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METRIC APPROXIMATIONS OF WREATH PRODUCTS

by Ben HAYES & Andrew W. SALE (*)

Abstract. — Given the large class of groups already known to be sofic, there is seemingly a shortfall in results concerning their permanence properties. We address this problem for wreath products, and in particular investigate the behaviour of more general metric approximations of groups under wreath products.

Our main result is the following. Suppose that H is a sofic group and G is a countable, discrete group. If G is sofic, hyperlinear, weakly sofic, or linear sofic, then $G \wr H$ is also sofic, hyperlinear, weakly sofic, or linear sofic respectively. In each case we construct relevant metric approximations, extending a general construction of metric approximations for $G \wr H$ that uses soficity of H.

Résumé. — On connait aujourd'hui de nombreux groupes sofiques. Néanmoins il existe peu de résultats concernant la stabilité de la propriété de soficité. Ce travail s'intéresse au produit en couronne de groupes sofiques mais aussi de groupes vérifiant des propriétés d'approximations métriques plus générales.

Considérons un groupe sofique H et un groupe dénombrable discret G. Notre résultat principal démontre que si G est sofique, hyperlinéaire, faiblement sofique ou linéairement sofique, alors $G \wr H$ est respectivement sofique, hyperlinéaire, faiblement sofique ou linéairement sofique. Grâce à la soficité de H nous construisons explicitement dans chacun des cas ci-dessus une approximation métrique pour $G \wr H$.

1. Introduction

Sofic groups, introduced by Gromov [13] and developed by Weiss [25], are a large class of groups that can be approximated, in some sense, by finite groups. We consider sofic groups, as well as several other classes of groups which can be similarly defined by metric approximations, namely weakly sofic groups (introduced by Glebsky and Rivera [12]), linear sofic groups

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(introduced by Arzhantseva and Păunescu [1]) and hyperlinear groups (implicitly defined by Connes and explicitly by Rădulescu [5, 22]).

Via their approximations, sofic, hyperlinear, linear sofic, and weakly sofic groups have applications to a wide area of fields. For example, sofic groups are relevant to ergodic theory [2, 16], topological dynamics, in particular Gottschalk's surjunctivity conjecture [13, 16], group rings and Kaplansky's direct finiteness conjecture [8] (also for linear sofic groups [1, Prop. 2.6]), and L^2–invariants [9, 17]. Hyperlinear groups are of interest in operator algebras, particularly the Connes embedding theorem [5], and in group theory, particularly for the Kevare conjecture [20, Cor. 10.4]. We refer the reader to [3, 20] for surveys on sofic and hyperlinear groups.

There are many examples of sofic groups, including all amenable groups, all residually finite groups, and all linear groups (by Malcev's Theorem). However, because of the weakness of the approximation by finite groups, few permanence properties of soficity are properly understood. Relatively straightforward examples include closure under direct product and increasing unions, and the soficity of residually sofic groups. More substantial results generally require some amenability assumption. For example, an amalgamated product of two sofic groups is know to be sofic if the amalgamated subgroup is amenable (see [6, 11, 19, 21]). This was extended to encompass the fundamental groups of all graphs of groups with sofic vertex groups and amenable edge groups [4]. In the same paper, it is shown that the graph product of sofic groups is sofic. Also, if H is a subgroup of G that is sofic and coamenable, then G is sofic too [10].

Our first result is a new permanence property for soficity that concerns wreath products. Recall that the wreath product of two groups G and H is the semidirect product $\bigoplus_H G \rtimes H$

Theorem 1.1. — *Let G, H be countable, discrete, sofic groups. Then $G \wr H$ is sofic.*

When G is abelian, Theorem 1.1 was proved by Păunescu [19], who used methods of analysis and the notion of sofic equivalence relations developed by Elek and Lippner [7]. By Elek and Szabo [10, Thm. 1] it follows that $G \wr H$ is sofic if G is sofic and H is amenable. From this, we may apply the work of Vershik and Gordon on local embeddability into finite groups to see that $G \wr H$ is sofic if G is sofic and H is locally embeddable into amenable groups (this follows from the proof of [24, Prop. 3]). We refer the reader to Holt and Rees [14] for other results on metric approximation of wreath products (e.g. for commutator-contractive length functions).

We remark that, using the Magnus embedding (see [23] for both the original and a modern geometric definition), Theorem 1.1 implies the following (in fact it follows from the weaker version of Păunescu, mentioned above [19]).

Corollary 1.2. — *Let N be a normal subgroup of a finite rank free group F, and let N' be the derived subgroup of N. If F/N is sofic, then F/N' is sofic.*

Proof. — The Magnus embedding is $F/N' \hookrightarrow \mathbb{Z}^r \wr (F/N)$. Since soficity passes to subgroups we therefore get the corollary from Theorem 1.1, or [19]. □

Sofic groups are also weakly sofic, linear sofic and hyperlinear, and we ask to what extent these properties are preserved by wreath products. Weakly sofic groups are a class of groups which can be approximated by finite groups in a weaker sense than sofic groups, namely one is allowed to approximate G by any finite group with any bi-invariant metric, instead of just permutation groups with the Hamming distance, as is the case for soficity (see Section 4 for precise definitions). Linear soficity and hyperlinearity are each classes of groups which can be approximated by linear groups. To be linear sofic requires approximation by general linear groups with respect to the rank metric, while hyperlinearity requires approximation by unitary groups in the normalized Hilbert–Schmidt distance.

Our techniques proving Theorem 1.1 generalize to give the following, broader result.

Theorem 1.3. — *Let G, H be countable, discrete groups and assume that H is sofic. Then:*

(i) *If G is sofic, then so is $G \wr H$,*
(ii) *If G is hyperlinear, then so is $G \wr H$,*
(iii) *If G is linear sofic, then so is $G \wr H$,*
(iv) *If G is weakly sofic, then so is $G \wr H$.*

The proof of Theorem 1.3 is constructive, and is almost entirely self-contained, the exceptions being the use of equivalent definitions of soficity, hyperlinearity, and linear soficity, and a result used for (iii) that concerns the behaviour of Jordan blocks under tensor products. The first step in the proof of Theorem 1.3 is a general result on metric approximations of groups, the proof of which is quantitative (see Proposition 3.3).

Part (iv) follows immediately from Proposition 3.3, while the other three parts require extra constructions.

We remark that the arguments in the matricial cases (ii), (iii) are more delicate, as each of these arguments require tensor products of operators. In (iii), for example, linear soficity of G allows us to find almost homomorphisms $\theta\colon G \to \mathrm{GL}_n(\mathbb{F})$, for some field $\mathbb{F}$, so that $\frac{1}{n}\operatorname{Rank}(\theta(g) - \mathrm{Id})$ is bounded away from zero for $g \in G \setminus \{1\}$. However, this property is not stable under taking tensor products: for example if $\frac{1}{n}\operatorname{Rank}(\theta(g) - 5\,\mathrm{Id})$ and $\frac{1}{n}\operatorname{Rank}(\theta(h) - \frac{1}{5}\,\mathrm{Id})$ are both small for some $g, h \in G$, then $\frac{1}{n^2}\operatorname{Rank}(\theta(g) \otimes \theta(h) - \mathrm{Id})$ will be small. Because of this issue, we have to remark that linear soficity in fact implies that we can find an almost homomorphism $\theta\colon G \to \mathrm{GL}_n(\mathbb{F})$ so that $\inf_{\lambda \in \mathbb{F}\setminus\{0\}} \frac{1}{n}\operatorname{Rank}(\theta(g) - \lambda\,\mathrm{Id})$ is bounded away from zero for $g \in G \setminus \{1\}$. A similar issue occurs in the hyperlinear case, where we find an almost homomorphism $\theta\colon G \to \mathcal{U}(n)$ so that $\theta(g)$ stays a bounded distance away from the scalar matrices (a result of Radulescu [22] enables us to do this for case (ii)). In each of these cases, forcing the image of our group elements to be far away from the scalars is a property that is stable under tensor products. This is a direct computation in the unitary case, whereas the argument that this is true in the general linear case is more involved (see Proposition 4.11).

The structure of the paper is as follows. Section 2 contains the definition of $\mathcal{C}$–approximable groups, and the definition of sofic groups that we use. This section also looks at how we may determine that a map from a wreath product to a group is almost multiplicative, and how we endow our wreath products with suitable metrics. Once this is established, we give the initial construction of the metric approximations of a wreath product in Section 3, before extending this to each of the specific cases of Theorem 1.3 in Section 4.

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2. Preliminaries

We begin with the necessary definitions, as well as a useful lemma to help us identify metric approximations in wreath products. We first establish some notation.

Notation 2.1. — Throughout we will use 1 to denote the identity element of a group (we expect the reader to be able to infer which group it comes from), except when we talk of the identity matrix, when we use Id.

The *metric approximations*, to which we have referred, can be defined as an embedding of a group into a metric ultraproduct of groups, each with a given bi-invariant metric. Such an embedding gives rise to a sequence of maps to the groups in the ultraproduct. It is these maps on which we focus our attention.

Before we define key properties of these maps, we remind the reader that a metric d on a group H is said to be *bi-invariant* if $d(axb, ayb) = d(x, y)$ for all $a, b, x, y \in H$. Throughout the paper we will work with such metrics and their corresponding length functions. We recall that a function $\ell\colon H \to [0, \infty)$ is a *length function* on H if:

- $\ell(h) = \ell(h^{-1})$ for all $h \in H$,
- $\ell(gh) \leqslant \ell(g) + \ell(h)$ for all $g, h \in H$.

We say that ℓ is *conjugacy-invariant* if also $\ell(xgx^{-1}) = \ell(g)$ for all $x, g \in H$.

A conjugacy-invariant length function ℓ on G defines a bi-invariant metric by $d(x, y) = \ell(y^{-1}x)$. Conversely if G has a bi-invariant metric d, then $\ell(x) = d(x, 1)$ is a conjugacy-invariant length function.

Notation 2.2. — When we switch between metrics and length functions we will pair them up with equivalent decorations on the notation. For example, a metric d' will correspond to a length function ℓ'.

Definition 2.3. — *Let H be a group with a bi-invariant metric d. Fix a group G and a function $\theta\colon G \to H$.*

(1) *Given $F \subseteq G$ and $\varepsilon > 0$ we say that θ is* (F, ε, d)–multiplicative *if $\theta(1) = 1$ and*

$$\max_{g,h \in F} d\big(\theta(gh), \theta(g)\theta(h)\big) < \varepsilon.$$

(2) *Given $F \subseteq G$ and a function $c\colon G \setminus \{1\} \to (0, \infty)$ we say that θ is* (F, c, d)–injective *if for all $g \in F \setminus \{1\}$*

$$d\big(\theta(g), 1\big) \geqslant c(g).$$

We remark that we will use the phrases *almost multiplicative* and *almost injective* to mean (F, ε, d)–multiplicative and (F, c, d)–injective respectively when we do not wish to specify F, ε, c and d.

Definition 2.4. — *Let $\mathcal{C}$ be a class of pairs (H, d), where H is a group and d a bi-invariant metric on H (the same group may appear multiple* *times in $\mathcal{C}$ with different metrics). We say that a group G is $\mathcal{C}$–*approximable *if there is a function $c\colon G\setminus\{1\}\to(0,\infty)$ so that for every finite $F\subseteq G$ and $\varepsilon>0$ there is a pair $(H,d)\in\mathcal{C}$ and an (F,ε,d)–multiplicative function $\theta\colon G\to H$ which is also (F,c,d)–injective.*

A special example of $\mathcal{C}$–approximable groups are sofic groups, where $\mathcal{C}$ consists of the finite symmetric groups paired with the normalized Hamming distance (see [9]).

