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MINIMAL MODEL PROGRAM FOR EXCELLENT SURFACES

by Hiromu TANAKA

ABSTRACT. — We establish the minimal model program for log canonical and \mathbb{Q} -factorial surfaces over excellent base schemes.

RÉSUMÉ. — Nous prouvons les résultats prédits par le programme des modèles minimaux pour des surfaces log canoniques et \mathbb{Q} -factorielles sur des schémas excellents.

1. Introduction

The Italian school of algebraic geometry in the early 20th century established the classification theory for smooth projective surfaces over the field of complex numbers, which was generalised by Kodaira, Shafarevich and Bombieri–Mumford. In particular, Shafarevich studied the minimal model theory for regular surfaces in the classical sense [29]. Kollár and Kovács proved, in their unpublished preprint [17], that the minimal model program holds for log canonical surfaces over algebraically closed fields by the viewpoint of the higher dimensional minimal model theory. Recently, Fujino and the author established the minimal model program for \mathbb{Q} -factorial surfaces over algebraically closed fields [6, 30].

The purpose of this paper is extending the minimal model program for surfaces over algebraically closed fields to the one for surfaces over excellent base schemes. More precisely, we prove the following theorem.

THEOREM 1.1 (Theorem 4.5, Theorem 4.15). — *Let B be a regular excellent separated scheme of finite dimension. Let $\pi : X \rightarrow S$ be a projective*

B-morphism from a quasi-projective integral normal *B*-scheme *X* of dimension two to a quasi-projective *B*-scheme *S*. Let Δ be an effective \mathbb{R} -divisor on *X*. Assume that one of the following properties holds.

(LC) (X, Δ) is log canonical.

(QF) *X* is \mathbb{Q} -factorial and, for the irreducible decomposition $\Delta = \sum_{i \in I} \delta_i D_i$, the inequality $0 \leq \delta_i \leq 1$ holds for any $i \in I$.

Then, there exist a sequence of projective birational *S*-morphisms

$$X =: X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{r-1}} X_r =: X^\dagger$$

and \mathbb{R} -divisors Δ_i and Δ^\dagger defined by

$$\Delta =: \Delta_0, \quad (\varphi_i)_*(\Delta_i) =: \Delta_{i+1}, \quad \Delta_r =: \Delta^\dagger$$

which satisfy the following properties.

- (1) Each X_i is a projective integral normal *S*-scheme of dimension two.
- (2) Each (X_i, Δ_i) satisfies (LC) or (QF) according as the above assumption.
- (3) $\rho(X_{i+1}/S) = \rho(X_i/S) - 1$ for any i .
- (4) $(X^\dagger, \Delta^\dagger)$ satisfies one of the following conditions.
 - (a) $K_{X^\dagger/B} + \Delta^\dagger$ is nef over *S*.
 - (b) There is a projective *S*-morphism $\mu : X^\dagger \rightarrow Z$ to a projective *S*-scheme *Z* such that $\mu_*\mathcal{O}_{X^\dagger} = \mathcal{O}_Z$, $\dim X^\dagger > \dim Z$, $-(K_{X^\dagger/B} + \Delta^\dagger)$ is μ -ample and $\rho(X^\dagger/S) - 1 = \rho(Z/S)$.

If (a) holds, then $(X^\dagger, \Delta^\dagger)$ is called a log minimal model over *S*. If (b) holds, then $\mu : X^\dagger \rightarrow Z$ is called a $(K_{X^\dagger/B} + \Delta^\dagger)$ -Mori fibre space over *S*.

Many results in minimal model theory depend on the lower dimensional facts, hence Theorem 1.1 might be applied to establish the three dimensional minimal model program over $\text{Spec } \mathbb{Z}$ or imperfect fields.

Also, it is useful to treat varieties over excellent base schemes even when one studies the minimal model theory over an algebraically closed field of positive characteristic. For instance, a known proof for the fact that 3-dimensional terminal singularities are isolated depends on the result that terminal excellent surfaces are regular (cf. [16, Corollary 2.30]).

As applications of Theorem 1.1, we also prove inversion of adjunction (Theorem 5.1) and the Kollár–Shokurov connectedness theorem (Theorem 5.2) for surfaces over excellent schemes. The former is a direct consequence of the minimal model program, whilst the latter needs some arguments.

Remark 1.2. — It is worth explaining why the schemes that appear in Theorem 1.1 are assumed to be not only Noetherian but also excellent. We

will frequently use the desingularisation theorem by Lipman which holds for quasi-excellent surfaces [21]. Furthermore, in order to define canonical divisors, it seems to be necessary to assume that the base scheme B has a dualising complex. In particular, B needs to be universally catenary (cf. [10, Ch. III, Section 10]). Moreover, following [16], we restrict ourselves to treating only regular base schemes.

1.1. Proof of Theorem 1.1

Let us overview some of the ideas of the proof of Theorem 1.1. The main strategy is to imitate the proof in characteristic zero. To this end, we establish appropriate vanishing theorems. After that, we prove the base point free theorem by applying the standard argument called the X-method. Some of the vanishing theorems we will establish are as follows.

THEOREM 1.3 (Theorem 3.3). — *Let B be a regular excellent separated scheme of finite dimension. Let $\pi : X \rightarrow S$ be a projective B -morphism from a two-dimensional quasi-projective klt pair (X, Δ) over B to a quasi-projective B -scheme S . Let D be a \mathbb{Q} -Cartier \mathbb{Z} -divisor on X such that $D - (K_{X/B} + \Delta)$ is π -ample. If $\dim \pi(X) \geq 1$, then the equation $R^i \pi_* \mathcal{O}_X(D) = 0$ holds for every $i > 0$.*

THEOREM 1.4 (Theorem 3.8). — *Let k be a field of characteristic $p > 0$. Let (X, Δ) be a two-dimensional projective klt pair over k . Let D be a \mathbb{Q} -Cartier \mathbb{Z} -divisor on X such that $D - (K_X + \Delta)$ is ample. Let N be a nef Cartier divisor on X with $N \neq 0$. Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that*

$$H^i(X, D + mN) = 0$$

for any integers i and m satisfying $i > 0$ and $m \geq m_0$.

Let us overview how to show the above vanishing theorems. Theorem 1.3 is well-known for the case where $\dim \pi(X) = 2$ (cf. [16, Theorem 10.4]). One of the essential properties in this case is that the intersection matrix of the exceptional locus is negative definite. For the remaining case i.e. $\dim \pi(X) = 1$, the intersection matrix of any fibre of π is not negative definite but semi-negative definite. Hence we can apply an almost parallel argument to the one of [16, Theorem 10.4] after replacing some strict inequality signs “ $>$ ” by “ \geq ”.

Theorem 1.4 is known if k is algebraically closed ([31, Theorem 2.11]). The proof of [31, Theorem 2.11] depends on Kawamata’s covering trick, which unfortunately seems to need the assumption that k is algebraically

closed (cf. [12, Lemma 1-1-2]). In order to overcome this technical difficulty, we make use of the notion of F -singularities. Actually, we first establish a similar vanishing theorem for strongly F -regular surfaces (Proposition 3.7) and second extend it to the klt case via log resolutions.

1.2. Related results

On the classification theory for smooth surfaces over algebraically closed fields, we refer to [2, 1]. Theories of log surfaces and normal surfaces have been developed by Iitaka, Kawamata, Miyanishi, Sakai and many others (cf. [24, 25]).

It is worth mentioning on the fact that Theorem 1.1 is known for some special cases.

- (1) B is an algebraically closed field.
- (2) X is regular and $\Delta = 0$.

For the former case (1), Theorem 1.1 is a known result (cf. [6, 8, 30]). Concerning the latter case (2), let us review that Theorem 1.1 follows from a combination of known results. Thanks to an analogue of Castelnuovo's contractibility criterion (cf. [16, Theorem 10.5]), we can repeatedly contract a (-1) -curve i.e. the curve E appearing in [16, Theorem 10.5], and this process will terminate. Since we can make use of the cone theorem (cf. Subsection 2.4), we deduce from the Riemann–Roch theorem that the program ends with either a minimal model or a Mori fibre space.

For the general case, we would like a replacement of Castelnuovo's contractibility criterion, which is nothing but the base point free theorem established by the vanishing theorems mentioned in Subsection 1.1.

Remark 1.5. — Theorem 1.1 was proved under the assumption that (X, Δ) is a log canonical surface over a field k of characteristic $p > 0$ by the author in an earlier draft. In [3], Birkar–Chen–Zhang independently obtained the same result under the assumption that (X, Δ) is a klt surface over k . Both the proofs depend on Keel's result [13] which differ from ours.

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2. Preliminaries

2.1. Notation

In this subsection, we summarise notation used in this paper.

- We will freely use the notation and terminology in [11] and [16].
- A ring A is *quasi-excellent* if A is a Noetherian G-ring such that for any finitely generated A -algebra B , its regular locus

$$\{\mathfrak{p} \in \text{Spec } B \mid B_{\mathfrak{p}} \text{ is a regular local ring}\}$$

is an open subset of $\text{Spec } B$ (cf. [23, §32]). A ring A is *excellent* if A is quasi-excellent and universally catenary.

- We say a scheme X is *regular* (resp. *Gorenstein*) if the local ring $\mathcal{O}_{X,x}$ at any point $x \in X$ is regular (resp. Gorenstein). We say a scheme X is *excellent* if X is quasi-compact and any point $x \in X$ admits an affine open neighbourhood $U \simeq \text{Spec } A$ for some excellent ring A .
- For a scheme X , its *reduced structure* X_{red} is the reduced closed subscheme of X such that the induced morphism $X_{\text{red}} \rightarrow X$ is surjective.
- For an integral scheme X , we define the *function field* $K(X)$ of X to be $\mathcal{O}_{X,\xi}$ for the generic point ξ of X .
- For a scheme S , we say X is a *variety over* S or an *S -variety* if X is an integral scheme that is separated and of finite type over S . We say X is a *curve over* S or an *S -curve* (resp. a *surface over* S or an *S -surface*) if X is an S -variety of dimension one (resp. two).
- Let Δ be an \mathbb{R} -divisor on a Noetherian normal scheme and let $\Delta = \sum_{i \in I} \delta_i \Delta_i$ be the irreducible decomposition. Take a real number a . We write $\Delta \leq a$ if $\delta_i \leq a$ for any $i \in I$. We set $\Delta \leq^a := \sum_{i \in I, \delta_i \leq a} \delta_i \Delta_i$. We define $\Delta \geq a$ and $\Delta \geq^a$ etc, in the same way.
- For a field k , let \bar{k} be an algebraic closure of k . If k is of characteristic $p > 0$, then we set $k^{1/p^\infty} := \bigcup_{e=0}^\infty k^{1/p^e} = \bigcup_{e=0}^\infty \{x \in \bar{k} \mid x^{p^e} \in k\}$.
- We say a field k of positive characteristic is *F -finite* if $[k : k^p] < \infty$.

