$. This shows that the kernel $A1_{U_i}$ is bounded from $L^p(V_i)$ to $L^{p'}(V_i)$ for all $p \in [1,2)$. This completes the proof of Theorem 1.6 in the case of low energy, $\boldsymbol{\lambda} \leqslant 1$.

5. Pseudodifferential operator microlocalization

According to Theorem 3.3, the spectral measure is a Lagrangian distribution associated to the Lagrangian submanifold Λ^{nd} (on ${}^{\Phi}T^*X_0^2$) and to the Lagrangian submanifold Λ^* (on ${}^0T^*X^2$). We next define the notion of microlocal support, which is a closed subset of ${}^{\Phi}T^*X_0^2$ giving the essential support "in phase space", for such distributions. It is a special case of the notion of semiclassical wavefront set, defined for example in [39, Section 8.4].

Definition 5.1. — *Let $u \in I^k(X_0^2, \Lambda^{nd}; {}^0\Omega^{1/2})$. We decompose $u = u_0 + u_+ + u_-$ as in Definition 3.2. We define the semiclassical wavefront set, or microlocal support,* $\mathrm{WF}_h(u)$, *of u by*

$$\mathrm{WF}_h(u_0) \cup T_+\Big(\mathrm{WF}_h((\rho_L\rho_R)^{\imath/h}u_+)\Big) \cup T_-\Big(\mathrm{WF}_h((\rho_L\rho_R)^{-\imath/h}u_-)\Big). \tag{5.1}$$

Similarly, for $u = u_+ + u_- \in I^k(X^2, \Lambda^*; {}^0\Omega^{1/2})$ as in Definition 3.2, we define

$$\mathrm{WF}_h(u) = T_+\Big(\mathrm{WF}_h\left((xx')^{\imath/h}u_+\right)\Big) \cup T_-\Big(\mathrm{WF}_h\left((xx')^{-\imath/h}u_-\right)\Big). \tag{5.2}$$

Here $T_\pm$ are the shift operators defined in (3.3).

Remark 5.2. — We note that in (5.1),

$$(\rho_L\rho_R)^{\pm\imath/h}u_\pm \in I^k(X_0^2, \tilde{\Lambda}^{nd}; \Omega^{1/2}),$$

so these wavefront sets are defined in the usual way [39, Section 8.4]. They may also be defined directly via an oscillatory integral representation for $(\rho_L\rho_R)^{\pm\imath/h}u_\pm$, that is, an expression of the form

$$h^{-m-(n+1)/2-k/2}\int e^{\imath\phi(Z,v)/h}a(Z,v,h)\,\mathrm{d}v + O(h^\infty), \tag{5.3}$$

where $v \in \mathbb{R}^k$, with a smooth, and we use Z for local coordinates on X_0^2, as described explicitly in Regions 1–5 in Section 3. This requires that ϕ locally parametrizes $\tilde{\Lambda}^{nd}$ (nondegenerately), i.e. the map ι from C_ϕ,

$$C_\phi = \{(Z,v) \mid d_v\phi(Z,v) = 0\}$$

to $\tilde{\Lambda}^{nd}$, given by

$$C_\phi \ni (Z, v) \mapsto \iota(Z, v) := (Z, d_Z\phi(Z, v)) \in \tilde{\Lambda}^{nd},$$

is a local diffeomorphism. The wavefront set (5.3) is then given by

$$\left\{ q \in \tilde{\Lambda}^{nd} \;\middle|\; \begin{array}{l} a(Z, v, h) \text{ is not } O(h^\infty) \text{ in a neighbourhood of} \\ (Z, v, 0), \text{ where } \iota(Z, v) = q \end{array} \right\}.$$

This wavefront set is a closed subset of $\tilde{\Lambda}^{nd}$; hence, $\mathrm{WF}_h(u)$ in (5.1) is a closed subset of Λ^{nd}. A similar remark can be made for (5.2).

We also remark that, strictly speaking, we should use a different notation such as ${}^0\mathrm{WF}_h(u)$ in (5.1), (5.2), to indicate that the wavefront set lies in a different bundle, e.g. in ${}^\Phi T^* X_0^2$ instead of $T^* X_0^2$ in (5.1). However, to avoid cumbersome notation, we use the simpler $\mathrm{WF}_h(u)$.

We recall that the Schwartz kernel of a semiclassical 0-pseudodifferential operator of order $(0, k)$ (the first index is the semiclassical order, the second the differential order) takes the form

$$h^{-(n+1)} \int e^{\imath(z-z')\cdot\zeta/h} a(z, \zeta, h)\, \mathrm{d}\zeta \tag{5.4}$$

(where a is a symbol of order k in ζ, uniformly in h) near the diagonal and away from the boundary of X_0^2, and

$$A = h^{-(n+1)} \int_{\mathbb{R}^{n+1}} e^{\imath\big((x-x')\lambda+(y-y')\cdot\mu\big)/(hx')} a(x, y, \lambda, \mu, h)\, \mathrm{d}\lambda\, \mathrm{d}\mu \tag{5.5}$$

(where a is a symbol of order k in (λ, μ), uniformly in h) near the boundary of the diagonal in X_0^2; away from the diagonal, the kernel is smooth and $O(h^\infty \rho_L^\infty \rho_R^\infty)$.

We wish to show that by composing with pseudodifferential operators acting on X, we can localize the microlocal support of $u \in I^m(\Lambda)$. More precisely, we shall establish

Proposition 5.3.

(1) *Suppose that $U \in I^m(\Lambda^{nd})$ and $A \in {}^0\Psi^{0,0}(X)$. Then $AU \in I^m(\Lambda^{nd})$ and we have*

$$\begin{aligned} \mathrm{WF}_h(AU) &\subset \pi_L^{-1}\big(\mathrm{WF}_h(A)\big) \cap \mathrm{WF}_h(U), \\ \mathrm{WF}_h(UA) &\subset \pi_R^{-1}\big(\mathrm{WF}_h(A)\big) \cap \mathrm{WF}_h(U). \end{aligned} \tag{5.6}$$

Here π_L, π_R is the left, resp. right projection from ${}^\Phi T^* X_0^2 \to {}^0 T^* X$, that is, the composite map

$${}^\Phi T^* X_0^2 \to \big({}^0 T^* X\big)^2 \to {}^0 T^* X \tag{5.7}$$

where the first map is induced by the blow-down map $\beta : X_0^2 \to X^2$, *and the second is the left, resp. right projection.*

(2) *Similarly, suppose that* $U^* \in I^m(\Lambda^*)$ *and* $A \in {}^0\Psi^{0,0}(X)$. *Then* $AU^* \in I^m(\Lambda^*)$ *and we have*

$$\begin{aligned} \mathrm{WF}_h(AU^*) &\subset \pi_L^{-1}\big(\mathrm{WF}_h(A)\big) \cap \mathrm{WF}_h(U^*), \\ \mathrm{WF}_h(U^*A) &\subset \pi_R^{-1}\big(\mathrm{WF}_h(A)\big) \cap \mathrm{WF}_h(U^*). \end{aligned} \tag{5.8}$$

In this case, π_L, π_R *denote the left, resp. right projection from* $({}^0T^*X)^2 \to {}^0T^*X$, *that is, the second arrow in* (5.7).

Proof. — The proofs of (5.6) and (5.8) are very similar, but (5.6) is more complicated due to the blowup. Because of this, we only prove (5.6).

The second statement in (5.6) follows from the first by switching the left and right variables. So we only prove the first. To do this, we write down local parametrizations of U, and check the statement (5.6) on each. We use local coordinates valid in Regions 1–5 as described in Section 3.

Region 1. — In this region, U has a local representation

$$U = h^{-m-k/2-(n+1)/2} \int_{\mathbb{R}^k} e^{\imath\phi(z,z',v)/h} b(z,z',v,h)\,\mathrm{d}v\,,$$

and A has a representation (5.4). The composition is given by an oscillatory integral

$$h^{-m-k/2-3(n+1)/2} \int e^{\imath((z-z'')\cdot\zeta + \phi(z'',z',v))/h} a(z,\zeta,h) b(z'',z',v,h)\,\mathrm{d}v\,\mathrm{d}\zeta\,\mathrm{d}z''.$$

We perform stationary phase in the variables (z'',ζ). We note that the Hessian in these variables is non-degenerate, as the matrix of second derivatives takes the form

$$\begin{pmatrix} * & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$$

which has nonzero determinant, irrespective of the top left entry. The stationary phase expansion then shows that this expression can be simplified to

$$h^{-m-k/2-(n+1)/2} \int e^{\imath\phi(z,z',v)/h} c(z,z',v,h)\,\mathrm{d}v + O(h^\infty)\,,$$

where c has an expansion

$$c(z,z',v,h) = \sum_{j=0}^{\infty} h^j Q_j\big(a(z,\zeta,h) b(z'',z',v,h)\big)\Big|_{z''=z,\zeta=-d_z\phi(z,z',v)}$$

where Q_j is a differential operator in the (z'',ζ) variables of degree $2j$. This shows that $AU \in I^m(\Lambda)$ and has microlocal support contained in $\mathrm{WF}_h(U)$

(since the amplitude c is $O(h^\infty)$ wherever $b = O(h^\infty)$). The microlocal support is also contained in the set

$$\{(z, z', v, h) \mid (z, d_z\phi(z, z', v)) \in \mathrm{WF}_h(A)\},$$

which is to say that the microlocal support is contained in $\mathrm{WF}_h(U) \cap \pi_L^{-1}\,\mathrm{WF}_h(A)$.

Region 2a. — In this region, U has a local representation

$$U = h^{-m-k/2-(n+1)/2} \int e^{\imath(\phi(x,y,z',v)\pm\log x)/h} b(x, y, z', v, h)\,\mathrm{d}v\,,$$

and A has a representation (5.5). The composition is given by an oscillatory integral

$$h^{-m-k/2-3(n+1)/2} \int e^{\imath\left((x-x'')\lambda/x+(y-y'')\cdot\mu/x+\phi(x'',y'',z',v)\pm\log x''\right)/h} \times a(x, y, \lambda, \mu, h) b(x'', y'', z', v, h)\,\mathrm{d}v\,\frac{\mathrm{d}x''\,\mathrm{d}y''}{x''^{n+1}}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,.$$

We change to coordinates $s'' = x''/x$ and $Y'' = (y - y'')/x$. In these coordinates we have

$$h^{-m-k/2-3(n+1)/2} \int e^{\imath\left((1-s'')\lambda+Y''\cdot\mu+\phi(xs'',y-xY'',z',v)\pm\log(xs'')\right)/h} \times a(xy, \lambda, \mu, h) b(xs'', y - xY'', z', v, h)\,\mathrm{d}v\,\frac{\mathrm{d}s''\,\mathrm{d}Y''}{s''^{n+1}}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,.$$

We then perform stationary phase in the variables (s, λ, Y, μ). There is a stationary point at

$$s'' = 1\,,\ Y'' = 0\,,\ \lambda = x\mathrm{d}_x\phi \pm 1\,,\ \mu = x\mathrm{d}_y\phi\,. \tag{5.9}$$

We check that the Hessian in these variables is non-degenerate at this critical point. The matrix of second derivatives takes the form

$$\begin{pmatrix} * & \mathrm{Id} & O(x) & 0 \\ \mathrm{Id} & 0 & 0 & 0 \\ O(x) & 0 & * & \mathrm{Id} \\ 0 & 0 & \mathrm{Id} & 0 \end{pmatrix}$$

which has nonzero determinant when x is small, irrespective of the starred entries. The stationary phase expansion then shows that this expression can be simplified to

$$h^{-m-k/2-(n+1)/2} \int e^{\imath(\phi(x,y,z',v)\pm\log x)/h} c(x, y, z', v, h)\,\mathrm{d}v + O(h^\infty)\,,$$

where $c(x,y,z',v,h)$ has an expansion

$$\sum_{j=0}^{\infty} h^j Q_j\big(a(x,y,\lambda,\mu,h)b(x's'',y-xY'',z',v,h)\big)\Big|_{x''=x,y''=y,\lambda=xd_x\phi\pm 1,\mu=xd_y\phi}$$

where Q_j is a differential operator in (s'',λ,Y'',μ) of degree $2j$. This shows that $AU \in I^m(\Lambda)$, and has microlocal support contained in $\mathrm{WF}_h(U)$ (since $c = O(h^\infty)$ wherever $b = O(h^\infty)$). The microlocal support is also contained in

$$\Big\{(x,y,z',v,h)\,\Big|\,(x,y,\pm 1 + xd_x\phi(x,y,z',v), xd_y\phi(x,y,z',v)) \in \mathrm{WF}_h(A)\Big\},$$

which (comparing with (5.9)) shows that the microlocal support of AU is also contained in $\pi_L^{-1}\,\mathrm{WF}_h(A)$.

Region 2b. — In this region, the calculation is similar to Region 1, so we omit the details.

Region 3. — In this region, the calculation is similar to Region 2a, so again we omit the details.

Region 4a. — Here we use the coordinates

$$s = \frac{x}{x'},\ x',\ y,\ Y = \frac{y'-y}{x'}.$$

In this region, U has a local representation

$$U = h^* \int e^{\imath(\phi(s,x',y,Y,v)\pm\log s)/h} b(s,x',y,Y,v,h)\,\mathrm{d}v,$$

and A has a representation (5.5). The composition is given by an oscillatory integral

$$\begin{aligned} &h^{-m-k/2-3(n+1)/2} \\ &\quad\times \int e^{\imath\big((x-x'')\lambda/x+(y-y'')\cdot\mu/x+\phi(x''/x',x',y',(y'-y'')/x',v)\pm\log(x''/x')\big)/h} \\ &\qquad\times a(x,y,\lambda,\mu,h)b(x''/x',x',y'',(y'-y'')/x',v,h)\,\mathrm{d}v\,\frac{\mathrm{d}x''\,\mathrm{d}y''}{x''^{n+1}}\mathrm{d}\lambda\mathrm{d}\mu. \end{aligned}$$

We introduce coordinates $Y'' = (y-y'')/x$, $s'' = x''/x$. The integral becomes

$$\begin{aligned} &h^{-m-k/2-3(n+1)/2}\int e^{\imath\big((1-s'')\lambda+Y''\cdot\mu+\phi(s''s,x',y'',sY''+Y,v)\pm\log(s''s)\big)/h} \\ &\qquad\times a(x,y,\lambda,\mu,h)b(s''s,x',y'',sY''+Y,v,h)\,\mathrm{d}v\,\frac{\mathrm{d}s''dY''}{s''^{n+1}}\mathrm{d}\lambda\mathrm{d}\mu. \end{aligned}$$

We perform stationary phase in the variables (s'', Y'', λ, μ). It is straightforward to check that the Hessian in these variables is nondegenerate at the stationary point

$$s'' = 1,\ Y'' = 0,\ \lambda = s\mathrm{d}_s\phi \pm 1,\ \mu = x\mathrm{d}_y\phi - \mathrm{d}_Y\phi. \tag{5.10}$$

We then get a stationary phase expansion, as in the previous regions, leading to the conclusion that AU has an expression

$$U = h^{-m-k/2-(n+1)/2} \int e^{\imath(\phi(s,x',y,Y,v)\pm\log s)/h} c(s,x',y,Y,v,h)\,\mathrm{d}v + O(h^\infty),$$

such that c is given in terms of a and b by a stationary phase expansion as in Regions 1 or 2a above. Thus, AU is a Lagrangian distribution in $I^m(\Lambda)$, and c is $O(h^\infty)$ wherever $b = O(h^\infty)$, and is supported where $(x, y, \lambda, \mu) \in \mathrm{WF}_h(A)$. It follows (using (5.10)) that $\mathrm{WF}_h(AU)$ is contained in $\mathrm{WF}_h(U) \cap \pi_L^{-1}(\mathrm{WF}_h(A))$.