Definition 2.5. — *Let A be a finite set. The* normalized Hamming distance*, denoted d_{Hamm}, on* $\mathrm{Sym}(A)$ *is defined by*

$$d_{\mathrm{Hamm}}(\pi,\tau)=\frac{1}{|A|}\left|\{a\in A:\pi(a)\neq\tau(a)\}\right|.$$

The corresponding length function is denoted ℓ_{Hamm}.

For our purposes, we will use an alternative (but equivalent) definition of soficity (see [9, Thm. 1]).

Definition 2.6. — *Let G be a countable discrete group, F a finite subset of G, and $\varepsilon>0$. Fix a finite set A and a function $\sigma\colon G\to\mathrm{Sym}(A)$. We say that σ is (F,ε)–*free *if*

$$\min_{g\in F\setminus\{1\}}\ell_{\mathrm{Hamm}}(\sigma(g))>1-\varepsilon.$$

*We say that σ is an (F,ε)–*sofic approximation *if it is $(F,\varepsilon,d_{\mathrm{Hamm}})$–multiplicative, and (F,ε)–free. Lastly, we say that G is* sofic *if for every finite $F\subseteq G$ and $\varepsilon>0$, there is a finite set A and an (F,ε)–sofic approximation $\sigma\colon G\to\mathrm{Sym}(A)$.*

Our aim is to start with approximations for G and H and use them to build approximations for the wreath product $G\wr H$. We will use permutational wreath products in our approximations, and we recall here the definitions. Note that all our wreath products are of the restricted variety, meaning we use direct sums rather than direct products.

Definition 2.7. — *Let X be a set on which H acts. The* permutational wreath product *is defined as*

$$G\wr_X H=\bigoplus_X G\rtimes H$$

where the action of $h\in H$ is given via $\alpha_h\in\mathrm{Aut}\left(\bigoplus_X G\right)$, defined by a coordinate shift:

$$\alpha_h\Big((g_x)_{x\in X}\Big)=(g_{h^{-1}x})_{x\in X}.$$

The regular wreath product $G \wr H$ is defined as above, taking $X = H$ with H acting on itself by left-multiplication.

A homomorphism $\varphi\colon G \wr H \to K$, for some group K, can be decomposed into a pair of homomorphisms $\varphi_1\colon \bigoplus_H G \to K$, $\varphi_2\colon H \to K$ which satisfy the following equivariance condition:

$$\varphi_2(h)\varphi_1(g) = \varphi_1(\alpha_h(g))\varphi_2(h), \text{ for all } h \in H,\ g \in \bigoplus_H G.$$

The following lemma gives an analogue to this for the case of almost multiplicative maps.

Lemma 2.8. — *Let G, H be countable, discrete groups, and let*

$$\mathrm{proj}_H\colon G \wr H \to H \text{ and } \mathrm{proj}_G\colon G \wr H \to \bigoplus_H G$$

be the natural projection maps (note that the latter is not a homomorphism). For a finite subset $F_0 \subseteq G \wr H$ define subsets

$$E_1 = \big\{\alpha_h(g) : h \in \mathrm{proj}_H(F_0) \cup \{1\}, g \in \mathrm{proj}_G(F_0)\big\},$$
$$E_2 = \mathrm{proj}_H(F_0).$$

Let $\varepsilon > 0$ and K be a group with a bi-invariant metric d. Suppose $\Theta\colon G \wr H \to K$ is a map with $\Theta(1) = 1$ such that

- *the restriction of Θ to $\bigoplus_H G$ is $(E_1, \varepsilon/6, d)$–multiplicative,*
- *the restriction of Θ to H is $(E_2, \varepsilon/6, d)$–multiplicative,*
- $\max_{g\in E_1, h\in E_2} d\big(\Theta(g,h), \Theta(g,1)\Theta(1,h)\big) < \varepsilon/6$,
- $\max_{g\in E_1, h\in E_2} d\left(\Theta(1,h)\Theta(g,1), \Theta(\alpha_h(g),1)\Theta(1,h)\right) < \varepsilon/6$.

Then Θ is (F_0, ε, d)–multiplicative.

Proof. — Note that if $(g,h), (g',h')$ are in F_0, then $g, g', \alpha_h(g') \in E_1$, and $h, h' \in E_2$. Applying the triangle inequality gives the result. Verification of this is left to the reader. □

In our construction we use maps to groups of the form $L \wr_B \mathrm{Sym}(B)$, for some group L endowed with a bi-invariant metric. To make sense of the notions of almost multiplicativity and almost injectivity we need a bi-invariant metric on this wreath product. The following proposition explains how we do this, using the language of length functions. This was described independently by Holt and Rees [14, §5].

Proposition 2.9. — *Let L be a group with a conjugacy-invariant length function ℓ and suppose that $\ell(g) \leqslant 1$ for all $g \in G$. For a finite set B, define $\tilde{\ell}$ on $L \wr_B \mathrm{Sym}(B)$ by*

$$\tilde{\ell}\big((k_b)_{b\in B}, \tau\big) = \ell_{\mathrm{Hamm}}(\tau) + \frac{1}{|B|} \sum_{\substack{b\in B\\ \tau(b)=b}} \ell(k_b).$$

Then $\tilde{\ell}$ is a conjugacy-invariant length function.

Proof. — We first show that $\tilde{\ell}$ is conjugacy-invariant. Fix $h = (h_b), k = (k_b) \in \bigoplus_B L$ and $\pi, \tau \in \mathrm{Sym}(B)$. Then

$$(k,\tau)^{-1}(h,\pi)(k,\tau) = \big(\alpha_{\tau^{-1}}(k^{-1}h)\alpha_{\tau^{-1}\pi}(k), \tau^{-1}\pi\tau\big).$$

Using the conjugacy-invariance of ℓ_{Hamm} we have:

$$\tilde{\ell}\big((k,\tau)^{-1}(h,\pi)(k,\tau)\big) = \ell_{\mathrm{Hamm}}(\pi) + \frac{1}{|B|} \sum_{\substack{b\in B\\ \tau^{-1}\pi\tau(b)=b}} \ell(k^{-1}_{\tau(b)} h_{\tau(b)} k_{\pi^{-1}\tau(b)}).$$

Note that if $\tau^{-1}\pi\tau(b) = b$, then $\tau(b) = \pi^{-1}\tau(b)$. We can use this to rewrite the summation term above, and then use the conjugacy invariance of ℓ to further simplify it:

$$\frac{1}{|B|} \sum_{\substack{b\in B\\ \pi\tau(b)=\tau(b)}} \ell(k^{-1}_{\tau(b)} h_{\tau(b)} k_{\tau(b)}) = \frac{1}{|B|} \sum_{\substack{b\in B\\ \pi\tau(b)=\tau(b)}} \ell(h_{\tau(b)}) = \frac{1}{|B|} \sum_{\substack{b\in B\\ \pi(b)=b}} \ell(h_b).$$

Thus we see that

$$\tilde{\ell}\big((k,\tau)^{-1}(h,\pi)(k,\tau)\big) = \ell_{\mathrm{Hamm}}(\pi) + \frac{1}{|B|} \sum_{\substack{b\in B\\ \pi(b)=b}} \ell(h_b) = \tilde{\ell}(h,\pi).$$

The proof that $\tilde{\ell}\big((k,\pi)^{-1}\big) = \tilde{\ell}(k,\pi)$ is similar.

We now prove the triangle inequality. Take h, k, π, τ as above. Then

$$\begin{aligned}\ell\big((k,\tau)(h,\pi)\big) &= \ell_{\mathrm{Hamm}}(\tau\pi) + \frac{1}{|B|} \sum_{\substack{b\in B\\ \pi(b)=\tau^{-1}(b)}} \ell(k_b h_{\tau^{-1}(b)})\\ &= \ell_{\mathrm{Hamm}}(\tau\pi) + \frac{1}{|B|} \sum_{\substack{b\in B\\ \pi(b)=\tau^{-1}(b)}} \ell(k_b h_{\pi(b)})\end{aligned}$$

Let $\hat{B} = \{b \in B : \pi(b) = \tau^{-1}(b) \neq b\}$. Then, using the fact that ℓ is bounded by 1 we get

$$\ell\big((k,\tau)(h,\pi)\big) \leqslant \ell_{\mathrm{Hamm}}(\tau\pi) + \frac{1}{|B|}\left(|\hat{B}| + \sum_{\substack{b\in B\\ \tau(b)=b}} \ell(k_b) + \sum_{\substack{b\in B\\ \pi(b)=b}} \ell(h_b)\right).$$

From the above we see that it is enough to show that

$$\ell_{\mathrm{Hamm}}(\tau\pi) + \frac{|\hat{B}|}{|B|} \leqslant \ell_{\mathrm{Hamm}}(\tau) + \ell_{\mathrm{Hamm}}(\pi)\,.$$

Using the definition of the Hamming distance, we need

$$|\{b : \pi(b) \neq \tau^{-1}(b)\}| + |\hat{B}| \leqslant |\{b \in B : \tau(b) \neq b\}| + |\{b \in B : \pi(b) \neq b\}|\,.$$

Since $\hat{B} \subseteq \{b \in B : \pi(b) = b\}$, we get that the above is the same as:

$$\begin{aligned}&|\{b : \pi(b) \neq \tau^{-1}(b)\}|\\ &\qquad\leqslant |\{b \in B : \tau^{-1}(b) \neq b\}| + |\{b \in B : \pi(b) \neq b, \pi(b) \neq \tau^{-1}(b)\}|\,,\end{aligned}$$

which we can deduce from the inclusion

$$\begin{aligned}\{b : \pi(b) \neq \tau^{-1}(b)\} \subseteq &\{b \in B : \tau^{-1}(b) \neq b\}\\ &\cup \{b \in B : \pi(b) \neq b, \pi(b) \neq \tau^{-1}(b)\}\,.\end{aligned}$$

This completes the proof of the triangle inequality and thus of Proposition 2.9. □

3. Construction of the Approximation

In the following we let G, H, K be groups, B a finite set, and we suppose that functions $\theta\colon G \to K$ and $\sigma\colon H \to \mathrm{Sym}(B)$ (not necessarily homomorphisms) are given.

3.1. Some intuition

To give some idea of the intuition behind the construction that follows, consider first how one can think of an element of a wreath product $G \wr H$. One may consider $(g,h) \in G \wr H$, where $g = (g_x)_{x\in H}$, as a journey through H, starting at the identity, finishing at h, and picking up elements of G at selected points of H (namely, pick up g_x at x whenever $g_x \neq 1$).

If both G, H are sofic, we wish to construct a finite model for $G \wr H$ using symmetric groups (here $K = \mathrm{Sym}(A)$). A sofic approximation, roughly speaking, gives us a finite set (A or B), inside of which a significant part of the set behaves like a prescribed finite subset of G or H respectively (see e.g. [13, p. 157], [8, Prop. 4.4]). We ultimately seek such a set for $G \wr H$, and first we may try to combine A and B in a way which mimics the wreath product of groups. However this approach leads to a problem.

The problem is that in the approximation of H using B there is no prescribed point in B representing the identity. Thus the "journey" through H from the identity to h will translate to a "journey" in B from β to $b = \sigma(h)\beta$, where the choice of β is arbitrary, and may be allowed to vary.

It is for this reason that, in the construction below, we use $\bigoplus_B K$, rather than just K, in the wreath product $(\bigoplus_B K) \wr_B \mathrm{Sym}(B)$ that we map into. This could be interpreted as using one copy of K for each choice of "identity vertex" in B.

