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ON THE MAXIMALITY OF THE TRIANGULAR SUBGROUP

by Jean-Philippe FURTER & Pierre-Marie POLONI (*)

Abstract. — We prove that the subgroup of triangular automorphisms of the complex affine $n$-space is maximal among all solvable subgroups of $\text{Aut}(\mathbb{A}^n_C)$ for every $n$. In particular, it is a Borel subgroup of $\text{Aut}(\mathbb{A}^n_C)$, when the latter is viewed as an ind-group. In dimension two, we prove that the triangular subgroup is a maximal closed subgroup and that nevertheless, it is not maximal among all subgroups of $\text{Aut}(\mathbb{A}^2_C)$. Given an automorphism $f$ of $\mathbb{A}^2_C$, we study the question whether the group generated by $f$ and the triangular subgroup is equal to the whole group $\text{Aut}(\mathbb{A}^2_C)$.

Résumé. — Nous montrons que le sous-groupe des automorphismes triangulaires est un sous-groupe résoluble maximal de $\text{Aut}(\mathbb{A}^n_C)$ pour tout $n$. Il forme ainsi un sous-groupe de Borel du ind-groupe $\text{Aut}(\mathbb{A}^n_C)$. En dimension deux, nous montrons que le sous-groupe triangulaire est un sous-groupe fermé maximal mais qu’il n’est néanmoins pas maximal parmi tous les sous-groupes de $\text{Aut}(\mathbb{A}^2_C)$. Un automorphisme $f$ de $\mathbb{A}^2_C$ étant donné, nous étudions la question suivante : le sous-groupe engendré par $f$ et par les automorphismes triangulaires est-il égal au groupe $\text{Aut}(\mathbb{A}^2_C)$ tout entier ?

1. Introduction

The main purpose of this paper is to study the Jonquières subgroup $B_n$ of the group $\text{Aut}(\mathbb{A}^n_C)$ of polynomial automorphisms of the complex affine $n$-space, i.e. its subgroup of triangular automorphisms. We will settle the titular question by providing three different answers, depending on to which properties the maximality condition is referring to.

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THEOREM 1.1.

(1) For every \( n \geq 2 \), the subgroup \( \mathcal{B}_n \) is maximal among all solvable subgroups of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \).

(2) The subgroup \( \mathcal{B}_2 \) is maximal among the closed subgroups of \( \text{Aut}(\mathbb{A}^2_\mathbb{C}) \).

(3) The subgroup \( \mathcal{B}_2 \) is not maximal among all subgroups of \( \text{Aut}(\mathbb{A}^2_\mathbb{C}) \).

Recall that \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) is naturally an ind-group, i.e. an infinite dimensional algebraic group. It is thus equipped with the usual ind-topology (see Section 2 for the definitions). In particular, since \( \mathcal{B}_n \) is a closed connected solvable subgroup of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \), the first statement of Theorem 1.1 can be interpreted as follows:

COROLLARY 1.2. — The group \( \mathcal{B}_n \) is a Borel subgroup of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \).

This generalizes a remark of Berest, Eshmatov and Eshmatov [4] stating that triangular automorphisms of \( \mathbb{A}^2_\mathbb{C} \), of Jacobian determinant 1, form a Borel subgroup (i.e. a maximal connected solvable subgroup) of the group \( \text{SAut}(\mathbb{A}^2_\mathbb{C}) \) of polynomial automorphisms of \( \mathbb{A}^2_\mathbb{C} \) of Jacobian determinant 1. Actually, the proofs in [4] also imply Corollary 1.2 in the case \( n = 2 \). Nevertheless, since they are based on results of Lamy [15], which use the Jung–van der Kulk–Nagata structure theorem for \( \text{Aut}(\mathbb{A}^2_\mathbb{C}) \), these arguments are specific to the dimension 2 and cannot be generalized to higher dimensions.

The Jonquières subgroup of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) is thus a good analogue of the subgroup of invertible upper triangular matrices, which is a Borel subgroup of the classical linear algebraic group \( \text{GL}_n(\mathbb{C}) \). Moreover, Berest, Eshmatov and Eshmatov strengthen this analogy when \( n = 2 \) by proving that \( \mathcal{B}_2 \) is, up to conjugacy, the only Borel subgroup of \( \text{Aut}(\mathbb{A}^2_\mathbb{C}) \). On the other hand, it is well known that there exist, if \( n \geq 3 \), algebraic additive group actions on \( \mathbb{A}^n_\mathbb{C} \) that cannot be triangularized [1, 21]. Therefore, we ask the following problem.

PROBLEM 1.3. — Show that Borel subgroups of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) are not all conjugate (\( n \geq 3 \)).

This problem turns out to be closely related to the question of the boundedness of the derived length of solvable subgroups of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \). We give such a bound when \( n = 2 \). More precisely, the maximal derived length of a solvable subgroup of \( \text{Aut}(\mathbb{A}^2_\mathbb{C}) \) is equal to 5 (see Proposition 3.14). As a consequence, we prove that the group \( \text{Aut}_z(\mathbb{A}^3_\mathbb{C}) \) of automorphisms of \( \mathbb{A}^3_\mathbb{C} \) fixing the last coordinate admits non-conjugate Borel subgroups (see
Corollary 3.22). Note that such a phenomenon has already been pointed out in [4].

The paper is organized as follows. Section 1 is the present introduction. In Section 2, we recall the definitions of ind-varieties and ind-groups given by Shafarevich and explain how the automorphism group of the affine $n$-space may be endowed with the structure of an ind-group.

In Section 3, we prove the first two statements of Theorem 1.1 and discuss the question, whether the ind-group $\text{Aut}(\mathbb{A}^n_k)$ does admit non-conjugate Borel subgroups. We then study the group of all automorphisms of $\mathbb{A}^n_k$ fixing the last variable, proving that it admits non-conjugate Borel subgroups. In the last part of Section 3, we give examples of maximal closed subgroups of $\text{Aut}(\mathbb{A}^n_k)$.

Finally, we consider $\text{Aut}(\mathbb{A}^2_k)$ as an “abstract” group in Section 4. We show that triangular automorphisms do not form a maximal subgroup of $\text{Aut}(\mathbb{A}^2_k)$. More precisely, after defining the affine length of an automorphism in Definition 4.1, we prove the following statement:

**Theorem 1.4.** — For any field $k$, the two following assertions hold.

1. If the affine length of an automorphism $f \in \text{Aut}(\mathbb{A}^2_k)$ is at least 1 (i.e. $f$ is not triangular) and at most 4, then the group generated by $B_2$ and $f$ satisfies

\[ \langle B_2, f \rangle = \text{Aut}(\mathbb{A}^2_k). \]

2. There exists an automorphism $f \in \text{Aut}(\mathbb{A}^2_k)$ of affine length 5 such that the group $\langle B_2, f \rangle$ is strictly included into $\text{Aut}(\mathbb{A}^2_k)$.

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## 2. Preliminaries: the ind-group of polynomial automorphisms

In [24, 25], Shafarevich introduced the notions of ind-varieties and ind-groups, and explained how to endow the group of polynomial automorphisms of the affine $n$-space with the structure of an ind-group. Since these two papers are well-known to contain several inaccuracies, we now recall the definitions from Shafarevich and describe the ind-group structure of the automorphism group of the affine $n$-space.

For simplicity, we assume in this section that $k$ is an algebraically closed field.
2.1. Ind-varieties and ind-groups

We first define the category of infinite dimensional algebraic varieties (ind-varieties for short).

**Definition 2.1** (Shafarevich [24]).

1. An ind-variety $V$ (over $k$) is a set together with an ascending filtration $V_{\leq 0} \subseteq V_{\leq 1} \subseteq V_{\leq 2} \subseteq \cdots \subseteq V$ such that the following holds:
   
   (a) $V = \bigcup_d V_{\leq d}$.
   
   (b) Each $V_{\leq d}$ has the structure of an algebraic variety (over $k$).
   
   (c) Each $V_{\leq d}$ is Zariski closed in $V_{\leq d+1}$.

2. A morphism of ind-varieties (or ind-morphism) is a map $\varphi : V \to W$ between two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ such that there exists, for every $d$, an $e$ for which $\varphi(V_{\leq d}) \subseteq W_{\leq e}$ and such that the induced map $V_{\leq d} \to W_{\leq e}$ is a morphism of varieties (over $k$).

In particular, every ind-variety $V$ is naturally equipped with the so-called ind-topology in which a subset $S \subseteq V$ is closed if and only if every subset $S_{\leq d} := S \cap V_{\leq d}$ is Zariski-closed in $V_{\leq d}$.

We remark that the product $V \times W$ of two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ has the structure of an ind-variety for the filtration $V \times W = \bigcup_d V_{\leq d} \times W_{\leq d}$.

**Definition 2.2.** — An ind-group is a group $G$ which is an ind-variety such that the multiplication $G \times G \to G$ and inversion $G \to G$ maps are morphisms of ind-varieties.

If $G$ is an abstract group, we denote by $D(G) = D^1(G)$ its (first) derived subgroup. It is the subgroup generated by all commutators $[g, h] := ghg^{-1}h^{-1}$, $g, h \in G$. The $n$-th derived subgroup of $G$ is then defined inductively by $D^n(G) = D^1(D^{n-1}(G))$ for $n \geq 1$, where by definition $D^0(G) = G$. A group $G$ is called solvable if $D^n(G) = \{1\}$ for some integer $n \geq 0$. Furthermore, the smallest such integer $n$ is called the derived length of $G$.

For later use, we state (and prove) the following results which are well-known for algebraic groups and which extend straightforwardly to ind-groups.

**Lemma 2.3.** — Let $H$ be a subgroup of an ind-group $G$. Then, the following assertions hold.

1. The closure $\overline{H}$ of $H$ is again a subgroup of $G$.  

(2) We have \( D(\overline{H}) \subseteq \overline{D(H)} \).
(3) If \( H \) is solvable, then \( \overline{H} \) is solvable too.

**Proof.**

(1). The proof for algebraic groups given in [11, Proposition 7.4A, p. 54] directly applies to ind-groups. This proof being very short, we give it here. Inversion being a homeomorphism, we get \( (\overline{H})^{-1} = \overline{H^{-1}} = \overline{H} \). Similarly, left translation by an element \( x \) of \( H \) being a homeomorphism, we get \( x\overline{H} = \overline{xH} = \overline{H} \), i.e. \( H\overline{H} \subseteq \overline{H} \). In turn, right translation by an element \( x \) of \( \overline{H} \) being a homeomorphism, we get \( \overline{H}x = \overline{Hx} \subseteq \overline{H}\overline{H} \subseteq \overline{H} = \overline{H} \). This says that \( \overline{H} \) is a subgroup.