Region 4b. — This is given by a rather similar calculation to Region 4a, which we omit.

Region 5. — Here we use the coordinates

$$s_1 = \frac{x}{y'_1 - y_1},\ s_2 = \frac{x'}{y'_1 - y_1},\ t = y'_1 - y_1,\ y', Z_j = \frac{y'_j - y_j}{y'_1 - y_1}, \quad j \geqslant 2.$$

In this region, U has a local representation

$$U = h^{-m-k/2-(n+1)/2} \int e^{\imath(\phi(s_1,s_2,t,y',Z,v)\pm\log(s_1s_2))/h} b(s,x',y,Y,v,h)\,\mathrm{d}v.$$

Writing $s'' = x/x''$ and $Y'' = (y-y'')/x$ as before, the composition is given by an oscillatory integral

$$\begin{aligned} h^{-m-\frac{k}{2}-\frac{3(n+1)}{2}} \int \exp\Bigg\{\frac{\imath}{h}\Bigg((1-s'')\lambda + Y''\cdot\mu \\ + \phi\left(\frac{s_1s''t}{t-xY''_1}, \frac{s_2t}{t-xY''_1}, t-xY''_1, y', \frac{tZ_j + xY''_j}{t-xY''_1}, v\right) \pm \log\frac{s_1s_2s''t^2}{(t-xY''_1)^2}\Bigg)\Bigg\} \\ \times\ a(x,y,\lambda,\mu,h) b\left(\frac{s_1s''t}{t-xY''_1}, \frac{s_2t}{t-xY''_1}, t-xY''_1, y', \frac{tZ_j + xY''_j}{t-xY''_1}, v, h\right) \\ \mathrm{d}v\,\frac{\mathrm{d}s''\,\mathrm{d}Y''}{s''^{n+1}}\mathrm{d}\lambda\mathrm{d}\mu. \end{aligned}$$

We perform stationary phase in the variables (s'', Y'', λ, μ). There is a critical point at

$$\begin{aligned} &s'' = 1\,,\ Y'' = 0\,,\ \lambda = s_1 d_{s_1}\phi\,,\\ &\mu = s_1\Big(s_{s_1}\phi + s_2 d_{s_2}\phi + Z\cdot d_Z\phi - t d_t\phi - s_1 d_{Z_1}\phi\Big)\,,\\ &\mu_j = -s_1 d_{Z_j}\phi\,,\ j \geqslant 2\,. \end{aligned} \tag{5.11}$$

It is straightforward to check that the Hessian in these variables is nondegenerate. We then get a stationary phase expansion, as in the previous regions, leading to the conclusion that AU has an expression

$$U = h^{-m-k/2-(n+1)/2} \int e^{\imath(\phi(s_1,s_2,t,y',Z,v)\pm\log(s_1 s_2))/h} \\ c(s_1,s_2,t,y',Z,v,h)\,\mathrm{d}v + O(h^\infty)\,,$$

such that c is given in terms of a and b by a stationary phase expansion. Thus, $AU \in I^m(\Lambda)$, and using the same reasoning as above, its microlocal support is contained in $\mathrm{WF}_h(U) \cap \pi_L^{-1}\,\mathrm{WF}_h(A)$. □

6. The spectral measure at high energy

In this section, we prove Theorem 1.3 for high energies, $\boldsymbol{\lambda} \geqslant 1$, which immediately implies also Theorem 1.5. Our first task is to choose an appropriate partition of the identity operator. This is done in exactly the same way as was done in [17] in the asymptotically conic case.

Before getting into the details we explain the advantage of using a partition of the identity. It is to microlocalize the spectral measure (taking advantage of the microlocal support estimate, Proposition 5.3) so that only the Lagrangian Λ^{nd} is relevant, while the other part, Λ^*, disappears. This is important in our pointwise estimate, as the Lagrangian Λ^{nd} locally projects diffeomorphically to the base manifold except at where it meets $N^*\operatorname{diag}_0$, i.e. the projection ${}^\Phi\pi$, restricted to Λ^{nd} has maximal rank except at the intersection with $N^*\operatorname{diag}_0$, which leads to the most favourable L^∞ estimates. (The drop in rank at the diagonal leads to the different form of the estimates for small $d(z,z')$ in Theorem 1.5.) By contrast, we cannot control the rank of the projection from Λ^* to the base (except by making additional geometric assumptions, such as nonpositive curvature of X°, which we do in Sections 8 and 9).

6.1. Partition of the identity

Our operators $Q_i(\boldsymbol{\lambda})$ will be semiclassical 0-pseudodifferential operators of order $(0,0)$, where the first index denotes the semiclassical order and the second, the differential order, and the semiclassical parameter is $h = \boldsymbol{\lambda}^{-1}$. In fact, all but $Q_0(\boldsymbol{\lambda})$ will have differential order $-\infty$.

First of all, we will choose Q_0 of order $(0,0)$ microlocally supported away from the characteristic variety of $h^2\Delta - 1$, say in the region $\{\sigma(h^2\Delta) \in [0,3/4] \cup [5/4,\infty)\}$, and microlocally equal to the identity in a smaller region, say $\{\sigma(h^2\Delta) \in [0,1/2] \cup [3/2,\infty)\}$. In light of the disjointness of semiclassical wavefront sets, the term $Q_0(\boldsymbol{\lambda}) \mathrm{d}E_P(\boldsymbol{\lambda}) Q_0(\boldsymbol{\lambda})$ has empty microlocal support, and is therefore $O(h^\infty)$. Taking into account the behaviour at the boundary, we find that

$$(6.1) \quad Q_0(\boldsymbol{\lambda}) \mathrm{d}E_P(\boldsymbol{\lambda}) Q_0(\boldsymbol{\lambda}) \in h^\infty (\rho_L \rho_R)^{n/2 - \imath/h} C^\infty(X_0^2 \times [0,h_0]) \\ + h^\infty (\rho_L \rho_R)^{n/2 + \imath/h} C^\infty(X_0^2 \times [0,h_0]) \,.$$

This clearly satisfies Theorem 1.3.

We next choose a cutoff function $\chi(x)$, equal to 1 for $x \leqslant \epsilon$ and 0 for $x \geqslant 2\epsilon$. We decompose the remainder $\mathrm{Id} - Q_0(\boldsymbol{\lambda})$ into $(\mathrm{Id} - Q_0(\boldsymbol{\lambda}))\chi(x)$ and $(\mathrm{Id} - Q_0(\boldsymbol{\lambda}))(1 - \chi(x))$, and further decompose these two pieces in the following way.

We divide the interval $[-3/2, 3/2]$ into a union of intervals B_i with overlapping interiors, and with diameter $\leqslant \beta$. We then decompose $(\mathrm{Id} - Q_0(\boldsymbol{\lambda}))\chi(x)$ into operators

$$Q_i(\boldsymbol{\lambda}), \dots, Q_{N_1}(\boldsymbol{\lambda})$$

such that each operator $Q_i(\boldsymbol{\lambda})$ has wavefront set contained in $\{\lambda(\lambda^2 + h^{ij}\mu_i\mu_j)^{-1/2} \subset B_i\}$.

Next, we decompose $(\mathrm{Id} - Q_0(\boldsymbol{\lambda}))(1 - \chi(x))$. The idea is still to decompose this operator into pieces, so that on each piece the microlocal support is small. Let $d(\,\cdot\,,\cdot)$ be the Sasaki distance on T^*X°. We break up $(\mathrm{Id} - Q_0(\boldsymbol{\lambda}))(1 - \chi(x))$ into a finite number of operators

$$Q_{N_1+1}(\boldsymbol{\lambda}), \dots, Q_{N_1+N_2}(\boldsymbol{\lambda}),$$

each of which is such that the microlocal support has diameter $\leqslant \eta$ with respect to the Sasaki distance on T^*X°. This is possible since the microlocal support of $(\mathrm{Id} - Q_0(\boldsymbol{\lambda}))(1 - \chi(x))$ is compact in T^*X°. We choose $\eta < \iota/4$, where ι is the injectivity radius of (X°, g).

We now prove a key property about the microlocal support of

$$Q_i(\boldsymbol{\sigma}) \mathrm{d}E_P(\boldsymbol{\lambda}) Q_i(\boldsymbol{\sigma})^*,$$

when ϵ, β and η are sufficiently small.

PROPOSITION 6.1. — *Suppose that ϵ, β and η are sufficiently small. Then, provided that $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ satisfy $\boldsymbol{\sigma} \in [(1-\delta)\boldsymbol{\lambda}, (1+\delta)\boldsymbol{\lambda}]$ for sufficiently small δ, the microlocal support of $Q_i(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda})Q_i(\boldsymbol{\sigma})^*$, $i \geqslant 0$, is a subset of Λ^{nd}.*

Remark 6.2. — In the composition $Q_i(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda})Q_i(\boldsymbol{\sigma})^*$, we view all operators as semiclassical operators with parameter $h = \boldsymbol{\lambda}^{-1}$. To do this for $Q_i(\boldsymbol{\sigma})$, we need to scale the fibre variables in the symbol by a factor of $\boldsymbol{\lambda}/\boldsymbol{\sigma}$. This is of little consequence as $\boldsymbol{\lambda}/\boldsymbol{\sigma}$ is close to 1 by assumption.

Proof. — Recall that Λ^{nd} consists of a neighbourhood U_1 of $\partial_{\mathrm{FF}}\Lambda$ in Λ, together with a neighbourhood U_2 of $\Lambda \cap N^* \operatorname{diag}_0$ in Λ.

First suppose that $i = 0$. By Proposition 5.3, the microlocal support of $Q_0(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda})Q_0(\boldsymbol{\sigma})^*$ is empty for sufficiently small δ, so the conclusion of Proposition 6.1 trivially holds.

Next suppose that $1 \leqslant i \leqslant N_1$. We claim that if ϵ, β and δ are sufficiently small, then the microsupport of $Q_i(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda})Q_i(\boldsymbol{\sigma})^*$ is contained in U_1.

Let $U_1' = U_1 \setminus \partial_{\mathrm{FF}}\Lambda$, i.e. a deleted neighbourhood of $\partial_{\mathrm{FF}}\Lambda$. Since the microlocal support is always a closed set it suffices to show that

$$\mathrm{WF}_h\left(Q_i(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda})Q_i(\boldsymbol{\sigma})^*\right) \setminus \{\rho_F = 0\} \text{ is contained in } U_1'.$$

By Proposition 5.3, this wavefront set is contained in

$$\left\{ (z,\zeta;z',-\zeta') \,\middle|\, \begin{array}{l} (z,\zeta) \text{ and } (z',\zeta') \text{ lie on the same geodesic,} \\ |\zeta|_g = |\zeta'|_g = 1,\ (z, \frac{\boldsymbol{\lambda}}{\boldsymbol{\sigma}}\zeta), (z', \frac{\boldsymbol{\lambda}}{\boldsymbol{\sigma}}\zeta') \in \mathrm{WF}_h(Q_i(\boldsymbol{\sigma})) \end{array} \right\}.$$

We prove the claim by contradiction. Suppose that the claim were false. Choose sequences β_k, δ_k and ϵ_k tending to zero as $k \to \infty$, and for each k, a partition of the identity $Q_i^{(k)}$ satisfying the conditions above relative to β_k, δ_k and ϵ_k. Then, if the claim is false for all $Q_i^{(k)}$, there are sequences of pairs of points $(x_k, y_k, \lambda_k, \mu_k)$, $(x_k', y_k', \lambda_k', \mu_k')$ in $\mathrm{WF}_h(Q_i^{(k)})$, lying on the same geodesic γ_k, with $x_k, x_k' \leqslant \epsilon_k$, $|\lambda_k - \lambda_k'| \leqslant \beta_k(1-\delta_k)^{-1}$, but the corresponding point $(x_k, y_k, \lambda_k, \mu_k; x_k', y_k', -\lambda_k', -\mu_k')$ not in U_1'. By compactness we can take a convergent subsequence, with $x_k \to 0$, $x_k' \to 0$, $y_k \to y_0$, $y_k' \to y_0'$, $\lambda_k, \lambda_k' \to \lambda_0$, $\mu_k \to \mu_0$, $\mu_k' \to \mu_0'$. Consider the limiting behaviour of the geodesic γ_k connecting $(x_k, y_k, \lambda_k, \mu_k)$ and $(x_k', y_k', -\lambda_k', -\mu_k')$. If $\lambda_0 \neq \pm 1$ then γ_k converges to a boundary bicharacteristic, that is, an integral curve of (3.4) contained in $\{x = 0\}$, and therefore takes the form

$$x(\tau) = 0,\ y(\tau) = y^*,\ \lambda(\tau) = \cos\tau,\ \mu(\tau) = \sin\tau\mu^*, \text{ where } \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sin\tau.$$

(It is straightforward to check that this satisfies the geodesic equations (3.4) in the parameter t.) Therefore, $(y_0, \lambda_0, \mu_0) = (y^*, \cos\tau, \sin\tau\mu^*)$ and $(y_0', \lambda_0', \mu_0') = (y^*, \cos\tau', \sin\tau'\mu^*)$ for some τ and τ'. Since $|\lambda_k - \lambda_k'| \to 0$, we have $\tau = \tau'$, and it then follows that $\mu_0 = \mu_0'$. This shows that the limiting point lies on $\partial_{\mathrm{FF}}\Lambda$, in fact over the fibre F_{y^*} of FF lying over y^* (over which point on this fibre depends on the limiting values of x/x' and $(y'-y)/x'$). Hence the sequence converging to it eventually lies in U_1', which is our desired contradiction.