- Let $f : X \rightarrow Y$ be a projective morphism of Noetherian schemes, where X is an integral normal scheme. For a Cartier divisor L on X , we say L is f -free if the induced homomorphism $f^*f_*\mathcal{O}_X(L) \rightarrow \mathcal{O}_X(L)$ is surjective. Let M be an \mathbb{R} -Cartier \mathbb{R} -divisor on X . We say M is f -ample (resp. f -semi-ample) if we can write $M = \sum_{i=1}^r a_i M_i$ for some $r \geq 1$, positive real numbers a_i and f -ample (resp. f -free) Cartier divisors M_i . We say M is f -big if we can write $M = A + E$ for some f -ample \mathbb{R} -Cartier \mathbb{R} -divisor A and effective \mathbb{R} -divisor E . We can define f -nef \mathbb{R} -divisors in the same way as in [16, Definition 1.4]. We say M is f -numerically-trivial, denoted by $M \equiv_f 0$, if M and $-M$ are f -nef.

2.1.1. Excellent schemes

If B is an excellent scheme, then any scheme X that is of finite type over B is again excellent (cf. [23, §32]). We can freely use resolution of singularities for excellent surfaces (cf. [21] and Subsection 2.1.7). Also excellent schemes satisfy reasonable dimension formulas (cf. [22, Section 8.2]).

2.1.2. Base schemes

We basically work over a base scheme B satisfying the following properties.

ASSUMPTION 2.1. — B is a scheme which is excellent, regular, separated, and of finite dimension.

2.1.3. Quasi-projective morphisms

Let $f : X \rightarrow S$ be a morphism of Noetherian schemes. We say f is projective (resp. quasi-projective) if there exists a closed immersion (resp. an immersion) $X \hookrightarrow \mathbb{P}_S^n$ over S for some non-negative integer n .

Remark 2.2. — We adopt the definition of projective morphisms by [11]. If there is an ample invertible sheaf on S in the sense of [11, p. 153], then our definition coincides with the one of Grothendieck (cf. [5, Section 5.5.1]).

2.1.4. Perturbation of nef and big divisors

We here give two remarks on fundamental properties of divisors, which is well-known for varieties over fields.

Remark 2.3. — Let $f : X \rightarrow Y$ be a projective morphism of Noetherian schemes, where X is an integral normal scheme. Let L be an \mathbb{R} -Cartier \mathbb{R} -divisor. Then the following are equivalent.

- (1) L is f -ample.
- (2) For any closed point $y \in Y$ and any integral closed subscheme V of X_y , it holds that $V \cdot L^{\dim V} > 0$.

Indeed, clearly (1) implies (2), hence it suffices to prove the opposite implication. If L is \mathbb{Q} -Cartier \mathbb{Q} -divisor, then it is well-known that (2) implies (1) (cf. [18, Proposition 1.41]). Let us consider the general case. Assume that L satisfies (2). We can write $L = \sum_{i=1}^n r_i L_i$ for real numbers r_1, \dots, r_n and Cartier divisors L_1, \dots, L_n . Thanks to the Nakai’s criterion for projective schemes over fields (cf. [19, Theorem 2.3.18]) and openness of amplitude (cf. [18, Proposition 1.41]), if q_i is a rational number sufficiently close to r_i for any $i \in \{1, \dots, n\}$, then also $L' := \sum_{i=1}^n q_i L_i$ satisfies (2). Thus, we can write $L = \sum_{j=1}^m s_j L'_j$ such that $s_j \in \mathbb{R}_{>0}$, $\sum_{j=1}^m s_j = 1$ and L'_j is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. Hence, L satisfies (1).

Remark 2.4. — Let $f : X \rightarrow Y$ be a projective morphism of Noetherian schemes, where X is an integral normal scheme. Let M be an f -nef and f -big \mathbb{R} -Cartier \mathbb{R} -divisor on X . Since M is f -big, we can write $M = A + E$ for some f -ample \mathbb{R} -divisor A and effective \mathbb{R} -divisor E . For any $\epsilon \in \mathbb{R}$ such that $0 < \epsilon \leq 1$, we obtain

$$M = A + E = (1 - \epsilon)M + \epsilon(A + E) = A_\epsilon + \epsilon E,$$

where $A_\epsilon := (1 - \epsilon)M + \epsilon A$. Since M is f -nef and A is f -ample, Remark 2.3 implies that A_ϵ is f -ample. Therefore, $M - \epsilon E$ is f -ample for any real number ϵ such that $0 < \epsilon \leq 1$.

2.1.5. Canonical sheaves

Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective normal B -variety, equipped with the structure morphism $\alpha : X \rightarrow B$. For the definition of the *canonical sheaf* $\omega_{X/B}$, we refer to [16, Definition 1.6]. Note that $\omega_{X/B}$ coincides with $\mathcal{H}^r(\alpha^! \mathcal{O}_B)$, where $\alpha^!$ is defined as in [10, Ch. III, Theorem 8.7] and r is the integer such that $\mathcal{H}^r(\alpha^! \mathcal{O}_B)|_{X^{\text{reg}}} \neq 0$ for

the regular locus X^{reg} of X . There is a \mathbb{Z} -divisor $K_{X/B}$, called the *canonical divisor*, such that $\omega_{X/B} \simeq \mathcal{O}_X(K_{X/B})$. Note that $K_{X/B}$ is uniquely determined up to linear equivalences.

LEMMA 2.5. — *Let $\alpha : X \xrightarrow{\beta} B' \xrightarrow{\theta} B$ be quasi-projective morphisms of Noetherian schemes such that*

- (1) B and B' satisfy Assumption 2.1, and
- (2) X and B' are normal B -varieties.

Then $\omega_{B'/B}$ is an invertible sheaf and there is an isomorphism of \mathcal{O}_X -modules: $\omega_{X/B} \simeq \omega_{X/B'} \otimes_{\mathcal{O}_X} \beta^ \omega_{B'/B}$.*

Proof. — Since B' is regular and in particular Gorenstein, it follows that $\omega_{B'/B}$ is an invertible sheaf. We have that

$$\alpha^! \mathcal{O}_B \simeq \beta^! \theta^! \mathcal{O}_B \simeq \beta^! (\omega_{B'/B}[r]) \simeq \beta^! \mathcal{O}_{B'} \otimes \beta^* \omega_{B'/B}[r]$$

for some integer r , where the first isomorphism holds by [10, Ch. III, Proposition 8.7(2)], the second one follows from [10, Ch. V, Example 2.2 and Theorem 3.1] and the third one holds by [10, Ch. III, Proposition 8.8(6)]. Taking the s -th cohomology sheaf \mathcal{H}^s of the both hand sides for an appropriate integer s , we have that

$$\omega_{X/B} \simeq \omega_{X/B'} \otimes \beta^* \omega_{B'/B},$$

as desired. □

2.1.6. Singularities of minimal model program

Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective normal B -variety. We say (X, Δ) is a *log pair* over B if Δ is an effective \mathbb{R} -divisor on X such that $K_{X/B} + \Delta$ is \mathbb{R} -Cartier. Although the book [16, Definition 2.8] gives the definition of terminal, canonical, klt, plt, dlt, log canonical singularities only for the case where Δ is a \mathbb{Q} -divisor, we can extend them for the case where Δ is an \mathbb{R} -divisor in the same way. Note that we always assume that Δ is effective although [16, Definition 2.8] does not impose this assumption.

2.1.7. Log resolutions

Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective surface over B . For a closed subset Z of X , we say $f : Y \rightarrow X$ is a *log*

resolution of (X, Z) if f is a projective birational B -morphism from a quasi-projective B -surface such that $f^{-1}(Z) \cup \text{Ex}(f)$ is a simple normal crossing divisor, where we refer to [16, Definition 1.7] for the definition of simple normal crossing divisors. By [21], there exists a log resolution of (X, Z) for any quasi-projective B -surface X and any closed subset Z of X . For an \mathbb{R} -divisor Δ , a log resolution of (X, Δ) means a log resolution of $(X, \text{Supp } \Delta)$.

2.1.8. Multiplier ideals

Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective normal surface over B . Let Δ be an \mathbb{R} -divisor such that $K_{X/B} + \Delta$ is \mathbb{R} -Cartier. For a log resolution $\mu : V \rightarrow X$ of (X, Δ) , we define the multiplier ideal of (X, Δ) by

$$\mathcal{J}_\Delta := \mu_* \mathcal{O}_V(\lceil K_{V/B} - \mu^*(K_{X/B} + \Delta) \rceil).$$

By the same argument as in [20, Theorem 9.2.18], we can show that the coherent sheaf \mathcal{J}_Δ does not depend on the choice of log resolutions. If Δ is effective, then it follows that $\mathcal{J}_\Delta \subset \mathcal{O}_X$.

2.2. Intersection numbers and Riemann–Roch theorem

We recall the definition of intersection numbers (cf. [15, Ch. I, Sections 1 and 2]).

DEFINITION 2.6.

- (1) Let C be a projective curve over a field k . Let M be an invertible sheaf on C . It is well-known that

$$\chi(C, mM) = \dim_k(H^0(C, mM)) - \dim_k(H^1(C, mM)) \in \mathbb{Z}[m]$$

and the degree of this polynomial is at most one (cf. [15, Ch. I, Section 1, Theorem in p. 295]). We define the degree of M over k , denoted by $\deg_k M$ or $\deg M$, to be the coefficient of m .

- (2) Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective B -scheme, and let $C \hookrightarrow X$ be a closed immersion such that C is a projective B -curve and the structure morphism $C \rightarrow B$ factors through $\text{Spec } k$ for a field k . In particular, C is projective over k (cf. [11, Ch. II, Theorem 7.6]). Let L be an invertible sheaf on X . We define the intersection number over k , denoted by $L \cdot_k C$ or $L \cdot C$, to be $\deg_k(L|_C)$.