(2). Fix an element \( y \) of \( H \). The map \( \phi: G \to G, \ x \mapsto [x,y] = xyx^{-1}y^{-1} \) being an ind-morphism, it is in particular continuous. Since \( H \) is obviously contained in \( \phi^{-1}(D(\overline{H})) \), we get \( \overline{H} \subseteq \phi^{-1}(\overline{D(H)}) \). Consequently, we have proven that

\[
\forall x \in \overline{H}, \forall y \in H, \ [x,y] \in \overline{D(H)}.
\]

In turn (and analogously), for each fixed element \( x \) of \( \overline{H} \), the map \( \psi: G \to G, \ y \mapsto [x,y] \) is continuous. Since \( H \) is included into \( \psi^{-1}(\overline{D(H)}) \), we get \( \overline{H} \subseteq \psi^{-1}(\overline{D(H)}) \) and thus

\[
\forall x,y \in \overline{H}, \ [x,y] \in \overline{D(H)}.
\]

This implies the desired inclusion.

(3). If \( H \) is solvable, it admits a sequence of subgroups such that

\[
H = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_n = \{1\} \quad \text{and} \quad D(H_i) \subseteq H_{i+1} \text{ for each } i.
\]

This yields \( \overline{H} = \overline{H}_0 \supseteq \overline{H}_1 \supseteq \cdots \supseteq \overline{H}_n = \{1\} \) and by (2) we get \( D(\overline{H}_i) \subseteq \overline{D(H_i)} \) for each \( i \). \( \square \)

### 2.2. Automorphisms of the affine \( n \)-space

As usual, given an endomorphism \( f \in \text{End}(\mathbb{A}^n_k) \), we denote by \( f^* \) the corresponding endomorphism of the algebra of regular functions \( \mathcal{O}(\mathbb{A}^n_k) = k[x_1, \ldots, x_n] \). Note that every endomorphism \( f \in \text{End}(\mathbb{A}^n_k) \) is uniquely determined by the polynomials \( f_i = f^*(x_i) \), \( 1 \leq i \leq n \).

In the sequel, we identify the set \( \mathcal{E}_n(k) := \text{End}(\mathbb{A}^n_k) \) with \( (k[x_1, \ldots, x_n])^n \). We thus simply denote by \( f = (f_1, \ldots, f_n) \) the element of \( \mathcal{E}_n(k) \) whose corresponding endomorphism \( f^* \) is given by

\[
f^*: \mathcal{O}(\mathbb{A}^n_k) \to \mathcal{O}(\mathbb{A}^n_k), \quad P(x_1, \ldots, x_n) \mapsto P \circ f = P(f_1, \ldots, f_n).
\]
The composition $g \circ f$ of two endomorphisms $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ is equal to

$$g \circ f = (g_1(f_1, \ldots, f_n), \ldots, g_n(f_1, \ldots, f_n)).$$

Note that for each nonnegative integer $d$, the following set is naturally an affine space (and therefore an algebraic variety!).

$$k[x_1, \ldots, x_n]_{\leq d} := \{ P \in k[x_1, \ldots, x_n], \ \deg P \leq d \}.$$

If $f = (f_1, \ldots, f_n) \in \mathcal{E}_n(k)$, we set $\deg f := \max_i\{\deg f_i\}$ and define

$$\mathcal{E}_n(k)_{\leq d} := \{ f \in \mathcal{E}_n(k), \ \deg f \leq d \}.$$

The equality $\mathcal{E}_n(k)_{\leq d} = (k[x_1, \ldots, x_n]_{\leq d})^n$ shows that $\mathcal{E}_n(k)_{\leq d}$ is naturally an affine space. Moreover, the filtration $\mathcal{E}_n(k) = \bigcup_d \mathcal{E}_n(k)_{\leq d}$ defines a structure of ind-variety on $\mathcal{E}_n(k)$.

We denote by $\mathcal{G}_n(k) = \text{Aut}(\mathbb{A}^n_k)$ the automorphism group of $\mathbb{A}^n_k$. The next result allows us to endow $\mathcal{G}_n(k)$ with the structure of an ind-variety.

**Lemma 2.4.** — Denote by $\mathcal{C}_n(k)$, resp. $\mathcal{J}_n(k)$, the set of elements $f$ in $\mathcal{E}_n(k)$ whose Jacobian determinant $\text{Jac}(f)$ is a constant, resp. a nonzero constant. Then, the following assertions hold:

1. The set $\mathcal{C}_n(k)$ is closed in $\mathcal{E}_n(k)$.
2. The set $\mathcal{J}_n(k)$ is open in $\mathcal{C}_n(k)$.
3. The set $\mathcal{G}_n(k)$ is closed in $\mathcal{J}_n(k)$.

**Proof.**

1. Since $\deg(\text{Jac}(f)) \leq n(\deg(f) - 1)$, the map $\text{Jac} : \mathcal{E}_n(k) \to k[x_1, \ldots, x_n]$ is an ind-morphism. By definition, $\mathcal{C}_n(k)$ is the preimage of the set $k$ which is closed in $k[x_1, \ldots, x_n]$.

2. The Jacobian morphism induces a morphism $\varphi : \mathcal{C}_n(k) \to k$, $f \mapsto \text{Jac}(f)$. By definition, $\mathcal{J}_n(k)$ is the preimage of the set $k^*$ which is open in $k$.

3. Set $\mathcal{J}_{n,0} := \{ f \in \mathcal{J}_n(k), \ f(0) = 0 \}$. Every element $f \in \mathcal{J}_{n,0}$ admits a formal inverse for the composition (see e.g. [7, Theorem 1.1.2]), i.e. a formal power series $g = \sum_{d \geq 1} g_d$, where each $g_d = (g_{d,1}, \ldots, g_{d,n})$ is a $d$-homogeneous element of $\mathcal{E}_n(k)$, meaning that $g_{d,1}, \ldots, g_{d,n}$ are $d$-homogeneous polynomials in $k[x_1, \ldots, x_n]$ such that

$$f \circ g = g \circ f = (x_1, \ldots, x_n) \quad \text{(as formal power series)}.$$

Furthermore, for each $d$, the map $\psi_d : \mathcal{J}_{n,0} \to \mathcal{E}_n(k)$ sending $f$ onto $g_d$ is a morphism because each coefficient of every component of $g_d$ can be expressed as a polynomial in the coefficients of the components of $f$ and in
the inverse \((\text{Jac } f)^{-1}\) of the polynomial \(\text{Jac } f\). Recall furthermore (see [2, Theorem 1.5]) that every automorphism \(f \in G_n(k)\) satisfies

\[(2.1) \quad \deg(f^{-1}) \leq (\deg f)^{n-1}.
\]

Therefore, an element \(f \in \mathcal{J}_n(k)_{\leq d}\) is an automorphism if and if \(\tilde{f} := f - f(0)\) is an automorphism. This amounts to saying that \(f\) is an automorphism if and only if \(\psi_e(\tilde{f}) = 0\) for all integers \(e > d^{n-1}\). These conditions being closed, we have proven that \(G_n(k)_{\leq d}\) is closed in \(\mathcal{J}_n(k)_{\leq d}\) for each \(d\), i.e. that \(G_n(k)\) is closed in \(\mathcal{J}_n(k)\). Note that when the field \(k\) has characteristic zero, the Jacobian conjecture (see for example [2, 7]) asserts that the equality \(G_n(k) = \mathcal{J}_n(k)\) actually holds. \(\square\)

Since the multiplication \(G_n(k) \times G_n(k) \to G_n(k)\) and inversion \(G_n(k) \to G_n(k)\) maps are morphisms (for the inversion, this again relies on the fundamental inequality \(2.1\)), we obtain that \(G_n(k)\) is an ind-group.

### 3. Borel subgroups

Throughout this section, we work over the field \(k = \mathbb{C}\) of complex numbers.

Note that the affine subgroup

\[A_n = \{f = (f_1, \ldots, f_n) \in G_n(\mathbb{C}) \mid \deg(f_i) = 1 \text{ for all } i = 1, \ldots, n\}\]

and the Jonquières (or triangular) subgroup

\[B_n = \{f = (f_1, \ldots, f_n) \in G_n(\mathbb{C}) \mid \forall i, f_i \in \mathbb{C}[x_i, \ldots, x_n]\}
\]

\[= \{f \in G_n(\mathbb{C}) \mid \forall i, f_i = a_i x_i + p_i, a_i \in \mathbb{C}^*, p_i \in \mathbb{C}[x_{i+1}, \ldots, x_n]\}\]

are both closed in \(G_n(\mathbb{C})\).

It is well known that the group \(G_n(\mathbb{C})\) is connected (see e.g. [25, proof of Lemma 4], [13, Proposition 2] or [22, Theorem 6]). The same is true for \(B_n\).

**Lemma 3.1.** — The groups \(G_n(\mathbb{C}) = \text{Aut}(\mathbb{A}^n_\mathbb{C})\) and \(B_n\) are connected.

**Proof.** — We say that a variety \(V\) is curve-connected if for all points \(x, y \in V\), there exists a morphism \(\varphi : C \to V\), where \(C\) is a connected curve (not necessarily irreducible) such that \(x\) and \(y\) both belong to the image of \(\varphi\). The same definition applies to ind-varieties.

We prove that \(G_n(\mathbb{C})\) and \(B_n\) are curve-connected. Let \(f\) be an element in \(G_n(\mathbb{C})\). We first consider the morphism \(\alpha : \mathbb{A}^n_\mathbb{C} \to G_n(\mathbb{C})\) defined by

\[\alpha(t) = f - tf(0, \ldots, 0)\]
which is contained in \( \mathcal{B}_n \) if \( f \) is triangular. Note that \( \alpha(0) = f \) and that the automorphism \( \tilde{f} := \alpha(1) \) fixes the origin of \( \mathbb{A}^n_\mathbb{C} \).

Therefore the morphism \( \beta: \mathbb{A}^1_\mathbb{C} \setminus \{0\} \to \mathcal{G}_n(\mathbb{C}), t \mapsto (t^{-1} \cdot \text{id}_{\mathbb{A}^n_\mathbb{C}}) \circ f \circ (t \cdot \text{id}_{\mathbb{A}^n_\mathbb{C}}) \) extends to a morphism \( \beta: \mathbb{A}^1_\mathbb{C} \to \mathcal{G}_n(\mathbb{C}) \) (with values in \( \mathcal{B}_n \) if \( f \), thus \( \tilde{f} \), is triangular) such that \( \beta(1) = \tilde{f} \) and such that \( \beta(0) \) is a linear map, namely the linear part of \( \tilde{f} \). This concludes the proof since \( \text{GL}_n(\mathbb{C}) \) (resp. the set of all invertible upper triangular matrices) is curve-connected. \( \square \)

Recall that the subgroup of upper triangular matrices in \( \text{GL}_n(\mathbb{C}) \) is solvable and has derived length \( \lceil \log_2(n) \rceil + 1 \), where \( \lceil x \rceil \) denotes the smallest integer greater than or equal to the real number \( x \) (see e.g. [26, p. 16]). In contrast, we have the following result.