3.2. The construction

The aim of this section is to define an approximation of $G \wr H$ into a wreath product that is, in some sense, smaller, or more controllable, than $G \wr H$. In Section 4 this approximation is then used to prove each part of Theorem 1.3, composing it with a further approximation into the specific type of group for each property. The only exception is part (iv), where weak soficity follows immediately from this construction and Proposition 3.3. Given a group K and a set B, we consider the wreath product

$$\left(\bigoplus_B K\right) \wr_B \mathrm{Sym}(B) = \bigoplus_B \left(\bigoplus_B K\right) \rtimes \mathrm{Sym}(B),$$

so $\pi \in \mathrm{Sym}(B)$ acts on $\bigoplus_B (\bigoplus_B K)$ by α_π where

$$\alpha_\pi\big((k_b)_{b\in B}\big) = (k_{\pi^{-1}(b)})_{b\in B}, \text{ if } k_b \in \textstyle\bigoplus_B K \text{ for all } b \in B.$$

Note that if we identify $k \in \bigoplus_B (\bigoplus_B K)$ with an element $(k_{b,\beta})_{b,\beta\in B}$ in $\bigoplus_{B\oplus B} K$, then

$$\alpha_\pi((k_{b,\beta})) = (k_{\pi^{-1}(b),\beta}).$$

Notation 3.1. — When we encounter sets of the form $\bigoplus_B(\bigoplus_B A)$, we will use roman subscripts to identify the outer index, and greek subscripts to identify the inner index. For example, for $a = (a_b)_{b\in B}$, we have each $a_b \in \bigoplus_B A$, expressed as $a_b = (a_{b,\beta})_{\beta\in B}$.

Given the maps $\theta\colon G \to K$ and $\sigma\colon H \to \mathrm{Sym}(B)$ and a finite subset E of H we define

$$\Theta : G \wr H \to \bigoplus_B K \wr_B \mathrm{Sym}(B)$$

by $\Theta(g,h) = (\theta_B(g), \sigma(h))$, where θ_B is a map we proceed to define below.

We use the finite subset $E \subset H$ to define a subset B_E of B, given as the intersection $B = B_1 \cap B_2$, where

$$\begin{aligned} B_1 &= \{b \in B : \sigma(h_1)^{-1}b \neq \sigma(h_2)^{-1}b \text{ for all } h_1, h_2 \in E, h_1 \neq h_2\},\\ B_2 &= \{b \in B : \sigma(h_1h_2)^{-1}b = \sigma(h_2)^{-1}\sigma(h_1)^{-1}b \text{ for all } h_1, h_2 \in E\}. \end{aligned}$$

If we consider an element of $\bigoplus_B(\bigoplus_B K)$ as a function $B\times B \to K$, where we follow the convention of Notation 3.1, then θ_B is the map defined by

$$(g_h)_{h\in H} \mapsto \begin{cases} (b,\beta) \mapsto \theta(g_{h_0}), & \text{if } b \in B_E, \text{ if } g_h = 1 \text{ for all } h \in H \setminus E,\\ & \qquad\qquad \text{and } h_0 \in H \text{ is so that } b = \sigma(h_0)\beta;\\ (b,\beta) \mapsto 1, & \text{otherwise.} \end{cases}$$

Referring back to the intuition of Section 3.1, we explain what happens when we fix β, the second coordinate in $B \times B$. This coordinate is the inner index for an element of $\bigoplus_B(\bigoplus_B K)$, and corresponds to a choice of a copy of K in the wreath product $\bigoplus_B K \wr_B \mathrm{Sym}(B)$. If $\beta \in \bigcap_{x\in E} \sigma(x)^{-1}B_E$, then θ_B restricts to a map that sends $(g_x)_{x\in E}$ to an element of $\bigoplus_B K$, where for each $x \in E$, the $\sigma(x)\beta$–coordinate is given by $\theta(g_x)$, and coordinates not of the form $\sigma(x)\beta$ for $x \in E$ are trivial. Thus, we can think of β as behaving as the chosen "identity vertex" in B. If an element (g,h) of $G \wr H$ is a journey through H, starting at 1 and picking up elements of G en route to h, then under Θ this turns into a collection of journeys through B, each starting at a suitable choice of β, and finishing at $\sigma(h)\beta$. The map θ_B tells you what elements of K to pick up along the way. If the original journey visited a vertex $x \in E$, then the image journey starting at β will pick up $\theta(g_x)$ at $\sigma(x)\beta$. This is visualized in Figure 3.1

We now give an equivalent definition of θ_B. This is necessary in order to establish further maps and notation which will be used later on. For $h \in H, b \in B$, define

$$\theta_b^{(h)}\colon G \to \bigoplus_B K$$

by $\theta_b^{(h)}(g) = (k_\beta)_{\beta\in B}$ where

$$k_\beta = \begin{cases} \theta(g), & \text{if } \beta = \sigma(h)^{-1}b,\\ 1, & \text{otherwise.} \end{cases}$$

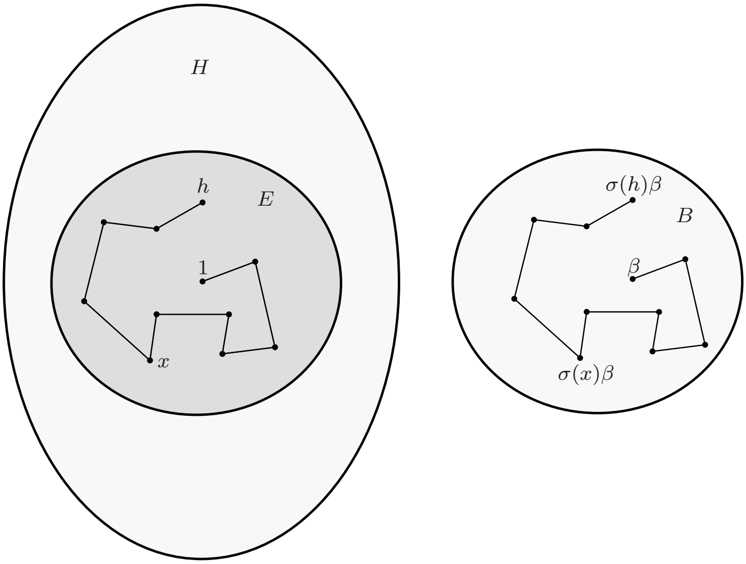

Figure 3.1. *The journey through H corresponding to (g,h) on the left, when $g \in \bigoplus_E H$, and the journey in the image of Θ corresponding to choosing β to play the role of the identity. The image $\Theta(g,h)$ will be made up of multiple such journeys, one for each suitable choice of β.*

Note that $\theta_b^{(h_1)}(g_1)$ and $\theta_b^{(h_2)}(g_2)$ commute if $b \in B_1, h_1, h_2 \in E, g_1, g_2 \in G$ and $h_1 \neq h_2$. Thus it makes sense to define, for $b \in B_E$,

$$\theta_b \colon \bigoplus_E G \to \bigoplus_B K$$

by

$$\theta_b\Big((g_h)_{h\in E}\Big) = \prod_{h\in E} \theta_b^{(h)}(g_h).$$

In our applications σ will be a sofic approximation, so we can think of B_E as making up the majority of B. Thus θ_b will be defined for "most" $b \in B$. We extend θ_b to be defined for all $b \in B$ by saying that θ_b maps everything to the identity for $b \in B \setminus B_E$.

Relating this to the intuition described above, for $g \in \bigoplus_E G$ and $\beta \in B$, the β–coordinate of $\theta_b(g)$ tells you what element of K to pick up at b if β is chosen as the "identity vertex".

We then obtain our equivalent definition of $\theta_B \colon \bigoplus_E G \to \bigoplus_B (\bigoplus_B K)$ by packaging all these maps together as a single map

$$\theta_B(g) = \big(\theta_b(g)\big)_{b \in B}$$

and extending θ_B to $\bigoplus_H G$ by declaring that $\theta_B(g) = 1$ if $g \in \bigoplus_H G$, but $g \notin \bigoplus_E G$.

We will prove that if K has a bi-invariant metric and θ and σ are almost multiplicative and almost injective, then Θ gives us our desired almost multiplicative and almost injective map. To do this, we need to use an appropriate bi-invariant length function on $(\bigoplus_B K) \wr_B \mathrm{Sym}(B)$.

Notation 3.2. — Let ℓ' be a conjugacy-invariant length function on $\bigoplus_B K$ such that $\ell' \leqslant 1$. Take $\tilde{\ell}$ to be the conjugacy-invariant length function on $(\bigoplus_B K) \wr_B \mathrm{Sym}(B)$ defined in Proposition 2.9. When ℓ' is the specific length function $\ell_{\max}$ defined on $\bigoplus_B K$ by

$$\ell_{\max}\big((k_b)_{b \in B}\big) = \max_{b \in B} \ell(k_b)\,,$$

where ℓ is a given length function on K such that $\ell \leqslant 1$, we denote the length function obtained from Proposition 2.9 by $\tilde{\ell}_{\max}$, to emphasise the specific choice of ℓ'.

In summary, we will be dealing with the following length functions, with corresponding metrics:

- ℓ on K, corresponding to d, and such that $\ell \leqslant 1$;
- ℓ' on $\bigoplus_B K$, corresponding to d', and such that $\ell' \leqslant 1$;
- $\ell_{\max}$ on $\bigoplus_B K$—we never refer to the corresponding metric;
- $\tilde{\ell}$ on $(\bigoplus_B K) \wr_B \mathrm{Sym}(B)$, corresponding to $\tilde{d}$;
- $\tilde{\ell}_{\max}$ on $(\bigoplus_B K) \wr_B \mathrm{Sym}(B)$, corresponding to $\tilde{d}_{\max}$.

Our aim is to prove the following.

Proposition 3.3. — *Let $F \subseteq G \wr H$ be finite and $\varepsilon > 0$. There are finite subsets $E_G \subseteq G$ and $E, E_H \subseteq H$, and an $\varepsilon' > 0$ with the following properties. Let*

- *$\sigma \colon H \to \mathrm{Sym}(B)$ be an (E_H, ε')–sofic approximation,*
- *$\theta : G \to K$ be a map,*
- *ℓ, d, ℓ', d', $\tilde{\ell}$, $\tilde{d}$, $\tilde{\ell}_{\max}$, and $\tilde{d}_{\max}$ be as described in Notation 3.2.*

Then $\Theta : G \wr H \to (\bigoplus_B K) \wr_B \mathrm{Sym}(B)$, as constructed above using E, θ and σ, has the following properties.

1. *Suppose the length function ℓ' on $\bigoplus_B K$ restricts to ℓ on each copy of K. If $\theta \colon G \to K$ is (E_G, ε', d)–multiplicative, then Θ is $(F, \varepsilon, \tilde{d})$–multiplicative.*

(2) *Let c be a map $c\colon G\setminus\{1\}\to(0,\infty)$. Define $c'\colon (G\wr H)\setminus\{1\}\to(0,\infty)$ by*

$$c'(g,h)=\begin{cases}\frac{1}{2}, & \text{if } h\neq 1\\ \max\limits_{x\in\operatorname{Supp}(g)}\frac{1}{2}c(g_x), & \text{if } h=1,\ g=(g_x)_{x\in H}.\end{cases}$$

Then, if $\theta\colon G\to K$ is (E_G,c,d)–injective then Θ is $(F,c',\tilde{d}_{\max})$–injective.

The remainder of this section is dedicating to proving Proposition 3.3. We will see below that, once E is given, the following upper bounds on ε' are sufficient:

$$\begin{aligned}&\text{for (1):}\quad \varepsilon'<\frac{\varepsilon}{48|E|^2},\\ &\text{for (2):}\quad \varepsilon'<\frac{1}{16\,|E|^2}\min\big\{c(g),1\mid g\in E_G\setminus\{1\}\big\}.\end{aligned}\tag{3.1}$$

As we see, the bounds on ε' depend only on ε and the set F.