The following is the Riemann–Roch theorem for curves.

THEOREM 2.7. — *Let k be a field. Let X be a projective curve over k .*

(1) *If L is a Cartier divisor on X , then*

$$\chi(X, L) = \deg_k(L) + \chi(X, \mathcal{O}_X).$$

(2) *Assume that X is regular. Let D be a \mathbb{Z} -divisor and let $D = \sum_{i \in I} a_i P_i$ be the prime decomposition. Then*

$$\deg_k(D) = \sum_{i \in I} a_i \dim_k(k(P_i))$$

where $k(P_i)$ is the residue field at P_i .

Proof. — The assertion (1) follows from Definition 2.6 (1). The assertion (2) holds by the same argument as the case where k is algebraically closed (cf. the proof of [11, Ch. IV, Theorem 1.3]). \square

As a corollary, we obtain a formula between $\deg(\omega_X)$ and $\chi(X, \mathcal{O}_X)$ for Gorenstein curves.

COROLLARY 2.8. — *Let k be a field. Let X be a projective Gorenstein curve over k . Then*

$$\deg_k(\omega_X) = 2(h^1(X, \mathcal{O}_X) - h^0(X, \mathcal{O}_X)) = -2\chi(X, \mathcal{O}_X).$$

Proof. — By Theorem 2.7(1), we obtain $\chi(X, \omega_X) = \deg(\omega_X) + \chi(X, \mathcal{O}_X)$. By Serre duality, we obtain $\chi(X, \omega_X) = -\chi(X, \mathcal{O}_X)$, which implies the assertion. \square

COROLLARY 2.9. — *Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective regular B -surface, and let $C \hookrightarrow X$ be a closed immersion over B such that C is a projective B -curve and the structure morphism $C \rightarrow B$ factors through $\text{Spec } k$ for a field k . Then,*

$$\begin{aligned} (K_{X/B} + C) \cdot_k C &= \deg_k \omega_{C/B} = \deg_k \omega_{C/k} \\ &= 2(\dim_k H^1(C, \mathcal{O}_C) - \dim_k H^0(C, \mathcal{O}_C)). \end{aligned}$$

Proof. — The first equality holds by the adjunction formula (cf. [16, (4.1.3)]), the second by Lemma 2.5, and the third by Corollary 2.8. \square

The following is the Riemann–Roch theorem for surfaces.

THEOREM 2.10. — *Let k be a field. Let X be a projective regular surface over k . Let D be a \mathbb{Z} -divisor on X . Then*

$$\chi(X, D) = \chi(X, \mathcal{O}_X) + \frac{1}{2} D \cdot_k (D - K_X).$$

Proof. — We can write $D = \sum_i A_i - \sum_j B_j$, where all A_i and B_j are prime divisors. Thus, it suffices to prove that if D satisfies the required equation, so do $D + C$ and $D - C$ for a prime divisor C on X . These can be checked directly by using Corollary 2.9. □

2.3. Negativity lemma

LEMMA 2.11. — *Let B be a scheme satisfying Assumption 2.1. Let $f : X \rightarrow Y$ be a projective birational B -morphism of quasi-projective normal B -surfaces. Let $-B$ be an f -nef \mathbb{R} -Cartier \mathbb{R} -divisor on X . Then the following hold.*

- (1) *B is effective if and only if f_*B is effective.*
- (2) *Assume that B is effective. For any point $y \in Y$, one of $f^{-1}(y) \subset \text{Supp } B$ and $f^{-1}(y) \cap \text{Supp } B = \emptyset$ holds.*

Proof. — We can apply the same argument as in [18, Lemma 3.39] by using [16, Theorem 10.1]. □

2.4. Cone theorem

Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective normal B -variety X to a quasi-projective B -scheme S . Let

$$Z(X/S)_{\mathbb{R}} := \bigoplus_C \mathbb{R}C,$$

where C runs over the projective B -curves in X such that $\pi(C)$ is one point. An element $\sum \alpha_C C$ of $Z(X/S)_{\mathbb{R}}$ is called an *effective 1-cycle* if each α_C is not negative. Let $N(X/S)_{\mathbb{R}} := Z(X/S)_{\mathbb{R}}/\cong$ be the quotient \mathbb{R} -vector space by numerical equivalence, i.e. for $\Gamma_1, \Gamma_2 \in Z(X/S)_{\mathbb{R}}$, the numerical equivalence $\Gamma_1 \equiv \Gamma_2$ holds if and only if $L \cdot \Gamma_1 = L \cdot \Gamma_2$ for any invertible sheaf L on X . For an element $\Gamma \in Z(X/S)_{\mathbb{R}}$, its numerical equivalence class is denoted by $[\Gamma]$. Let $\text{NE}(X/S)$ be the subset of $N(X/S)_{\mathbb{R}}$ consisting of the numerical equivalence classes of effective 1-cycles. If $N(X/S)_{\mathbb{R}}$ is a finite dimensional \mathbb{R} -vector space, then we set $\overline{\text{NE}}(X/S)$ to be the closure of $\text{NE}(X/S)$ with respect to the Euclidean topology. In this case, we set

$$\overline{\text{NE}}(X/S)_{L \geq 0} := \{[\Gamma] \in \overline{\text{NE}}(X/S) \mid L \cdot \Gamma \geq 0\}$$

for any \mathbb{R} -Cartier \mathbb{R} -divisor L on X .

Remark 2.12. — For the Stein factorisation $\pi : X \rightarrow T \rightarrow S$ of π , we have that $N(X/S)_{\mathbb{R}} = N(X/T)_{\mathbb{R}}$ and $\text{NE}(X/S) = \text{NE}(X/T)$.

LEMMA 2.13. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective normal B -surface to a quasi-projective B -scheme. Then the following hold.*

- (1) *If $\dim \pi(X) \geq 1$, then there exist finitely many projective B -curves C_1, \dots, C_q such that each $\pi(C_i)$ is one point and*

$$\text{NE}(X/S) = \sum_{i=1}^q \mathbb{R}_{\geq 0}[C_i].$$

- (2) *$N(X/S)_{\mathbb{R}}$ is a finite dimensional \mathbb{R} -vector space.*

Proof. — Replacing π by its Stein factorisation, we may assume that $\pi_*\mathcal{O}_X = \mathcal{O}_S$.

We show (1). We only treat the case where $\dim S = 1$ because the proofs are the same. Since the generic fibre $X \times_S K(S)$ is geometrically irreducible (cf. [32, Lemma 2.2]), there is a non-empty open subset S^0 of S such that X_s is an irreducible curve over $k(s)$ for any $s \in S^0$. Since $S \setminus S^0$ is a finite set, $\text{NE}(X/S)$ is generated by only finitely many B -curves C_1, \dots, C_q . Thus (1) holds.

We show (2). If $\dim S \geq 1$, then the assertion holds by (1). If $\dim S = 0$, then the assertion follows from [15, Ch. IV, Section 1, Proposition 4]. In any case, (2) holds. □

THEOREM 2.14. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective normal B -surface to a quasi-projective B -scheme. Let A be a π -ample \mathbb{R} -Cartier \mathbb{R} -divisor on X and let Δ be an effective \mathbb{R} -divisor on X such that $K_{X/B} + \Delta$ is \mathbb{R} -Cartier. Then there exist finitely many projective B -curves C_1, \dots, C_r on X that satisfy the following properties:*

- (1) *$\pi(C_i)$ is one point for any $1 \leq i \leq r$, and*
- (2) *$\overline{\text{NE}}(X/S) = \overline{\text{NE}}(X/S)_{K_{X/B} + \Delta + A \geq 0} + \sum_{i=1}^r \mathbb{R}_{\geq 0}[C_i]$.*

Proof. — Replacing π by its Stein factorisation, we may assume that $\pi_*\mathcal{O}_X = \mathcal{O}_S$. If $\dim S \geq 1$, then the assertion holds by Lemma 2.13. Thus we may assume that $\dim S = 0$, i.e. $S = \text{Spec } k$ where k is a field. If k is of characteristic $p > 0$, then the assertion follows from [32, Theorem 7.5]. If k is an algebraically closed field of characteristic zero, then the claim holds by [6, Theorem 1.1]. If k is a field of characteristic zero (not necessarily algebraically closed), then we can apply the same argument as in [32, Theorem 7.5] using [6, Theorem 1.1]. We are done. □

Remark 2.15. — We use the same notation as in the statement of Theorem 2.14. Assume that (X, Δ) is log canonical. If B is the affine spectrum of an algebraically closed field, then C_i can be chosen to be a curve such that $-(K_X + \Delta) \cdot C_i \leq 3$ (cf. [7, Proposition 3.8], [30, Proposition 3.15]). The author does not know whether a similar result can be formulated in this generality (cf. [32, Example 7.3]).

3. Vanishing theorems of Kawamata–Viehweg type

3.1. Relative case

A key result in this subsection is Proposition 3.2. As explained in Subsection 1.1, we can apply the same argument as in [16, Theorem 10.4]. More specifically, our proofs of Lemma 3.1 and Proposition 3.2 are quite similar to the ones of [16, Lemma 10.3.3] and [16, Theorem 10.4] respectively. However we give proofs of them for the sake of completeness.