**Lemma 3.2.** — The group \( \mathcal{B}_n \) is solvable of derived length \( n + 1 \).

**Proof.** — For each integer \( k \in \{0, \ldots, n\} \), denote by \( U_k \) the subgroup of \( \mathcal{B}_n \) whose elements are of the form \( f = (f_1, \ldots, f_n) \) where \( f_i = x_i \) for all \( i > k \) and \( f_i = x_i + p_i \) with \( p_i \in \mathbb{C}[x_{i+1}, \ldots, x_n] \) for all \( i \leq k \). We will prove \( D(\mathcal{B}_n) = U_n \) and \( D^j(U_n) = U_{n-j} \) for all \( j \in \{0, \ldots, n\} \).

For this, we consider the dilatation \( d(j, \lambda_j) \) and the elementary automorphism \( e(j, q_j) \) which are defined for every integer \( j \in \{1, \ldots, n\} \), every nonzero constant \( \lambda_j \in \mathbb{C}^* \) and every polynomial \( q_j \in \mathbb{C}[x_{j+1}, \ldots, x_n] \) by

\[
d(j, \lambda_j) = (g_1, \ldots, g_n) \quad \text{and} \quad e(j, q_j) = (h_1, \ldots, h_n),
\]

where \( g_j = \lambda_j x_j \), \( h_j = x_j + q_j \) and \( g_i = h_i = x_i \) for \( i \neq j \). Note that an element \( f \in U_k \) as above is equal to

\[
f = e(k, p_k) \circ \cdots \circ e(2, p_2) \circ e(1, p_1).
\]

In particular, this tells us that \( U_k \) is generated by the elements \( e(j, q_j) \), \( j \leq k \), \( q_j \in \mathbb{C}[x_{j+1}, \ldots, x_n] \).

The inclusion \( D(\mathcal{B}_n) \subseteq U_n \) is straightforward and left to the reader. The converse inclusion \( U_n \subseteq D(\mathcal{B}_n) \) follows from the equality

\[
[e(j, q_j), d(j, \lambda_j)] = e(j, (1 - \lambda_j)q_j).
\]

Finally, we prove \( D^j(U_n) = U_{n-j} \) by proving that the equality \( D(U_{k+1}) = U_k \) holds for all \( k \in \{0, \ldots, n-1\} \). The inclusion \( D(U_{k+1}) \subseteq U_k \) is straightforward and left to the reader. To prove the converse inclusion, let us introduce the map \( \Delta_i: \mathbb{C}[x_{i+1}, \ldots, x_n] \to \mathbb{C}[x_{i+1}, \ldots, x_n], q \mapsto q(x_{i+1}, \ldots, x_n) - q(x_{i+1}, \ldots, x_{i+1}) \). Note that \( \Delta_i \) is surjective and that

\[
[e(j, q_j), e(j+1, 1)] = e(j, \Delta_{j+1}(q_j))
\]

for all \( j \in \{1, \ldots, n-1\} \) and all \( q_j \in \mathbb{C}[x_{j+1}, \ldots, x_n] \). This implies \( U_k \subseteq D(U_{k+1}) \) and concludes the proof. \( \square \)
3.1. Triangular automorphisms form a Borel subgroup.

In this section, we prove the first two statements of Theorem 1.1 from the introduction. For this, we need the following result.

**Proposition 3.3.** — Let \( n \geq 2 \) be an integer. If a closed subgroup of \( \text{Aut}(A^n_C) \) strictly contains \( B_n \), then it also contains at least one linear automorphism that is not triangular.

**Proof.** — Let \( H \) be a closed subgroup of \( \text{Aut}(A^n_C) \) strictly containing \( B_n \). We first prove that \( H \) contains an automorphism whose linear part is not triangular. Let \( f = (f_1, \ldots, f_n) \) be an element in \( H \setminus B_n \). Then, there exists at least one component \( f_i \) of \( f \) that depends on an indeterminate \( x_j \) with \( j < i \), i.e. such that \( \frac{\partial f_i}{\partial x_j}(c) \neq 0 \). Now, choose \( c = (c_1, \ldots, c_n) \in A^n_C \) such that \( \frac{\partial f_i}{\partial x_j}(c) \neq 0 \) and consider the translation \( t_c := (x_1 + c_1, \ldots, x_n + c_n) \in B_n \).

Since \( f_i(x + c) = f_i(c) + \sum_k \frac{\partial f_i}{\partial x_k}(c)x_k + (\text{terms of higher order}) \), the linear part \( l \) of \( f \circ t_c \) is not triangular because it corresponds to the (non-triangular) invertible matrix \( \left( \frac{\partial f_i}{\partial x_k}(c) \right)_{ik} \). Composing on the left hand side by another translation \( t' \), we obtain an element \( g := t' \circ f \circ t \in H \) which fixes the origin of \( A^n_C \) and whose linear part is again \( l \).

For every \( \varepsilon \in C^* \), set \( h_\varepsilon := (\varepsilon x_1, \ldots, \varepsilon x_n) \in B_n \). We can finally conclude by noting that

\[
\lim_{\varepsilon \to 0} h_\varepsilon^{-1} \circ g \circ h_\varepsilon = l \in H,
\]

where the limit means that the ind-morphism \( \varphi: C^* \to \text{Aut}(A^n_C) \), \( \varepsilon \mapsto h_\varepsilon^{-1} \circ g \circ h_\varepsilon \) extends to a morphism \( \psi: C \to \text{Aut}(A^n_C) \) such that \( \psi(0) = l \).

Since we have \( \psi(\varepsilon) \in H \) for each \( \varepsilon \in C^* \), it is clear that \( \psi(0) \) must also belong to \( H \). Indeed, note that the set \( \{ \varepsilon \in C, \psi(\varepsilon) \in H \} \) is Zariski-closed in \( C \). \( \square \)

**Proposition 3.4.** — Let \( n \geq 2 \) be an integer. Then, the Jonquières group \( B_n \) is maximal among all solvable subgroups of \( \text{Aut}(A^n_C) \).

**Proof.** — Suppose by contradiction that there exists a solvable subgroup \( H \) of \( \text{Aut}(A^n_C) \) that strictly contains \( B_n \). Up to replacing \( H \) by its closure \( \overline{H} \) (see Lemma 2.3), we may assume that \( H \) is closed. By Proposition 3.3, the group \( H \cap A_n \) strictly contains \( B_n \cap A_n \). But since \( B_n \cap A_n \) is a Borel subgroup of \( A_n \), this prove that \( H \cap A_n \) is not solvable, thus that \( H \) itself is not solvable. Notice that we have used the fact that every Borel subgroup of...
a connected linear algebraic group is a maximal solvable subgroup. Indeed, every parabolic subgroup (i.e. a subgroup containing a Borel subgroup) of a connected linear algebraic group is necessarily closed and connected. See e.g. [11, Corollary B of Theorem (23.1), p. 143]. □

In dimension two, we establish another maximality property of the triangular subgroup which is actually stronger than the above one (see Remark 3.7 below).

**Proposition 3.5.** — The Jonquières group $B_2$ is maximal among the closed subgroups of $\text{Aut}(\mathbb{A}_2^2)$.

**Proof.** — Let $H$ be a closed subgroup of $\text{Aut}(\mathbb{A}_2^2)$ strictly containing $B_2$. By Proposition 3.3 above, $H$ contains a linear automorphism which is not triangular. This implies that $H$ contains all linear automorphisms, hence $A_2$, and it is therefore equal to $\text{Aut}(\mathbb{A}_2^2)$. Recall indeed that the subgroup $B_2 = B_2 \cap \text{GL}_2(\mathbb{C})$ of invertible upper triangular matrices is a maximal subgroup of $\text{GL}_2(\mathbb{C})$, since the Bruhat decomposition expresses $\text{GL}_2(\mathbb{C})$ as the disjoint union of two double cosets of $B_2$, which are namely $B_2$ and $B_2 \circ f \circ B_2$, where $f$ is any element of $\text{GL}_2(\mathbb{C}) \setminus B_2$. □

**Remark 3.6.** — Proposition 3.5 can not be generalized to higher dimension, since, if $n \geq 3$, then $B_n$ is strictly contained into the (closed) subgroup of automorphisms of the form $f = (f_1, \ldots, f_n)$ such that $f_n = a_n x_n + b_n$ for some $a_n, b_n \in \mathbb{C}$ with $a_n \neq 0$.

**Remark 3.7.** — Proposition 3.5 implies Proposition 3.4 for $n = 2$. Indeed, suppose that $B_2$ is strictly included into some solvable subgroup $H$ of $\text{Aut}(\mathbb{A}_2^2)$. Up to replacing $H$ by $\overline{H}$ (see Lemma 2.3), we may further assume that $H$ is closed. By Proposition 3.5, we would thus get that $H = \text{Aut}(\mathbb{A}_2^2)$. But this is a contradiction because the group $\text{Aut}(\mathbb{A}_2^2)$ is obviously not solvable, since it contains the linear group $\text{GL}(2, \mathbb{C})$ which is not solvable.

By Proposition 3.4, we can say that the triangular group $B_n$ is a Borel subgroup of $\text{Aut}(\mathbb{A}_n^2)$. This was already observed, in the case $n = 2$ only, by Berest, Eshmatov and Eshmatov in the nice paper [4] in which they obtained the following strong results. (In [4], these results are stated for the group $\text{SAut}(\mathbb{A}_2^2)$ of polynomial automorphisms of $\mathbb{A}_2^2$ of Jacobian determinant 1, but all the proofs remain valid for $\text{Aut}(\mathbb{A}_2^2)$.)

**Theorem 3.8 ([4]).**

1. All Borel subgroups of $\text{Aut}(\mathbb{A}_2^2)$ are conjugate to $B_2$.
2. Every connected solvable subgroup of $\text{Aut}(\mathbb{A}_2^2)$ is conjugate to a subgroup of $B_2$. 