We remark that $\Theta(1,1)=1$ by construction. We first explain how to define the sets E, E_G and E_H.

Let $F\subseteq G\wr H$ be finite and $\varepsilon>0$. Define projections $\operatorname{proj}_G\colon G\wr H\to\bigoplus_H G$ and $\operatorname{proj}_H\colon G\wr H\to H$ by $\operatorname{proj}_G(g,h)=g$ and $\operatorname{proj}_H(g,h)=h$. Let E_1,E_2 be as in Lemma 2.8 for the finite set $F_0=F\cup\{1\}\cup F^{-1}$:

$$E_1=\big\{\alpha_h(g):h\in\operatorname{proj}_H(F_0),g\in\operatorname{proj}_G(F_0)\big\}\subseteq\bigoplus_E G,\quad E_2=\operatorname{proj}_H(F_0)\subseteq H.$$

Recall that for $g=(g_x)_{x\in H}\in\bigoplus_H G$ the support of g, denoted $\operatorname{Supp}(g)$, is the set of $x\in H$ with $g_x\neq 1$. We set

$$E=E_2\cup\bigcup_{\substack{g\in E_1\\ h\in E_2}}h\operatorname{Supp}(g),\ \ E_G=\big\{g_x\in G:(g_x)\in E_1,x\in H\big\},\ \ E_H=E^{-1}E.$$

Since E_2 contains the identity, it follows that E and E^{-1} are both subsets of E_H.

Let K be as in Proposition 3.3 and let $\theta\colon G\to K$ be (E_G,ε',d)–multiplicative and (E_G,c,d)–injective, where ε' is controlled by the bounds in (3.1) above. Let $\sigma\colon H\to\operatorname{Sym}(B)$ be a (E_H,ε')–sofic approximation. Recall the set B_E is defined from E as the intersections of sets B_1,B_2 (which depend only on E). Lemma 3.4 confirms that, since σ is a sofic approximation, B_E makes up a significant proportion of the set B.

Lemma 3.4. — *Let $\kappa > 0$. If $\varepsilon' < \frac{\kappa}{4|E|^2}$ then $|B \setminus B_E| \leqslant \kappa |B|$.*

Proof. — Note that

$$B \setminus B_1 = \bigcup_{\substack{h_1,h_2 \in E \\ h_1 \neq h_2}} \{b \in B : \sigma(h_1)^{-1} b = \sigma(h_2)^{-1} b\}.$$

Since $E_H \supseteq E \cup E^{-1}$, by (E_H, ε')–soficity of σ we have

$$d_{\mathrm{Hamm}}(\sigma(h_2)^{-1}, \sigma(h_2^{-1})) < \varepsilon'.$$

Thus for $h_1 \neq h_2$, we have

$$\begin{aligned} \frac{\left|\{b \in B : \sigma(h_1)^{-1} b = \sigma(h_2)^{-1} b\}\right|}{|B|} &= d_{\mathrm{Hamm}}(\sigma(h_1)^{-1}, \sigma(h_2)^{-1}) \\ &= 1 - \ell_{\mathrm{Hamm}}(\sigma(h_1)\sigma(h_2)^{-1}) \\ &< 1 - \ell_{\mathrm{Hamm}}(\sigma(h_1)\sigma(h_2^{-1})) + \varepsilon' \\ &\leqslant 1 - \ell_{\mathrm{Hamm}}(\sigma(h_1 h_2^{-1})) + 2\varepsilon' \\ &< 3\varepsilon', \end{aligned}$$

where in the last two lines we again use that $E_H \supseteq E \cup E^{-1} \cup E^{-1}E$. Thus

$$\frac{|B \setminus B_1|}{|B|} \leqslant 3\,|E|^2\,\varepsilon'.$$

Similarly, $(E_H, \varepsilon', d_{\mathrm{Hamm}})$–multiplicativity of σ gives

$$\frac{|B \setminus B_2|}{|B|} \leqslant \sum_{h_1,h_2 \in E} \Big(1 - d_{\mathrm{Hamm}}\big(\sigma(h_1 h_2), \sigma(h_1)\sigma(h_2)\big)\Big) \leqslant |E|^2\,\varepsilon'.$$

This proves the lemma. □

Use the set $E \subset H$ and the maps θ, σ to define the maps θ_B, Θ, as constructed at the start of this section.

3.3. Part (1) of Proposition 3.3

We claim that if ε' is sufficiently small, then the map Θ is $(F, \varepsilon, \tilde{d})$–multiplicative.

Take $\kappa > 0$ so that $\kappa < \frac{\varepsilon}{12}$, and take $\varepsilon' > 0$ satisfying the hypothesis of Lemma 3.4, so we will have $\varepsilon' < \frac{\varepsilon}{48|E|^2}$.

We now apply Lemma 2.8, verifying below the four necessary conditions to show that Θ is $(F, \varepsilon, \tilde{d})$–multiplicative. We first check that it is $(E_1, \varepsilon/6, \tilde{d})$–multiplicative when restricted to $\bigoplus_H G$. Recall that throughout Proposition 3.3 we assume that $\ell \leqslant 1$ and $\ell' \leqslant 1$, while in part (1) we assume furthermore that ℓ' restricts to ℓ on each copy of K. Let $g, g' \in E_1$ with $g = (g_x)_{x\in H}, g' = (g'_x)_{x\in H}$. Since $E_1 \subset \bigoplus_E G$, we may apply θ_b to g, g', and gg'. Then

$$\begin{aligned}\tilde{d}(\theta_B(g)\theta_B(g'), \theta_B(gg')) &= \frac{1}{|B|}\sum_{b\in B} d'\big(\theta_b(g)\theta_b(g'), \theta_b(gg')\big)\\ &\leqslant \kappa + \frac{1}{|B|}\sum_{b\in B_E} d'\big(\theta_b(g)\theta_b(g'), \theta_b(gg')\big).\end{aligned}$$

By the definitions of θ_b and of E_1, we realise that each component of $\theta_b(gg')^{-1}\theta_b(g)\theta_b(g')$ is either 1 or $\theta(g_x g'_x)^{-1}\theta(g_x)\theta(g'_x)$, for $x \in E$. Thus

$$\begin{aligned}\tilde{d}(\theta_B(g)\theta_B(g'), \theta_B(gg')) &\leqslant \kappa + \frac{1}{|B|}\sum_{b\in B_E}\sum_{x\in E} d\big(\theta(g_x)\theta(g'_x), \theta(g_x g'_x)\big)\\ &\leqslant \kappa + |E|\,\varepsilon',\end{aligned}$$

where in the last line we use that θ is (E_G, ε', d)–multiplicative. Since $\kappa + |E|\,\varepsilon' < \frac{\varepsilon}{6}$, we see that Θ is $(E_1, \varepsilon/6, \tilde{d})$–multiplicative.

The fact that the restriction to H is $(E_2, \varepsilon/6, \tilde{d})$–multiplicative is more straightforward. Indeed, for $h, h' \in E_2$ we have

$$\tilde{d}(\sigma(hh'), \sigma(h)\sigma(h')) = d_{\mathrm{Hamm}}(\sigma(hh'), \sigma(h)\sigma(h')) < \varepsilon',$$

where we note that we can use the multiplicative property of σ since $E_2 \subseteq E_H$.

By construction, the third condition of Lemma 2.8, bounding the distance between $\Theta(g, h)$ and $\Theta(g, 1)\Theta(1, h)$, is automatically satisfied by Θ, since these elements are equal.

We finish part (1) by verifying the bound on

$$\tilde{d}\big(\Theta(1, h)\Theta(g, 1), \Theta(\alpha_h(g), 1)\Theta(1, h)\big) \text{ for } g \in E_1, h \in E_2\,.$$

We have

$$\begin{aligned}&\tilde{d}\big(\Theta(1, h)\Theta(g, 1), \Theta(\alpha_h(g), 1)\Theta(1, h)\big)\\ &\qquad= \tilde{d}\big((\alpha_{\sigma(h)}(\theta_B(g)), \sigma(h)), (\theta_B(\alpha_h(g)), \sigma(h))\big)\\ &\qquad= \frac{1}{|B|}\sum_{b\in B} d'\big(\theta_{\sigma(h)^{-1}b}(g), \theta_b(\alpha_h(g))\big)\\ &\qquad= \frac{1}{|B|}\sum_{b\in B} d'\big(\theta_b(g), \theta_{\sigma(h)b}(\alpha_h(g))\big)\,.\end{aligned}$$

Using Lemma 3.4, and that $\ell' \leqslant 1$, we can disregard what happens for b outside of both B_E and $\sigma(h)^{-1}B_E$ for a controlled cost. This gives us the following upper bound for the above distance:

$$2\kappa + \frac{1}{|B|} \sum_{b\in B_E\cap\sigma(h)^{-1}B_E} d'\big(\theta_b(g), \theta_{\sigma(h)b}(\alpha_h(g))\big).$$

Since $\mathrm{Supp}(\alpha_h(g)) = h\,\mathrm{Supp}(g)$, and E contains both $\mathrm{Supp}(g)$ and $h\,\mathrm{Supp}(g)$, it follows that for every $b \in B_E \cap \sigma(h)^{-1}(B_E)$ we have

$$\theta_{\sigma(h)b}(\alpha_h(g)) = \prod_{x\in h\,\mathrm{Supp}(g)} \theta^{(x)}_{\sigma(h)b}(g_{h^{-1}x}) = \prod_{x\in \mathrm{Supp}(g)} \theta^{(hx)}_{\sigma(h)b}(g_x).$$

Note that we have used that $\theta(1) = 1$ to restrict the number of terms in the product. We use that for $h \in E$ (and hence for $h \in E_2$) and $b \in B_E \cap \sigma(h)^{-1}B_E$ we have that $\theta^{(hx)}_{\sigma(h)b}(g) = \theta^{(x)}_b(g)$. Inserting this into the above equation we see that

$$\theta_{\sigma(h)b}(\alpha_h(g)) = \prod_{x\in \mathrm{Supp}(g)} \theta^{(x)}_b(g_x) = \theta_b(g).$$

Returning to the above inequality, we have shown that

$$\frac{1}{|B|} \sum_{b\in B_E\cap\sigma(h)^{-1}B_E} d'(\theta_b(g), \theta_{\sigma(h)b}(\alpha_h(g))) = 0$$

so

$$\tilde{d}\big(\Theta(1,h)\Theta(g,1), \Theta(\alpha_h(g),1)\Theta(1,h)\big) < 2\kappa < \frac{\varepsilon}{6}.$$

This completes the proof of part (1) of Proposition 3.3.

3.4. Part (2) of Proposition 3.3

We now show that Θ is $(F, c', \tilde{d}_{\max})$–injective, when the length function ℓ' on $\bigoplus_B K$ is $\ell_{\max}$.