LEMMA 3.1. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective regular B -surface to a quasi-projective B -curve with $\pi_*\mathcal{O}_X = \mathcal{O}_S$. Let $s \in S$ be a closed point and let $\pi^*(s) = \sum_{1 \leq i \leq n} c_i C_i$ be the irreducible decomposition with $c_i \in \mathbb{Z}_{>0}$. Let $D = \sum_{1 \leq i \leq n} d_i C_i$ be an \mathbb{R} -divisor on X . If $D \not\leq 0$, then there exists an integer j such that $1 \leq j \leq n$, $d_j > 0$ and $C_j \cdot D \leq 0$.*

Proof. — By $D \not\leq 0$, there exists the maximum real number $a \in \mathbb{R}_{>0}$ such that $c_i - ad_i \geq 0$ holds for any $i \in \{1, \dots, n\}$. Then we have that $\sum_{1 \leq i \leq n} (c_i - ad_i)C_i \geq 0$ and $c_j - ad_j = 0$ for some $j \in \{1, \dots, n\}$ with $d_j > 0$. Therefore, we obtain

$$-aC_j \cdot D = C_j \cdot (\pi^*(s) - aD) = C_j \cdot \sum_{i \neq j} (c_i - ad_i)C_i \geq 0,$$

which implies $C_j \cdot D \leq 0$. □

PROPOSITION 3.2. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective regular B -surface X to a quasi-projective B -scheme S . Let Δ be an \mathbb{R} -divisor on X with $0 \leq \Delta < 1$ and let L be a Cartier divisor such that $L - (K_{X/B} + \Delta)$ is π -nef and π -big. If $\dim \pi(X) \geq 1$, then the equation $R^i \pi_* \mathcal{O}_X(L) = 0$ holds for every $i > 0$.*

Proof. — Set $K_X := K_{X/B}$. By perturbing Δ , we may assume Δ is a \mathbb{Q} -divisor such that $L - (K_X + \Delta)$ is π -ample (cf. Remark 2.4). We set $A := L - (K_X + \Delta)$.

By taking the Stein factorisation of π , we may assume that $\pi_*\mathcal{O}_X = \mathcal{O}_S$. If $\dim S = 2$, then the assertion follows from [16, Theorem 10.4]. Thus we may assume that $\dim S = 1$. Then the assertion is clear for $i > 1$. We prove $R^1\pi_*\mathcal{O}_X(L) = 0$.

Fix a closed point $s \in S$ and let C_1, \dots, C_n be the prime divisors on X contracting to s , that is, $\text{Supp}(\pi^{-1}(s)) = \text{Supp}(C_1 \cup \dots \cup C_n)$. Each C_i is a projective curve over $k(s)$, hence the intersection number over $k(s)$ of an \mathbb{R} -divisor on X and C_i is defined in Definition 2.6. We set $h^i(C_i, F) := \dim_{k(s)} H^i(C_i, F)$ for any coherent sheaf F on C_i . Note that $C_i^2 \leq 0$ for any $i \in \{1, \dots, n\}$. We consider the following:

CLAIM. — *The equation $H^1(Z, L|_Z) = 0$ holds for any $r_i \in \mathbb{Z}_{\geq 0}$ and $Z := \sum_{i=1}^n r_i C_i$ with $\sum_{i=1}^n r_i \geq 1$.*

If this claim holds, then the assertion in the proposition follows from the theorem on formal functions [11, Ch. III, Theorem 11.1]. We prove this claim by the induction on $\sum_{i=1}^n r_i$.

If $\sum_{i=1}^n r_i = 1$, that is, $Z = C_i$ holds for some i , then

$$h^1(C_i, L|_{C_i}) = h^0(C_i, \omega_{C_i} \otimes L^{-1}|_{C_i}) = 0,$$

where the second equation holds by the following calculation

$$\begin{aligned} \deg_{k(s)}(\omega_{C_i} \otimes L^{-1}|_{C_i}) &= (K_X + C_i - L) \cdot_{k(s)} C_i \\ &= (K_X + C_i - (K_X + \Delta + A)) \cdot_{k(s)} C_i \\ &= ((C_i - \Delta) - A) \cdot_{k(s)} C_i \\ &< 0. \end{aligned}$$

Thus we are done for the case where $\sum_{i=1}^n r_i = 1$.

We assume $\sum_{i=1}^n r_i > 1$. For every i with $r_i \geq 1$, we obtain the following short exact sequence:

$$0 \rightarrow \mathcal{O}_{C_i} \otimes \mathcal{O}_X(- (Z - C_i)) \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z - C_i} \rightarrow 0.$$

Thus, by the induction hypothesis, it is sufficient to find an index j satisfying $r_j \geq 1$ and

$$h^1(C_j, \mathcal{O}_X(L - (Z - C_j))|_{C_j}) = 0.$$

Note that

$$h^1(C_j, \mathcal{O}_X(L - (Z - C_j))|_{C_j}) = h^0(C_j, \omega_{C_j} \otimes \mathcal{O}_X(-L + Z - C_j)|_{C_j}).$$

For every i , we see

$$\begin{aligned} \deg_{k(s)}(\omega_{C_i} \otimes \mathcal{O}_X(-L + Z - C_i)|_{C_i}) &= (K_X + C_i - L + Z - C_i) \cdot_{k(s)} C_i \\ &= (K_X - (K_X + \Delta + A) + Z) \cdot_{k(s)} C_i \\ &< (Z - \Delta) \cdot_{k(s)} C_i. \end{aligned}$$

Since $Z - \Delta \not\leq 0$, we can apply Lemma 3.1 to $D = Z - \Delta$. Then, we can find an index j satisfying $r_j \geq 1$ and

$$(Z - \Delta) \cdot C_j \leq 0.$$

This implies $h^1(C_j, \mathcal{O}_X(L - (Z - C_j))|_{C_j}) = 0$, as desired. □

THEOREM 3.3. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a two-dimensional quasi-projective klt pair (X, Δ) over B to a quasi-projective B -scheme. Let D be a \mathbb{Q} -Cartier \mathbb{Z} -divisor on X such that $D - (K_{X/B} + \Delta)$ is π -nef and π -big. Assume that one of the following conditions holds.*

- (1) $\dim \pi(X) \geq 1$.
- (2) $\dim \pi(X) = 0$ and for the Stein factorisation $\pi : X \rightarrow T \rightarrow S$ of π , T is the affine spectrum of a field of characteristic zero.

Then the equation $R^i \pi_* \mathcal{O}_X(D) = 0$ holds for every $i > 0$.

Proof. — Replacing π by its Stein factorisation, we may assume that $\pi_* \mathcal{O}_X = \mathcal{O}_S$. If (2) holds, then the assertion follows from [12, Theorem 1-2-5] after taking the base change to the algebraic closure. If (1) holds, then we can apply the same proof as in [12, Theorem 1-2-5] by using Proposition 3.2 instead of [12, Theorem 1-2-3]. □

THEOREM 3.4. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective normal B -surface to a quasi-projective B -scheme. Let Δ be an \mathbb{R} -divisor on X such that $K_{X/B} + \Delta$ is \mathbb{R} -Cartier and let L be a Cartier divisor on X such that $L - (K_{X/B} + \Delta)$ is π -nef and π -big. Assume that one of the following conditions holds.*

- (1) $\dim \pi(X) \geq 1$.
- (2) $\dim \pi(X) = 0$ and for the Stein factorisation $\pi : X \rightarrow T \rightarrow S$ of π , T is the affine spectrum of a field of characteristic zero.

Then the equation $R^i \pi_*(\mathcal{O}_X(L) \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta) = 0$ holds for every $i > 0$.

Proof. — We can apply the same argument as in [31, Theorem 2.10] by using Theorem 3.3 instead of [31, Corollary 2.7]. □

3.2. Global case of positive characteristic

In this subsection, we establish a vanishing theorem of Kawamata–Viehweg type for surfaces of positive characteristic (Theorem 3.8). The heart of this subsection is Proposition 3.7. For this, we need some auxiliary results: Lemma 3.5 and Lemma 3.6. Our strategy is similar to but modified from [31, Section 2].

LEMMA 3.5. — *Let k be a field. Let X be a projective normal variety over k . Let M be a coherent sheaf on X and let A be an ample \mathbb{R} -Cartier \mathbb{R} -divisor on X . Then there exists a positive integer $r(M, A)$ such that*

$$H^i(X, M \otimes_{\mathcal{O}_X} \mathcal{O}_X(rA + N)) = 0$$

for any $i > 0$, any real number $r \geq r(M, A)$ and any nef \mathbb{R} -Cartier \mathbb{R} -divisor N such that $rA + N$ is Cartier.

Proof. — The assertion follows from the same argument as in [31, Theorem 2.2]. For the Fujita vanishing theorem in our setting, we refer to [14, Theorem 1.5]. □

LEMMA 3.6. — *Let k be a field. Let X be a projective normal variety over k . Let M be a coherent sheaf on X and let N be a nef \mathbb{R} -Cartier \mathbb{R} -divisor on X with $N \neq 0$. Then there exists a positive integer r_0 such that*

$$H^{\dim X}(X, M \otimes_{\mathcal{O}_X} \mathcal{O}_X(rN + N')) = 0$$

for any real number $r \geq r_0$ and any nef \mathbb{R} -Cartier \mathbb{R} -divisor N' such that $rN + N'$ is Cartier.

Proof. — We may assume that k is an infinite field by taking the base change to its algebraic closure if k is a finite field. Since X is projective, there exists a surjection $\bigoplus_{j=1}^{\ell} M_j \rightarrow M$ where each M_j is an invertible sheaf. Thus, we may assume that M is an invertible sheaf. Taking a sufficiently ample hyperplane section H of X , where H is a projective normal variety ([28, Theorem 7']), we obtain

$$0 \rightarrow \mathcal{O}_X(M + N'') \rightarrow \mathcal{O}_X(H + M + N'') \rightarrow \mathcal{O}_X(H + M + N'')|_H \rightarrow 0.$$

for any nef Cartier divisor N'' on X . By the Fujita vanishing theorem ([14, Theorem 1.5]), we see $H^i(X, H + M + N'') = 0$ for any $i > 0$ and nef Cartier divisor N'' . Thus, it follows that

$$\begin{aligned} H^{\dim X}(X, M + N'') &\simeq H^{\dim X-1}(H, \mathcal{O}_X(H + M + N'')|_H) \\ &= H^{\dim H}(H, \mathcal{O}_X(H + M + N'')|_H) \end{aligned}$$

for any nef Cartier divisor N'' on X . Repeating this process for $N'' := rN + N'$, we may assume $\dim X = 1$. Then N is ample. The assertion follows from Lemma 3.5. \square

In the following proposition, we use the notion of F -singularities. For definitions and basic properties, we refer to [4, Section 2.3] (cf. [26]). Although [4] works over an algebraically closed field of positive characteristic, the same argument works over F -finite fields.

PROPOSITION 3.7. — *Let k be an F -finite field of characteristic $p > 0$. Let X be a projective normal variety over k . Assume that one of the following two conditions holds.*

- (1) (X, Δ) is sharply F -pure where Δ is an effective \mathbb{Q} -divisor on X such that $(p^{e_1} - 1)(K_X + \Delta)$ is Cartier for some $e_1 \in \mathbb{Z}_{>0}$.
- (2) (X, Δ) is strongly F -regular, where Δ is an effective \mathbb{R} -divisor on X such that (X, Δ) is \mathbb{R} -Cartier.