If $\lambda_0 = \pm 1$ then the limiting geodesic could be an interior bicharacteristic. In this case we must have $\lambda_0, \lambda_0' \in \{\pm 1\}$, i.e the points $(x_0, y_0, \lambda_0, \mu_0)$ and $(x_0', y_0', -\lambda_0', \mu_0')$ are both an endpoint of this bicharacteristic. However the condition that the difference $|\lambda - \lambda'| \to 0$ along the sequence means that either both λ_0, λ_0' are $+1$ or both are -1. Thus, the limiting points $(x_0, y_0, \lambda_0, \mu_0)$ and $(x_0', y_0', -\lambda_0', \mu_0')$ are again equal in this case, with $x_0 = x_0' = 0$, $\mu_0 = \mu_0' = 0$, which shows that the limiting point lies on $\partial_{\mathrm{FF}}\Lambda$, hence the sequence converging to it eventually lies in U_1', again producing a contradiction.

We next claim that if $N_1 + 1 \leqslant i \leqslant N_1 + N_2$, then for η sufficiently small, the wavefront set of $Q_i(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda})Q_i(\boldsymbol{\sigma})^*$ is contained in U_2. The argument is similar. Choose a sequence η_k tending to zero as $k \to \infty$, and for each k, a partition of the identity $Q_i^{(k)}$ satisfying the conditions above relative to η_k. Then, if the claim is false for all $Q_i^{(k)}$, then we could find a sequence $(z_k, \zeta_k), (z_k', \zeta_k')$ on the same geodesic, with $(z_k, \zeta_k), (z_k', \zeta_k') \in \mathrm{WF}_h(Q_i)$, with each $(z_k, \zeta_k; z_k', -\zeta_k')$ not in U_2. Using compactness we can extract a convergent subsequence from the (z_k, ζ_k), converging to (z_0, ζ_0). Since $\eta_k \to 0$ the sequence (z_k', ζ_k') also converges to (z_0, ζ_0). But the point $(z_0, \zeta_0, z_0, -\zeta_0)$ is in $N^* \operatorname{diag}_0$, and U_2 is a neighbourhood of $N^* \operatorname{diag}_0$ in Λ, so this gives us the contradiction. □

We now assume that ϵ, δ, η have been chosen small enough that the conclusion of Proposition 6.1 is valid.

6.2. Pointwise estimates for microlocalized spectral measure near the diagonal

In this section we show that an element U of $I^{-1/2}(X_0^2, \Lambda^{nd}; {}^0\Omega^{1/2})$ satisfies (1.14) and (1.15) in the region $d(z, z') \leqslant 1$.

We divide this into the case where we work away from the boundary of diag_0, and near a point on the boundary of the diagonal. The first case, localizing away from the boundary of the diagonal, has been treated in [19, Proposition 1.3] (this was done in the context of asymptotically conic manifolds, but away from the boundary, one "cannot tell" whether one is on an asymptotically conic or asymptotically hyperbolic manifold, so the argument applies directly).(9)

Thus, it remains to deal with the case where $\mathrm{d}E_P(\boldsymbol{\lambda})$ is microlocalized to a neighbourhood of a point $q \in \partial_{\mathrm{FF}} N^* \operatorname{diag}_0 \cap \Lambda^{nd}$. In this case, any parametrization of the Lagrangian Λ^{nd} must have at least n integrated variables, since the rank of the projection from Λ^{nd} to the base X_0^2 drops by n at $N^* \operatorname{diag}_0$.

The following result is essentially taken from [19].

Proposition 6.3. — *Let q be a point in $\partial_{\mathrm{FF}} N^* \operatorname{diag}_0 \cap \Lambda^{nd}_{\pm}$, and let $U \in I^{-1/2}(X_0^2, \Lambda^{nd}; {}^0\Omega^{1/2})$. Then microlocally near q, U can be represented as an oscillatory integral of the form*

$$h^{-n} \int_{\mathbb{R}^n} e^{\imath \Psi(x', y, r, w, v)/h} a(x', y, r, w, v, h) \, \mathrm{d}v \,, \tag{6.2}$$

where the coordinates (r, w), $w = (w_1, \ldots, w_n)$ define the boundary, that is, $\operatorname{diag}_0 = \{r = 0, w = 0\}$ and the differentials $\mathrm{d}r$ and $\mathrm{d}w_i$ are linearly independent. Here also, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, and a is smooth and compactly supported in all variables. Moreover, we may assume that Ψ has the properties

$$\begin{cases} (a)\ \mathrm{d}_{v_j} \Psi = w_j + O(r) \,, \\ (b)\ \Psi = \sum_{j=1}^n v_j \partial_{v_j} \Psi + O(r) \,, \\ (c)\ \mathrm{d}^2_{v_j v_k} \Psi = r A_{jk}, \ \text{where } A \text{ is a smooth, nondegenerate matrix} \\ (d)\ \mathrm{d}_v \Psi = 0 \implies \Psi = \pm d(z, z') \,. \end{cases} \tag{6.3}$$

Proof. — In this region, $(x', y, s = x/x', Y = (y' - y)/x')$ furnish local coordinates on X_0^2. In these coordinates, the diagonal is defined by $s = 1, Y = 0$. We will write (r, w) for a suitable rotation of the coordinates $(s - 1, Y)$. Let $(\xi', \eta, \rho, \kappa)$ be the dual coordinates to (x', y, r, w). We claim that, for some rotation (r, w) of the $(s - 1, Y)$ coordinates, we have $\mathrm{d}\rho|_{N^* \operatorname{diag}_0 \cap \Lambda^{nd}} = 0$ at q. This follows from the fact that $N^* \operatorname{diag}_0 \cap \Lambda^{nd}$ projects to diag_0 with n-dimensional fibres (in fact $N^* \operatorname{diag}_0 \cap \Lambda^{nd}$ is an S^n-bundle over diag_0), so the $n+1$ differentials $\mathrm{d}\rho, \mathrm{d}\kappa_1, \ldots, \mathrm{d}\kappa_n$ span an n-dimensional space in $T_q(N^* \operatorname{diag}_0 \cap \Lambda^{nd})$ for each $q \in N^* \operatorname{diag}_0 \cap \Lambda^{nd}$.

Secondly, we claim $\mathrm{d}r|_{\Lambda^{nd}} \neq 0$ at q. To see this, we observe that there must be a vector $V \in T_q \Lambda^{nd}$ tangent to Λ^{nd} but not tangent to $N^* \operatorname{diag}_0$.

(9) Notice that the dimension was denoted n in [19], instead of $n+1$, as here, when comparing [19, (1.13)] with (1.14).

Therefore, V must have a non-zero ∂_{w_j} or ∂_r component. Also, the vectors ∂_{κ_i} are tangent to Λ^{nd} at q, since $T_q(N^* \operatorname{diag}_0 \cap \Lambda^{nd})$ is the codimension 1 subspace of $T_q(N^* \operatorname{diag}_0)$ given by the vectors annihilated by $\mathrm{d}\rho$. Thus, as Λ^{nd} is Lagrangian, we have

$$\omega(V, \partial_{\kappa_i}) = \Big(\mathrm{d}y \wedge \mathrm{d}\eta + \mathrm{d}r \wedge \mathrm{d}\rho + \mathrm{d}w \wedge \mathrm{d}\kappa + \mathrm{d}x' \wedge \mathrm{d}\xi'\Big)(V, \partial_{\kappa_i}) = 0 .$$

This implies that $\mathrm{d}w_j(V) = 0$ for each j, which implies V has a non-zero ∂_r component, as claimed.

This shows that (x', y, r, κ) furnish coordinates on Λ^{nd} locally near q. Thus, we can express the remaining coordinates, restricted to Λ^{nd}, as smooth functions of these:

$$\begin{aligned} w &= W(x', y, r, \kappa), \quad & \rho &= R(x', y, r, \kappa), \\ \xi' &= \Xi'(x', y, r, \kappa), \quad & \eta &= H(x', y, r, \kappa). \end{aligned} \tag{6.4}$$

Also, using the fact that Λ^{nd} is Lagrangian, the form

$$d\Big(\xi' \mathrm{d}x' + \eta \cdot \mathrm{d}y + \rho \mathrm{d}r + \sum_j \kappa_j \mathrm{d}w_j\Big) = 0 \text{ on } \Lambda^{nd} . \tag{6.5}$$

It follows that there is a function $f(x', y, r, \kappa)$ on Λ^{nd}, defined near q, such that

$$\Xi' \mathrm{d}x' + H \cdot \mathrm{d}y + R \mathrm{d}r + \sum_j \kappa_j \mathrm{d}W_j = \mathrm{d}f . \tag{6.6}$$

Notice that $\Lambda^{nd} \cap \{r = 0\} = \Lambda^{nd} \cap N^* \operatorname{diag}_0$, and at $N^* \operatorname{diag}_0$ we have $\xi' = 0, \eta = 0, w = 0$. Therefore, at $r = 0$, we have

$$\frac{\partial f}{\partial x'} = 0 , \quad \frac{\partial f}{\partial y} = 0 , \quad \frac{\partial f}{\partial \kappa} = 0 .$$

It follows that f is constant when $r = 0$. Since f is undetermined up to a constant, we may assume that $f = 0$ when $r = 0$; that is, $f(x', y, r, \kappa) = r\tilde{f}(x', y, r, \kappa)$.

We claim that the function

$$\Psi = \sum_{j=1}^{n} \big(w_i - W_i(x', y, r, v)\big) v_i + f(x', y, r, v)$$

locally parametrizes the Lagrangian Λ^{nd} near q, and satisfies properties (a)–(d) above.

To check that Ψ parametrizes Λ^{nd}, we set $\mathrm{d}_v \Psi = 0$. This implies that

$$w_i - W_i(x', y, r, v) = \sum_i v_j \frac{\partial W_j}{\partial v_i} - \frac{\partial f}{\partial v_i}. \tag{6.7}$$

On the other hand, the 1-form identity (6.6) shows that the functions W, R, Ξ', H and f satisfy the identities

$$\begin{aligned}
\Xi'(x', y, r, v) = -\sum_j v_j \frac{\partial W_j}{\partial x'}(x', y, r, v) + \frac{\partial f}{\partial x'}(x', y, r, v) &= \frac{\partial \Psi}{\partial x'}, \\
H_k = -\sum_j v_j \frac{\partial W_j}{\partial y_k} + \frac{\partial f}{\partial y_k} &= \frac{\partial \Psi}{\partial y_k}, \\
R = -\sum_j v_j \frac{\partial W_j}{\partial r} + \frac{\partial f}{\partial r} &= \frac{\partial \Psi}{\partial r}, \\
\sum_j v_j \frac{\partial W_j}{\partial v_i} &= \frac{\partial f}{\partial v_i}.
\end{aligned} \tag{6.8}$$

The last of these identities shows that the RHS of (6.7) vanishes. We therefore find that the Lagrangian parametrized by Ψ is

$$\begin{aligned}
&\left\{(x', y, r, w; \xi', \eta, \rho, \kappa) \,\middle|\, \xi' = d_{x'}\Psi,\ \eta = d_y\Psi,\ \rho = d_r\Psi,\ \kappa = d_w\Psi\right\} \\
&\quad = \left\{(x', y, r, w; \xi', \eta, \rho, \kappa) \,\middle|\, \xi' = \Xi',\ \eta = H',\ \rho = R,\ w = W\right\} \\
&\quad = \Lambda^{nd}.
\end{aligned} \tag{6.9}$$

It follows that, microlocally near q, the spectral measure may be written as an oscillatory integral with phase function Ψ, as in (6.2), where the power of h is given by $-m - N/4 - k/2 = -n$ where $m = -1/2$ is the order of the Lagrangian distribution, $N = 2(n+1)$ is the spatial dimension and $k = n$ is the number of integrated variables.

Conditions (a) and (b) are easily verified, using the fact that W and f are $O(r)$. To check condition (c), we write

$$d^2_{vv}\Psi = rA + O(r^2),$$

where A is an $n \times n$ matrix function of (r, x', y, v). We claim that A is invertible at q. It suffices to check $\det \mathrm{d}^2_{vv}\Psi \geqslant cr^n$ for some $c > 0$ near q. On one hand, since Ψ is a phase function parametrizing Λ^{nd} nondegenerately in a neighbourhood of q, then we have a local diffeomorphism

$$\{(r, x', y, v)\} \longrightarrow \{(r, x', y, \Psi'_r, \Psi'_{x'}, \Psi'_y, \Psi'_v)\}.$$

The determinant of the differential of the map

$$\{(r, x', y, \Psi'_r, \Psi'_{x'}, \Psi'_y, \Psi'_v)\} \longrightarrow \{(r, x', y, \Psi'_v)\} \tag{6.10}$$

is thus equal to the determinant of the differential of the map

$$\{(r, x', y, v)\} \longrightarrow \{(r, x', y, \Psi'_v)\},$$

which is simply $\det d^2_{vv}\Psi$. On the other hand, it is obvious that the determinant of the differential of map (6.10) equals the determinant of the projection map

$$\pi : \{(r, x', y, \Psi'_r, \Psi'_{x'}, \Psi'_y) : \Psi'_v = 0\} \longrightarrow \{(r, x', y)\}$$

from Λ^{nd} to X_0^2. Since $\det \mathrm{d}\pi \geqslant cr^n$ near q [10, Proposition 19], we get

$$\det d^2_{vv}\Psi \geqslant cr^n \quad \text{near } q\,,$$

which implies A is invertible.

To check (d), we notice that if $r = 0$ and $\mathrm{d}_v\Psi = 0$, then $\Psi = 0$. Also, $\Lambda^{nd} \cap \{r = 0\} = \Lambda^{nd} \cap N^* \operatorname{diag}_0$, so this is at the diagonal, i.e. $d(z, z') = 0$, so this agrees with (d). On the other hand, if $r \neq 0$, then condition (c) says that $\mathrm{d}^2_{vv}\Psi$ is nondegenerate. In this case we can eliminate the extra v variables, and reduce to a function of (x', y, r, w) that parametrizes the Lagrangian Λ^{nd} locally. (This is an analytic reflection of the geometric fact that the projection $\pi : {}^{\Phi}T^*X_0^2 \to X_0^2$ has full rank restricted to Λ^{nd}, for $r \neq 0$.) But then [10, Proposition 16] implies that the value of Ψ, when $\mathrm{d}_v\Psi = 0$, is equal to $d(z, z') + c$ on the forward half of Λ^{nd} (with respect to geodesic flow), and $-d(z, z') + c'$ on the backward half of Λ^{nd}. Condition (b) implies that $c = c' = 0$. This completes the proof of Proposition 6.3. $\square$

Using this we now show

Proposition 6.4. — *Suppose $U \in I^{-1/2}(X_0^2, \Lambda^{nd}; {}^0\Omega^{1/2})$. Then, U can be written in the form* (1.14) *with the amplitude functions $b_\pm$ satisfying* (1.15) *in the region $d(z, z') \leqslant 1$.*

Proof. — We estimate the integral (6.2) by dividing into three cases, depending on the relative size of r, $|w|$ and h.