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Recall that there exist, for every $n \geq 3$, connected solvable subgroups of $\text{Aut}(\mathbb{A}_n^c)$ that are not conjugate to subgroups of $\mathcal{B}_n$ [1, 21]. Hence, the second statement of the above theorem does not hold for $\text{Aut}(\mathbb{A}_n^c)$, $n \geq 3$. Similarly, we believe that not all Borel subgroups of $\text{Aut}(\mathbb{A}_n^c)$ are conjugate to $\mathcal{B}_n$ if $n \geq 3$. This would be clearly the case, if we knew that the following question has a positive answer.

**Question 3.9.** — Is every connected solvable subgroup of $\text{Aut}(\mathbb{A}_n^c)$, $n \geq 3$, contained into a maximal connected solvable subgroup?

The natural strategy to attack the above question would be to apply Zorn’s lemma, as we do in the proof of the following general proposition.

**Proposition 3.10.** — Let $G$ be a group endowed with a topology. Suppose that there exists an integer $c > 0$ such that every solvable subgroup of $G$ is of derived length at most $c$. Then, every solvable (resp. connected solvable) subgroup of $G$ is contained into a maximal solvable (resp. maximal connected solvable) subgroup.

**Proof.** — Let $H$ be a solvable (resp. connected solvable) subgroup of $G$. Denote by $\mathcal{F}$ the set of solvable (resp. connected solvable) subgroups of $G$ that contain $H$. Our hypothesis, on the existence of the bound $c$, implies that the poset $(\mathcal{F}, \subseteq)$ is inductive. Indeed, if $(H_i)_{i \in I}$ is a chain in $\mathcal{F}$, i.e. a totally ordered family of $\mathcal{F}$, then the group $\bigcup_i H_i$ is solvable, because we have that

$$D^j\left(\bigcup_i H_i\right) = \bigcup_i D^j(H_i)$$

for each integer $j \geq 0$. Moreover, if all $H_i$ are connected, then so is their union. Thus, $\mathcal{F}$ is inductive and we can conclude by Zorn’s lemma. □

**Remark 3.11.** — Proposition 3.10 does not require any compatibility conditions between the group structure and the topology on $G$. Let us moreover recall that an algebraic group (and all the more an ind-group) is in general not a topological group.

We are now left with another concrete question.

**Definition 3.12.** — Let $G$ be a group. We set

$$\psi(G) := \sup\{l(H) \mid H \text{ is a solvable subgroup of } G\} \in \mathbb{N} \cup \{+\infty\},$$

where $l(H)$ denotes the derived length of $H$.

**Question 3.13.** — Is $\psi(\text{Aut}(\mathbb{A}_n^c))$ finite?
Recall that $\psi(\text{GL}(n, \mathbb{C}))$ is finite. This classical result has been first established in 1937 by Zassenhaus [27, Satz 7] (see also [16]). More recently, Martelo and Ribón have proved in [17] that $\psi((O_{\text{ana}}(\mathbb{C}^n), 0)) < +\infty$, where $(O_{\text{ana}}(\mathbb{C}^n), 0)$ denotes the group of germs of analytic diffeomorphisms defined in a neighbourhood of the origin of $\mathbb{C}^n$.

Our next result answers Question 3.13 in the case $n = 2$.

**Proposition 3.14.** — We have $\psi(\text{Aut}(\mathbb{A}^2_\mathbb{C})) = 5$.

**Proof.** — The proof relies on a precise description of all subgroups of $\text{Aut}(\mathbb{A}^2_\mathbb{C})$, due to Lamy, that we will recall below. Using this description, the equality $\psi(\text{Aut}(\mathbb{A}^2_\mathbb{C})) = 5$ directly follows from the equality $\psi(\mathcal{A}_2) = 5$ that we will establish in the next section (see Proposition 3.16). The description of all subgroups of $\text{Aut}(\mathbb{A}^2_\mathbb{C})$ given by Lamy uses the amalgamated structure of this group, generally known as the theorem of Jung, van der Kulk and Nagata: The group $\text{Aut}(\mathbb{A}^2_\mathbb{C})$ is the amalgamated product of its subgroups $\mathcal{A}_2$ and $\mathcal{B}_2$ over their intersection

$$\text{Aut}(\mathbb{A}^2_\mathbb{C}) = \mathcal{A}_2 \ast_{\mathcal{A}_2 \cap \mathcal{B}_2} \mathcal{B}_2.$$ 

In the discussion below, we will use the Bass–Serre tree associated to this amalgamated structure. We refer the reader to [23] for details on Bass–Serre trees in full generality and to [15] for details on the particular tree associated to the above amalgamated structure. That latter tree is the tree whose vertices are the left cosets $g \circ \mathcal{A}_2$ and $h \circ \mathcal{B}_2$, $g, h \in \text{Aut}(\mathbb{A}^2_\mathbb{C})$. Two vertices $g \circ \mathcal{A}_2$ and $h \circ \mathcal{B}_2$ are related by an edge if and only if there exists an element $k \in \text{Aut}(\mathbb{A}^2_\mathbb{C})$ such that $g \circ \mathcal{A}_2 = k \circ \mathcal{A}_2$ and $h \circ \mathcal{B}_2 = k \circ \mathcal{B}_2$, i.e. if and only if $g^{-1} \circ h \in \mathcal{A}_2 \circ \mathcal{B}_2$. The group $\text{Aut}(\mathbb{A}^2_\mathbb{C})$ acts on the Bass–Serre tree by left translation: For all $g, h \in \text{Aut}(\mathbb{A}^2_\mathbb{C})$, we set $g \circ (h \circ \mathcal{A}_2) = (g \circ h) \circ \mathcal{A}_2$ and $g \circ (h \circ \mathcal{B}_2) = (g \circ h) \circ \mathcal{B}_2$. Each element of $\text{Aut}(\mathbb{A}^2_\mathbb{C})$ satisfies one property of the following alternative:

1. It is triangularizable, i.e. conjugate to an element of $\mathcal{B}_2$. This is the case where the automorphism fixes at least one point on the Bass–Serre tree.
2. It is a Hénon automorphism, i.e. it is conjugate to an element of the form

$$g = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k,$$

where $k \geq 1$, each $a_i$ belongs to $\mathcal{A}_2 \setminus \mathcal{B}_2$ and each $b_i$ belongs to $\mathcal{B}_2 \setminus \mathcal{A}_2$. This is the case where the automorphism acts without fixed points, but preserves a (unique) geodesic of the Bass–Serre tree on which it acts as a translation of length $2k$. 


Furthermore, according to [15, Theorem 2.4], every subgroup $H$ of $\text{Aut}(A_2^2)$ satisfies one and only one of the following assertions:

1. It is conjugate to a subgroup of $A_2$ or of $B_2$.
2. Every element of $H$ is triangularizable and $H$ is not conjugate to a subgroup of $A_2$ or of $B_2$. In that case, $H$ is Abelian.
3. The group $H$ contains some Hénon automorphisms (i.e. non triangularizable automorphisms) and all those have the same geodesic on the Bass–Serre tree. The group $H$ is then solvable.
4. The group $H$ contains two Hénon automorphisms having different geodesics. Then, $H$ contains a free group with two generators.

Let $H$ be now a solvable subgroup of $\text{Aut}(A_2^2)$. If we are in case (1), then we may assume that $H$ is a subgroup of $A_2$ or of $B_2$. Since $\psi(A_2) = 5$ and $\psi(B_2) = 3$ (the group $B_2$ being solvable of derived length 3), this settles this case. In case (2), $H$ is Abelian hence of derived length at most 1. In case (3), there exists a geodesic $\Gamma$ which is globally fixed by every element of $H$. Therefore, we may assume without restriction that

$$H = \{ f \in \text{Aut}(A_2^2), \ f(\Gamma) = \Gamma \}.$$ 

Note that $D^2(H)$ is included into the group $K$ that fixes pointwise the geodesic $\Gamma$. Up to conjugation, we may assume that $\Gamma$ contains the vertex $B_2$, i.e. that $K$ is included into $B_2$. By [15, Proposition 3.3], each element of $\text{Aut}(A_2^2)$ fixing an unbounded set of the Bass–Serre tree has finite order. If $f, g \in K$, their commutator is of the form $(x + p(y), y + c)$. This latter automorphism being of finite order, it must be equal to the identity, showing that $K$ is Abelian. Therefore, we get $D^3(H) = \{1\}$.

Finally, we cannot be in case (4), because a free group with two generators is not solvable.

From Propositions 3.10 and 3.14, we get at once the following result, which also follows from Theorem 3.8 above.

**Corollary 3.15.** — Every solvable connected subgroup of $\text{Aut}(A_2^2)$ is contained into a Borel subgroup.

### 3.2. Proof of the equality $\psi(A_2) = 5$.

Recall that Newman [20] has computed the exact value $\psi(\text{GL}(n, \mathbb{C}))$ for all $n$. It turns out that $\psi(\text{GL}(n, \mathbb{C}))$ is equivalent to $5 \log_9(n)$ as $n$ goes to infinity (see [26, Theorem 3.10]). Let us give a few particular values for
\(\psi(\text{GL}(n, \mathbb{C}))\) taken from [20].

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 18 & 26 & 34 & 66 & 74 \\
\hline
\psi(\text{GL}(n, \mathbb{C})) & 1 & 4 & 5 & 6 & 7 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
\end{array}
\]

We now consider the affine group \(A_n\). On the one hand, observe that \(A_n\) is isomorphic to a subgroup of \(\text{GL}(n+1, \mathbb{C})\). Hence, \(\psi(A_n) \leq \psi(\text{GL}(n+1, \mathbb{C}))\). On the other hand, we have the short exact sequence

\[1 \to \mathbb{C}^n \to A_n \xrightarrow{L} \text{GL}(n+1, \mathbb{C}) \to 1,\]

where \(L: A_n \to \text{GL}(n, \mathbb{C})\) is the natural morphism sending an affine transformation to its linear part. Thus, if \(H\) is a solvable subgroup of \(A_n\), we have a short exact sequence

\[1 \to H \cap (\mathbb{C}^n) \to H \xrightarrow{L} L(H) \to 1.\]

Since \(L(H)\) is solvable of derived length at most \(\psi(\text{GL}(n, \mathbb{C}))\) and since \(H \cap (\mathbb{C}^n)\) is Abelian, this implies that \(l(H) \leq \psi(\text{GL}(n, \mathbb{C})) + 1\). Therefore, we have proved the general formula

\[\psi(\text{GL}(n, \mathbb{C})) \leq \psi(A_n) \leq \min\{\psi(\text{GL}(n, \mathbb{C})) + 1, \psi(\text{GL}(n+1, \mathbb{C}))\}\].