In order to get (2) we will need to further restrict the size of κ (and hence also of ε'). We take κ small enough so that, in addition to having $\kappa < \frac{\varepsilon}{12}$, we also have

$$\kappa < \frac{1}{4} \min\big\{c(g), 1 \mid g \in E_G \setminus \{1\}\big\}.$$

First suppose $(g,h) \in F$. If $h \neq 1$, then

$$\tilde{\ell}_{\max}\big((\theta_B(g), \sigma(h))\big) \geqslant \ell_{\mathrm{Hamm}}(\sigma(h)) \geqslant 1 - \varepsilon' \geqslant 1/2 = c'(g,h).$$

We may therefore assume that $h=1$. Let $g=(g_x)_{x\in E}$. We then have that

$$\begin{aligned}\tilde{\ell}_{\max}\big((\theta_B(g),1)\big) &= \frac{1}{|B|}\sum_{b\in B}\ell_{\max}(\theta_b(g))\\ &\geqslant -\kappa+\frac{1}{|B|}\sum_{b\in B_E}\ell_{\max}(\theta_b(g))\end{aligned}$$

using Lemma 3.4 to obtain the inequality. Since for $b\in B_E$ the components of $\theta_b(g)$ are either 1 or $\theta(g_x)$ for some $x\in E$, we get $\ell_{\max}(\theta_b(g)) = \max_{x\in E}\ell(\theta(g_x))$. Hence

$$\begin{aligned}\tilde{\ell}_{\max}\big((\theta_B(g),1)\big) &\geqslant -\kappa+\frac{|B_E|}{|B|}\max_{x\in E}\ell(\theta(g_x))\\ &\geqslant -\kappa+(1-\kappa)\max_{x\in \mathrm{Supp}(g)}c(g_x)\end{aligned}$$

where the last inequality follows from Lemma 3.4 and the fact that θ is (E_G,c,d)–injective. By the choices of κ and E_G, we get

$$\begin{aligned}&-\kappa+(1-\kappa)\max_{x\in \mathrm{Supp}(g)}c(g_x)\\ &\qquad\geqslant \frac{-1}{4}\max_{x\in \mathrm{Supp}(g)}c(g_x)+\left(1-\frac{1}{4}\right)\max_{x\in \mathrm{Supp}(g)}c(g_x)=c'(g,1)\,.\end{aligned}$$

This verifies that Θ is $(F,c',\tilde{d}_{\max})$–injective, and thus completes the proof of Proposition 3.3.

Remark 3.5. — Our proof can in fact be subtly modified to give a stronger version of Proposition 3.3, that is reminiscent of the notion of strong discrete $\mathcal{C}$–approximations of Holt–Rees [14]. Namely, for any $\eta>0$ we can improve the conclusion of part (2) to say that Θ is $(F,c',\tilde{d}_{\max})$–injective, where c' is given by

$$c'(g,h)=\begin{cases}(1-\eta)\,, & \text{if } h\neq 1\\ \max\limits_{x\in \mathrm{Supp}(g)}(1-\eta)c(g_x)\,, & \text{if } h=1,\ g=(g_x)_{x\in H}\,.\end{cases}$$

For this improved version, the parameters E,E_H,E_G,ε' will depend upon η. We have elected to not give this improved version in order to simplify the statement of the proposition and its proof.

4. Applications of Proposition 3.3

In this section, we use Proposition 3.3 to prove Theorem 1.3. Part (iv) of Theorem 1.3 follows immediately from Proposition 3.3, so we focus on proving the remaining three parts. Each of parts (i), (ii), (iii) are proved below in separate subsections. We recall that the aim is to show that, for H a countable, discrete, sofic group, the wreath product $G \wr H$ is respectively sofic, hyperlinear, or linear sofic, whenever G is such a group.

4.1. Proof of part (i): Sofic

We restate and prove our soficity result for wreath products.

THEOREM 4.1. — *Let G, H be countable, discrete, sofic groups. Then $G \wr H$ is sofic.*

Proof. — In order to show that $G \wr H$ is sofic, we show that $G \wr H$ is $\mathcal{C}$-approximable, where $\mathcal{C}$ is the class of symmetric groups with the normalized Hamming distance. To do this we compose the map Θ from Section 3.2 with a second map Ψ, as described below.

Let $F \subseteq G \wr H$ be finite and $\varepsilon > 0$. Let E_G, E, E_H and $\varepsilon' > 0$ be as in Proposition 3.3 for F, ε. Define c on $G \setminus \{1\}$ by $c(g) = \frac{1}{2}$, and so

$$c' \colon G \wr H \setminus \{1\} \to (0, 1/2]$$

as constructed in Proposition 3.3, is either $1/2$ if $h \neq 1$, or $1/4$ otherwise.

Since G, H are sofic we can find corresponding sofic approximations. For H we take $\sigma \colon H \to \mathrm{Sym}(B)$, for a finite set B, to be an (E_H, ε')–sofic approximation; for G we take $\theta \colon G \to \mathrm{Sym}(A)$, for a finite set A, to be an (E_G, ε')–sofic approximation. Note that, since $\varepsilon' < 1/2$ (see (3.1) following Proposition 3.3), the (E_G, ε')–free condition of θ implies that it is $(E_G, c, d_{\mathrm{Hamm}})$–injective.

With these maps, let $\Theta \colon G \wr H \to (\bigoplus_B \mathrm{Sym}(A)) \wr_B \mathrm{Sym}(B)$ be the map constructed in Section 3, with $K = \mathrm{Sym}(A)$. We now explain how we embed $(\bigoplus_B \mathrm{Sym}(A)) \wr_B \mathrm{Sym}(B)$ into $\mathrm{Sym}\left(\bigoplus_B A \oplus B\right)$. First, define

$$\Phi \colon \bigoplus_B \mathrm{Sym}(A) \to \mathrm{Sym}\left(\bigoplus_B A\right)$$

by the diagonal action

$$\Phi((\pi_\beta)_{\beta \in B}) : (a_\beta)_{\beta \in B} \mapsto (\pi_\beta(a_\beta))_{\beta \in B}, \quad \text{for } \pi_\beta \in \mathrm{Sym}(A), (a_\beta)_{\beta \in B} \in \bigoplus_B A.$$

Then, use Φ to define the embedding

$$\Psi \colon \left(\bigoplus_B \mathrm{Sym}(A)\right) \wr_B \mathrm{Sym}(B) \to \mathrm{Sym}\left(\bigoplus_B A \oplus B\right)$$

by

$$\Psi(\pi,\tau)\colon (a,b)\mapsto (\Phi(\pi_{\tau(b)})(a),\tau(b))$$

for $\pi\in\bigoplus_B\left(\bigoplus_B \mathrm{Sym}(A)\right)$, $\tau\in\mathrm{Sym}(B)$, $a\in\bigoplus_B A$, and $b\in B$. A routine computation reveals that Ψ is a homomorphism.

Let $\ell',\ell_{\max}$ be the conjugacy-invariant length functions on $\bigoplus_B \mathrm{Sym}(A)$ given by

$$\ell'(\pi)=\ell_{\mathrm{Hamm}}(\Phi(\pi)),$$
$$\ell_{\max}(\pi)=\max_{\beta\in B}\ell_{\mathrm{Hamm}}(\pi_\beta)$$

for $\pi=(\pi_\beta)_{\beta\in B}\in\bigoplus_B\mathrm{Sym}(A)$. Then take $\tilde{d},\tilde{d}_{\max}$ to be the bi-invariant metrics as constructed in Proposition 2.9 from the length functions $\ell',\ell_{\max}$ on $\bigoplus_B\mathrm{Sym}(A)$.

Because Ψ is a homomorphism, for $\pi_1,\pi_2\in\bigoplus_B\left(\bigoplus_B\mathrm{Sym}(A)\right),\tau_1,\tau_2\in\mathrm{Sym}(B)$, we have:

$$d_{\mathrm{Hamm}}(\Psi(\pi_1,\tau_1),\Psi(\pi_2,\tau_2))=\tilde{d}((\pi_1,\tau_1),(\pi_2,\tau_2)). \tag{4.1}$$

It thus follows directly from Proposition 3.3 that $\Psi\circ\Theta$ is $(F,\varepsilon,d_{\mathrm{Hamm}})$–multiplicative.

We now show that $\Psi\circ\Theta$ is (F,c',d_{Hamm})–injective.
Let $\pi\in\bigoplus_B\left(\bigoplus_B\mathrm{Sym}(A)\right)$ and $\tau\in\mathrm{Sym}(B)$. Write $\pi=(\pi_b)_{b\in B}$ for $\pi_b\in\bigoplus_B\mathrm{Sym}(A)$ and, for a fixed $b\in B$, let $\pi_b=(\pi_{b,\beta})_{\beta\in B}$. For each $b\in B$ such that $\tau(b)=b$ we then have

$$\begin{aligned}\ell_{\mathrm{Hamm}}(\Phi(\pi_b))&=1-\frac{1}{|A|^{|B|}}\left|\{(a_\beta)_{\beta\in B}\mid \pi_{b,\beta}a_\beta=a_\beta\}\right|\\&=1-\frac{1}{|A|^{|B|}}\prod_{\beta\in B}\left|\{a\in A\mid \pi_{b,\beta}a=a\}\right|\\&=1-\prod_{\beta\in B}\big(1-\ell_{\mathrm{Hamm}}(\pi_{b,\beta})\big)\end{aligned}$$

which implies

$$\tilde{\ell}\big((\pi,\tau)\big)=\ell_{\mathrm{Hamm}}(\tau)+\frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}}\left[1-\prod_{\beta\in B}\big(1-\ell_{\mathrm{Hamm}}(\pi_{b,\beta})\big)\right].$$

Since $0\leqslant\ell_{\mathrm{Hamm}}(\pi_{b,\beta})\leqslant 1$ we have for each $b\in B$:

$$\prod_{\beta\in B}\big(1-\ell_{\mathrm{Hamm}}(\pi_{b,\beta})\big)\leqslant 1-\max_{\beta\in B}\ell_{\mathrm{Hamm}}(\pi_{b,\beta})$$

and inserting this into the above expression for $\tilde{d}$ shows that

$$\tilde{\ell}((\pi,\tau)) \geqslant \tilde{\ell}_{\max}((\pi,\tau)).$$

Combining equation (4.1) with the preceding inequality, we get for each $g \in F \setminus \{1\}$

$$\ell_{\mathrm{Hamm}}\big(\Psi(\Theta(g))\big) = \tilde{\ell}\big(\Theta(g)\big) \geqslant \tilde{\ell}_{\max}\big(\Theta(g)\big) \geqslant c'(g)$$

since Proposition 3.3 implies Θ is $(F, c', \tilde{d}_{\max})$–injective. This shows that $\Psi \circ \Theta$ is $(F, c', d_{\mathrm{Hamm}})$–injective. Hence we have shown that $G \wr H$ is $\mathcal{C}$–approximable, where $\mathcal{C}$ is the class of symmetric groups equipped with the Hamming distance and this means that $G \wr H$ is sofic. □

We remark that one can use the improved version of Proposition 3.3, as per Remark 3.5, to show that $\Psi \circ \Theta$ as considered in the above proof is an (F, ε)–sofic approximation provided $\theta\colon G \to \mathrm{Sym}(A)$ and $\sigma\colon H \to \mathrm{Sym}(B)$ are sufficiently good sofic approximations. In this way one can in fact directly show that $G \wr H$ has arbitrarily good sofic approximations.

4.2. Proof of part (ii): Hyperlinear

In this section, we deduce hyperlinearity of $G \wr H$, assuming that G is hyperlinear and H is sofic. Hyperlinear groups are defined by admitting a metric approximation to unitary groups, $\mathcal{U}(n)$, paired with the normalized Hilbert–Schmidt metric.

Let $\mathrm{tr}\colon M_n(\mathbb{C}) \to \mathbb{C}$ be the normalized trace:

$$\mathrm{tr}(A) = \frac{1}{n}\sum_{j=1}^{n} A_{jj}$$

where $A = (A_{ij}) \in M_n(\mathbb{C})$.

Definition 4.2. — *The* normalized Hilbert–Schmidt norm *on* $M_n(\mathbb{C})$ *is defined by*

$$\|A\|_2 = \mathrm{tr}(A^*A)^{1/2}, \quad \textit{for } A \in M_n(\mathbb{C}).$$

The normalized Hilbert–Schmidt metric *on* $\mathcal{U}(n)$ *is therefore given by*

$$d_{\mathrm{HS}}(U,V) = \|U - V\|_2\,, \quad \textit{for } U, V \in \mathcal{U}(n).$$

The corresponding length function is denoted ℓ_{HS}.