Case 1. $|r| \leqslant h$. — Since $r = 0$ at $\Lambda^{nd} \cap N^* \operatorname{diag}_0$, and $\mathrm{d}r \neq 0$ there, $|r|$ is comparable to $d(z, z')$. So in this case, we have $d(z, z') = O(h)$. Thus, we need to show (comparing to (6.2))

$$\left(h\frac{\mathrm{d}}{\mathrm{d}h}\right)^j \left(e^{\mp \imath d(z,z')/h} \int_{\mathbb{R}^n} e^{\imath\Psi(x',y,r,w,v)/h} a(x', y, r, w, v, h)\, \mathrm{d}v\right) = O(1)\,.$$

For $j = 0$ this is trivial. Consider $j = 1$. We claim that this is also $O(1)$. This differential operator is certainly harmless when applied to the amplitude, a. When applied to the exponential, it brings down a factor $\imath(\pm d(z, z') + \Psi)/h$, which using (b) and $r, d(z, z') = O(h)$, we write $\imath h^{-1} v_i \Psi_{v_i} + O(1)$. The term $\imath h^{-1} v_i \Psi_{v_i}$ times the exponential $e^{\imath\Psi/h}$ is equal to $v_i d_{v_i} e^{\imath\Psi/h}$. We integrate by parts, shifting the v_i derivative to the amplitude a. In this way we see that the result of applying $h\partial_h$ to the expression is still $O(1)$. A similar argument applies to repeated applications of $h\partial_h$. Thus this term takes the form (1.14).

Case 2. $|r| \leqslant c|w|$ for some small constant c. — In this case, there must be some w_j such that $|r| \leqslant c|w_j|$. Then $\mathrm{d}_{v_j}\phi = w_j + O(r) \neq 0$ in a neighbourhood of q, provided c is sufficiently small. We can then integrate by parts arbitrarily many times in v_j, obtaining infinite order vanishing in h. The same is true for any number of $h\partial_h$ derivatives applied to (6.2). Thus this term satisfies (1.14).

Case 3. $|r| \geqslant h$ and $|r| \geqslant c|w|$, with c as in Case 2. — The idea for this region is to use a stationary phase estimate, as in [19, Section 4]. We follow this proof almost verbatim; the changes required here are mostly notational. In this case, we show a representation of the form (1.14), (1.15). Notice that if $\mathrm{d}_v\Psi = 0$ and $|r| \geqslant h$ then locally there are two sheets $\Lambda^{nd}_\pm$ of Λ^{nd} above X_0^2 (see Proposition 3.1). On $\Lambda^{nd}_\pm$ we divide by $e^{\pm id(z,z')/h}$ and show an estimate of the form (1.15). The argument for each is the same, so we only describe the argument for Λ^{nd}_+. Thus, we define, with $d = d(z,z')$,

$$b(x',y,r,w,h) = e^{-id/h}h^{-n}\int e^{\imath\Psi(x',y,r,w,v)/h}a(x',y,r,w,v,h)\,\mathrm{d}v\,, \tag{6.11}$$

and seek to prove the estimate

$$\Big|(h\partial_h)^\alpha b\Big| \leqslant C\big(1+\frac{|r|}{h}\big)^{-n/2}, \tag{6.12}$$

since in Case 3, we have $|r| \sim |(r,w)| \sim d$.

Define

$$\widetilde{\Psi}(x',y,r,w,v) = r^{-1}\big(\Psi(x',y,r,w,v) - d(z,z')\big)\,, \tag{6.13}$$

and let $\tilde{\boldsymbol{\lambda}} = r/h$. Notice that this function $\widetilde{\Psi}$ is C^∞ in v, and (6.3) implies that all v-derivatives (of all orders) are uniformly bounded in the region $|r| \geqslant h$ and $|r| \geqslant c|w|$. Then the LHS of (6.12) is

$$\begin{aligned} &h^\alpha\partial_h^\alpha b(x',y,r,w,v,h) \\ &\qquad = \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\tilde{\boldsymbol{\lambda}}^\beta\int e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}(x',y,r,w,v)}\widetilde{\Psi}^\beta\big(h^\gamma\partial_h^\gamma a\big)(x',y,r,w,v,h)\,\mathrm{d}v\,. \end{aligned}$$

Therefore, we need to show that, for any β, we have

$$\left|\int e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}(x',y,r,w,v)}(\tilde{\boldsymbol{\lambda}}\widetilde{\Psi})^\beta a(x',y,r,w,v,h)\,\mathrm{d}v\right| \leqslant C\tilde{\boldsymbol{\lambda}}^{-\frac{n}{2}}. \tag{6.14}$$

We now fix (x',y,r,w) with $r \geqslant h$. We use a cutoff function Υ to divide the v integral into two parts: one on the support of Υ, in which $|d_v\widetilde{\Psi}| \geqslant \tilde{\epsilon}/2$, and the other on the support of $1-\Upsilon$, in which $|d_v\widetilde{\Psi}| \leqslant \tilde{\epsilon}$. On the support of Υ, we integrate by parts in v and gain any power of $\tilde{\boldsymbol{\lambda}}^{-1}$, proving (6.14). On the support of $1-\Upsilon$, we make a change of variable to θ coordinates:

$$(v_1,\dots,v_n) \to (\theta_1,\dots,\theta_n), \quad \theta_i = d_{v_i}\widetilde{\Psi},\ i=1,\dots,n. \tag{6.15}$$

By property (c) of Proposition 6.3,

$$\frac{\partial\theta_j}{\partial v_k} = d^2_{v_j v_k}\widetilde{\Psi} = \pm A_{jk}\,,$$

where A_{jk} is nondegenerate. This shows that this change of variables is locally nonsingular, provided $\tilde{\epsilon}$ is sufficiently small. Thus, for each point v in the support of $1-\Upsilon$, there is a neighbourhood in which we can make this change of variables. Using the compactness of the support of a in (6.11), there are a finite number of neighbourhoods covering $\operatorname{supp}\Upsilon$ and the v-support of a.

On each such neighbourhood U, we define $\mathcal{B}_\delta := \{\theta : |\theta| \leqslant \delta\}$. Choose a C^∞ function $\chi_{\mathcal{B}_\delta}(\theta)$ which is equal to 1 when on the set $\mathcal{B}_\delta$ and 0 outside $\mathcal{B}_{2\delta}$, and with derivatives bounded by

$$\left|\nabla^{(j)}_\theta \chi_{\mathcal{B}_\delta}(\theta)\right| \leqslant C\delta^{-j}.$$

Here δ is a parameter which we will eventually choose to be $\tilde{\boldsymbol{\lambda}}^{-1/2}$; however, for now we leave its value free. Consider the integral (6.14) after changing variables and with the cutoff function $\chi_{\mathcal{B}_\delta}(\theta)$ inserted (where we stipulate $\delta \leqslant \tilde{\epsilon}/2$, which means that $1-\Upsilon = 1$ on $\operatorname{supp}\chi_{\mathcal{B}_\delta}(\theta)$):

$$\left| \int e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}(x',y,r,w,\theta)} \big(\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}\big)^\beta a(x',y,r,w,\theta,h)\chi_{\mathcal{B}_\delta}(\theta) \frac{\mathrm{d}\theta}{|A^{-1}(x',y,r,w,\theta)|} \right|.$$

Using property (d) of Proposition 6.3, we see that $\widetilde{\Psi} = 0$ when $\theta = 0$. Also, from (6.15), we have $\mathrm{d}_\theta\widetilde{\Psi} = 0$ when $\theta = 0$. Hence $\widetilde{\Psi} = O(|\theta|^2)$. Hence

$$\left| \tilde{\boldsymbol{\lambda}}^\beta \int e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}(x',y,r,w,\theta)} \widetilde{\Psi}^\beta a(x',y,r,w,\theta,h)\chi_{\mathcal{B}_\delta}(\theta) \frac{\mathrm{d}\theta}{|A^{-1}(x',y,w,\theta)|} \right| \leqslant C(\tilde{\boldsymbol{\lambda}}\delta^2)^\beta\delta^n\,. \tag{6.16}$$

It remains to treat the integral with the cutoff $(1-\chi_{\mathcal{B}_\delta}(\theta))$. Notice that $|d_\theta\widetilde{\Psi}|$ is comparable to $|\theta|$ since $\mathrm{d}_\theta\widetilde{\Psi} = 0$ when $\theta = 0$, and

$$\mathrm{d}^2_{\theta_i\theta_j}\widetilde{\Psi} = \sum_{k,l}(A^{-1})_{il}(A^{-1})_{jk}\mathrm{d}^2_{v_k v_l}\widetilde{\Psi}$$

is nondegenerate when $\theta = 0$. We define the differential operator L by

$$L = \frac{-id_\theta\widetilde{\Psi}\cdot\partial_\theta}{\tilde{\boldsymbol{\lambda}}\big|d_\theta\widetilde{\Psi}\big|^2}\,.$$

Then the adjoint operator ${}^tL$ is given by

$$ {}^tL = -L + \frac{\imath}{\tilde{\boldsymbol{\lambda}}}\left(\frac{\Delta_\theta \widetilde{\Psi}}{|d_\theta \widetilde{\Psi}|^2} - 2\frac{\mathrm{d}^2_{\theta_j\theta_k}\widetilde{\Psi}\,\mathrm{d}_{\theta_j}\widetilde{\Psi}\,\mathrm{d}_{\theta_k}\widetilde{\Psi}}{|d_\theta\widetilde{\Psi}|^4}\right). $$

We have chosen L such that $Le^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}} = e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}}$. So we introduce N factors of L applied to the exponential $e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}}$ and integrate by parts N times to obtain

$$ \left|\int e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}(x',y,r,w,\theta)}(\tilde{\boldsymbol{\lambda}}\widetilde{\Psi})^\beta a(x',y,r,w,\theta,h)(1-\chi_{\mathcal{B}_\delta}(\theta))(1-\Upsilon)\,\mathrm{d}\theta\right| \\ \leqslant C\int\left|({}^tL)^N\big((\tilde{\boldsymbol{\lambda}}\widetilde{\Psi})^\beta a(x',y,r,w,\theta,h)(1-\chi_{\mathcal{B}_\delta}(\theta))(1-\Upsilon)\big)\right|\mathrm{d}\theta. $$

Inductively we find, using $|d_\theta\widetilde{\Psi}|\sim|\theta|$, that

$$ \left|({}^tL)^N\big((\tilde{\boldsymbol{\lambda}}\widetilde{\Psi})^\beta a(1-\chi_{\mathcal{B}_\delta})(1-\Upsilon)\big)\right| \leqslant C\tilde{\boldsymbol{\lambda}}^{-N+\beta}\max\{|\theta|^{2\beta-2N},|\theta|^{2\beta-N}\delta^{-N}\}. $$

Choosing N large enough, we get

$$ \left|\int e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}(x',y,r,w,\theta)}(\tilde{\boldsymbol{\lambda}}\widetilde{\Psi})^\beta a(x',y,r,w,\theta,h)(1-\chi_{\mathcal{B}_\delta})(1-\Upsilon)\,\mathrm{d}\theta\right| \\ \leqslant \tilde{\boldsymbol{\lambda}}^{-N+\beta}\int_{|\theta|\geqslant\delta}\big(|\theta|^{2\beta-2N}+|\theta|^{2\beta-N}\delta^{-N}\big)\mathrm{d}\theta \\ \leqslant C\tilde{\boldsymbol{\lambda}}^{-N+\beta}\delta^{2\beta-2N}\delta^n. \tag{6.17} $$

We choose $\delta = \tilde{\boldsymbol{\lambda}}^{-1/2}$ to balance the two estimates (6.16) and (6.17). We finally obtain

$$ \left|\int e^{\imath\tilde{\boldsymbol{\lambda}}\widetilde{\Psi}(x',y,r,w,\theta)}(\tilde{\boldsymbol{\lambda}}\widetilde{\Psi})^\beta a(x',y,r,w,\theta,h)(1-\Upsilon)\,\mathrm{d}\theta\right| \leqslant C\tilde{\boldsymbol{\lambda}}^{-n/2}, $$

which proves (6.14) as desired. □

Proof of Theorem 1.3 for high energies, $\boldsymbol{\lambda}\geqslant 1$. — We express the spectral measure as a sum of 4 types of terms, (1)–(4), as in Theorem 3.3. Then, using Proposition 6.1, we see that the microlocalized spectral measure $Q_i(\boldsymbol{\lambda})\mathrm{d}E_P(\boldsymbol{\lambda})Q_i(\boldsymbol{\lambda})^*$ has microsupport contained in Λ^{nd}, so the terms of type (3) can be disregarded. Clearly, terms of type (2) and (4) satisfy the conclusion of Theorem 1.3, so it is only necessary to consider the terms of type (1).

For terms of type (1), Proposition 6.4 shows that Theorem 1.3 is satisfied. On the other hand, for large distance, we know from Proposition 3.1 that Λ^{nd}_+ and Λ^{nd}_- both project diffeomorphically to an open set $V\subset X_0^2\setminus\mathrm{diag}_0$ under ${}^\Phi\pi$, and therefore, according to [10, Proposition 20], the Lagrangian submanifolds $\Lambda^{nd}_\pm$ are parametrized by the distance function, $\pm d(z,z')$. The amplitude has a classical expansion in powers of $h = \boldsymbol{\lambda}^{-1}$, as shown by the construction of [10, Section 4], and the leading power of h is $h^{1/2-(n+1)/2} = h^{-n/2}$, as claimed. $\square$

Corollary 6.5. — *Let δ be a small positive number. For high energies, $\boldsymbol{\lambda} \geqslant 1$, one has*

$$\left| Q_k(\boldsymbol{\lambda}) \left(\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\sigma}} \right)^j \mathrm{d}E_P(\boldsymbol{\sigma}) \right) Q_k^*(\boldsymbol{\lambda})(z,z') \right| \leqslant \begin{cases} C\boldsymbol{\lambda}^{n-j}(1 + d(z,z')\boldsymbol{\lambda})^{-n/2+j}, & \text{for } d(z,z') \leqslant 1 \\ C\boldsymbol{\lambda}^{n/2} d(z,z')^j e^{-nd(z,z')/2}, & \text{for } d(z,z') \geqslant 1 \end{cases}$$

provided $\boldsymbol{\sigma}$ is in $[(1-\delta)\boldsymbol{\lambda}, (1+\delta)\boldsymbol{\lambda}]$.