For \(n = 2\), this yields \(\psi(A_2) = 4\) or \(5\). We shall now prove that \(A_2\) contains solvable subgroups of derived length 5 (see Lemma 3.19 below), hence the following desired result.

**Proposition 3.16.** — The maximal derived length of a solvable subgroup of the affine group \(A_2\) is 5, i.e. we have \(\psi(A_2) = 5\).

As explained above, it still remains to provide an example of a solvable subgroup of \(A_2\) of derived length 5. In that purpose, recall that the group \(\text{PSL}(2, \mathbb{C})\) contains a subgroup isomorphic to the symmetric group \(S_4\) and that all such subgroups are conjugate (see for example [3]).

**Definition 3.17.** — The binary octahedral group \(2O\) is the pre-image of the symmetric group \(S_4\) by the \((2 : 1)\)-cover \(\text{SL}(2, \mathbb{C}) \to \text{PSL}(2, \mathbb{C})\).

The following result is also well-known.

**Lemma 3.18.** — The derived length of the binary octahedral group \(G = 2O\) is 4.

**Proof.** — Using the short exact sequence

\[0 \to \{\pm I\} \to G \xrightarrow{\pi} S_4 \to 0,\]

we get \(\pi(D^2G) = D^2(\pi(G)) = D^2(S_4) = V_4\), where \(V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2\) is the Klein group. One could also easily check that \(\pi^{-1}(V_4)\) is isomorphic to
the quaternion group \( \mathbb{Q}_8 \). The equality \( \pi(D^2G) = V_4 \) is then sufficient for showing that \( D^2G = \pi^{-1}(V_4) \). Indeed, if \( D^2G \) was a strict subgroup of \( \pi^{-1}(V_4) \cong \mathbb{Q}_8 \), it would be cyclic, hence \( \pi(D^2G) = V_4 \) would be cyclic too. A contradiction. Since \( D^2G \cong \mathbb{Q}_8 \) has derived length 2, this shows us that the derived length of \( G \) is \( 2 + 2 = 4 \).

**Lemma 3.19.** — Consider the pre-image \( L^{-1}(G) \cong G \rtimes \mathbb{C}^2 \) of the binary octahedral group \( G := 2O \subseteq \text{SL}(2, \mathbb{C}) \) by the natural morphism \( L: \mathcal{A}_2 \to \text{GL}(2, \mathbb{C}) \) sending an affine transformation onto its linear part. Then, the derived length of \( L^{-1}(G) \) is equal to 5.

**Proof.** — By Lemma 3.18, the derived length of \( G \) is 4. The short exact sequence

\[
1 \to \mathbb{C}^2 \to G \rtimes \mathbb{C}^2 \to G \to 1
\]

implies that the derived length of \( G \rtimes \mathbb{C}^2 \) is at most \( 4 + 1 = 5 \). Moreover, the strictly decreasing sequence \( G = D^0(G) > D^1(G) > D^2(G) > D^3(G) > D^4(G) = 1 \) shows that the group \( D^2(G) \) is non-Abelian and in particular non-cyclic. By Lemma 3.20 below, we thus have \( D^i(G \rtimes \mathbb{C}^2) = D^i(G) \rtimes \mathbb{C}^2 \) for every \( i \leq 3 \). But since \( D^3(G) \) is non-trivial, the group \( D^3(G \rtimes \mathbb{C}^2) = D^3(G) \rtimes \mathbb{C}^2 \) strictly contains the subgroup \((\mathbb{C}^2, +)\) of translations and cannot be Abelian, because the group \( \mathbb{C}^2 \) is its own centralizer in \( \mathcal{A}_2 \).

Finally, we get \( D^4(G \rtimes \mathbb{C}^2) \neq 1 \), proving that the derived length of \( G \rtimes \mathbb{C}^2 \) is indeed 5. \( \square \)

**Lemma 3.20.** — Let \( H \) be a finite non-cyclic subgroup of \( \text{GL}(2, \mathbb{C}) \). Then the derived subgroup of \( L^{-1}(H) = H \rtimes \mathbb{C}^2 \subseteq \mathcal{A}_2 \) is the group \( D(H) \rtimes \mathbb{C}^2 \).

**Proof.** — Set \( K := D(H \rtimes \mathbb{C}^2) \cap \mathbb{C}^2 \). Note that \( K \) contains the commutator [\( \text{id} + v, h \)] for all \( v \in \mathbb{C}^2 \), \( h \in H \), i.e. it contains all elements \( h \cdot v - v \). It is enough to show that these vectors generate \( \mathbb{C}^2 \). Indeed, it would then imply that there exist \( h_1, v_1, h_2, v_2 \) such that the vectors \( h_1 \cdot v_1 - v_1 \) and \( h_2 \cdot v_2 - v_2 \) are linearly independent. But then, \( K \) would also contain the vectors \( h_1 \cdot (\lambda_1 v_1) - (\lambda_1 v_1) + h_2 \cdot (\lambda_2 v_2) - \lambda_2 v_2 \) for any \( \lambda_1, \lambda_2 \in \mathbb{C} \), proving that \( K = \mathbb{C}^2 \). Therefore, let us assume by contradiction that there exists a non-zero vector \( w \in \mathbb{C}^2 \) such that \( h \cdot v - v \) is a multiple of \( w \) for all \( h \in H \), \( v \in \mathbb{C}^2 \). Take \( w' \in \mathbb{C}^2 \) such that \((w, w')\) is a basis of \( \mathbb{C}^2 \). In this basis, any element of \( H \) admits a matrix of the form

\[
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}.
\]
Therefore, by the theory of representations of finite group, we may assume, up to conjugation, that each element of $H$ admits a matrix of the form
\[
\begin{pmatrix}
a & 0 \\
0 & 1
\end{pmatrix}.
\]
This would imply that $H$ is isomorphic to a finite subgroup of $\mathbb{C}^*$, hence that it is cyclic. A contradiction. \hfill $\square$

### 3.3. An ind-group with nonconjugate Borel subgroups.

In this section, we consider the subgroup \(\text{Aut}_z(\mathbb{A}_3^3)\) of \(\text{Aut}(\mathbb{A}_3^3)\) consisting of all automorphisms \(f = (f_1, f_2, z)\) fixing the last coordinate of \(\mathbb{A}_3^3 = \text{Spec}(\mathbb{C}[x, y, z])\). Since it is clearly a closed subgroup, it is also an ind-group. Note that \(\text{Aut}_z(\mathbb{A}_3^3)\) is naturally isomorphic to a subgroup of \(\text{Aut}(\mathbb{A}_2^2(z))\).

In its turn, the field \(\mathbb{C}(z)\) can be embedded into the field \(\mathbb{C}\), so that the group \(\text{Aut}(\mathbb{A}_2^2(z))\) is isomorphic to a subgroup of \(\text{Aut}(\mathbb{A}_2^2)\). Therefore, by Proposition 3.14, we get
\[
\psi(\text{Aut}_z(\mathbb{A}_3^3)) \leq \psi(\text{Aut}(\mathbb{A}_2^2(z))) \leq \psi(\text{Aut}(\mathbb{A}_2^2)) = 5.
\]
Recall moreover that \(\text{Aut}_z(\mathbb{A}_3^3)\) contains nontriangularizable additive group actions [1]. Let us briefly describe the example given by Bass. Consider the following locally nilpotent derivation of \(\mathbb{C}[x, y, z]\):
\[
\Delta = -2y\partial_x + z\partial_y.
\]
Then, the derivation \((xz + y^2)\Delta\) is again locally nilpotent. We associate it with the morphism
\[
(\mathbb{C}, +) \to \text{Aut}_\mathbb{C}(\mathbb{C}[x, y, z]), \quad t \mapsto \exp(t(xz + y^2)\Delta).
\]
The automorphism of \(\mathbb{A}_3^3\) corresponding to \(\exp(t(xz + y^2)\Delta)\) is given by
\[
f_t := (x - 2ty(xz + y^2) - t^2z(xz + y^2)^2, y + tz(xz + y^2), z) \in \text{Aut}(\mathbb{A}_3^3).
\]
For \(t = 1\), we get the famous Nagata automorphism. Note that the fixed point set of the corresponding \((\mathbb{C}, +)\)-action on \(\mathbb{A}_3^3\) is the hypersurface \(\{xz + y^2 = 0\}\) which has an isolated singularity at the origin. On the other hand, the fixed point set of a triangular \((\mathbb{C}, +)\)-action on \(\mathbb{A}_3^3\)
\[
t \mapsto g_t = \exp(t(a(y, z)\partial_x + b(z)\partial_y)) \in \text{Aut}(\mathbb{A}_3^3)
\]
is the set \(\{a(y, z) = b(z) = 0\}\), which is isomorphic to a cylinder \(\mathbb{A}_1^3 \times Z\) for some variety \(Z\). This implies that the \((\mathbb{C}, +)\)-action \(t \mapsto f_t\) is not triangularizable.
By Proposition 3.10, it follows that $\text{Aut}_z(A^3_\mathbb{C})$ contains Borel subgroups that are not conjugate to a subgroup of the group
$$\mathcal{B}_z = \{(f_1, f_2, z) \in \text{Aut}(A^3_\mathbb{C}) \mid f_1 \in \mathbb{C}[x, y, z], f_2 \in \mathbb{C}[y, z]\}$$
of triangular automorphisms of $\text{Aut}_z(A^3_\mathbb{C})$.

**Proposition 3.21.** The group $\mathcal{B}_z$ is a Borel subgroup of $\text{Aut}_z(A^3_\mathbb{C})$.

**Proof.** With the same proof as for Lemma 3.1, we obtain easily that $\mathcal{B}_z$ is connected. It is also solvable, since it can be seen as a subgroup of the Jonquières subgroup of $\text{Aut}(A^2_{\mathbb{C}(z)})$, which is solvable.

Now, we simply follow the proof of Proposition 3.3. Let $H \subset \text{Aut}_z(A^3_\mathbb{C})$ be a closed subgroup strictly containing $\mathcal{B}_z$ and take an element $f$ in $H \setminus \mathcal{B}_z$, i.e. an element $f = (f_1, f_2, z)$ with $f_2 \in \mathbb{C}[x, y, z] \setminus \mathbb{C}[y, z]$. Arguing as before, we can find suitable translations $t_c = (x+c_1, y+c_2, z)$ and $t_{c'} = (x+c_1', y+c_2', z)$ such that the automorphism $g = t_c \circ f \circ t_{c'}$ fixes the point $(0, 0, 0)$ and is of the form $g = (g_1, g_2, z)$ with $g_2 = xc(z) + yd(z) + h(x, y, z)$ for some $c(z), d(z) \in \mathbb{C}[z], c(z) \neq 0$, and some polynomial $h(x, y, z)$ belonging to the ideal $(x^2, xy, y^2)$ of $\mathbb{C}[x, y, z]$.