Definition 4.3. — *We say a group is* hyperlinear *if it is* $\mathcal{C}$*–approximable, where* $\mathcal{C}$ *is the class of unitary groups, paired with the normalized Hilbert–Schmidt metrics.*

We will need that our approximations $\theta\colon G \to \mathcal{U}(n)$ not only map $\theta(g)$ far away from Id for $g \neq 1$, but that in fact $\theta(g)$ is far away from the unit circle $S^1 = \{\lambda\,\mathrm{Id} : |\lambda| = 1\}$ in $\mathcal{U}(n)$. To put this in a framework where we can take advantage of Proposition 3.3, we use the following set-up.

Define $\overline{d}_{\mathrm{HS}}$, a bi-invariant metric on $\mathcal{U}(n)/S^1$, by

$$\overline{d}_{\mathrm{HS}}(US^1, VS^1) = \inf_{\lambda\in S^1} d_{\mathrm{HS}}(\lambda U, V), \quad \text{for } U, V \in \mathcal{U}(n).$$

Let $\overline{\ell}_{\mathrm{HS}}$ denote the corresponding length function. We will abuse this notation and write $\overline{d}_{\mathrm{HS}}(U,V)$. Note that we can directly use the normalized trace to calculate $\overline{\ell}_{\mathrm{HS}}(U)$ as follows:

$$\overline{\ell}_{\mathrm{HS}}(U)^2 = \inf_{\lambda\in S^1} \|U - \lambda\,\mathrm{Id}\|_2^2 = \inf_{\lambda\in S^1} 2 - 2\,\mathrm{re}(\overline{\lambda}\,\mathrm{tr}(U)) = 2 - 2\,|\mathrm{tr}(U)|\,.$$

In light of this, we get the following reformulation of a result of Rădulescu in [22] which gives an equivalent definition of hyperlinearity.

Proposition 4.4. — *Let G be a group and $c\colon G\setminus\{1\} \to (0,\sqrt{2})$ any function.*

Then G is hyperlinear if and only if for every $\varepsilon > 0$ and any finite $F \subseteq G$ there is a positive integer n and a function $\theta\colon G \to \mathcal{U}(n)$ which is $(F,\varepsilon,d_{\mathrm{HS}})$–multiplicative and so that $q\circ\theta$ is $(F,c,\overline{d}_{\mathrm{HS}})$–injective, where $q\colon \mathcal{U}(n) \to \mathcal{U}(n)/S^1$ is the quotient map.

Theorem 4.5. — *Let H be a countable, discrete, sofic group and G a countable, discrete, hyperlinear group. Then $G \wr H$ is hyperlinear.*

Proof. — We proceed in an analogous manner as for Theorem 4.1, when we dealt with soficity. In particular, we show that $G\wr H$ is $\mathcal{C}$–approximable, where $\mathcal{C}$ is as in Definition 4.3. The necessary maps to demonstrate this will be constructed as a composition, starting with Θ from Proposition 3.3 followed by an appropriate embedding into a unitary group.

Step 1: Setting the scene. — Let $F \subseteq G\wr H$ be finite and $\varepsilon > 0$. Let E_G, E, E_H and $\varepsilon' > 0$ be as Proposition 3.3 for F,ε. Let $c\colon G\setminus\{1\} \to (0,1/2]$ be given by $c(g) = \frac{1}{2}$ for $g \in G\setminus\{1\}$ and let $c'\colon G\wr H\setminus\{1\} \to (0,1/2]$ be the map constructed in Proposition 3.3. Since H is sofic we can find an (E_H,ε')–sofic approximation $\sigma\colon H \to \mathrm{Sym}(B)$ for some finite set B. Since G is hyperlinear we apply Proposition 4.4 to find an $(E_G,\varepsilon,d_{\mathrm{HS}})$–multiplicative map $\theta\colon G \to \mathcal{U}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$ so that $q\circ\theta$ is $(E_G,c,\overline{d}_{\mathrm{HS}})$–injecitve.

Let $\Theta\colon G\wr H \to (\bigoplus_B \mathcal{U}(\mathcal{H}))\wr_B \rtimes \mathrm{Sym}(B)$ be the map constructed from θ,σ and E in Section 3. Similarly construct $\bar{\Theta} : G\wr H \to \left(\bigoplus_B \mathcal{U}(\mathcal{H})/S^1\right)\wr_B \mathrm{Sym}(B)$ from $q\circ\theta,\sigma$ and E.

Define

$$\Phi\colon \bigoplus_B \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H}^{\otimes B})$$

by

$$\Phi\colon (V_\beta)_{\beta\in B} \mapsto \bigotimes_{\beta\in B} V_\beta, \quad \text{for } (V_\beta)_{\beta\in B} \in \bigoplus_B \mathcal{U}(\mathcal{H}).$$

We now define

$$\Psi\colon \left(\bigoplus_B \mathcal{U}(\mathcal{H})\right)\wr_B \mathrm{Sym}(B) \to \mathcal{U}\left(\bigoplus_B \left(\mathcal{H}^{\otimes B}\right)\right)$$

by

$$\Psi((U_b)_{b\in B},\tau)\colon (\xi_b)_{b\in B} \mapsto \left(\Phi(U_b)\left(\xi_{\tau^{-1}(b)}\right)\right)_{b\in B}$$

for $(\xi_b)_{b\in B} \in \bigoplus_B \left(\mathcal{H}^{\otimes B}\right)$, $(U_b)_{b\in B} \in \bigoplus_B \bigoplus_B \mathcal{U}(\mathcal{H})$, and $\tau\in\mathrm{Sym}(B)$. The collection of maps we have is summarized in Figure 4.1.

$$\begin{tikzcd}
G\wr H \arrow[r, "\Theta"] \arrow[dr, "\bar{\Theta}"'] & \left(\bigoplus_B \mathcal{U}(\mathcal{H})\right)\wr_B \mathrm{Sym}(B) \arrow[r, "\Psi"] \arrow[d] & \mathcal{U}\left(\bigoplus_B \left(\mathcal{H}^{\otimes B}\right)\right) \\
 & \left(\bigoplus_B \mathcal{U}(\mathcal{H})/S^1\right)\wr_B \mathrm{Sym}(B) &
\end{tikzcd}$$

Figure 4.1. A plan of the maps involved.

Let $\tilde{d},\tilde{d}_{\max}$ be the bi-invariant metrics on $(\bigoplus_B \mathcal{U}(\mathcal{H}))\wr_B \mathrm{Sym}(B)$ and $\left(\bigoplus_B \mathcal{U}(\mathcal{H})/S^1\right)\wr_B \mathrm{Sym}(B)$, respectively, induced by Proposition 2.9 from the length functions $\ell',\ell_{\max}$ on $\bigoplus_B \mathcal{U}(\mathcal{H})$ and $\bigoplus_B \mathcal{U}(\mathcal{H})/S^1$, respectively, which are given by

$$\ell'(V) = \frac{1}{2}\ell_{\mathrm{HS}}(\Phi(V)),$$

$$\ell_{\max}(\bar{V}) = \max_{\beta\in B} \frac{\overline{\ell}_{\mathrm{HS}}(q(V_\beta))}{\sqrt{2}},$$

for $V=(V_\beta)_{\beta\in B}\in\bigoplus_B \mathcal{U}(\mathcal{H})$, and $\bar{V} = (q(V_\beta))_{\beta\in B} \in \bigoplus_B \mathcal{U}(\mathcal{H})/S^1$. Note that d_{HS} is bounded by 2, whereas $\overline{d}_{\mathrm{HS}}$ is bounded by $\sqrt{2}$.

Step 2: A formula for $d_{\mathrm{HS}}(\Psi(U,\tau),\mathrm{Id})$. — We aim to bound the d_{HS}–distance from a point in the image of Ψ to the identity in terms of the $\tilde{d}$–distance for its pre-image. To this end, we first observe that the matrix representation for $\Psi(U,\tau)$ will be a block permutation matrix, with blocks corresponding to elements of B. The matrix will have a non-zero block in the (b,b)–position precisely when $\tau(b)=b$. Thus we get

$$\mathrm{tr}\left(\Psi(U,\tau)\right) = \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \mathrm{tr}\left(\Phi(U_b)\right), \quad \text{where } U=(U_b)_{b\in B}\,.$$

This implies that

$$\|\Psi(U,\tau)-\mathrm{Id}\|_2^2 = 2-2\,\mathrm{re}(\mathrm{tr}(\Psi(U,\tau))) = 2-\frac{2}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \mathrm{re}(\mathrm{tr}(U_b))\,.$$

By the definition of the Hamming metric we can rewrite the right-hand side as

$$2\ell_{\mathrm{Hamm}}(\tau)+\frac{2}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} 1-\mathrm{re}(\mathrm{tr}(\Phi(U_b)))\,.$$

Hence

$$\|\Psi(U,\tau)-\mathrm{Id}\|_2^2 = 2\ell_{\mathrm{Hamm}}(\tau)+\frac{2}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \|\Phi(U_b)-\mathrm{Id}\|_2^2\,. \tag{4.2}$$

Step 3: Almost multiplicativity. — Since $\|\Phi(U_b)-\mathrm{Id}\|_2\leqslant\sqrt{2}$, we can get an upper bound of

$$\|\Psi(U,\tau)-\mathrm{Id}\|_2^2 \leqslant 2\ell_{\mathrm{Hamm}}(\tau)+\frac{4}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \|\Phi(U_b)-\mathrm{Id}\|_2 \leqslant 8\tilde{d}((U,\tau),1)\,.$$

In summary, since Ψ is a homomorphism, for $(U_1,\tau_1),(U_2,\tau_2)\in(\bigoplus_B\mathcal{U}(\mathcal{H}))\wr_B\mathrm{Sym}(B)$ we have shown

$$d_{\mathrm{HS}}(\Psi(U_1,\tau_1),\Psi(U_2,\tau_2)) \leqslant 2\sqrt{2}\tilde{d}((U_1,\tau_1),(U_2,\tau_2))^{1/2}.$$

From Proposition 3.3 we know that Θ is $(F,\varepsilon,\tilde{d})$–multiplicative. With this, the above inequality then implies that $\Psi\circ\Theta$ is $(F,2\sqrt{2\varepsilon},d_{\mathrm{HS}})$–multiplicative.