7. Factorization of Spectral Measure

We shall prove high energy restriction theorem by our microlocalized spectral measure estimates obtained above. To do so we need a factorization of spectral measure in terms of Poisson operator.

First of all, we review the Poisson operator. Melrose [28, p. 103], following [26, 27], determined the structure of the generalized eigenfunctions of Laplacian with respect to 0-metric: for any $\boldsymbol{\lambda} \in \mathbb{R} \setminus \{0\}$ and each $f \in C^\infty(\partial M)$ there is a unique solution of $(\Delta - n^2/4 - \boldsymbol{\lambda}^2)u = 0$ of the form

$$u = x^{\imath\boldsymbol{\lambda}} x^{n/2} f + x^{-\imath\boldsymbol{\lambda}} x^{n/2} f_- + u',$$

where $f_- \in C^\infty(\partial M)$ and $u' \in L^2(M)$. The Poisson operators, investigated by Melrose and Zworski [30], Hassell and Vasy [18] in the case of scattering metrics, and Joshi and Sá Barreto [24], Graham and Zworski [13], Guillarmou [14] in the case of 0-metrics, are maps from boundary data to generalized eigenfunctions. More precisely, Graham and Zworski showed that, given a choice of boundary defining function x on X, there is a unique family of operators $\mathcal{P}(\zeta)$ satisfying

- For $\{\operatorname{Re}\boldsymbol{\zeta} \geqslant n/2\} \setminus \{\boldsymbol{\zeta} = n/2\}$, the Poisson operator is meromorphic for $\operatorname{Re}\boldsymbol{\zeta} > n/2$ and continues up to $\{\operatorname{Re}\boldsymbol{\zeta} = n/2\} \setminus \{\boldsymbol{\zeta} = n/2\}$;
- $\mathcal{P}(\boldsymbol{\zeta}) : C^\infty(\partial M) \longrightarrow C^\infty(M \setminus \partial M)$,
- $(\Delta - \boldsymbol{\zeta}(n - \boldsymbol{\zeta}))\mathcal{P}(\boldsymbol{\zeta}) \equiv 0$,
- $\mathcal{P}(\boldsymbol{\zeta})f = x^{n-\boldsymbol{\zeta}} f + o(x^{n-\boldsymbol{\zeta}})$, if $\operatorname{Re}\boldsymbol{\zeta} > \frac{n}{2}$,
- $\mathcal{P}(\boldsymbol{\zeta})f = x^{n-\boldsymbol{\zeta}} f + x^{\boldsymbol{\zeta}} f_- + O(x^{n/2+1})$ if $\operatorname{Re}\boldsymbol{\zeta} = n/2, s \neq n/2$, where $f_- \in C^\infty(\partial M)$.

Like the scattering counterpart in [18, Lemma 5.2], we show

Proposition 7.1. — *For any* $\boldsymbol{\lambda} \in [n^2/4, \infty)$*, the spectral measure* $\mathrm{d}E_P(\boldsymbol{\lambda})$ *on an asymptotically hyperbolic manifold* X *of dimension* $n+1$ *can be factorized in terms of the Poisson operator as*

$$\mathrm{d}E_P(\boldsymbol{\lambda}) = -\frac{1}{2\pi}\mathcal{P}\Big(n/2 - \boldsymbol{\lambda}\imath\Big)\mathcal{P}^*\Big(n/2 - \boldsymbol{\lambda}\imath\Big). \tag{7.1}$$

Proof. — It suffices to show that the operators on the LHS and RHS of (7.1) agree acting on an arbitrary $f \in C_c^\infty(X^\circ)$.

Stone's formula gives

$$\mathrm{d}E_P(\boldsymbol{\lambda})f = \frac{\boldsymbol{\lambda}}{\pi\imath}\Big(R\big(n/2 + \imath\boldsymbol{\lambda}\big) - R\big(n/2 - \imath\boldsymbol{\lambda}\big)\Big)f. \tag{7.2}$$

The resolvent expression (3.2) yields

$$R\big(n/2 \pm \imath\boldsymbol{\lambda}\big)f = x^{n/2\pm\imath\boldsymbol{\lambda}}f_\pm,$$

where $f_\pm \in C^\infty(X)$. Let $v_\pm = f_\pm|_{\partial X}$ be the restriction to the boundary of $f_\pm$. Then, from the properties of the Poisson operator listed above, we see that

$$\mathrm{d}E_P(\boldsymbol{\lambda})f = \frac{\boldsymbol{\lambda}}{\pi\imath}\mathcal{P}(n/2 - \imath\boldsymbol{\lambda})(v_+).$$

Thus, to finish the proof, we must show that $v_+ = (2\imath\boldsymbol{\lambda})^{-1}\mathcal{P}(n/2 - \imath\boldsymbol{\lambda})^* f$. This is equivalent to showing that

$$2\imath\boldsymbol{\lambda}\langle v_+, w\rangle_{L^2(\partial X)} = \langle f, \mathcal{P}(n/2 - \imath\boldsymbol{\lambda})w\rangle_{L^2(X)} \tag{7.3}$$

for all $w \in C^\infty(\partial X)$. To do this, we use the pairing formula from [13]:

Proposition 7.2 (Pairing formula). — *For* $\operatorname{Re}\boldsymbol{\zeta} = n/2$ *and* u_1, u_2 *in the form of*

$$u_j = x^{n-\boldsymbol{\zeta}}v_j + x^{\boldsymbol{\zeta}}v_{j,-} + O(x^{n/2+1}), \quad \textit{with}\, v_j, v_{j,-} \in C^\infty(\partial M)$$

such that

$$(\Delta - \boldsymbol{\zeta}(n - \boldsymbol{\zeta}))u_j = r_j \in C_c^\infty(M),$$

we have

$$(2\boldsymbol{\zeta} - n)\int_{\partial M}(v_1\bar{v}_2 - v_{1,-}\bar{v}_{2,-})\,\mathrm{d}h = \int_M (u_1\bar{r}_2 - r_1\bar{u}_2)\,\mathrm{d}g.$$

We apply the pairing formula to

$$u_1 = R\Big(\frac{n}{2} + \imath\boldsymbol{\lambda}\Big)f \quad \text{and} \quad u_2 = \mathcal{P}\Big(\frac{n}{2} - \imath\boldsymbol{\lambda}\Big)w, \quad \boldsymbol{\zeta} = \frac{n}{2} + \imath\boldsymbol{\lambda}$$

with $r_1 = f$ and $r_2 = 0$ and obtain (7.3), as required. □

8. Restriction theorem at high energy

We now prove the restriction theorem, Theorem 1.6, at high energies, $\boldsymbol{\lambda} \geqslant 1$.

The first step, following [17, Section 3], is to reduce the restriction theorem (1.20) to

$$\begin{aligned}(8.1)\quad &\|Q_i(\boldsymbol{\lambda})\mathrm{d}E_P(\boldsymbol{\lambda})Q_i^*(\boldsymbol{\lambda})\|_{L^p\to L^{p'}}\\ &\qquad\leqslant \begin{cases} C\boldsymbol{\lambda}^{(n+1)(1/p-1/p')-1}, & 1\leqslant p\leqslant \frac{2(n+2)}{n+4},\\ C\boldsymbol{\lambda}^{n(1/p-1/2)}, & \frac{2(n+2)}{n+4}\leqslant p<2.\end{cases}\end{aligned}$$

In fact, given (8.1), the factorization (7.1) and the TT^* trick yields

$$\|Q_i(\boldsymbol{\lambda})\mathcal{P}(n/2-\imath\boldsymbol{\lambda})\|_{L^2\to L^{p'}}\leqslant \begin{cases} C\boldsymbol{\lambda}^{(n+1)(1/2-1/p')-1/2}, & 1\leqslant p\leqslant \frac{2(n+2)}{n+4},\\ C\boldsymbol{\lambda}^{n(1/p-1/2)/2}, & \frac{2(n+2)}{n+4}\leqslant p<2,\end{cases}$$

for all $\boldsymbol{\lambda}$. We sum them up over i and deduce the global estimates

$$\|\mathcal{P}(n/2-\imath\boldsymbol{\lambda})\|_{L^2\to L^{p'}}\leqslant \begin{cases} C\boldsymbol{\lambda}^{(n+1)(1/2-1/p')-1/2}, & 1\leqslant p\leqslant \frac{2(n+2)}{n+4},\\ C\boldsymbol{\lambda}^{n(1/p-1/2)/2}, & \frac{2(n+2)}{n+4}\leqslant p<2,\end{cases}$$

for all $\boldsymbol{\lambda}$. Then, using the TT^* trick again but in reverse, we find that

$$\begin{aligned}&\|\mathcal{P}(n/2-\imath\boldsymbol{\lambda})\mathcal{P}^*(n/2-\imath\boldsymbol{\lambda})\|_{L^p\to L^{p'}}\\ &\qquad\leqslant \begin{cases} C\boldsymbol{\lambda}^{(n+1)(1/p-1/p')-1}, & 1\leqslant p\leqslant \frac{2(n+2)}{n+4},\\ C\boldsymbol{\lambda}^{n(1/p-1/2)}, & \frac{2(n+2)}{n+4}\leqslant p<2,\end{cases}\end{aligned}$$

which, by (7.1), is precisely Theorem 1.6.

To prove (8.1), we follow the argument in [17] and [9], and apply complex interpolation to the analytic (in the parameter $a\in\mathbb{C}$) family of operators

$$\phi(P/\boldsymbol{\lambda})\chi_+^a(\boldsymbol{\lambda}-P).$$

Here ϕ is a smooth function supported on $(1-\delta,1+\delta)$ and equal to 1 on $(1-\delta/2,1+\delta/2)$ for some positive δ, and χ_+^a is an entire family of distributions, defined for $\operatorname{Re} a>-1$ by

$$\chi_+^a=\frac{x_+^a}{\Gamma(a+1)}\quad\text{with}\quad x_+^a=\begin{cases}x^a & \text{if } x\geqslant 0,\\ 0 & \text{if } x<0,\end{cases}\qquad \operatorname{Re} a>-1.$$

When $\operatorname{Re} a>0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\chi_+^a=\chi_+^{a-1},$$

and using this identity, we extend χ_+^a to the entire complex a-plane. Since $\chi_+^0 = H(x)$, we have $\chi_+^{-1} = \delta_0$, and more generally $\chi_+^{-k} = \delta_0^{(k-1)}$. Therefore,

$$\chi_+^0(\boldsymbol{\lambda} - P) = E_P\big((0, \boldsymbol{\lambda}]\big) \quad \text{and} \quad \chi_+^{-k}(\boldsymbol{\lambda} - P) = \Big(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\Big)^{k-1} \mathrm{d}E_P(\boldsymbol{\lambda}). \tag{8.2}$$

Moreover, for any $\mu, \nu \in \mathbb{C}$, it is shown in [20, p. 86] that

$$\chi_+^\mu * \chi_+^\nu = \chi_+^{\mu+\nu+1}.$$

Using this identity, and the fact that the $\boldsymbol{\lambda}$-derivatives of the spectral measure are well-defined and obey kernel estimates as in Theorem 1.5, we define, following [17], operators $\chi_+^a(\boldsymbol{\lambda} - P)$. For $k \in \mathbb{N}$ and $-(k+1) < \operatorname{Re} a < 0$, we define

$$\begin{aligned}\chi_+^a(\boldsymbol{\lambda} - P) &= \chi_+^{k+a} * \chi_+^{-(k+1)}(\boldsymbol{\lambda} - P)\\ &= \int_0^{\boldsymbol{\lambda}} \frac{\boldsymbol{\sigma}^{k+a}}{\Gamma(k+a+1)} \Big(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\Big)^k \mathrm{d}E_P(\boldsymbol{\lambda} - \boldsymbol{\sigma})\, \mathrm{d}\boldsymbol{\sigma}.\end{aligned}$$

A standard application of Stein's complex interpolation theorem [33] yields

Proposition 8.1. — *Suppose that, for $s \in \mathbb{R}$, we have*

$$\big\|Q_i(\boldsymbol{\lambda}) \phi(P/\boldsymbol{\lambda}) \chi_+^{\imath s}(\boldsymbol{\lambda} - P) Q_i^*(\boldsymbol{\lambda})\big\|_{L^2(X) \to L^2(X)} \leqslant C_1 e^{C(1+|s|)},$$

and for some $\beta > 0$,

$$\big\|Q_i(\boldsymbol{\lambda}) \phi(P/\boldsymbol{\lambda}) \chi_+^{-\beta+\imath s}(\boldsymbol{\lambda} - P) Q_i^*(\boldsymbol{\lambda})\big\|_{L^1(X) \to L^\infty(X)} \leqslant C_2 e^{C(1+|s|)}.$$

Then, the spectral measure $\mathrm{d}E_P(\boldsymbol{\lambda}) = \chi_+^{-1}(\boldsymbol{\lambda} - P)$ is bounded from

$$L^p(X) \to L^{p'}(X), \ \text{for } p = 2\beta/(\beta+1),$$

with an operator norm bound

$$\big\|\mathrm{d}E_P(\boldsymbol{\lambda})\big\|_{L^p(X) \to L^{p'}(X)} \leqslant C'(C)\, C_1^{(\beta-1)/\beta}\, C_2^{1/\beta}. \tag{8.3}$$

Therefore, to prove (8.1), for $\boldsymbol{\lambda} \geqslant 1$, we need to establish the estimates

$$\big\|Q_i(\boldsymbol{\lambda}) \phi(P/\boldsymbol{\lambda}) \chi_+^{\imath s}(\boldsymbol{\lambda} - P) Q_i^*(\boldsymbol{\lambda})\big\|_{L^2 \to L^2} \leqslant C_1 e^{C(1+|s|)}, \tag{8.4}$$

and for $p \in [1, 2(n+2)/(n+4)]$, we require

$$\big\|Q_i(\boldsymbol{\lambda}) \phi(P/\boldsymbol{\lambda}) \chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda} - P) Q_i^*(\boldsymbol{\lambda})\big\|_{L^1 \to L^\infty} \leqslant C_2 \boldsymbol{\lambda}^{n/2} e^{C(1+|s|)}, \tag{8.5}$$

while for $p \in [2(n+2)/(n+4), 2)$, we require

$$\big\|Q_i(\boldsymbol{\lambda}) \phi(P/\boldsymbol{\lambda}) \chi_+^{-j-1+\imath s}(\boldsymbol{\lambda} - P) Q_i^*(\boldsymbol{\lambda})\big\|_{L^1 \to L^\infty} \leqslant C_2 \boldsymbol{\lambda}^{n/2} e^{C(1+|s|)}, \tag{8.6}$$

for all $j \in \mathbb{Z}, \ j \geqslant \frac{n}{2}$.