Conjugating this $g$ by the automorphism $(tx, ty, z) \in H, t \neq 0$, and taking the limit when $t$ goes to 0, we obtain an element of the form $(a(z)x + b(z)y, c(z) x + d(z)y, z)$ with $c(z) \neq 0$ in $H$. By Lemma 3.23 below, this implies that the group $H$ is not solvable. □

**Corollary 3.22.** The ind-group $\text{Aut}_z(A^3_\mathbb{C})$ contains non-conjugate Borel subgroups.

In the course of the proof of Proposition 3.21, we have used the following lemma that we prove now.

**Lemma 3.23.** The subgroup $B_2(\mathbb{C}[z])$ of upper triangular matrices of $\text{GL}_2(\mathbb{C}[z])$ is a maximal solvable subgroup.

**Proof.** For every $\alpha \in \mathbb{C}$, denote by $\text{ev}_\alpha : \text{GL}_2(\mathbb{C}[z]) \to \text{GL}_2(\mathbb{C})$ the evaluation map that associates to an element $M(z) \in \text{GL}_2(\mathbb{C}[z])$ the constant matrix $M(\alpha)$ obtained by replacing $z$ by $\alpha$. Let $H$ be a subgroup of $\text{GL}_2(\mathbb{C}[z])$ strictly containing the group $B_2(\mathbb{C}[z])$. By definition, $H$ contains a non-triangular matrix, i.e. a matrix of the form
$$M = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \text{ with } c \neq 0.$$Choose a complex number $\alpha$ such that $c(\alpha) \neq 0$. Then, the group $\text{ev}_\alpha(H)$ contains the upper triangular constant matrices $B_2(\mathbb{C})$ and a non-triangular matrix. Therefore, $\text{ev}_\alpha(H) = \text{GL}_2(\mathbb{C})$ and $H$ is not solvable. □
Remark 3.24. — By Nagao’s theorem (see [18] or e.g. [23, Chapter II, no. 1.6]), we have an amalgamated product structure
\[ \text{GL}_2(\mathbb{C}[z]) = \text{GL}_2(\mathbb{C}) \ast_{\text{B}_2(\mathbb{C})} \text{B}_2(\mathbb{C}[z]). \]
However, contrarily to the case of \( \text{Aut}(\mathbb{A}^2) \), the group \( \text{B}_2(\mathbb{C}[z]) \) is not a maximal closed subgroup. Indeed, for every complex number \( \alpha \), this group is strictly included into the group \( \text{ev}_\alpha^{-1}(\text{B}_2(\mathbb{C})) \).

3.4. Maximal closed subgroups

In this section, we mainly focus on the following question.

Question 3.25. — What are the maximal closed subgroups of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \)?

First of all, it is easy to observe that, since the action of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) on \( \mathbb{A}^n_\mathbb{C} \) is infinite transitive, i.e. \( m \)-transitive for all integers \( m \geq 1 \), the stabilizers of a finite number of points are examples of maximal closed subgroups.

Proposition 3.26. — For every finite subset \( \Delta \) of \( \mathbb{A}^n_\mathbb{C} \), \( n \geq 2 \), the group \( \text{Stab}(\Delta) = \{ f \in \text{Aut}(\mathbb{A}^n_\mathbb{C}), f(\Delta) = \Delta \} \) is a maximal subgroup of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \). Furthermore, it is closed.

Proof. — Let \( \Delta = \{ a_1, \ldots, a_k \} \) be a finite subset of \( \mathbb{A}^n_\mathbb{C} \). Let \( f \in \text{Aut}(\mathbb{A}^n_\mathbb{C}) \setminus \text{Stab}(\Delta) \). We will prove that \( \langle \text{Stab}(\Delta), f \rangle = \text{Aut}(\mathbb{A}^n_\mathbb{C}) \), where \( \langle \text{Stab}(\Delta), f \rangle \) denotes the subgroup of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) that is generated by \( \text{Stab}(\Delta) \) and \( f \). We will use repetitively the well-known fact that \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) acts \( 2k \)-transitively on \( \mathbb{A}^n_\mathbb{C} \).

We first observe that \( \langle \text{Stab}(\Delta), f \rangle \) contains an element \( g \) such that \( g(\Delta) \cap \Delta = \emptyset \). To see this, denote by \( m := |\Delta \cap f(\Delta)| \) the cardinality of the set \( \Delta \cap f(\Delta) \). Up to composing it by an element of \( \text{Stab}(\Delta) \), we can suppose that \( f \) fixes the points \( a_1, \ldots, a_m \) and maps \( a_{m+1}, \ldots, a_k \) outside \( \Delta \). If \( m \geq 1 \), then we consider an element \( \alpha \in \text{Stab}(\Delta) \) that maps the point \( a_m \) onto \( a_{m+1} \) and sends all points \( f(a_{m+1}), \ldots, f(a_k) \) outside the set \( f^{-1}(\Delta) \). Remark that \( g = f \circ \alpha \circ f \) is an element of \( \langle \text{Stab}(\Delta), f \rangle \) with \( |\Delta \cap g(\Delta)| < m \).

By descending induction on \( m \), we can further suppose that \( |\Delta \cap g(\Delta)| = 0 \) as desired.

Now, consider any \( \varphi \in \text{Aut}(\mathbb{A}^n_\mathbb{C}) \). Let us prove that \( \varphi \) belongs to the subgroup \( \langle \text{Stab}(\Delta), g \rangle \). Take an element \( \beta \in \text{Stab}(\Delta) \) such that \( \beta(\varphi(\Delta)) \cap g^{-1}(\Delta) = \emptyset \). Then, \( g(\beta(\varphi(\Delta)) \cap \Delta = \emptyset \) and we can find an element \( \gamma \in \text{Stab}(\Delta) \) such that \( \gamma \circ \beta(\varphi(\Delta)) = f \cdot \varphi(\Delta) \).
Stab(Δ) such that (γ ◦ g ◦ β ◦ ϕ)(aᵢ) = g(aᵢ) for all i. We have ϕ = β⁻¹ ◦ g⁻¹ ◦ γ⁻¹ ◦ g ◦ δ ∈ Stab(Δ), g, where δ := g⁻¹ ◦ (γ ◦ g ◦ β ◦ ϕ) is an element of Stab(Δ), proving that (Stab(Δ), g) is equal to the whole group Aut(An). Therefore, the group Stab(Δ) is actually maximal in Aut(An). Finally, note that for each point a ∈ An the evaluation map evₐ : Aut(An) → An, f ↦→ f(a), is an ind-morphism. Since ∆ is a closed subset of An the equality

\[ \text{Stab}(\Delta) = \bigcap_i (\text{ev}_{a_i})^{-1}(\Delta) \]

implies that Stab(Δ) is closed in Aut(An).

□

Besides the above examples and the triangular subgroup B₂, the only other maximal closed subgroup of Aut(An), n ≥ 2, contains strictly the affine subgroup A₂. The fact that A₂ is maximal among all closed subgroups of Aut(An) is a particular case of the following recent result of Edo [5].

(We recall that the so-called tame subgroup of Aut(An) is its subgroup generated by An and B₂.)

**Theorem 3.27** ([5]). — If a closed subgroup of Aut(An), n ≥ 2, contains strictly the affine subgroup A₂, then it also contains the whole tame subgroup, hence its closure. In particular, for n = 2, the affine group A₂ is maximal among the closed subgroups of Aut(An).

**Remark 3.28.** — Note that Theorem 3.27 does not allow us to settle the question of the (non) maximality of Aₙ among the closed subgroups of Aut(An) when n ≥ 3. Indeed, on the one hand, it was recently shown that, in dimension 3, the tame subgroup is not closed (see [6]). But, on the other hand, it is still unknown whether it is dense in Aut(An) or not. For n ≥ 4, the three questions, whether the tame subgroup is closed, whether it is dense, or even whether it is a strict subgroup of Aut(An), are all open.

Let us finally remark that the affine group A₂ is not a maximal among all abstract subgroups of Aut(An). Indeed, using the amalgamated structure

\[ \text{Aut}(\mathcal{A}_2^n) = A_2 \ast_{A_2 \cap B_2} B_2 \]

and following [8], we can define the multidegree (or polydegree) of any automorphism f ∈ Aut(An) in the following way. If f admits an expression

\[ f = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \circ a_{k+1}, \]

where each aᵢ belongs to A₂, each bᵢ belongs to B₂ and aᵢ /∈ B₂ for 2 ≤ i ≤ k, bᵢ /∈ A₂ for 1 ≤ i ≤ k, the multidegree of f is defined as the finite sequence (possibly empty) of integers at least equal to 2:

\[ \text{mdeg}(f) = (\text{deg } b_1, \text{deg } b_2, \ldots, \text{deg } b_k). \]
Then, the subgroup $M_r := \langle A_2, (B_2)^{\leq r} \rangle \subseteq \text{Aut}(\mathbb{A}^2_{\mathbb{C}})$ coincides with the set of automorphisms whose multidegree is of the form $(d_1, \ldots, d_k)$ for some $k$ with $d_1, \ldots, d_k \leq r$. We thus have a strictly increasing sequence of subgroups 

$A_2 = M_1 < M_2 < \cdots < M_d < \cdots,$

demonstrating in particular that $A_2$ is not a maximal abstract subgroup.

4. Non-maximality of the Jonquières subgroup in dimension 2

Throughout this section, we work over an arbitrary ground field $k$.

Recall that by the famous Jung–van der Kulk–Nagata theorem [12, 14, 19], the group $\text{Aut}(\mathbb{A}^2_k)$, of algebraic automorphisms of the affine plane, is the amalgamated free product of its affine subgroup

$A = \{(ax + by + c, a'x + b'y + c') \in \text{Aut}(\mathbb{A}^2_k) \mid a, b, c, a', b', c' \in \mathbb{k}\}$

and its Jonquières subgroup

$B := \{(ax + p(y), b'y + c') \in \text{Aut}(\mathbb{A}^2_k) \mid a, b', c' \in \mathbb{k}, p(y) \in \mathbb{k}[y]\}$

above their intersection. Therefore, every element $f \in \text{Aut}(\mathbb{A}^2_k)$ admits a reduced expression as a product of the form

$(\ast) \quad f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1},$

where $a_1, \ldots, a_n$ belong to $A \setminus A \cap B$, and $t_1, \ldots, t_{n+1}$ belong to $B$ with $t_2, \ldots, t_n \notin A \cap B$.