Step 4: Almost injectivity. — Let $V=(V_\beta)_{\beta\in B}\in\bigoplus_B\mathcal{U}(\mathcal{H})$. For each β, we have that

$$|\mathrm{tr}(V_\beta)| = 1-\left(\frac{\overline{\ell}_{\mathrm{HS}}(q(V_\beta))}{\sqrt{2}}\right)^2.$$

Thus

$$|\mathrm{tr}(\Phi(V))| = \prod_{\beta\in B} |\mathrm{tr}(V_\beta)| \leqslant 1 - \max_{\beta\in B}\left(\frac{\overline{\ell}_{\mathrm{HS}}(q(V_\beta))}{\sqrt{2}}\right)^2 = 1 - \ell_{\max}(\bar{V})^2$$

where $\bar{V} = (q(V_\beta))_{\beta\in B}$. Since $2 - 2\,|\mathrm{tr}(\Phi(V))| \leqslant \|\Phi(V) - \mathrm{Id}\|_2^2$, we get $\ell_{\max}(\bar{V})^2 \leqslant \frac{1}{2}\,\|\Phi(V) - \mathrm{Id}\|_2^2$. Inserting this into equation (4.2) and arguing as in Section 4.1 we see that, if $\bar{U}_b = (q(U_{b,\beta}))_{\beta\in B}$, then

$$\begin{aligned}\|\Psi(U,\tau) - \mathrm{Id}\|_2^2 &\geqslant 2\ell_{\mathrm{Hamm}}(\tau) + \frac{2}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \ell_{\max}(\bar{U}_b)^2\\ &\geqslant 2\ell_{\mathrm{Hamm}}(\tau)^2 + 2\left(\frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \ell_{\max}(\bar{U}_b)\right)^2\\ &\geqslant \left(\ell_{\mathrm{Hamm}}(\tau) + \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \ell_{\max}(\bar{U}_b)\right)^2\\ &= \tilde{d}_{\max}\big((\bar{U},\tau),1\big)^2\end{aligned}$$

where $\bar{U} = (\bar{U}_b)_{b\in B}$. As $q\circ\theta$ is $(E_G, c, \overline{d}_{\mathrm{HS}})$–injective, it follows by Proposition 3.3 that $\bar{\Theta}$ is $(F, c', \tilde{d}_{\max})$–injective. Thus, for (U,τ) in the image of Θ, it follows that $(\bar{U},\tau)$ is in the image of $\bar{\Theta}$, and

$$\ell_{\mathrm{HS}}(\psi(U,\tau))^2 = \|\Psi(U,\tau) - \mathrm{Id}\|_2^2 \geqslant (c'(x))^2.$$

Thus $\Psi\circ\Theta$ is (F, c', d_{HS})–injective and $(F, 2\sqrt{\varepsilon}, d_{\mathrm{HS}})$–multiplicative. As $\varepsilon > 0$ is arbitrary the proof is complete. □

As with soficity, one can use the improved version of Proposition 3.3 from Remark 3.5 to strengthen the bounds in the above results. In particular this will show that

$$\min_{x\in F\setminus\{1\}} \overline{\ell}_{\mathrm{HS}}(\Psi(\Theta(x))) \geqslant 1 - \varepsilon\,,$$

provided $\sigma\colon H \to \mathrm{Sym}(B)$ is a sufficiently good sofic approximation and θ satisfies

$$\min_{g\in E} \overline{\ell}_{\mathrm{HS}}(\theta(g)) > 1 - \kappa\,,$$

for a sufficiently large E and a sufficiently small κ. In this manner, we can directly verify the conclusion of Proposition 4.4 for $G \wr H$ if H is sofic and G is hyperlinear.

4.3. Proof of part (iii): Linear Sofic

We recall the following definition due to Arzhantseva and Păunescu [1].

Definition 4.6. — *Let $\mathbb{F}$ be a field. Define a bi-invariant metric d_{rk}, with corresponding length function ℓ_{rk}, on $\mathrm{GL}_n(\mathbb{F})$ by*

$$d_{\mathrm{rk}}(A,B) = \frac{1}{n}\operatorname{Rank}(A-B).$$

We say that a group is linear sofic over $\mathbb{F}$ *if it is $\mathcal{C}$-approximable, where $\mathcal{C}$ consists of all general linear groups $\mathrm{GL}_n(\mathbb{F})$, each paired with the metric d_{rk}.*

In this section we use Proposition 3.3 to show that $G \wr H$ is linear sofic if G is linear sofic and H is sofic. Proving that the map we constructed is sufficiently injective turns out to be trickier than in any of the other cases. As in the case of hyperlinear groups, we will need that our linear sofic approximation $\theta\colon G \to \mathrm{GL}_n(\mathbb{F})$ does not just satisfy that $\frac{1}{n}\operatorname{Rank}(\theta(g)-\mathrm{Id})$ is bounded away from 0 for $g \neq 1$, but in fact we need

$$\min_{\lambda\in\mathbb{K}^\times}\frac{1}{n}\operatorname{Rank}_{\mathbb{K}}(\theta(g)-\lambda\,\mathrm{Id}) > 0,$$

where $\operatorname{Rank}_{\mathbb{K}}$ indicates that we are computing dimension over the algebraic closure $\mathbb{K}$ of $\mathbb{F}$. Thus we use the following definition.

Definition 4.7. — *Let $\mathbb{F}$ be a field, for $A, B \in \mathrm{GL}_n(\mathbb{F})$, we let*

$$\overline{d}_{\mathrm{rk}}(A,B) = \min_{\lambda\in\mathbb{K}^\times}\frac{1}{n}\operatorname{Rank}(A-\lambda B)$$

and $\overline{\ell}_{\mathrm{rk}}$ denote the corresponding length function.

Note that, since $\frac{1}{n}\operatorname{Rank}_{\mathbb{F}}(A-B) = \frac{1}{n}\operatorname{Rank}_{\mathbb{K}}(A-B)$ for $A, B \in \mathrm{GL}_n(\mathbb{F})$, we have $d_{\mathrm{rk}}(A,B) \geqslant \overline{d}_{\mathrm{rk}}(A,B)$.

We will then use the following fact, which is a consequence of an equivalent characterization of linear soficity given by Arzhantseva–Păunescu [1, Thm. 5.10].

Proposition 4.8. — *Let G be a linear sofic group over the field $\mathbb{F}$ and let $\mathbb{K}$ denote the algebriac closure of $\mathbb{F}$.*

Then, for any $\delta \in (0,\frac{1}{8})$ and any finite $F \subseteq G$, there is a positive integer n and a function $\theta\colon G \to \mathrm{GL}_n(\mathbb{F})$ which is $(F,\delta,d_{\mathrm{rk}})$–multiplicative, and so that $q\circ\theta$ is $(F,c,\overline{d}_{\mathrm{rk}})$–injective, where $c(g) = \frac{1}{8}-\delta$ for all $g \in G$, and $q\colon \mathrm{GL}_n(\mathbb{F}) \to \mathrm{PGL}_n(\mathbb{K})$ is the canoncial map given by composing the natural inclusion $\mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{K})$ with the quotient map $\mathrm{GL}_n(\mathbb{K}) \to \mathrm{PGL}_n(\mathbb{K})$.

Proof. — By [1, Thm. 5.10], it follows that there exists a function

$$\theta_0\colon G \to \mathrm{GL}_m(\mathbb{F})$$

for some $m \in \mathbb{N}$, which is $(F, \delta, d_{\mathrm{rk}})$–multiplicative and so that $d_{\mathrm{rk}}(\theta_0(g) - \mathrm{Id}) \geqslant \frac{1}{4} - 2\delta$ for all $g \in F \setminus \{1\}$. Now consider

$$\theta\colon G \to \mathrm{GL}_{2m}(\mathbb{F})$$

given in matrix block form by

$$\theta(g) = \begin{bmatrix} \theta_0(g) & 0 \\ 0 & \mathrm{Id} \end{bmatrix}.$$

Fix $\lambda \in \mathbb{K}^\times$ and $g \in F \setminus \{1\}$. If $\lambda \neq 1$, then we see that $d_{\mathrm{rk}}(\theta(g), \lambda\,\mathrm{Id}) \geqslant \frac{1}{2}$. On the other hand, if $\lambda = 1$ then

$$\frac{1}{2m}\,\mathrm{Rank}_{\mathbb{K}}(\theta(g) - \lambda\,\mathrm{Id}) = \frac{1}{2} \cdot \left[\frac{1}{m}\,\mathrm{Rank}_{\mathbb{F}}(\theta_0(g) - \mathrm{Id})\right] \geqslant \frac{1}{8} - \delta\,.$$

Thus θ is the required function. □

In order to use Proposition 3.3 to prove that $G \wr H$ is linear sofic, we will need to use tensor products of matrices. The main fact we will need is that if $A \in \mathrm{GL}_n(\mathbb{F}), B \in \mathrm{GL}_k(\mathbb{F})$ and $\overline{\ell}_{\mathrm{rk}}(A), \overline{\ell}_{\mathrm{rk}}(B)$ are both bounded away from zero, then $\overline{\ell}_{\mathrm{rk}}(A \otimes B)$ is also bounded away from zero. We formulate this precisely in the Proposition 4.11 below, whose proof uses similar ideas to [1, Lem. 5.4, Prop. 5.8].

Let $J_\alpha(A)$ denote the number of Jordan blocks in the Jordan normal form of A associated to the eigenvalue α. If α is not an eigenvalue then we set $J_\alpha(A) = 0$. Given a number α and a positive integer n we let $J(\alpha, n)$ denote the standard $n \times n$ Jordan block with eigenvalue α. In characteristic zero, the following is a classic result explaining how Jordan blocks behave under tensor products, known as the Clebsch–Gordan formula (and in fact one can even say what the precise Jordan block decomposition of $J(\alpha, n) \otimes J(\beta, k)$ is, though we will not need this). See, for example, [18, Thm. 2]. For positive characteristic, this result is a consequence of [15, Thm. 2.2.2].

Theorem 4.9. — *Let $\mathbb{K}$, be an algebraically closed field, α, β be nonzero elements of $\mathbb{K}$, and n, k be positive integers. Then*

$$J_{\alpha\beta}\big(J(\alpha, n) \otimes J(\beta, k)\big) = \min\{n, k\}.$$

We will use this to prove the following.

Lemma 4.10. — *Let $\mathbb{K}$ be an algebraically closed field and take $A \in \mathrm{GL}_n(\mathbb{K})$ and $B \in \mathrm{GL}_k(\mathbb{K})$.*

Then, for each $\lambda \in \mathbb{K}$,

$$J_\lambda(A \otimes B) \leqslant \min\left\{k \max_{\alpha\in\mathbb{K}} J_\alpha(A), n \max_{\beta\in\mathbb{K}} J_\beta(B)\right\}.$$

Proof. — Let us first prove that $J_\lambda(A \otimes B) \leqslant k \max_{\alpha\in\mathbb{K}} J_\alpha(A)$, as the other inequality will follow by symmetry. First, assuming that A and B have unique eigenvalues α and β respectively, the result of the lemma becomes

$$J_{\alpha\beta}(A \otimes B) \leqslant kJ_\alpha(A)\,. \tag{4.3}$$

Both sides of the above inequality are additive under taking direct sums of matrices with the given eigenvalues. So we may assume that A and B are one Jordan block, in which case (4.3) follows from Theorem 4.9.

Now suppose the eigenvalues of A and B are not necessarily unique. Since $\mathbb{K}$ is algebraically closed, up to conjugacy we may write A and B as direct sums

$$A = \bigoplus_{\alpha\in\mathbb{F}} A_\alpha\,, \quad B = \bigoplus_{\beta\in\mathbb{K}} B_\beta$$

where A_α is the direct sum of all Jordan blocks of A associated to eigenvalue α, and similarly for B_β. Suppose A_α is $n_\alpha \times n_\alpha$ and B_β is $k_\beta \times k_\beta$. Then

$$A \otimes B = \bigoplus_{\alpha,\beta\in\mathbb{K}} A_\alpha \otimes B_\beta\,,$$

which leads to the following, using (4.3):

$$\begin{aligned} J_\lambda(A \otimes B) = \sum_{\alpha\beta=\lambda} J_\lambda(A_\alpha \otimes B_\beta) &\leqslant \sum_{\alpha\beta=\lambda} k_\beta J_\alpha(A) \\ &\leqslant \sum_{\beta\in\mathbb{K}} k_\beta \max_{\alpha\in\mathbb{K}} J_\alpha(A) = k \max_{\alpha\in\mathbb{K}} J_\alpha(A)\,. \end{aligned}$$

This completes the proof. □

To see how the normalized rank metric $\overline{d}_{\mathrm{rk}}$ behaves under tensor products, we remark that $\mathrm{Rank}(A-\alpha\,\mathrm{Id}) = n - J_\alpha(A)$ for every $\alpha \in \mathbb{K}$, implying $\overline{d}_{\mathrm{rk}}(A, \mathrm{Id}) = \inf_{\lambda\in\mathbb{K}} \left(1 - \frac{1}{n} J_\lambda(A)\right)$. The following is thus an immediate consequence of this fact, and of Lemma 4.10.