Estimate (8.4) follows immediately from the sup bound on the multiplier $\chi_+^{\imath s}$:

$$|\chi_+^{\imath s}(t)| \leqslant \Big|\frac{1}{\Gamma(\imath s)}\Big| \leqslant e^{\pi|s|/2}.$$

For the remaining two estimates, we invoke [17, Lemma 3.3], which we repeat here:

Lemma 8.2. — *Suppose that $k \in \mathbb{N}$, that $-k < a < b < c$ and that $b = \theta a + (1-\theta)c$. Then there exists a constant C such that for any C^{k-1} function $f\colon \mathbb{R} \to \mathbb{C}$ with compact support, one has*

$$\|\chi_+^{b+\imath s} * f\|_\infty \leqslant C(1+|s|)e^{\pi|s|/2}\|\chi_+^a * f\|_\infty^\theta \|\chi_+^c * f\|_\infty^{1-\theta}$$

for all $s \in \mathbb{R}$.

Before proving (8.5) and (8.6), we first rewrite $\phi(P/\boldsymbol{\lambda})\chi_+^{\beta+\imath s}(\boldsymbol{\lambda} - P)$ as a convolution.

$$\begin{aligned}
&\phi(P/\boldsymbol{\lambda})\chi_+^{\beta+\imath s}(\boldsymbol{\lambda} - P)\\
&\quad= \int \phi(\boldsymbol{\sigma}/\boldsymbol{\lambda})\chi_+^{\beta+\imath s+k-1} * \chi_+^{-k}(\boldsymbol{\lambda} - \boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\sigma})\mathrm{d}\boldsymbol{\sigma}\\
&\quad= \iint \phi(\boldsymbol{\sigma}/\boldsymbol{\lambda})\chi_+^{\beta+\imath s+k-1}(\alpha)\chi_+^{-k}(\boldsymbol{\lambda} - \boldsymbol{\sigma} - \alpha)\mathrm{d}E_P(\boldsymbol{\sigma})\,\mathrm{d}\boldsymbol{\sigma}\mathrm{d}\alpha\\
&\quad= \boldsymbol{\lambda}^{\beta+\imath s+1}\iint \phi(\boldsymbol{\sigma})\chi_+^{\beta+\imath s+k-1}(\alpha)\chi_+^{-k}(1 - \boldsymbol{\sigma} - \alpha)\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\,\mathrm{d}\boldsymbol{\sigma}\mathrm{d}\alpha\\
&\quad= \boldsymbol{\lambda}^{\beta+\imath s+1}\int \chi_+^{\beta+\imath s+k-1}(\alpha)\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\big)\Big|_{\boldsymbol{\sigma}=1-\alpha}\mathrm{d}\boldsymbol{\sigma}\mathrm{d}\alpha\\
&\quad= \boldsymbol{\lambda}^{\beta+\imath s+1}\Big(\chi_+^{\beta+\imath s+k-1} * \big(\phi(\,\cdot\,)\mathrm{d}E_P(\boldsymbol{\lambda}\,\cdot)\big)^{(k-1)}\Big)(1).
\end{aligned}$$

We now break the proof up into two cases, n even and n odd (and remind the reader that the dimension of X is $n+1$). We write $n = 2k$ in the former case, and $n = 2k+1$ in the second, with $k \in \mathbb{N}$ in both cases.

First we take the case $n = 2k$ is even. In this case, we take $\beta = -n/2-1 = -k-1$ above, and apply microlocalizing operators on the left and the right. In the following calculations, the operators (that is, their Schwartz kernels) are evaluated at the point (z, z'), which we do not always indicate in notation; also we write $Q(\boldsymbol{\lambda})$ for $Q_i(\boldsymbol{\lambda})$, where i is arbitrary. We obtain

$$\begin{aligned}
(8.7)\quad &Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda} - P)Q^*(\boldsymbol{\lambda})\\
&\quad= \boldsymbol{\lambda}^{-k+\imath s}\Big(\chi_+^{-2+\imath s}(\,\cdot\,) * Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\Big(\phi(\,\cdot\,)\mathrm{d}E_P(\boldsymbol{\lambda}\,\cdot\,)\Big)Q^*(\boldsymbol{\lambda})\Big)(1).
\end{aligned}$$

We now apply Lemma 8.2 with $b=-2$, $a=-3$, $c=-1$ and obtain

$$\begin{aligned}
&\left|Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\right|\\
&\lesssim \boldsymbol{\lambda}^{-k}\sup_{\Lambda}\left|\int \chi_+^{-1}(\alpha)Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda-\alpha}Q^*(\boldsymbol{\lambda})\,\mathrm{d}\alpha\right|^{1/2}\\
&\qquad\times\sup_{\Lambda}\left|\int \chi_+^{-3}(\alpha)Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda-\alpha}Q^*(\boldsymbol{\lambda})\,\mathrm{d}\alpha\right|^{1/2}\\
&\lesssim \boldsymbol{\lambda}^{-k}\sup_{\Lambda}\left|Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda}Q^*(\boldsymbol{\lambda})\right|^{1/2}\\
&\qquad\times\sup_{\Lambda}\left|Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k+1}}{\mathrm{d}\boldsymbol{\sigma}^{k+1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda}Q^*(\boldsymbol{\lambda})\right|^{1/2}.
\end{aligned}$$

We now plug in Corollary 6.5 and get

$$\begin{aligned}
&\left|\Big(Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\Big)(z,z')\right|\\
&\qquad\lesssim \boldsymbol{\lambda}^{-n/2}\sup_{\Lambda\in\boldsymbol{\lambda}\operatorname{supp}\phi}\left(\sum_{l=0}^{k-1}\Lambda^{n-l}(1+\boldsymbol{\lambda}\Lambda d(z,z'))^{-n/2+l}\right)^{1/2}\\
&\qquad\qquad\times\sup_{\Lambda\in\boldsymbol{\lambda}\operatorname{supp}\phi}\left(\sum_{l=0}^{k+1}\Lambda^{n-l}(1+\boldsymbol{\lambda}\Lambda d(z,z'))^{-n/2+l}\right)^{1/2}\lesssim\boldsymbol{\lambda}^{n/2},
\end{aligned}$$

provided $d(z,z')$ is small. On the other hand, if $d(z,z')$ is large, Corollary 6.5 gives

$$\begin{aligned}
&\left|\Big(Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\Big)(z,z')\right|\\
&\qquad\lesssim \boldsymbol{\lambda}^{-n/2}\sup_{\Lambda\in\boldsymbol{\lambda}\operatorname{supp}\phi}\left|\sum_{l=0}^{k-1}\Lambda^{n/2}d(z,z')^l e^{-nd(z,z')}\right|^{1/2}\\
&\qquad\qquad\times\sup_{\Lambda\in\boldsymbol{\lambda}\operatorname{supp}\phi}\left|\sum_{l=0}^{k+1}\Lambda^{n/2}d(z,z')^l e^{-nd(z,z')}\right|^{1/2}\lesssim 1.
\end{aligned}$$

In all these estimates, the dependence of the implicit constant on s is as in Lemma 8.2, consistent with the dependence claimed in (8.5).

When $n = 2k+1$, the proof is almost identical. Instead of (8.7) we use the identity

$$
\begin{aligned}
&Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda}) \\
&\quad = \boldsymbol{\lambda}^{-k+\imath s}\left(\chi_+^{-3/2+\imath s}(\,\cdot\,) * Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\Big(\phi(\,\cdot\,)\mathrm{d}E_P(\boldsymbol{\lambda}\,\cdot\,)\Big)Q^*(\boldsymbol{\lambda})\right)(1).
\end{aligned}
$$

We then apply Lemma 8.2 with $b = -3/2$, $a = -2$, $c = -1$ and obtain

$$
\begin{aligned}
&\left|Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\right| \\
&\qquad = \Bigg|\boldsymbol{\lambda}^{-n/2+\imath s}\int \chi_+^{-3/2+\imath s}(\alpha) \\
&\qquad\qquad Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\Big)\Big|_{\boldsymbol{\sigma}=1-\alpha}Q^*(\boldsymbol{\lambda})\,\mathrm{d}\alpha\Bigg| \\
&\qquad \lesssim \boldsymbol{\lambda}^{-n/2}\sup_{\Lambda}\left|Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k-1}}{\mathrm{d}\boldsymbol{\sigma}^{k-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda}Q^*(\boldsymbol{\lambda})\right|^{1/2} \\
&\qquad\qquad \times \sup_{\Lambda}\left|Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{k}}{\mathrm{d}\boldsymbol{\sigma}^{k}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda}Q^*(\boldsymbol{\lambda})\right|^{1/2}.
\end{aligned}
$$

We then follow the same argument as before to show that

$$
\left|\Big(Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-n/2-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\Big)(z,z')\right| \lesssim \boldsymbol{\lambda}^{n/2}.
$$

To prove (8.6), we use the identity

$$
\begin{aligned}
&Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-j-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda}) \\
&\quad = \boldsymbol{\lambda}^{-j+\imath s}\left(\chi_+^{-2+\imath s}(\,\cdot\,) * Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{j-1}}{\mathrm{d}\boldsymbol{\sigma}^{j-1}}\Big(\phi(\,\cdot\,)\mathrm{d}E_P(\boldsymbol{\lambda}\,\cdot\,)\Big)Q^*(\boldsymbol{\lambda})\right)(1).
\end{aligned}
$$

We then apply Lemma 8.2 with $b = -2$, $a = -3$, $c = -1$ and obtain, as before,

$$\begin{aligned}
&\boldsymbol{\lambda}^j \left| Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-j-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\right| \\
&\lesssim \sup_\Lambda \left| \int \chi_+^{-1}(\alpha)Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{j-1}}{\mathrm{d}\boldsymbol{\sigma}^{j-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda-\alpha} Q^*(\boldsymbol{\lambda})\,\mathrm{d}\alpha\right|^{1/2} \\
&\qquad \sup_\Lambda \left| \int \chi_+^{-3}(\alpha)Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{j-1}}{\mathrm{d}\boldsymbol{\sigma}^{j-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda-\alpha} Q^*(\boldsymbol{\lambda})\,\mathrm{d}\alpha\right|^{1/2} \\
&\lesssim \sup_\Lambda \left| Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{j-1}}{\mathrm{d}\boldsymbol{\sigma}^{j-1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda} Q^*(\boldsymbol{\lambda})\right|^{1/2} \\
&\qquad \times \sup_\Lambda \left| Q(\boldsymbol{\lambda})\frac{\mathrm{d}^{j+1}}{\mathrm{d}\boldsymbol{\sigma}^{j+1}}\Big(\phi(\boldsymbol{\sigma})\mathrm{d}E_P(\boldsymbol{\lambda}\boldsymbol{\sigma})\Big)\Big|_{\boldsymbol{\sigma}=\Lambda} Q^*(\boldsymbol{\lambda})\right|^{1/2}.
\end{aligned}$$

As in the first case, the spectral measure estimates give

$$\begin{aligned}
&\left|\Big(Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-j-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\Big)(z,z')\right| \\
&\quad\lesssim \boldsymbol{\lambda}^{-j} \sup_{\Lambda\in\boldsymbol{\lambda}\operatorname{supp}\phi} \left|\sum_{l=0}^{j-1} \Lambda^{n-l}(1+\boldsymbol{\lambda}\Lambda d(z,z'))^{-n/2+l}\right|^{1/2} \\
&\qquad\qquad \times \sup_{\Lambda\in\lambda\operatorname{supp}\phi} \left|\sum_{l=0}^{j+1} \Lambda^{n-l}(1+\boldsymbol{\lambda}\Lambda d(z,z'))^{-n/2+l}\right|^{1/2} \lesssim \boldsymbol{\lambda}^{n-j},
\end{aligned}$$

provided $d(z,z')$ is small. On the other hand, if $d(z,z')$ is large,

$$\begin{aligned}
&\left|\Big(Q(\boldsymbol{\lambda})\phi(P/\boldsymbol{\lambda})\chi_+^{-j-1+\imath s}(\boldsymbol{\lambda}-P)Q^*(\boldsymbol{\lambda})\Big)(z,z')\right| \\
&\qquad\lesssim \boldsymbol{\lambda}^{-j} \sup_{\Lambda\in\boldsymbol{\lambda}\operatorname{supp}\phi} \left|\sum_{l=0}^{j-1} \Lambda^{n/2} d(z,z')^l e^{-nd(z,z')}\right|^{1/2} \\
&\qquad\qquad \left| \sup_{\Lambda\in\boldsymbol{\lambda}\operatorname{supp}\phi} \sum_{l=0}^{j+1} \Lambda^{n/2} d(z,z')^l e^{-nd(z,z')}\right|^{1/2} \lesssim \boldsymbol{\lambda}^{n/2-j}.
\end{aligned}$$

Remark 8.3. — Notice that it is here that we gain an advantage by working on an asymptotically hyperbolic rather than conic space: the exponential decay $e^{-nd/2}$ in the large distance estimate kills the polynomial growth d^j caused by j differentiations of the phase function $e^{\pm\imath\boldsymbol{\lambda}d}$, so there is no limit to the number of differentiations that we can consider. This is what allows us to obtain a result for p in the range $2(n+2)/(n+4) < p < 2$.

On an asymptotically conic manifold, by contrast, if we differentiate more than $(\dim X - 1)/2$ times, we get a growing kernel as $d(z, z') \to \infty$, and no $L^1 \to L^\infty$ estimate is possible.

9. Spectral multipliers

In this section we prove Theorem 1.8, assuming that (X°, g) is a Cartan–Hadamard manifold, as well as being asymptotically hyperbolic and non-trapping, with no resonance at the bottom of the continuous spectrum.

9.1. A geometric lemma

In order to adapt the proof from Section 2, we need to establish comparability between the Riemannian measure on hyperbolic space, and the Riemannian measure on (X°, g), as expressed in polar coordinates. Recall that on a Cartan–Hadamard manifold, the exponential map from T_pX, $p \in X$ to X is a diffeomorphism from T_pX to X. Thus, the metric on X can be expressed globally in polar normal coordinates based at p. Let r be the distance, and $\omega \in S^n$, be polar normal coordinates based at p.

Lemma 9.1. — *Suppose X is an asymptotically hyperbolic Cartan–Hadamard manifold, and let $p \in X$ be any point. The Riemannian measure on X can be expressed in the form*

$$m_p(r, \omega)(\sinh r)^n \mathrm{d}r\mathrm{d}\omega\,,$$

where $m_p(r, \omega)$ is uniformly bounded on $X \times X$ (that is, uniform in p as well as in (r, ω)).