**Definition 4.1.** — *The number $n$ of affine non-triangular automorphisms appearing in such an expression for $f$ is unique. We call it the affine length of $f$ and denote it by $\ell_A(f)$.*

**Remark 4.2.** — Instead of counting affine elements to define the length of an automorphism of $\mathbb{A}^2$, one can of course also consider the Jonquières elements and define the triangular length $\ell_B(f)$ of every $f \in \text{Aut}(\mathbb{A}^2_k)$. Actually, this is the triangular length, that one usually uses in the literature. Let us in particular recall that this length map $\ell_B : \text{Aut}(\mathbb{A}^2_{\mathbb{C}}) \to \mathbb{N}$ is lower semicontinuous [9], when considering $\text{Aut}(\mathbb{A}^2_{\mathbb{C}})$ as an ind-group. Since

$\ell_A(f) = \max_{b_1, b_2 \in B} \ell_B(b_1 \circ f \circ b_2) - 1$

for every $f \in \text{Aut}(\mathbb{A}^2_k)$ and since the supremum of arbitrarily many lower semicontinuous maps is lower semicontinuous, we infer that $\ell_A$ has also this property.
Proposition 4.3. — The affine length map $\ell_A: \text{Aut}(A^2_k) \to \mathbb{N}$ is lower semicontinuous.

The next result shows that the Jonquières subgroup is not a maximal subgroup of $\text{Aut}(A^2_k)$.

Proposition 4.4. — Let $p \in k[y]$ be a polynomial that fulfils the following property:

\[(WG) \quad \forall \alpha, \beta, \gamma \in k, \quad \text{deg}[p(y) - \alpha p(\beta y + \gamma)] \leq 1 \implies \alpha = \beta = 1 \text{ and } \gamma = 0,\]

and consider the following elements of $\text{Aut}(A^2_k)$:

$$\sigma = (y, x), \quad t = (-x + p(y), y), \quad f = (\sigma \circ t)^2 \circ \sigma \circ (t \circ \sigma)^2.$$ 

Then, the subgroup generated by $B$ and $f$ is a strict subgroup of $\text{Aut}(A^2_k)$, i.e. $\langle B, f \rangle \neq \text{Aut}(A^2_k)$.

Remark 4.5. — Polynomials satisfying the above property (WG) are called weakly general in [10], where a stronger notion of a general polynomial is also given (see [10, Definition 15, p. 585]). In particular, by [10, Example 65, p. 608], the polynomial $q = y^5 + y^4$ is weakly general if $k$ is a field of characteristic zero.

Moreover, the polynomial $q = y^{2p} - y^{2p-1}$ is weakly general if $\text{char}(k) = p > 0$. This follows directly from the fact that the coefficients of $y^{2p}$, $y^{2p-1}$ and $y^{2p-2}$ in the polynomial $q(y) - \alpha q(\beta y + \gamma)$ are equal to $1 - \alpha \beta^{2p}$, $1 - \alpha \beta^{2p-1}$ and $-\alpha \beta^{2p-2} \gamma$, respectively.

Proof of Proposition 4.4. — Remark that $\sigma$ and $t$, hence $f$, are involutions. Therefore, every element $g \in \langle B, f \rangle$ can be written as

$$g = b_1 \circ f \circ b_2 \circ f \circ \cdots \circ b_k \circ f \circ b_{k+1},$$

where the elements $b_i$ belong to $B$ and where we can assume without restriction that $b_2, \ldots, b_k$ are different from the identity (otherwise, the expression for $g$ could be shortened using that $f^2 = \text{id}$).

In order to prove the proposition, it is enough to show that no element $g$ as above is of affine-length equal to 1. Note that $\ell_A(g) = 0$ if $k = 0$ and that $\ell_A(g) = \ell_A(f) = 5$ if $k = 1$. It remains to consider the case where $k \geq 2$. 

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For this, let us define four subgroups $B_0, \ldots, B_3$ of $B$ by

- $B_0 = B$,
- $B_1 = A \cap B = \{(ax + by + c, b'y + c') \mid a, b, c, b', c' \in k, a, b' \neq 0\}$,
- $B_2 = (A \cap B) \cap [\sigma \circ (A \cap B) \circ \sigma]$
  $= \{(ax + c, b'y + c') \mid a, b, c', c' \in k, a, b' \neq 0\}$,
- $B_3 = \{(x, y + c') \mid c' \in k\}.$

Note that $B = B_0 \supseteq B_1 \supseteq B_2 \supseteq B_3.$ We will now give a reduced expression of $u_i := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2$ for each $i \in \{2, \ldots, k\}.$ We do it by considering successively the four following cases:

1. $b_i \in B_0 \setminus B_1$;  
2. $b_i \in B_1 \setminus B_2$;  
3. $b_i \in B_2 \setminus B_3$;  
4. $b_i \in B_3 \setminus \{\text{id}\}$.

Case 1. — $b_i \in B_0 \setminus B_1$.

Since $b_i \in B \setminus A$, the element $u_i$ admits the following reduced expression

$$u_i = (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2.$$  

Case 2. — $b_i \in B_1 \setminus B_2$.

Since $\tilde{b}_i := \sigma \circ b_i \circ \sigma \in A \setminus B$, the element $u_i$ has the following reduced expression

$$u_i = t \circ \sigma \circ t \circ \tilde{b}_i \circ t \circ \sigma \circ t.$$  

Case 3. — $b_i \in B_2 \setminus B_3$.

Let us check that $\overline{b}_i := t \circ \sigma \circ b_i \circ \sigma \circ t \in B \setminus A.$ We are in the case where $b_i = (ax + c, b'y + c')$ with $(a, c, b') \neq (1, 0, 1).$ A direct calculation gives that

$$\overline{b}_i = (b'x + p(ay + c) - b'p(y) - c', ay + c).$$

By the assumption made on $p$, we have that $\deg[p(ay + c) - b'p(y)] \geq 2,$ hence that $\overline{b}_i \in B \setminus A.$ Therefore $u_i$ admits the following reduced expression

$$u_i = t \circ \sigma \circ \overline{b}_i \circ \sigma \circ t.$$  

Case 4. — $b_i \in B_3 \setminus \{\text{id}\}$.

Let us check that $\tilde{b}_i := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2 \in B \setminus A.$ We are in the case where $b_i = (x, y + c')$ with $c' \in \mathbb{C}^*.$ Using the computation in case 3 with $(a, c, b') = (1, 0, 1),$ we then obtain that

$$\tilde{b}_i = t \circ \sigma \circ (x - c', y) \circ \sigma \circ t = t \circ (x, y - c') \circ t$$

$$= (x + p(y - c') - p(y), y - c') \in B \setminus A.$$  

Therefore, the element $u_i$ has the following reduced expression

$$u_i = \tilde{b}_i.$$
Finally we obtain a reduced expression for an element \( g \in \langle B, f \rangle \) from the above study of cases, since we can express
\[
g = b_1 \circ f \circ b_2 \circ f \circ \cdots \circ b_k \circ f \circ b_{k+1} = b_1 \circ (\sigma \circ t)^2 \circ \sigma \circ u_2 \circ \sigma \circ \cdots \circ \sigma \circ u_k \circ \sigma \circ (t \circ \sigma)^2 \circ b_{k+1}.
\]
In particular, observe that \( \ell_A(g) \geq 6 \) if \( k \geq 2 \). This concludes the proof. \( \square \)

Note that the element \( f \) such that \( \langle B, f \rangle \neq \text{Aut}(A^2_k) \), that we constructed in Proposition 4.4, is of affine-length \( \ell_A(f) = 5 \). Our next result shows that 5 is precisely the minimal length for elements \( f \in \text{Aut}(A^2_k) \setminus B \) with that property.

**Proposition 4.6.** — Suppose that \( f \in \text{Aut}(A^2_k) \) is an automorphism of affine length \( \ell \) with \( 1 \leq \ell \leq 4 \). Then, the subgroup generated by \( B \) and \( f \) is equal to the whole group \( \text{Aut}(A^2_k) \), i.e. \( \langle B, f \rangle = \text{Aut}(A^2_k) \).

In order to prove the above proposition, it is useful to remark that we can impose extra conditions on the elements \( t_1, \ldots, t_{n+1}, a_1, \ldots, a_n \) appearing in a reduced expression (*) of an automorphism \( f \in \text{Aut}(A^2_k) \). We do it in Proposition 4.10 below. First, we need to introduce some notations.

**Notation 4.7.** — In the sequel, we will denote, as in the proof of Proposition 4.4, by \( \sigma \) the involution \( \sigma = (y, x) \in \text{Aut}(A^2_k) \) and by \( B_2 \) the subgroup
\[
B_2 = \{(ax + c, b'y + c') \in \text{Aut}(A^2_k) | a, c, b', c' \in k \} \subset A \cap B.
\]
Moreover, we denote by \( I \) the subset
\[
I = \{(-x + p(y), y) \in \text{Aut}(A^2_k) | p(y) \in k[y], \deg p(y) \geq 2 \} \subset B \setminus A \cap B.
\]
Note that the elements of \( I \) are all involutions.

**Lemma 4.8.** — The followings hold:

1. \( B_2 \circ \sigma = \sigma \circ B_2 \).
2. \( B \setminus A \cap B = I \circ B_2 = B_2 \circ I = B_2 \circ I \circ B_2 \).
3. \( A \setminus A \cap B \subset (A \cap B) \circ \sigma \circ (A \cap B) \).