Proposition 4.11. — *Let* $\mathbb{F}$ *be a field. Let* $n, k \in \mathbb{N}$ *and* $A \in \mathrm{GL}_n(\mathbb{F})$, $B \in \mathrm{GL}_k(\mathbb{F})$. *Then*

$$\overline{d}_{\mathrm{rk}}(A \otimes B, \mathrm{Id}) \geqslant \max\left\{\overline{d}_{\mathrm{rk}}(A, \mathrm{Id}), \overline{d}_{\mathrm{rk}}(B, \mathrm{Id})\right\}.$$

Theorem 4.12. — *Let* G *be a linear sofic group over the field* $\mathbb{F}$ *and* H *be a sofic group. Then* $G \wr H$ *is linear sofic over* $\mathbb{F}$.

Proof. — The structure of the proof is analogous to that of Theorem 4.5. We compose the map $\Theta : G \wr H \to (\bigoplus_B \mathrm{GL}_n(\mathbb{F})) \wr_B \mathrm{Sym}(B)$ from Proposition 3.3 with a map Ψ giving us a map from $G \wr H$ to a linear group. We verify that $\Psi \circ \Theta$ satisfies the required almost multiplicativity and almost injectivity conditions.

Step 1: Setting the scene. — Recall that $q\colon \mathrm{GL}_n(\mathbb{F}) \to \mathrm{PGL}_n(\mathbb{K})$ denotes the composition of the canonical inclusion $\mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K}$ is the algebraic closure of $\mathbb{F}$, with the quotient map $\mathrm{GL}_n(\mathbb{K}) \to \mathrm{PGL}_n(\mathbb{K})$.

Take a finite subset F of $G \wr H$ and $\varepsilon > 0$. Define $c\colon G \setminus \{1\} \to (0, \infty)$ to take the value $\frac{1}{16}$ for all $g \neq 1$. Let $E_G \subseteq G, E, E_H \subseteq H$, $c' : G \setminus \{1\} \to (0, \infty)$, and $\varepsilon' > 0$ all be as determined by F, ε, and c in Proposition 3.3. Note that from (3.1) in the proof of Proposition 3.3, we know that $\varepsilon' < \frac{1}{16^2} < \frac{1}{16}$. Thus, taking $\delta = \varepsilon'$ in Proposition 4.8 gives us a map $\theta\colon G \to \mathrm{GL}_n(\mathbb{F})$ that is $(E_G, \varepsilon', d_{\mathrm{rk}})$–multiplicative and is such that $q \circ \theta$ is $(E_G, c, \overline{d}_{\mathrm{rk}})$–injective.

Let $\sigma\colon H \to \mathrm{Sym}(B)$, for some finite set B, be an (E_H, ε')–sofic approximation and take

$$\Theta\colon G \wr H \to \left(\bigoplus_B \mathrm{GL}_n(\mathbb{F}) \right) \wr_B \mathrm{Sym}(B)$$

to be the map constructed from θ, σ, and E in Section 3. Meanwhile, let

$$\bar{\Theta}\colon G \wr H \to \left(\bigoplus_B \mathrm{PGL}_n\left(\mathbb{K}\right) \right) \wr_B \mathrm{Sym}(B)$$

be the map constructed using $q \circ \theta$ in place of θ.

We now describe how to embed the image of Θ into a linear group. First define

$$\Phi\colon \bigoplus_B \mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}\left((\mathbb{F}^n)^{\otimes B} \right)$$

by

$$\Phi\colon (X_\beta)_{\beta \in B} \mapsto \bigotimes_{\beta \in B} X_\beta, \quad \text{for } (X_\beta)_{\beta \in B} \in \bigoplus_B \mathrm{GL}_n(\mathbb{F}).$$

Using Φ, we define

$$\Psi\colon \left(\bigoplus_B \mathrm{GL}_n(\mathbb{F}) \right) \wr_B \mathrm{Sym}(B) \to \mathrm{GL}\left(\bigoplus_B \left((\mathbb{F}^n)^{\otimes B} \right) \right)$$

by

$$\Psi((A_b)_{b \in B}, \tau)\colon (\xi_b)_{b \in B} \mapsto \left(\Phi(A_b)\left(\xi_{\tau^{-1}(b)} \right) \right)_{b \in B}$$

for $(\xi_b)_{b\in B} \in \bigoplus_B \left((\mathbb{F}^n)^{\otimes B}\right)$, $(A_b)_{b\in B} \in \bigoplus_B \bigoplus_B \mathrm{GL}_n(\mathbb{F})$, and $\tau \in \mathrm{Sym}(B)$.

The collection of maps we have is summarized in Figure 4.2.

$$\begin{tikzcd} G \wr H \arrow[r, "\Theta"] \arrow[dr, "\bar{\Theta}"'] & \left(\bigoplus_B \mathrm{GL}_n(\mathbb{F})\right) \wr_B \mathrm{Sym}(B) \arrow[r, "\Psi"] \arrow[d] & \mathrm{GL}\left(\bigoplus_B \left((\mathbb{F}^n)^{\otimes B}\right)\right) \\ & \left(\bigoplus_B \mathrm{PGL}_n(\mathbb{K})\right) \wr_B \mathrm{Sym}(B) & \end{tikzcd}$$

Figure 4.2. A plan of the maps involved.

Let $\widetilde{d}, \tilde{d}_{\max}$ be the bi-invariant metrics on the wreath products $(\bigoplus_B \mathrm{GL}_n(\mathbb{F})) \wr_B \mathrm{Sym}(B)$, and $(\bigoplus_B \mathrm{PGL}_n(\mathbb{K})) \wr_B \mathrm{Sym}(B)$, respectively, obtained by applying Proposition 2.9 to the length functions $\ell', \ell_{\max}$ on $\bigoplus_B \mathrm{GL}_n(\mathbb{F})$, and $\bigoplus_B \mathrm{PGL}_n(\mathbb{K})$, respectively, given by

$$\ell'(X) = \ell_{\mathrm{rk}}(\Phi(X)),$$
$$\ell_{\max}(\bar{X}) = \max_{\beta\in B} \overline{\ell}_{\mathrm{rk}}(\Phi((X_\beta)_{\beta\in B})),$$

where

$$X = (X_\beta)_{\beta\in B} \in \bigoplus_B \mathrm{GL}_n(\mathbb{F}), \quad \text{and} \quad \bar{X} = (q(X_\beta))_{\beta\in B} \in \bigoplus_B \mathrm{PGL}_n(\mathbb{K}).$$

Step 2: A formula for $\ell_{\mathrm{rk}}(\Psi(A,\tau))$. — We wish to show that $\Psi\circ\Theta$ is almost multiplicative and almost injective. To do this we need a good handle on $\ell_{\mathrm{rk}}(\Psi(A,\tau))$ when (A,τ) is in the image of Θ.

Write $A = (A_b)_{b\in B}$ with $A_b \in \bigoplus_B \mathrm{GL}_n(\mathbb{F})$. The kernel of $\Psi(A,\tau) - \mathrm{Id}$ is given by

$$\left\{ (\xi_b)_{b\in B} \in \bigoplus_{\substack{b\in B\\ \tau(b)\neq b}} (\mathbb{F}^n)^{\otimes B} : \Phi(A_{\tau(b)})(\xi_b) = \xi_{\tau(b)} \right\} \oplus \left(\bigoplus_{\substack{b\in B\\ \tau(b)=b}} \ker(\Phi(A_b) - \mathrm{Id}) \right).$$

Focusing on the left term in the above direct sum, if we pick a cycle $(b_1\ b_2 \dots b_k)$ of τ, with $k \geqslant 2$, then ξ_{b_1} determines ξ_{b_i} for $i = 2,\dots,k$.

Thus each cycle of length greater than 1 contributes exactly $n^{|B|}$ to the dimension of the kernel. Let $\mathrm{cyc}_0(\tau)$ be the number of cycles of length at least two in the cycle decomposition of τ. From the above discussion we see that the dimension of $\ker(\Psi(A,\tau)-\mathrm{Id})$ is

$$n^{|B|}\,\mathrm{cyc}_0(\tau) + \sum_{\substack{b\in B\\ \tau(b)=b}} \dim(\ker(\Phi(A_b)-\mathrm{Id}))\,.$$

It follows that

$$\begin{aligned}\ell_{\mathrm{rk}}(\Psi(A,\tau)) &= 1 - \frac{\dim(\ker(\Psi(A,\tau)-\mathrm{Id})}{n^{|B|}\,|B|}\\ &= 1 - \frac{\mathrm{cyc}_0(\tau)}{|B|} - \sum_{\substack{b\in B\\ \tau(b)=b}} \frac{1-\ell_{\mathrm{rk}}(\Phi(A_b))}{|B|}\,.\end{aligned}$$

Since

$$\ell_{\mathrm{Hamm}}(\tau) = 1 - \frac{|\{b\in B : \tau(b)=b\}|}{|B|}$$

we get

$$\ell_{\mathrm{rk}}(\Psi(A,\tau)) = \ell_{\mathrm{Hamm}}(\tau) - \frac{\mathrm{cyc}_0(\tau)}{|B|} + \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \ell_{\mathrm{rk}}(\Phi(A_b))\,. \tag{4.4}$$

Step 3: Almost multiplicativity. — Equation (4.4) implies that

$$\ell_{\mathrm{rk}}(\Psi(A,\tau),) \leqslant \tilde{\ell}((A,\tau))\,.$$

Bi-invariance implies that for $(A_1,\tau_1),(A_2,\tau_2)\in(\bigoplus_B \mathrm{GL}_n(\mathbb{F}))\wr_B \mathrm{Sym}(B)$ we have:

$$d_{\mathrm{rk}}(\Psi(A_1,\tau_1),(A_2,\tau_2)) \leqslant \tilde{d}((A_1,\tau_1),(A_2,\tau_2))\,.$$

Thus $(F,\varepsilon,d_{\mathrm{rk}})$–multiplicativity of $\Psi\circ\Theta$ follows from the $(F,\varepsilon,\tilde{d})$–multiplicativity of Θ.

Step 4: Almost injectivity. — While for almost multiplicativity we used the almost multiplicativity of Θ, for almost injectivity we will use the almost injectivity of $\bar{\Theta}$.

Elementary calculations yield

$$\ell_{\mathrm{Hamm}}(\tau) = \frac{|B| - |b\in B\colon \tau(b)=b|}{|B|} \geqslant \frac{2\,\mathrm{cyc}_0(\tau)}{|B|}\,.$$

Using this in (4.4), we get that

$$\ell_{\mathrm{rk}}(\Psi(A,\tau)) \geqslant \frac{1}{2}\ell_{\mathrm{Hamm}}(\tau) + \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=b}} \ell_{\mathrm{rk}}(\Phi(A_b))\,.$$

By repeated applications of Proposition 4.11 we have, for each $b \in B$,

$$\ell_{\mathrm{rk}}(\Phi(A_b)) \geqslant \max_{\beta\in B} \overline{\ell}_{\mathrm{rk}}(A_{b,\beta})\,.$$

This implies that

$$\ell_{\mathrm{rk}}(\Psi(A,\tau)) \geqslant \frac{1}{2}\tilde{\ell}_{\max}((\bar{A},\tau))\,.$$

where $\bar{A} = ((q(A_{b,\beta})_{\beta\in B})_{b\in B}$. If (A,τ) lies in the image of Θ then $(\bar{A},\tau)$ lies in the image of $\bar{\Theta}$. Then, $(F, c', \tilde{d}_{\max})$–injectivity of $\bar{\theta}$, coupled with the above inequality, gives us $(F, \frac{c'}{2}, d_{\mathrm{rk}})$–injectivity of $\Psi \circ \Theta$. ☐

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