Proof. — This result can be extracted from the resolvent construction in [10]. Recall that in that paper, the outgoing resolvent $(h^2\Delta - h^2n^2/4 - (1 - \imath 0))^{-1}$ was shown to be a sum of terms, the principal one of which is a semiclassical intersecting Lagrangian distribution

$$I^{1/2}(X_0^2, (N^* \operatorname{diag}_0, \Lambda_+); {}^0\Omega^{1/2}).$$

Here $\Lambda_+ = \Lambda_+^{nd} \cup \Lambda_+^*$ is the closure of the forward bicharacteristic relation, in a certain sense. In the case of a Cartan–Hadamard manifold, the projection ${}^\Phi\pi : {}^\Phi T^* X_0^2 \to X_0^2$ restricts to a diffeomorphism from $\Lambda_+ \setminus N^* \operatorname{diag}_0$ to $X_0^2 \setminus \operatorname{diag}_0$; that is, except over the diagonal, Λ_+ projects diffeomorphically to the base X_0^2. We also point out that there is no need to decompose Λ_+

into pieces $\Lambda_+^{nd} \cup \Lambda_+^*$ as was done in [10] to deal with geodesics that might "return" to the front face FF; this is not possible for Cartan–Hadamard manifolds.

The Lagrangian Λ_+ can be given coordinates as follows: first, we use coordinates (z', ω) for $\Lambda_+ \cap N^* \operatorname{diag}_0$, where z' is a coordinate in X° (corresponding to the right variable in X_0^2) and $\omega \in S^n$ is a coordinate on the unit tangent bundle in $T_{z'}X^\circ$, with respect to the metric g. Then by definition Λ_+ is the flowout from $\Lambda_+ \cap N^* \operatorname{diag}_0$ by bicharacteristic flow, which coincides with geodesic flow in this case. Let r denote the function on Λ_+ equal to the time taken to flow to that point from $\Lambda_+ \cap N^* \operatorname{diag}_0$ by the left geodesic flow. This gives us (z', r, ω) as coordinates on Λ_+. Then, using the projection ${}^\Phi\pi$ to the base, (r, ω) may be identified with polar normal coordinates based at z'.

Now consider the principal symbol at Λ_+. By [10], if we use coordinates (z, z') arising from $X \times X$ on Λ_+ (away from $N^* \operatorname{diag}_0$), then the principal symbol is $\sim (\rho_L \rho_R)^{n/2}$ times $|\mathrm{d}g(z)\mathrm{d}g'(z')|^{1/2}$, where $\mathrm{d}g$ ($\mathrm{d}g'$) indicate the Riemannian measure in the left (right) variables, and we use the notation $a \sim b$ to mean that $C^{-1}b \leqslant a \leqslant Cb$ for some uniform C. Next we recall from Proposition 3.4 that the distance r on X_0^2 is such that $e^{-nr/2} \sim (\rho_L \rho_R)^{n/2}$ for $r \geqslant 1$. It follows that the principal symbol is comparable to

$$e^{-nr/2} |\mathrm{d}g \, \mathrm{d}g'|^{1/2} \tag{9.1}$$

on Λ_+, for $r \geqslant 1$.

On the other hand, the principal symbol a satisfies the transport equation

$$\mathcal{L}_{\partial_r} a = 0 \,,$$

in the coordinates (z', r, ω). Since a is a half-density, it must take the form

$$\Big| b(z', \omega) \mathrm{d}z' \mathrm{d}r \mathrm{d}\omega \Big|^{1/2} .$$

We can compute $b(z', \omega)$ by comparing with the symbol of the resolvent at $N^* \operatorname{diag}$. Using coordinates (z', ω, τ), where τ is the norm on $T^*_{z'}X$ with respect to the metric g (that is, (ω, τ) are polar coordinates in $T^*_{z'}X$), this symbol is $(\tau^2 - 1)^{-1} |\mathrm{d}g' \tau^n \mathrm{d}\tau \mathrm{d}\omega|^{1/2}$. A simple calculation shows that $\{\tau, r\} = 1$. Using [10, (B.2)], we find that

$$a = c \Big| \mathrm{d}g' \mathrm{d}r \mathrm{d}\omega \Big|^{1/2}, \quad c > 0 \text{ constant.} \tag{9.2}$$

Comparing (9.1) and (9.2), we find that

$$\mathrm{d}g \sim e^{nr} \mathrm{d}r \mathrm{d}\omega \quad \text{for } r \geqslant 1 \,,$$

which completes the proof. □

So, let $F \in H^s([-1,1])$, $s > (n+1)/2$, be an even function. We consider the operator $F(\alpha P)$, where $\alpha \in (0,1]$. To analyze this operator, we break the Schwartz kernel into two pieces using the characteristic function $\chi_{d(z,z')\leqslant 1}$. The near-diagonal piece $F(\alpha P)\chi_{d(z,z')\leqslant 1}$ can be treated using the methods from [17]; this operator essentially satisfies Theorem 1.2. The far-from-diagonal piece, $F(\alpha P)(1-\chi_{d(z,z')\leqslant 1})$, can be treated rather like the case of hyperbolic space studied in Section 2.

9.2. Near diagonal part of $F(\alpha P)$

Theorem 1.2 does not apply directly in the current setting, since the volume of balls of radius ρ are not comparable to ρ^{n+1} for large ρ on asymptotically hyperbolic manifolds; instead, the volume grows as $e^{n\rho}$ as $\rho \to \infty$. However, it is certainly the case that the volume of balls of radius $\rho \leqslant 1$ is comparable to ρ^{n+1}. This follows from the Bishop–Gromov inequality: if the sectional curvatures are between 0 and $-\kappa$, say, then the volume of any ball of radius ρ is bounded by the volume in Euclidean space, and the volume on a simply connected space of constant curvature $-\kappa$.

The place where this volume comparability was used in [17] was in the proof of the following Lemma, which we modify so as to apply to our near-diagonal operator.

Lemma 9.2 ([17, Lemma 2.7]). — *Suppose that (X,d,μ) is a metric measure space, with metric d and measure μ, such that the balls of radius $\rho \leqslant 1$ have measure comparable to ρ^{n+1}. Assume that S is an integral operator, bounded from $L^p(X)$ to $L^q(X)$ for some $1 \leqslant p < q \leqslant \infty$. Let $S\chi_{d(z,z')\leqslant s}$, be the integral operator given by the integral kernel of S times the characteristic function of $\{(z,z') \mid d(z,z') \leqslant s\}$, for some $s \leqslant 1$. Then*

$$\|S\chi_{d(z,z')\leqslant s}\|_{L^p\to L^p} \leqslant Cs^{(n+1)(1/p-1/q)}\|S\|_{L^p\to L^q}\,.$$

Proof. — We omit the proof, which is a trivial modification of the proof of [17, Lemma 2.7]. □

Using this lemma we prove a modified version of Theorem 1.2 in an abstract setting.

Proposition 9.3. — *Let (X,d,μ) be as in Lemma 9.2. Suppose L is a positive self-adjoint operator with finite propagation speed on $L^2(X)$. If the restriction estimate*

$$\|\mathrm{d}E_{\sqrt{L}}(\boldsymbol{\lambda})\|_{L^p\to L^{p'}} \leqslant \begin{cases} C & \text{when } \boldsymbol{\lambda} \text{ is small,} \\ C\boldsymbol{\lambda}^{(n+1)(1/p-1/p')-1} & \text{when } \boldsymbol{\lambda} \text{ is large} \end{cases} \tag{9.3}$$

holds for $1 \leqslant p \leqslant 2(n+2)/(n+4)$, *then spectral multipliers localized near the diagonal are uniformly bounded in* $0 < \alpha < 1$, *in the sense*

$$\sup_{0<\alpha<1} \|F(\alpha\sqrt{L})\chi_{d(z,z')\leqslant 1}\|_{L^p\to L^p} \leqslant C\|F\|_{H^s}\,,$$

where $F \in H^s(\mathbb{R})$ *is an even function with* $s > (n+1)(1/p-1/2)$ *supported in* $[-1,1]$.

Remark 9.4. — Note that asymptotically hyperbolic spaces satisfy the measure property in Lemma 9.2. This enables Proposition 9.3 to be applied to the Laplacian on such spaces.

Proof. — We follow the proof of [17, Section 2]. Suppose η is an even smooth function compactly supported on $(-4,4)$, satisfying

$$\sum_{l\in\mathbb{Z}} \eta(2^{-l}t) = 1 \quad \text{for all } t \neq 0\,.$$

Thus we take a partition of unity for $F(\boldsymbol{\lambda})$, say $F(\boldsymbol{\lambda}) = F_0 + \sum_{l>0} F_l(\boldsymbol{\lambda})$, where

$$\begin{aligned} F_0(\boldsymbol{\lambda}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{l\leqslant 0} \eta(2^{-l}t)\hat{F}(t)\cos(t\boldsymbol{\lambda})\,\mathrm{d}t \\ F_l(\boldsymbol{\lambda}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(2^{-l}t)\hat{F}(t)\cos(t\boldsymbol{\lambda})\,\mathrm{d}t \qquad\qquad \text{for } l > 0\,. \end{aligned}$$

By virtue of finite speed of propagation of $\cos(tP)$ [7], i.e.

$$\operatorname{supp}\,\cos(tP) \subset \{d(z,z') \leqslant |t|\}\,,$$

the kernel of $F_l(\alpha P)\chi_{d(z,z')\leqslant 1}$ is supported on

$$\{d(z,z') \leqslant 2^{l+2}\alpha\}$$

as $\eta(2^{-l}t)$ is supported on $(-2^{l+2}, 2^{l+2})$.

By Lemma 9.2,

$$\begin{aligned} \|F(\alpha P)\chi_{d(z,z')\leqslant 1}\|_{L^p\to L^p} &\leqslant \sum_{l\geqslant 0} \|F_l(\alpha P)\chi_{d(z,z')\leqslant 1}\|_{L^p\to L^p} \\ &\leqslant C\sum_{l\geqslant 0} (2^l\alpha)^{(n+1)(1/p-1/2)}\|F_l(\alpha P)\|_{L^p\to L^2}\,. \end{aligned} \tag{9.4}$$

We take a further decomposition

$$F_l(\alpha P) = \psi F_l(\alpha P) + (1-\psi)F_l(\alpha P)$$

by a cutoff function ψ supported on $(-4,4)$ such that $\psi(\boldsymbol{\lambda}) = 1$ for $\boldsymbol{\lambda} \in (-2,2)$.

Then a T^*T argument reduces $\|\psi F_l(\alpha P)\|_{L^p\to L^2}$ to the restriction estimates.

$$\begin{aligned}
&\|\psi F_l(\alpha P)\|^2_{L^p\to L^2}\\
&\quad= \||\psi F_l|^2(\alpha P)\|_{L^p\to L^{p'}}\\
&\quad\leqslant \int_0^{4/\alpha} |\psi F_l(\alpha\boldsymbol{\lambda})|^2 \|\mathrm{d}E_P(\boldsymbol{\lambda})\|_{L^p\to L^{p'}}\,\mathrm{d}\boldsymbol{\lambda}\\
&\quad\leqslant \frac{C}{\alpha}\int_0^{4} |\psi F_l(\boldsymbol{\lambda})|^2 \|\mathrm{d}E_P(\boldsymbol{\lambda}/\alpha)\|_{L^p\to L^{p'}}\,\mathrm{d}\boldsymbol{\lambda}\\
&\quad\leqslant \frac{C}{\alpha}\int_0^{\alpha} |\psi F_l(\boldsymbol{\lambda})|^2\,\mathrm{d}\boldsymbol{\lambda} + \frac{C}{\alpha}\int_\alpha^4 |\psi F_l(\boldsymbol{\lambda})|^2\left(\frac{\boldsymbol{\lambda}}{\alpha}\right)^{(n+1)(1/p-1/p')-1}\mathrm{d}\boldsymbol{\lambda},
\end{aligned}$$

where we used (9.3) in the last line. So

$$\begin{aligned}
\|\psi F_l(\alpha P)\|_{L^p\to L^2} &\leqslant C\alpha^{-((n+1)/2)(1/p-1/p')}\|\psi F_l\|_2\\
&= C\alpha^{-(n+1)(1/p-1/2)}\|\psi F_l\|_2\,.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\sum_{l\geqslant 0}(2^l\alpha)^{(n+1)(1/p-1/2)}\|\psi F_l(\alpha P)\|_{L^p\to L^2}\\
&\quad\leqslant \sum_{l\geqslant 0} 2^{l(n+1)(1/p-1/2)}\|\psi F_l\|_2 \leqslant C\|F\|_{B^{(n+1)(1/p-1/2)}_{1,2}} \leqslant C\|F\|_{H^s}
\end{aligned}\tag{9.5}$$

for $s > (n+1)(1/p-1/2)$.

We next treat the terms involving $(1-\psi)F_l$. This works exactly as in [17, Section 2].