Let L be a Cartier divisor on X such that $L - (K_X + \Delta)$ is ample, and let N be a nef \mathbb{R} -Cartier \mathbb{R} -divisor on X with $N \neq 0$. Then there exists a positive integer r_0 such that the equation

$$H^i(X, L + rN + N') = 0$$

holds for any $i \geq \dim X - 1$, real number $r \geq r_0$ and nef \mathbb{R} -Cartier \mathbb{R} -divisor such that $rN + N'$ is Cartier.

Proof. — By Lemma 3.6, we may assume $i = \dim X - 1$. Thanks to [4, Lemma 4.1], we may assume that (1) holds. It follows from (1) that we get an exact sequence (cf. [4, Definition 2.7]):

$$0 \rightarrow B_e \rightarrow F_*^e(\mathcal{O}_X(-(p^e - 1)(K_X + \Delta))) \rightarrow \mathcal{O}_X \rightarrow 0,$$

where B_e is a coherent sheaf depending on $e \in e_1\mathbb{Z}_{>0}$. For any nef Cartier divisor N'' , we obtain the exact sequence:

$$\begin{aligned} H^{\dim X - 1}(X, p^e(L + N'') - (p^e - 1)(K_X + \Delta)) \\ \rightarrow H^{\dim X - 1}(X, L + N'') \\ \rightarrow H^{\dim X}(X, B_e \otimes_{\mathcal{O}_X} \mathcal{O}_X(L + N'')). \end{aligned}$$

By the Fujita vanishing theorem ([14, Theorem 1.5]), we can find $e_2 \in e_1\mathbb{Z}_{>0}$ such that

$$\begin{aligned} H^{\dim X - 1}(X, p^{e_2}(L + N'') - (p^{e_2} - 1)(K_X + \Delta)) \\ = H^{\dim X - 1}(X, p^{e_2}N'' + L + (p^{e_2} - 1)(L - (K_X + \Delta))) \\ = 0 \end{aligned}$$

for every nef Cartier divisor N'' . By Lemma 3.6, we can find an integer $r_0 > 0$, depending on e_2 , such that

$$H^{\dim X}(X, B_{e_2} \otimes_{\mathcal{O}_X} \mathcal{O}_X(L + rN + N'')) = 0$$

for every $r \geq r_0$ and nef \mathbb{R} -Cartier \mathbb{R} -divisor such that $rN + N'$ is Cartier. We are done by substituting $rN + N'$ for N'' . □

THEOREM 3.8. — *Let k be a field of characteristic $p > 0$. Let (X, Δ) be a two-dimensional projective klt pair over k . Let D be a \mathbb{Q} -Cartier \mathbb{Z} -divisor on X such that $D - (K_X + \Delta)$ is nef and big. Let N and M_1, \dots, M_q be nef \mathbb{R} -Cartier \mathbb{R} -divisors on X with $N \neq 0$. Then there exists a positive integer r_0 such that*

$$H^i \left(X, L + rN + \sum_{j=1}^q s_j M_j \right) = 0$$

for any $i > 0$ and real numbers r, s_1, \dots, s_q such that $r \geq r_0, s_j \geq 0$ and $rN + \sum_{j=1}^q s_j M_j$ is Cartier.

Proof. — We treat the following three cases separately.

- (1) k is F -finite, X is regular and Δ is simple normal crossing.
- (2) X is regular and Δ is simple normal crossing.
- (3) The case without any additional assumptions.

For the case (1), the assertion in the theorem follows from Proposition 3.7 and the fact that (X, Δ) is strongly F -regular (cf. [27, Proposition 6.18]).

Assume that the conditions in (2) hold. Take models over a finitely generated field over \mathbb{F}_p , that is, we can find an intermediate field $\mathbb{F}_p \subset k_0 \subset k$ with k_0 finitely generated over \mathbb{F}_p and $X_0, \Delta_0, D_0, N_0, M_{j,0}$ over k_0 whose base changes and pull-backs are X, Δ, D, N, M_j respectively, and these satisfy the same properties as in the statement. Since k_0 is F -finite, it follows from the case (1) that

$$\begin{aligned} \dim_k H^i \left(X, D + rN + \sum_{j=1}^q s_j M_j \right) \\ = \dim_{k_0} H^i \left(X_0, D_0 + rN_0 + \sum_{j=1}^q s_j M_{j,0} \right) = 0 \end{aligned}$$

for numbers i, r, s_j as in the statement. Thus the assertion in the theorem follows for the case (2).

For the case (3), we can apply the same proof as in [12, Theorem 1-2-5] by using the case (2) instead of [12, Theorem 1-2-3]. □

THEOREM 3.9. — *Let k be a field of characteristic $p > 0$. Let X be a projective normal surface over k and let Δ be an \mathbb{R} -divisor such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let L be a Cartier divisor on X such that $L - (K_X + \Delta)$ is nef and big. Let N and M_1, \dots, M_q be nef \mathbb{R} -Cartier \mathbb{R} -divisors on X with $N \neq 0$. Then there exists a positive integer r_0 such that*

$$H^i \left(X, \mathcal{O}_X \left(L + rN + \sum_{j=1}^q s_j M_j \right) \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta \right) = 0$$

for any $i > 0$ and real numbers r, s_1, \dots, s_q such that $r \geq r_0, s_j \geq 0$ and $rN + \sum_{j=1}^q s_j M_j$ is Cartier.

Proof. — We can apply the same argument as in [31, Theorem 2.9] by using Theorem 3.8 (resp. Theorem 3.3) instead of [31, Theorem 2.6] (resp. [31, Theorem 2.7]). □

4. Minimal model program

4.1. Base point free theorem

In Section 3, we established some vanishing theorems of Kawamata–Viehweg type. By applying standard arguments called the X-method, we show the base point free theorem of Kawamata–Shokurov type (Theorem 4.2). We start with a non-vanishing theorem of Shokurov type.

THEOREM 4.1. — *Let X be a projective regular variety over a field k with $\dim X \leq 2$. Let Δ be an \mathbb{R} -divisor on X such that $\lfloor \Delta \rfloor \leq 0$ and $\{\Delta\}$ is simple normal crossing. Let D be a nef Cartier divisor on X . Assume that the following conditions hold.*

- (1) $aD - (K_X + \Delta)$ is nef and big for some positive real number a .
- (2) If k is of positive characteristic and $\dim X = 2$, then $D \neq 0$.

Then there exists a positive integer m_0 such that $H^0(X, mD - \lfloor \Delta \rfloor) \neq 0$ for any integer m with $m \geq m_0$.

Proof. — If k is of characteristic zero, then the assertion follows from [12, Theorem 2-1-1] by taking the base change to its algebraic closure. Thus we may assume that k is of positive characteristic.

We first treat the case where $\dim X = 1$. If $D \neq 0$, then the assertion follows from the Serre vanishing theorem. Thus we may assume that $D \equiv 0$. We have that

$$h^0(X, mD - \lfloor \Delta \rfloor) = \chi(X, mD - \lfloor \Delta \rfloor) = \chi(X, -\lfloor \Delta \rfloor) = h^0(X, -\lfloor \Delta \rfloor) \geq 1,$$

where the second equation follows from the Riemann–Roch theorem (cf. Theorem 2.7) and the first and third equations hold by the Kodaira vanishing theorem for curves.

Thus we may assume that $\dim X = 2$. Since $D \neq 0$ and D is nef, it follows from Serre duality that $H^2(X, mD - \iota\Delta_{\perp}) = 0$ for any $m \gg 0$. By the Riemann–Roch theorem (cf. Theorem 2.10), we have that

$$\begin{aligned} h^0(X, mD - \iota\Delta_{\perp}) &\geq \frac{1}{2}(mD - \iota\Delta_{\perp}) \cdot (mD - \iota\Delta_{\perp} - K_X) + \chi(X, \mathcal{O}_X) \\ &\geq \frac{1}{2}mD \cdot (aD - 2\iota\Delta_{\perp} - K_X) + \frac{1}{2}(-\iota\Delta_{\perp}) \cdot (-\iota\Delta_{\perp} - K_X) + \chi(X, \mathcal{O}_X) \end{aligned}$$

for any $m \gg 0$. Thus it suffices to show that $D \cdot (aD - 2\iota\Delta_{\perp} - K_X) > 0$, which holds by

$$D \cdot (aD - 2\iota\Delta_{\perp} - K_X) \geq D \cdot (aD - (K_X + \Delta)) > 0.$$

We are done. □

THEOREM 4.2. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective normal B -surface to a quasi-projective B -scheme. Assume that there are an effective \mathbb{R} -divisor Δ on X and a nef Cartier divisor D on X which satisfy the following properties:*

- (1) $\iota\Delta_{\perp} = 0$,
- (2) $K_{X/B} + \Delta$ is \mathbb{R} -Cartier,
- (3) $D - (K_{X/B} + \Delta)$ is π -nef and π -big, and
- (4) if T is the affine spectrum of a field of positive characteristic for the Stein factorisation $\pi : X \rightarrow T \rightarrow S$ of π , then $D \neq 0$.

Then there exists a positive integer b_0 such that bD is π -free for any integer b with $b \geq b_0$.

Proof. — Replacing π by its Stein factorisation, we may assume that $\pi_*\mathcal{O}_X = \mathcal{O}_S$. By perturbing Δ , we may assume that Δ is a \mathbb{Q} -divisor such that $K_{X/B} + \Delta$ is \mathbb{Q} -Cartier, and $D - (K_{X/B} + \Delta)$ is π -ample.

First we treat the case where $\dim S = 0$, i.e. $S = \text{Spec } k$ for a field k . If k is of characteristic zero, then the assertion follows from [6, Theorem 13.1] possibly after replacing k by its algebraic closure. If k is of positive characteristic, then the assertion follows from the same argument as in [31, Theorem 3.2] after replacing [31, Theorem 2.6] and [31, Theorem 2.9] by Theorem 3.8 and Theorem 3.9, respectively.