**Remark 4.9.** — In particular, Assertion (3) implies that the group generated by \( \sigma \) and all triangular automorphisms is equal to the whole \( \text{Aut}(A^2_k) \), i.e. \( \langle B, \sigma \rangle = \text{Aut}(A^2_k) \).
Proof. — The first assertion is an easy consequence of the following equalities:

\[(ax + c, b'y + c') \circ \sigma = (ay + c, b'x + c') = \sigma \circ (b'x + c', ay + c).\]

Let us now prove the second assertion. It is easy to check that \(I \circ B_2 = B_2 \circ I = B_2 \circ I \circ B_2 \subset B \setminus A \cap B\). On the other hand, let \(f = (ax + p(y), b'y + c')\) be an element of \(B \setminus A \cap B\). Then \(f\) belongs to \(I \circ B_2\), since we can write

\[f = \left(-x + p\left(\frac{y - c'}{b'}\right), y\right) \circ (-ax, b'y + c').\]

It remains to prove the last assertion. For this, it suffices to write, given an element \(f = (ax + by + c, a'x + b'y + c')\) of \(A \setminus A \cap B\) with \(a' \neq 0\), that

\[f = (ax + by + c, a'x + b'y + c') = (x + a' - ay + c, y + c') \circ \sigma \circ (a'x + b'y, \frac{ba' - ab'}{a'}y).\]

\[\blacksquare\]

Proposition 4.10. — Let \(f \in \text{Aut}(A_k^2)\) be an automorphism of affine length \(\ell = n + 1\) with \(n \geq 0\). Then there exist triangular automorphisms \(\tau_1, \tau_2 \in B\) and triangular involutions \(i_1, \ldots, i_n \in I\) such that

\[(**) \quad f = \tau_1 \circ \sigma \circ i_1 \circ \sigma \circ \cdots \circ \sigma \circ i_n \circ \sigma \circ \tau_2.\]

In particular, the inverse of \(f\) is given by

\[f^{-1} = \tau_2^{-1} \circ \sigma \circ i_n \circ \sigma \circ \cdots \circ \sigma \circ i_1 \circ \sigma \circ \tau_1^{-1}.\]

Proof. — Let \(f\) be an automorphism of affine length \(\ell = n + 1\). By definition,

\[f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1},\]

for some \(a_1, \ldots, a_n \in A \setminus A \cap B, t_1, t_{n+1} \in B\) and \(t_2, \ldots, t_n \in B \setminus A \cap B\). Using Assertion (3) of Lemma 4.8, we may replace every \(a_i\) by \(\sigma\). The proposition then follows from Assertions (1) and (2) of Lemma 4.8. \(\blacksquare\)

We can now proceed to the proof of Proposition 4.6.

Proof of Proposition 4.6.

Case \(\ell = 1\). — Let \(f \in B\) with \(\ell_A(f) = 1\). By Proposition 4.10, we can write \(f = \tau_1 \circ \sigma \circ \tau_2\) for some \(\tau_1, \tau_2 \in B\). Thus, \(\langle B, f \rangle = \langle B, \sigma \rangle = \text{Aut}(A_k^2)\) follows from Remark 4.9.

The proofs for affine length \(\ell = 2, 3, 4\) will be based on explicit computations. In particular, it will be useful to observe that all \(i = (-x + p(y), y) \in I\)
satisfy that

(4.1) \[ i \circ (x + 1, y) \circ i = (x - 1, y), \]
(4.2) \[ \sigma \circ i \circ (x + 1, y) \circ i = (x - 1, y) \]

and

(4.3) \[ i \circ (x, y - 1) \circ i \circ (-x, y + 1) = (-x + (p(y) - p(y + 1)), y). \]

Case \( \ell = 2. \) — Let \( f \in B \) with \( \ell_A(f) = 2. \) By Proposition 4.10, we can suppose that \( f = \sigma \circ i \circ \sigma \) for some involution \( i = (-x + p(y), y) \in I. \) Consider the elements \( b_1 = \sigma \circ (x, y - 1) \circ \sigma \) and \( b_2 = \sigma \circ (-x, y + 1) \circ \sigma \) of \( B_2. \) Since

\[ f \circ b_1 \circ f \circ b_2 = \sigma \circ i \circ (x, y - 1) \circ i \circ (-x, y + 1) \circ \sigma, \]

it follows from Equality (4.3) above that the automorphism \( \sigma \circ (-x + (p(y) - p(y + 1)), y) \circ \sigma \) belongs to \( \langle B, f \rangle. \) By induction, we thus obtain an element in \( \langle B, f \rangle \) of the form \( \sigma \circ (-x + q(y), y) \circ \sigma \) with \( \deg(q) = 1. \) This element is in fact an element of \( A \setminus A \cap B \) and has therefore affine length 1. This implies that \( \langle B, f \rangle = \text{Aut}(A^2_k). \)

Case \( \ell = 3. \) — Let \( f \in B \) with \( \ell_A(f) = 3. \) By Proposition 4.10, we can suppose that \( f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \) for some \( i_1 = (-x + p_1(y), y), i_2 = (-x + p_2(y), y) \in I. \) We first use Equality (4.2), which implies that

(4.4) \[ \sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma = (x, y - 1), \]

where \( b \) denotes the element \( b = \sigma \circ (x + 1, y) \circ \sigma \in B_2. \) Hence, denoting by \( b' \) the element \( b' = \sigma \circ (-x, y + 1) \circ \sigma \) in \( B_2 \) and using Equalities (4.3) and (4.4), we obtain that

\[ f \circ b \circ f^{-1} \circ b' = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \circ b' \]
\[ = \sigma \circ i_1 \circ (x, y - 1) \circ i_1 \circ \sigma \circ b' \]
\[ = \sigma \circ i_1 \circ (x, y - 1) \circ i_1 \circ (-x, y + 1) \circ \sigma \]
\[ = \sigma \circ (-x + (p_1(y) - p_1(y + 1)), y) \circ \sigma \]

is an element of affine length 2 (or 1 in the case where \( \deg(p_1) = 2), \) which belongs to \( \langle B, f \rangle. \) Consequently, \( \langle B, f \rangle = \text{Aut}(A^2_k). \)

Case \( \ell = 4. \) — Let \( f \in B \) with \( \ell_A(f) = 4. \) By Proposition 4.10, we can suppose that \( f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma \) for some \( i_j = (-x + p_j(y), y) \in I, \)
Letting $b = \sigma \circ (x + 1, y) \circ \sigma$ as above, one get that
\[
    f \circ b \circ f^{-1} = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma \circ b \circ \sigma \circ i_3 \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\
    = \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\
    = \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ (-x, y + 1) \\
    \cdots \circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\
    = \sigma \circ i_1 \circ \sigma \circ i_2' \circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\
    = \sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ (x - 1, -y) \circ i_1 \circ \sigma \\
    = \sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ (x + 1, -y) \circ \sigma \\
    = \sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ \sigma \circ (-x, y + 1),
\]
where $i_2' = (-x + p_2'(y), y)$ and $i_1' = (-x + p_1'(y), y)$ for the polynomials $p_2'(y) = p_2(y) - p_2(y + 1)$ and $p_1'(y) = p_1(-y)$, respectively. In particular, $\langle B, f \rangle$ contains the element $\sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ \sigma$. Since $\deg(p_2') = \deg(p_2) - 1$, we obtain by induction an element in $\langle B, f \rangle$ of the form $\sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ \sigma$ with $\widetilde{t}_2 = (-x + \tilde{p}_2(y), y)$ and $\deg(\tilde{p}_2) = 1$. Since $\sigma \circ \widetilde{t}_2 \circ \sigma$ is an element of $A \setminus A \cap B$, the above $\sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ \sigma$ is an automorphism of affine length 3, and the proposition follows.

To conclude, let us emphasize that, as pointed to us by S. Lamy, our results concerning the non-maximality of $B$ are related to those of [10] about the existence of normal subgroups for the group $\text{SAut}(\mathbb{A}^2_\mathbb{C})$ of automorphisms of the complex affine plane whose Jacobian determinant is equal to 1. Indeed, the subgroup $\langle B, f \rangle$, generated by $B$ and a given automorphism $f$, is contained into the subgroup $B \circ (f)_N = \{ h \circ g \mid h \in B, g \in \langle f \rangle_N \}$, where $\langle f \rangle_N$ denotes the normal subgroup of $\text{Aut}(\mathbb{A}^2_\mathbb{C})$ that is generated by $f$.

Combined with Proposition 4.6, the above observation gives us a short proof of the following result.

**Theorem 4.11 ([10, Theorem 1]).** — If $f \in \text{SAut}(\mathbb{A}^2_\mathbb{C})$ is of affine length at most 4 and $f \neq \text{id}$, then the normal subgroup $\langle f \rangle_N$ generated by $f$ in $\text{SAut}(\mathbb{A}^2_\mathbb{C})$ is equal to the whole group $\text{SAut}(\mathbb{A}^2_\mathbb{C})$.

**Proof.** — The case where $f$ is a triangular automorphism being easy to treat (see [10, Lemma 30, p. 590]), suppose that $f \in \text{SAut}(\mathbb{A}^2_\mathbb{C})$ is of affine length at most 4 and at least 1. By Proposition 4.6, we have $\langle B, f \rangle = \text{Aut}(\mathbb{A}^2_\mathbb{C})$. Since the group $B \circ (f)_N$ contains $B$ and $f$, we get $B \circ (f)_N = \text{Aut}(\mathbb{A}^2_\mathbb{C})$. In particular, the element $(-y, x)$ can be written as $(-y, x) = b \circ g$ for some $b \in B$ and $g \in \langle f \rangle_N$. Consequently, $\langle f \rangle_N$ contains the element $g = b^{-1} \circ (-y, x)$ which is of affine length 1.
Remark that the Jacobian determinant of $b$ is equal to 1. Therefore, we can write $b^{-1} = (ax + P(y), a^{-1}y + c)$ for some $a \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $P(y) \in \mathbb{C}[y]$. Thus, $g$ is given by

$$g = (-ay + P(x), a^{-1}x + c).$$

Next, we consider the translation $\tau = (x + 1, y)$ and compute the commutator $[\tau, g] = \tau \circ g \circ \tau^{-1} \circ g^{-1}$, which is an element of $\langle f \rangle_N$. Since

$$[\tau, g] = (x + 1, y) \circ (-ay + P(x), a^{-1}x + c) \circ (x + 1, y) \circ (ay - ac, -a^{-1}x + a^{-1}P(ay - ac))$$

$$= (x - P(ay - ac) + P(ay - ac - 1) + 1, y - a^{-1})$$

is a triangular automorphism different from the identity, the theorem follows directly from [10, Lemma 30, p. 590].

On the other hand, we can retrieve the fact that the Jonquières subgroup is not a maximal subgroup of $\text{Aut}(\mathbb{A}^2_\mathbb{C})$ as a corollary of [10, Theorem 2]. Indeed, the latter produces elements $f \in \text{SAut}(\mathbb{A}^2_\mathbb{C})$ of affine length $\ell_A(f) = 7$ such that $\langle f \rangle_N \neq \text{SAut}(\mathbb{A}^2_\mathbb{C})$. In particular, by [10, Theorem 1] above, the identity is the only automorphism of affine length smaller than or equal to 4 contained in $\langle f \rangle_N$. Therefore, since $\langle B, f \rangle \subset B \circ \langle f \rangle_N$, the subgroup $\langle B, f \rangle$ does not contain any non-triangular automorphism of affine length $\leq 4$. Consequently, $\langle B, f \rangle$ is a strict subgroup of $\text{Aut}(\mathbb{A}^2_\mathbb{C})$.

**BIBLIOGRAPHY**


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