Using restriction estimates as above, we have

$$\begin{aligned}
&\big\|(1-\psi)F_l(\alpha P)\big\|^2_{L^p\to L^2}\\
&\qquad= \big\|\,|(1-\psi)F_l|^2(\alpha P)\big\|_{L^p\to L^{p'}}\\
&\qquad\leqslant \frac{C}{\alpha}\int_2^\infty \Big|(1-\psi)(\boldsymbol{\lambda})F_l(\boldsymbol{\lambda})\Big|^2\left(\frac{\boldsymbol{\lambda}}{\alpha}\right)^{(n+1)(1/p-1/p')-1}\mathrm{d}\boldsymbol{\lambda}
\end{aligned}$$

where we used the fact that $\boldsymbol{\lambda}\geqslant 2$ on the support of $1-\psi$. Note that

$$(1-\psi(\boldsymbol{\lambda}))F_l(\boldsymbol{\lambda}) = \frac{1-\psi(\boldsymbol{\lambda})}{2\pi}\int_{\mathbb{R}}\int_0^1 e^{\imath t(\boldsymbol{\lambda}-\boldsymbol{\lambda}')}\eta(2^{-l}t)F(\boldsymbol{\lambda}')\,\mathrm{d}\boldsymbol{\lambda}'\mathrm{d}t\tag{9.6}$$

and

$$e^{\imath t(\boldsymbol{\lambda}-\boldsymbol{\lambda}')} = \frac{1}{\imath^N(\boldsymbol{\lambda}-\boldsymbol{\lambda}')^N}\frac{\mathrm{d}^N}{\mathrm{d}t^N}e^{\imath t(\boldsymbol{\lambda}-\boldsymbol{\lambda}')},$$

where $\boldsymbol{\lambda}-\boldsymbol{\lambda}' \geqslant \boldsymbol{\lambda}/2$ for $\boldsymbol{\lambda} \in \operatorname{supp} 1-\psi$ and $\boldsymbol{\lambda}' \in \operatorname{supp} F$. Using this identity in (9.6) and integrating by parts in t yields

$$\Big|\big((1-\psi)F_l\big)(\boldsymbol{\lambda})\Big| \leqslant C\boldsymbol{\lambda}^{-N}2^{-N(l-1)}\|F\|_2$$

for any $N \in \mathbb{Z}_+$. Taking N sufficiently large, we obtain

$$\sum_l (2^l\alpha)^{(n+1)(1/p-1/2)}\big\|\big((1-\psi)F_l\big)(\alpha P)\big\|_{L^p\to L^2} \leqslant C\|F\|_{L^2} \leqslant C\|F\|_{H^s}. \tag{9.7}$$

Combining (9.5) and (9.7) yields

$$\sum_l (2^l\alpha)^{(n+1)(1/p-1/2)}\big\|F_l(\alpha P)\big\|_{L^p\to L^2} \leqslant C\|F\|_{H^s},$$

and together with (9.4) this proves the Proposition. □

9.3. Away from the diagonal on asymptotically hyperbolic manifolds

It remains to treat the kernel $F(\alpha P)\chi_{d(z,z')\geqslant 1}$. We will show that

$$F(\alpha P)\chi_{d(z,z')\geqslant 1} \text{ maps } L^p + L^2 \text{ to } L^2,$$

with an operator norm uniform in $\alpha \in (0,1]$. It suffices to show that $F(\alpha P)\chi_{d(z,z')\geqslant 1}$ maps $L^2 \to L^2$, and $L^1 \to L^2$, with operator norms uniform in α. The first statement follows from the fact that $F \in H^{(n+1)/2} \implies F \in L^\infty$, together with the result of Section 9.2. In fact, we have proved

$$\sup_{0<\alpha<1} \|F(\alpha\sqrt{L})\chi_{d(z,z')\leqslant 1}\|_{L^p\to L^p} \leqslant C\|F\|_{H^{(n+1)/2}}$$

provided $1 \leqslant p \leqslant 2(n+2)/(n+4)$. Noting the spectral multiplier is symmetric, we conclude that this operator is L^p bounded for $1 < p < \infty$. Consequently, $F(\alpha\sqrt{L})\chi_{d(z,z')\geqslant 1}$ is L^2 bounded. So in the remainder of this subsection, we show boundedness from L^1 to L^2, with an operator norm uniform in α.

Let $K_\alpha(z,z')$ be the Schwartz kernel of $F(\alpha P)\chi_{d(z,z')\geqslant 1}$. By Minkowski's inequality, the $L^1 \to L^2$ operator norm is bounded by

$$\sup_{z'} \left(\int \big|K_\alpha(z,z')\big|^2 \,\mathrm{d}\mu_z\right)^{1/2}.$$

Using the spectral theorem we have

$$K(z,z') = \int_0^\infty F(\boldsymbol{\lambda})\mathrm{d}E_P(\boldsymbol{\lambda})(z,z') \ \cdot \chi_{\{d(z,z')>1\}}(z,z').$$

We use coordinates (z', r, ω) as in Section 9.1. Using Lemma 9.1, we may estimate the Riemannian measure by $Ce^{nr}\mathrm{d}r\mathrm{d}\omega$. Therefore, it suffices to bound

$$\int_{\{r>1\}} \left| \int_0^\infty F(\alpha\boldsymbol{\lambda})\mathrm{d}E_P(\boldsymbol{\lambda})(r,\omega,z')\,\mathrm{d}\boldsymbol{\lambda}\right|^2 e^{nr}\,\mathrm{d}r\mathrm{d}\omega\,.$$

Using (1.14), we expand the kernel of the spectral measure as follows:

$$\begin{aligned}(9.8)\quad & \mathrm{d}E_P(\boldsymbol{\lambda})(z,z') \\ &= \sum_{\pm} e^{\pm\imath\boldsymbol{\lambda}r}\left(\sum_{j=0}^{[n/2]} \boldsymbol{\lambda}^{n/2-j} b_{\pm,j}(z',r,\omega)e^{-nr/2} + c(\boldsymbol{\lambda},z',r,\omega)e^{-nr/2}\right) \\ &\qquad + (\rho_L\rho_R)^{n/2+\imath\boldsymbol{\lambda}}\, a_+ + (\rho_L\rho_R)^{n/2-\imath\boldsymbol{\lambda}}\, a_- \\ &\qquad + (xx')^{n/2+\imath\boldsymbol{\lambda}}\,\tilde{a}_+ + (xx')^{n/2-\imath\boldsymbol{\lambda}}\,\tilde{a}_-\end{aligned}$$

where $b_{\pm,j}$ and c are bounded, and where $a_\pm, \tilde{a}_\pm$ are as in Theorem 1.3. Here, c is smooth in $\boldsymbol{\lambda}$ at $\boldsymbol{\lambda} = 0$ (due to our assumption that the resolvent kernel is holomorphic at the bottom of the spectrum), and decays as $O(\boldsymbol{\lambda}^{-1/2})$ as $\boldsymbol{\lambda} \to \infty$ for n odd, or $O(\boldsymbol{\lambda}^{-1})$ as $\boldsymbol{\lambda} \to \infty$ for n even. Moreover, c obeys symbolic estimates as $\boldsymbol{\lambda} \to \infty$, so $|d_{\boldsymbol{\lambda}} c| = O(\boldsymbol{\lambda}^{-3/2})$ as $\boldsymbol{\lambda} \to \infty$ when n is odd, or $O(\boldsymbol{\lambda}^{-2})$ when n is even.

We now consider a single term $b_{\pm,j}$ in (9.8). Thus, we need to estimate

$$(9.9)\qquad \int_{\{r>1\}} \left| \int_0^\infty F(\alpha\boldsymbol{\lambda})e^{\pm\imath\boldsymbol{\lambda}r}\boldsymbol{\lambda}^{n/2-j} b_{\pm,j}(r,\omega,z')e^{-nr/2}\,\mathrm{d}\boldsymbol{\lambda}\right|^2 e^{nr}\,\mathrm{d}r\mathrm{d}\omega$$

uniformly in α and z'. Arbitrarily choosing the sign $+$, using the uniform boundedness of $b_{j,\pm}$, and simplifying, it is enough to uniformly bound

$$(9.10)\qquad \int_{\{r>1\}} \left| \int_0^\infty F(\alpha\boldsymbol{\lambda})e^{\imath\boldsymbol{\lambda}r}\boldsymbol{\lambda}^{n/2-j}\,\mathrm{d}\boldsymbol{\lambda}\right|^2 \mathrm{d}r\,.$$

To estimate this, we prove the following lemma.

Lemma 9.5. — *Suppose that $F \in H^{(n+1)/2}(\mathbb{R})$ and*

$$G(\boldsymbol{\lambda}) = \theta(\boldsymbol{\lambda})\boldsymbol{\lambda}^m\phi(\boldsymbol{\lambda})\,,$$

where $\phi \in C_c^\infty(\mathbb{R})$, θ is the Heaviside function, and where $0 < m \leqslant n/2$. Then $\hat{F} * \hat{G}$ satisfies

$$(9.11)\qquad \int_{r\geqslant R} \left|(\hat{F} * \hat{G})(r)\right|^2 \mathrm{d}r = O(R^{-(2m+1)})\,.$$

Proof. — We first observe that $|\hat{G}(r)| \leqslant \langle r\rangle^{-m-1}$. Indeed, since the function $\theta(\boldsymbol{\lambda})\boldsymbol{\lambda}^m$ is homogeneous of degree m, the Fourier transform is homogeneous of degree $-1-m$, and hence is $O(\langle r\rangle^{-1-m})$ as $r \to \infty$. The Fourier transform $\hat{G}$ is therefore this homogeneous function convolved with $\hat{\phi}$. As $\hat{\phi} \in \mathcal{S}(\mathbb{R})$, $\hat{G}$ is L^∞, and still decays as $O(\langle r\rangle^{-1-m})$ as $r \to \infty$.

It therefore suffices to show that

$$\int_{r \geqslant R} \left| \int |\hat{F}(r-s)| \langle s\rangle^{-m-1} \, \mathrm{d}s \right|^2 \mathrm{d}r = O(R^{-(2m+1)}).$$

We break the RHS into

$$\int_{r \geqslant R} \Bigg| \int_{|s| \leqslant r/2} |\hat{F}(r-s)| \langle s\rangle^{-m-1} \, \mathrm{d}s + \int_{|s| \geqslant r/2} |\hat{F}(r-s)| \langle s\rangle^{-m-1} \, \mathrm{d}s \Bigg|^2 \mathrm{d}r. \tag{9.12}$$

Using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we estimate this by

$$2 \int_{r \geqslant R} \Bigg| \int_{|s| \leqslant r/2} |\hat{F}(r-s)| \langle s\rangle^{-m-1} \, \mathrm{d}s \Bigg|^2 \mathrm{d}r + 2 \int_{r \geqslant R} \Bigg| \int_{|s| \geqslant r/2} |\hat{F}(r-s)| \langle s\rangle^{-m-1} \, \mathrm{d}s \Bigg|^2 \mathrm{d}r. \tag{9.13}$$

The first of these terms we treat as follows. We apply Cauchy–Schwarz to the inner integral, obtaining

$$2 \int_{r \geqslant R} \left(\int_{|s| \leqslant r/2} \langle s\rangle^{-m-1} \, \mathrm{d}s \right) \left(\int_{|s'| \leqslant r/2} |\hat{F}(r-s')|^2 \langle s'\rangle^{-m-1} \, \mathrm{d}s' \right) \mathrm{d}r. \tag{9.14}$$

The s integral just gives a constant. In the second integral, we change variable to $r' = r - s'$, and note that $r' \geqslant r - r/2 \geqslant R/2$. The s' integral again gives a constant, and we get an upper bound of the form

$$2C \int_{r' \geqslant R/2} |\hat{F}(r')|^2 \, \mathrm{d}r'. \tag{9.15}$$

We can insert a factor $(2R)^{-(2m+1)}\langle r'\rangle^{n+1}$, since $r' \geqslant 2R$ and $n+1 \geqslant 2m+1$. This finally gives an estimate of the form $CR^{-(2m+1)}\|F\|^2_{H^{(n+1)/2}}$ for the first term of (9.13).

For the second term of (9.13), we estimate $\langle s\rangle^{-m-1} \leqslant C\langle r\rangle^{-m-1}$. This allows us to estimate this term by

$$2\|\hat{F}\|^2_{L^1} \int_{r \geqslant R} \langle r\rangle^{-2m-2} \, \mathrm{d}r \leqslant C\|F\|^2_{H^{1/2+\epsilon}} R^{-(2m+1)} \text{ for any } \epsilon > 0. \tag{9.16}$$

This completes the proof. □

We return to (9.10), which we write in the form

$$\alpha^{-n+2j}\int_{\{r>1\}}\left|\int_0^\infty F(\alpha\boldsymbol{\lambda})e^{\imath\boldsymbol{\lambda}r}(\alpha\boldsymbol{\lambda})^{n/2-j}\,\mathrm{d}\boldsymbol{\lambda}\right|^2\mathrm{d}r\,. \tag{9.17}$$

We change variables to $\boldsymbol{\lambda}'=\alpha\boldsymbol{\lambda}$ and $r'=r/\alpha$. We also choose $\phi\in C_c^\infty(\mathbb{R})$ to be identically 1 on the support of F, and write

$$G(\boldsymbol{\lambda}')=\theta(\boldsymbol{\lambda}'){\boldsymbol{\lambda}'}^{n/2-j}\phi(\boldsymbol{\lambda}')\,.$$

The integral becomes

$$\alpha^{-n-1+2j}\int_{\{r'>1/\alpha\}}\left|\int_{-\infty}^\infty F(\boldsymbol{\lambda}')G(\boldsymbol{\lambda}')e^{\imath\boldsymbol{\lambda}'r'}\,\mathrm{d}\boldsymbol{\lambda}'\right|^2\mathrm{d}r'. \tag{9.18}$$

The $\boldsymbol{\lambda}'$ integral gives us $(\hat{F}*\hat{G})(r')$. Applying Lemma 9.5 with $m=n/2-j$ and $R=\alpha^{-1}$, we see that (9.18) is bounded uniformly in α, as required.

We next consider the terms involving c, $a_\pm$, and $\tilde{a}_\pm$. The argument for all these terms is similar, so just consider c. In this case, we need a uniform bound on

$$\int_{r>1}\left|\int_0^\infty F(\alpha\boldsymbol{\lambda})e^{\pm\imath\boldsymbol{\lambda}r}c(\boldsymbol{\lambda},z',r,\omega)\,\mathrm{d}\boldsymbol{\lambda}\right|^2\mathrm{d}r\,. \tag{9.19}$$

We use the identity

$$e^{\pm\imath\boldsymbol{\lambda}r}=\pm\frac{1}{ir}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}e^{\pm\imath\boldsymbol{\lambda}r}.$$

We integrate by parts. This gives us

$$\int_{r>1}\left|\int_0^\infty e^{\pm\imath\boldsymbol{\lambda}r}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}\Big(F(\alpha\boldsymbol{\lambda})c(\boldsymbol{\lambda},z',r,\omega)\Big)\,\mathrm{d}\boldsymbol{\lambda}\right|^2\frac{1}{r^2}\,\mathrm{d}r\,. \tag{9.20}$$

When the derivative falls on F, we get $\alpha F'(\alpha\boldsymbol{\lambda})$. Since $F\in H^1(\mathbb{R})$, with compact support, the function $\alpha F'(\alpha\boldsymbol{\lambda})$ is L^1, with L^1 norm uniformly bounded in $\boldsymbol{\lambda}$. Since c is uniformly bounded, this gives us a uniform bound on the $\boldsymbol{\lambda}$ integral in (9.20). When the derivative falls on c, using the symbol estimates, we find that $\mathrm{d}_{\boldsymbol{\lambda}}c$ is integrable in $\boldsymbol{\lambda}$, and then we can use the fact that $F\in L^\infty(\mathbb{R})$ to see that in this case also, the $\boldsymbol{\lambda}$ integral in (9.20) is uniformly bounded. Finally, the r integral is convergent, so that establishes the uniform bound on (9.19).

Using the inequalities

$$xx'\leqslant C\rho_L\rho_R\leqslant C'e^{-r}, \tag{9.21}$$

(where the second inequality follows from Proposition 3.4), the same argument works for the $a_\pm$ and $\tilde{a}_\pm$ terms.

This completes the proof of Theorem 1.8.

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Xi CHEN
Shanghai Center for Mathematical Sciences
Fudan University
Shanghai 200433 (China)
xi_chen@fudan.edu.cn

Andrew HASSELL
Mathematical Sciences Institute
Australian National University
Canberra 2601 (Australia)
andrew.hassell@anu.edu.au