Second we consider the remaining case, i.e. $\dim S \geq 1$. If (X, Δ) is klt, then we can apply the same argument as in [12, Theorem 3-1-1] after replacing [12, Theorem 1-2-3] and [12, the Non-Vanishing Theorem] by Theorem 3.3 and Theorem 4.1, respectively. Thus we may assume that (X, Δ) is not klt. Since the problem is local on S , we may assume that S is affine. It suffices to show that $\mathcal{O}_X(bD)$ is generated by its global sections for any large integer b , which follows from the same argument as in [31, Theorem 3.2] by using Theorem 3.3 and Theorem 3.4 instead of [31, Theorem 2.6] and [31, Theorem 2.9], respectively. \square

Remark 4.3. — We can not drop the assumption (4) in the statement of Theorem 4.2 (cf. [33, Theorem 1.4]).

4.2. Minimal model program for \mathbb{Q} -factorial surfaces

The purpose of this subsection is to show the minimal model program for \mathbb{Q} -factorial surfaces (Theorem 4.5). To this end, we establish a contraction theorem of extremal rays.

THEOREM 4.4. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a quasi-projective normal B -surface X to a quasi-projective B -scheme S . Let Δ be an \mathbb{R} -divisor on X such that $0 \leq \Delta \leq 1$ and $K_{X/B} + \Delta$ is \mathbb{R} -Cartier. Let R be an extremal ray of $\overline{NE}(X/S)$ which is $(K_{X/B} + \Delta)$ -negative. Then there exists a projective S -morphism $f : X \rightarrow Y$ to a projective S -scheme Y that satisfies the following properties.*

- (1) $f_*\mathcal{O}_X = \mathcal{O}_Y$.
- (2) For any projective S -curve C on X such that $\pi(C)$ is one point, $f(C)$ is one point if and only if $[C] \in R$.
- (3) If $\dim Y \geq 1$, then the sequence

$$0 \rightarrow \text{Pic } Y \xrightarrow{f^*} \text{Pic } X \xrightarrow{[C]} \mathbb{Z}$$

is exact for any projective S -curve C on X such that $f(C)$ is one point.

- (4) $\rho(Y/S) = \rho(X/S) - 1$.
- (5) If X is \mathbb{Q} -factorial, then Y is \mathbb{Q} -factorial and the fibre $f^{-1}(y)$ of any closed point $y \in Y$ is irreducible.

Proof. — First of all, we will show that there exists a morphism $f : X \rightarrow Y$ satisfying (1) and (2), and that the properties (3)–(5) hold if $\dim Y = 0$. After that, we will prove that the properties (3)–(5) hold if $\dim Y \geq 1$.

Replacing π by its Stein factorisation, we may assume that $\pi_*\mathcal{O}_X = \mathcal{O}_S$. By perturbing Δ , we may assume that Δ is a \mathbb{Q} -divisor. By the cone theorem (Theorem 2.14), we can find a π -nef Cartier divisor L on X such that $\overline{\text{NE}}(X/S) \cap L^\perp = R$ and that $L - (K_{X/B} + \Delta)$ is π -ample. If $\dim S = 0$ (i.e. $S = \text{Spec } k$ for some field k) and $D \equiv 0$, then we can check that the structure morphism $X \rightarrow Y := \text{Spec } k$ satisfies all the properties (1)–(5). Otherwise, by Theorem 4.2, there exists a projective S -morphism $f : X \rightarrow Y$ that satisfies (1) and (2). From now on, we assume that $\dim Y \geq 1$.

We show (3). Let M be a Cartier divisor on X such that $M \cdot C = 0$. It suffices to show that there is a Cartier divisor M_Y on Y such that $M = f^*M_Y$. We can find a π -nef Cartier divisor L on X such that $\overline{\text{NE}}(X/S) \cap L^\perp = R$ and that $L - (K_{X/B} + \Delta)$ is π -ample. Replacing L by its multiple, we may assume that $L + M$ is π -nef and $L + M - (K_{X/B} + \Delta)$ is π -ample. Since bL and $b(L + M)$ are π -free for any large integer b , we have that the divisors $bL, (b + 1)L, b(L + M), (b + 1)(L + M)$ are the pull-backs of Cartier divisors on Y . Thus also M is the pull-back of a Cartier divisor on Y , hence (3) holds. The assertion (4) directly follows from (3).

We show (5). Assume that X is \mathbb{Q} -factorial. If $\dim Y = 1$, then Y is regular and in particular \mathbb{Q} -factorial. If $\dim Y = 2$, then we can apply the same argument as in [18, Corollary 3.18] by using (3) instead of [18, Corollary 3.7(4)]. In any case, Y is \mathbb{Q} -factorial. We prove that any fibre of f is irreducible. We only treat the case where $\dim Y = 1$. Let C_1 and C_2 be two curves contained in $f^{-1}(y)$ such that $C_1 \neq C_2$ and $C_1 \cap C_2 \neq \emptyset$. Since $C_1 \cdot C_2 > 0$ and $C_1^2 < 0$, this contradicts $\rho(X/Y) = 1$. Thus (5) holds. \square

THEOREM 4.5. — *Theorem 1.1 holds if (X, Δ) satisfies the condition (QF).*

Proof. — The assertion follows directly from Theorem 2.14 and Theorem 4.4. \square

4.3. Minimal model program for log canonical surfaces

In this subsection, we establish the minimal model program for log canonical surfaces. Since we have already got the cone theorem and the contraction theorem, it only remains to show that numerically log canonical surfaces are log canonical (Theorem 4.13). Actually it is necessary to establish this result in order to formulate the log canonical minimal model program without the \mathbb{Q} -factorial condition. Such a result is well-known if the base scheme B is an algebraically closed field ([18, Proposition 4.11]). The proof

of [18, Proposition 4.11] depends on the classification of log canonical surface singularities, whilst our strategy avoid using it.

DEFINITION 4.6. — *Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective normal B -surface and let Δ is an \mathbb{R} -divisor on X . We say a pair (X, Δ) is numerically log canonical if the following properties hold.*

- (1) Δ is effective.
- (2) For an arbitrary projective birational morphism $f : Y \rightarrow X$ and the f -exceptional \mathbb{R} -divisor E uniquely determined by

$$K_{Y/B} + f_*^{-1}\Delta + E \equiv_f 0,$$

the inequality $f_*^{-1}\Delta + E \leq 1$ holds. Note that E is uniquely defined since the intersection matrix of the exceptional locus is negative definite ([16, Theorem 10.1]).

It is obvious that a numerically log canonical pair (X, Δ) is log canonical if and only if (X, Δ) is a log pair, i.e. $K_{X/B} + \Delta$ is \mathbb{R} -Cartier.

THEOREM 4.7. — *Let B be a scheme satisfying Assumption 2.1. Let X be a quasi-projective normal B -surface and let Δ be an effective \mathbb{R} -divisor on X . For the irreducible decomposition $\Delta = \sum_{i \in I} d_i D_i$, we set $\Delta_1 := \sum_{i \in I} \min\{d_i, 1\} D_i$. Then there exists a projective birational B -morphism $f : Y \rightarrow X$ such that*

- (1) $(Y, f_*^{-1}\Delta_1 + E)$ is a two-dimensional \mathbb{Q} -factorial dlt pair over B , where E is the reduced divisor on X with $\text{Supp } E = \text{Ex}(f)$,
- (2) $K_{Y/B} + f_*^{-1}\Delta_1 + E$ is f -nef, and
- (3) $K_{Y/B} + f_*^{-1}\Delta + E + E' \equiv_f 0$ for some f -exceptional effective \mathbb{R} -divisor E' .

A projective birational morphism satisfying the above properties (1)–(3) is called a dlt blowup of (X, Δ) .

Proof. — Let $\Delta_2 := \Delta - \Delta_1$. We take a log resolution $\mu : W \rightarrow X$ of (X, Δ) . We can write

$$K_{W/B} + \mu_*^{-1}\Delta_1 + E_W \equiv_\mu -\mu_*^{-1}\Delta_2 + E'_W$$

where E_W is the μ -exceptional reduced divisor with $\text{Supp } E_W = \text{Ex}(\mu)$ and E'_W is an μ -exceptional \mathbb{R} -divisor. We run a $(K_{W/B} + \mu_*^{-1}\Delta_1 + E_W)$ -MMP over X (Theorem 4.5), and set $f : Y \rightarrow X$ to be the end result. Let $\mu_Y : W \rightarrow Y$ be the induced morphism, $E := (\mu_Y)_* E_W$ and $E' := -(\mu_Y)_* E'_W$. Then (1) and (2) hold. By the negativity lemma (Lemma 2.11), we get

$$K_{W/B} + \mu_*^{-1}\Delta + E_W - E'_W = \mu_Y^*(K_{Y/B} + f_*^{-1}\Delta + E + E'),$$

which implies $K_{Y/B} + f_*^{-1} \Delta_1 + E \equiv_f -f_*^{-1} \Delta_2 - E'$. It follows again from the negativity lemma (Lemma 2.11) that E' is effective, hence (3) holds. \square

Remark 4.8. — We use the same notation as in Theorem 4.7.

- (a) If (X, Δ) is a log pair i.e. $K_{X/B} + \Delta$ is \mathbb{R} -Cartier, then Theorem 4.7(3) implies that $K_{Y/B} + f_*^{-1} \Delta + E + E' = f^*(K_{X/B} + \Delta)$.
- (b) (X, Δ) is numerically log canonical if and only if $\Delta = \Delta_1$ and E' in Theorem 4.7(3) is zero.

LEMMA 4.9. — *Let B be a scheme satisfying Assumption 2.1. Let (X, Δ) be a numerically log canonical quasi-projective B -surface. Assume that $x \in X$ is a unique non-regular closed point of X and that $(X \setminus \{x\}, \Delta|_{X \setminus \{x\}})$ is dlt. Then one of the following assertions hold.*

- (1) X is \mathbb{Q} -factorial.
- (2) $x \notin \text{Supp } \Delta$ and there exists a projective birational B -morphism

$$g : Z \rightarrow X$$

from a normal \mathbb{Q} -factorial quasi-projective B -surface Z such that $E := \text{Ex}(g)$ is a prime divisor and that $(K_{Z/B} + E) \cdot E = 0$. Moreover there exists a dlt blowup $f : Y \rightarrow X$ of (X, Δ) (cf. Theorem 4.7) that factors through g .

Proof. — Let $f : Y \rightarrow X$ be a dlt blowup whose existence is guaranteed in Theorem 4.7. Since $(X \setminus \{x\}, \Delta|_{X \setminus \{x\}})$ is dlt, possibly after replacing Y by the scheme obtained by gluing $X \setminus \{x\}$ and $f^{-1}(X^0)$ for sufficiently small open neighbourhood X^0 of $x \in X$, we may assume that $f(\text{Ex}(f)) = \{x\}$. For the sum $\sum_{i=1}^r E_i$ of the f -exceptional prime divisors E_i , we can write as follows (cf. Remark 4.8):

$$K_{Y/B} + f_*^{-1} \Delta + \sum_{i=1}^r E_i \equiv_f 0.$$

CLAIM. — *There exists a projective birational B -morphism $g : Z \rightarrow X$ of quasi-projective normal B -surfaces which satisfies the following properties:*

- (1) Z is \mathbb{Q} -factorial,
- (2) $E := \text{Ex}(g)$ is a prime divisor,
- (3) $(K_{Z/B} + g_*^{-1} \Delta + E) \cdot E = 0$, and
- (4) f factors through g .

Proof of Claim. — We run a $(K_{Y/B} + f_*^{-1} \Delta + \sum_{i=2}^r E_i)$ -MMP over X (Theorem 4.5). We show that the end result $g : Z \rightarrow X$ satisfies (1)–(4).

Let $h : Y \rightarrow Z$ be the induced birational morphism. It is clear that (1) and (4) hold. Since $K_{Y/B} + g_*^{-1}\Delta + \sum_{i=2}^r E_i \equiv_f -E_1$, each step of this MMP contracts E_i for some $i \in \{2, \dots, r\}$. In particular, we get $h_*E_1 \neq 0$. Furthermore, all of E_2, \dots, E_r are contracted, since $h_*E_1 \neq 0$, $-h_*E_1$ is g -nef and $g^{-1}(x)$ is connected. Therefore, (2) holds and $E = h_*E_1$. It follows from $K_{Y/B} + f_*^{-1}\Delta + \sum_{i=1}^r E_i \equiv_f 0$ that

$$K_{Y/B} + f_*^{-1}\Delta + \sum_{i=1}^r E_i = h^*(K_{Z/B} + g_*^{-1}\Delta + E).$$

Thus (3) holds. □

Assuming that $x \in \text{Supp } \Delta$, it suffices to show that X is \mathbb{Q} -factorial. By (1)–(3), the projective birational morphism $g : Z \rightarrow X$ is the contraction of a $(K_{Z/B} + \frac{1}{2}g_*^{-1}\Delta + E)$ -negative extremal ray of $\overline{NE}(Z/X)$. It follows from Theorem 4.4 that X is \mathbb{Q} -factorial, as desired. It completes the proof of Lemma 4.9. □

Remark 4.10. — Since a pair (Z, E) , appearing in Lemma 4.9(2), is numerically log canonical and Z is \mathbb{Q} -factorial, (Z, E) is log canonical.

COROLLARY 4.11. — *Let B be a scheme satisfying Assumption 2.1. Let (X, Δ) be a two-dimensional dlt pair over B . Then X is \mathbb{Q} -factorial.*

Proof. — The assertion directly follows from Lemma 4.9. □

LEMMA 4.12. — *Let (C, D) be a one-dimensional projective semi log canonical pair over a field k in the sense of [16, Definition-Lemma 5.10]. If C is irreducible and $K_C + D \equiv 0$, then there exists a positive integer m such that $\mathcal{O}_C(m(K_C + D)) \simeq \mathcal{O}_C$.*

Proof. — Since C is semi log canonical, the singularities of C is at worst nodal. In particular C is Gorenstein. If $D \neq 0$, then ω_C^{-1} is ample and the assertion follows from [16, Lemma 10.6]. Thus we may assume that $D = 0$. Then we get $\omega_C \simeq \mathcal{O}_C$ by

$$h^0(X, \omega_X) = h^1(X, \omega_X) = h^0(X, \mathcal{O}_X) \neq 0,$$

where the first (resp. second) equation follows from Corollary 2.8 (resp. Serre duality). □

THEOREM 4.13. — *Let B be a scheme satisfying Assumption 2.1. Let (X, Δ) be a numerically log canonical quasi-projective B -surface. Then the following assertions hold.*

- (1) $K_{X/B}$ and all the irreducible components of Δ are \mathbb{Q} -Cartier. In particular, (X, Δ) is log canonical, that is, $K_{X/B} + \Delta$ is \mathbb{R} -Cartier.

(2) *There exists an open subset $X^0 \subset X$ such that $\text{Supp } \Delta \subset X^0$ and that X^0 is \mathbb{Q} -factorial.*

Proof. — By Lemma 4.9, we may assume that $\Delta = 0$ and $x \in X$ is a unique non-regular point. It suffices to show that $K_{X/B}$ is \mathbb{Q} -Cartier. Let

$$f : Y \xrightarrow{h} Z \xrightarrow{g} X$$

be birational morphisms as in Lemma 4.9(2). We have that

$$K_{Y/B} + E_Y = h^*(K_{Z/B} + E) \equiv_f 0$$

where E_Y is the reduced divisor with $\text{Supp } E_Y = \text{Ex}(f)$.

CLAIM. — *There exists a positive integer m such that $\mathcal{O}_Z(m(K_{Z/B} + E))|_E \simeq \mathcal{O}_E$.*

We now show Claim. It follows from [16, Theorem 2.31] that E is nodal and hence Gorenstein. Let $(E, \text{Diff}_E(0))$ be a pair defined in [16, Definition 4.2]. Since E is nodal and (Z, E) is log canonical (Remark 4.10), it follows from [16, Lemma 4.8] that $(E, \text{Diff}_E(0))$ is semi log canonical in the sense of [16, Definition-Lemma 5.10]. Then Claim holds by $(K_{Z/B} + E) \cdot E = 0$ and Lemma 4.12.

Since there is a natural morphism $E_Y \rightarrow E$, Claim implies that $\mathcal{O}_Y(m(K_{Y/B} + E_Y))|_{E_Y} \simeq \mathcal{O}_{E_Y}$ for sufficiently divisible $m \in \mathbb{Z}_{>0}$. The exact sequence:

$$0 \rightarrow \mathcal{O}_Y(-E_Y) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{E_Y} \rightarrow 0,$$

induces an exact sequence:

$$\begin{aligned} f_*\mathcal{O}_Y(m(K_{Y/B} + E_Y)) &\rightarrow f_*(\mathcal{O}_Y(m(K_{Y/B} + E_Y))|_{E_Y}) \\ &\rightarrow R^1 f_*\mathcal{O}_Y(m(K_{Y/B} + E_Y) - E_Y) = 0 \end{aligned}$$

where the last equation follows from Theorem 3.3. Thus, $K_{Y/B} + E_Y$ is f -semi-ample, which implies that there is a \mathbb{Q} -Cartier \mathbb{Q} -divisor L on X such that $K_{Y/B} + E_Y = f^*L$. Taking the push-forward f_* , it follows that $K_{X/B}$ is \mathbb{Q} -Cartier. □

Remark 4.14. — If B is essentially of finite type over a field of characteristic zero then Theorem 4.13 was known to follow from [9, Corollary 1.6].

THEOREM 4.15. — *Theorem 1.1 holds if (X, Δ) satisfies the condition (LC).*

Proof. — The assertion follows directly from Theorem 2.14, Theorem 4.4 and Theorem 4.13. □

5. Miscellaneous results

In this section, we prove the inversion of adjunction (Theorem 5.1) and the Kollár–Shokurov connectedness theorem (Theorem 5.2) for surfaces over excellent schemes.

5.1. Inversion of adjunction

THEOREM 5.1. — *Let B be a scheme satisfying Assumption 2.1. Let $(X, C + D)$ be a two-dimensional quasi-projective log pair over B , where C is a reduced divisor that has no common components with an effective \mathbb{R} -divisor D . Let C^N be the normalisation of C and let $D_{C^N} := \text{Diff}_{C^N}(D)$ (cf. [16, Definition 2.34]). Then the following hold.*

- (1) $(X, C + D)$ is log canonical around C if and only if (C^N, D_{C^N}) is log canonical.
- (2) $(X, C + D)$ is plt around C if and only if (C^N, D_{C^N}) is klt.

Proof. — For both of (1) and (2), we only show the if-part since the opposite implications follow from [16, Lemma 4.8].

Since the problem is local on X , we fix a closed point $x \in C \subset X$ around which we work. In particular, we may assume that all the irreducible components of $C + D$ contain x . Let $f : Y \rightarrow X$ be a dlt blowup of $(X, \Delta := C + D)$ as in Theorem 4.7 and we use the same notation as there. Let $\Delta_2 := \Delta - \Delta_1$. We have that

$$K_{Y/B} + f_*^{-1}\Delta_1 + E = f^*(K_{X/B} + C + D) - (f_*^{-1}\Delta_2 + E')$$

is f -nef (cf. Remark 4.8).

We show (1). We may assume that $f(\text{Ex}(f)) = \{x\}$. Assume that $(X, C + D)$ is not log canonical around x , which is equivalent to $f_*^{-1}\Delta_2 + E' \neq 0$ (Remark 4.8). Since $-(f_*^{-1}\Delta_2 + E')$ is f -nef, it follows from the negativity lemma (Lemma 2.11) that $\text{Supp } E'$ contains $f^{-1}(x)$. This implies that (C^N, D_{C^N}) is not log canonical by [16, Proposition 4.5(2)], hence (1) holds.

We show (2). Assume that (C^N, D_{C^N}) is klt. It suffices to show that $(X, C + D)$ is plt. It follows from (1) that $(X, C + D)$ is log canonical, hence $\Delta = \Delta_1$, $\Delta_2 = 0$ and $E' = 0$.

We now prove that f is an isomorphism. We assume by contradiction that $E \neq 0$. Since we work around x , we may assume that $f(\text{Ex}(f)) = \{x\}$. Since Theorem 4.7(1) implies $\text{Supp } E = \text{Ex}(f)$, we have that $f_*^{-1}C$ intersects E . Since $E \neq 0$, we get $\llcorner D_{C^N} \lrcorner \neq 0$ by [16, Proposition 4.5(6)], which

contradicts the fact that (C^N, D_{C^N}) is klt. Thus f is an isomorphism, as desired.

Since f is an isomorphism, $(X, C + D)$ is dlt. If $\lfloor D \rfloor \neq 0$, then (C^N, D_{C^N}) is not klt again by [16, Proposition 4.5 (6)]. Thus we get $\lfloor D \rfloor = 0$, hence (2) holds. □

5.2. Connectedness theorem

THEOREM 5.2. — *Let B be a scheme satisfying Assumption 2.1. Let $\pi : X \rightarrow S$ be a projective B -morphism from a two-dimensional quasi-projective log pair (X, Δ) over B to a quasi-projective B -scheme S with $\pi_* \mathcal{O}_X = \mathcal{O}_S$. Let $\text{Nklt}(X, \Delta)$ be the reduced closed subscheme of X consisting of the non-klt points of (X, Δ) . If $-(K_{X/B} + \Delta)$ is π -nef and π -big, then any fibre of the induced morphism $\text{Nklt}(X, \Delta) \rightarrow S$ is either empty or geometrically connected.*

Proof. — We divide the proof into three steps.

STEP 5.3. — *There exists an effective \mathbb{R} -divisor Δ' such that (X, Δ') is a log pair over B , $\Delta' \geq \Delta$, $\text{Nklt}(X, \Delta) = \text{Nklt}(X, \Delta')$, and $-(K_{X/B} + \Delta')$ is π -ample.*

Proof of Step 5.3. — Since $N := -(K_{X/B} + \Delta)$ is π -big, we can write

$$N = A + D$$

for some π -ample \mathbb{R} -Cartier \mathbb{R} -divisor A and an effective \mathbb{R} -divisor D . Thus for any rational number $0 < \epsilon < 1$, we have that

$$-(K_{X/B} + \Delta) = N = (1 - \epsilon)N + \epsilon A + \epsilon D,$$

where $(1 - \epsilon)N + \epsilon A$ is π -ample. Thus it suffices to find a rational number $0 < \epsilon < 1$ such that $\text{Nklt}(X, \Delta) = \text{Nklt}(X, \Delta + \epsilon D)$. We can find such a number ϵ by taking a log resolution of $(X, \Delta + D)$. □

STEP 5.4. — *If $\dim S \geq 1$, then the assertion in the theorem holds.*

Proof of Step 5.4. — By Step 5.3, we may assume that $-(K_{X/B} + \Delta)$ is π -ample. We have an exact sequence

$$0 \rightarrow \mathcal{J}_\Delta \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_W \rightarrow 0$$

where W is the closed subscheme corresponding to the multiplier ideal \mathcal{J}_Δ . Since the support of W is equal to the non-klt locus of (X, Δ) , we get

$W_{\text{red}} = \text{Nklt}(X, \Delta)$. Since $-(K_{X/B} + \Delta)$ is π -ample, it follows from Theorem 3.4 that $R^1\pi_*\mathcal{J}_\Delta = 0$, which implies that the induced homomorphism

$$\rho : \mathcal{O}_S = \pi_*\mathcal{O}_X \rightarrow \pi_*\mathcal{O}_W$$

is surjective. Since ρ factors through $\mathcal{O}_{\pi(W)}$, we have that

$$\mathcal{O}_{\pi(W)} \rightarrow \mathcal{O}_V = \pi_*\mathcal{O}_W$$

is surjective, where $W \rightarrow V \rightarrow \pi(W)$ is the Stein factorisation of $W \rightarrow \pi(W)$. Thus $V \rightarrow \pi(W)$ is a surjective closed immersion, hence a universal homeomorphism. In particular, any fibre of $W \rightarrow \pi(W)$ is geometrically connected. Since a surjective closed immersion $\text{Nklt}(X, \Delta) \rightarrow W$ is a universal homeomorphism, it completes the proof of Step 5.4. \square

STEP 5.5. — *If $\dim S = 0$, then the assertion in the theorem holds.*

Proof of Step 5.5. — Since $\dim S = 0$, we have that $S = \text{Spec } k$ for a field k . Taking the base change to the separable closure of k , we may assume that k is separably closed. In particular, it suffices to show that $\text{Nklt}(X, \Delta)$ is connected. By replacing (X, Δ) by its dlt blowup as in Theorem 4.7, we may assume that $(X, \Delta^{<1})$ is klt and $\text{Nklt}(X, \Delta) = \text{Supp } \Delta^{\geq 1}$ (cf. Remark 4.8). Set $A := -(K_X + \Delta)$. By Step 5.3, we may assume that A is ample. We run a $(K_X + \Delta^{<1} + A)$ -MMP: $f : X \rightarrow Y$. Since $-(K_X + \Delta^{<1} + A) = \Delta^{\geq 1} \neq 0$, the program ends with a Mori fibre space $\rho : Y \rightarrow T$. We set $\Delta_Y := f_*\Delta$. We now prove:

CLAIM. — *The following hold.*

- (1) $f(\text{Supp } \Delta^{\geq 1}) = \text{Supp } \Delta_Y^{\geq 1}$.
- (2) *If $\text{Nklt}(Y, \Delta_Y)$ is connected, then $\text{Nklt}(X, \Delta)$ is connected.*
- (3) $\text{Nklt}(Y, \Delta_Y) = \text{Supp } \Delta_Y^{\geq 1}$.

Since f is the composition of the extremal contractions, we only prove Claim under the assumption that $E := \text{Ex}(f)$ is irreducible. In particular, $\Delta^{\geq 1}$ is f -ample.

We show (1). The inclusion $f(\text{Supp } \Delta^{\geq 1}) \supset \text{Supp } \Delta_Y^{\geq 1}$ is clear. It suffices to show that $f(E) \in \text{Supp } \Delta_Y^{\geq 1}$. Since $\Delta^{\geq 1}$ is f -ample, there is an irreducible component C of $\text{Supp } \Delta^{\geq 1}$ such that $C \cdot E > 0$. It follows from $E^2 < 0$ that $C \neq E$. Thus, we get $f(E) \in \text{Supp } \Delta_Y^{\geq 1}$, hence (1) holds.

We show (2). We assume by contradiction that $\text{Supp } \Delta_Y^{\geq 1}$ is connected but $\text{Supp } \Delta^{\geq 1}$ is not. Thanks to (1), we can find a point $y \in \text{Supp } \Delta_Y^{\geq 1}$ such that $f^{-1}(y) \cap \text{Supp } \Delta^{\geq 1}$ is not connected. This contradicts Step 5.4. Thus (2) holds.

The assertion (3) follows from (1) and the fact that the MMP $f : X \rightarrow Y$ can be considered as a $(K_X + \Delta)$ -MMP, because the ampleness of A is preserved under the MMP. It completes the proof of Claim.

If $\dim T = 0$, then we have $\rho(Y) = 1$, hence $\text{Nklt}(Y, \Delta_Y) = \text{Supp } \Delta_Y^{\geq 1}$ is clearly connected. Thus we may assume that $\dim T = 1$. Assuming that $\text{Nklt}(Y, \Delta_Y)$ is not connected, let us derive a contradiction. Let D_1 and D_2 be distinct connected components of $\text{Nklt}(Y, \Delta_Y)$. Since $\Delta_Y^{\geq 1}$ is ρ -ample, $\text{Nklt}(Y, \Delta_Y) = \text{Supp } \Delta_Y^{\geq 1}$ dominates T . In particular, we may assume that D_1 dominates T . On the other hand, Step 5.4 implies that $\text{Nklt}(Y, \Delta_Y) \cap \rho^{-1}(t)$ is connected for any closed point $t \in T$. In particular, D_2 does not dominate T . Since D_2 is connected, we have that $D_2 \subset \rho^{-1}(t_0)$ for some closed point $t_0 \in T$. However, this implies that $\text{Nklt}(Y, \Delta_Y) \cap \rho^{-1}(t_0)$ has at least two connected components: D_2 and a connected component of $D_1 \cap \rho^{-1}(t_0)$. This is a contradiction. It completes the proof of Step 5.5. \square

Thus the assertion in Theorem holds by Step 5.4 and Step 5.5. \square

BIBLIOGRAPHY

- [1] L. BĂDESCU, *Algebraic surfaces*, Universitext, Springer, 2001, Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author, xiii+258 pages.
- [2] A. BEAUVILLE, *Complex algebraic surfaces*, London Mathematical Society Lecture Note Series, vol. 68, Cambridge University Press, 1983, Translated from the French by R. Barlow, N. I. Shepherd-Barron and M. Reid, iv+132 pages.
- [3] C. BIRKAR, Y. CHEN & L. ZHANG, “Iitaka’s $C_{n,m}$ conjecture for 3-folds over finite fields”, <https://arxiv.org/abs/1507.08760v2>, 2015.
- [4] P. CASCINI, H. TANAKA & C. XU, “On base point freeness in positive characteristic”, *Ann. Sci. Éc. Norm. Supér.* **48** (2015), no. 5, p. 1239-1272.
- [5] B. FANTECHI, L. GÖTTSCHE, L. ILLUSIE, S. L. KLEIMAN, N. NITSURE & A. VISTOLI, *Fundamental algebraic geometry: Grothendieck’s FGA explained*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, 2005, x+339 pages.
- [6] O. FUJINO, “Fundamental theorems for the log minimal model program”, *Publ. Res. Inst. Math. Sci.* **47** (2011), no. 3, p. 727-789.
- [7] ———, “Minimal model theory for log surfaces”, *Publ. Res. Inst. Math. Sci.* **48** (2012), no. 2, p. 339-371.
- [8] O. FUJINO & H. TANAKA, “On log surfaces”, *Proc. Japan Acad. Ser. A* **88** (2012), no. 8, p. 109-114.
- [9] C. D. HACON & C. XU, “On finiteness of B-representations and semi-log canonical abundance”, in *Minimal models and extremal rays (Kyoto, 2011)*, Advanced Studies in Pure Mathematics, vol. 70, Mathematical Society of Japan, 2016, p. 361-377.
- [10] R. HARTSHORNE, *Residues and duality*, Lecture Notes in Mathematics, vol. 20, Springer, 1966, with an appendix by P. Deligne, vii+423 pages.
- [11] ———, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977, xvi+496 pages.

- [12] Y. KAWAMATA, K. MATSUDA & K. MATSUKI, “Introduction to the minimal model problem”, in *Algebraic geometry (Sendai, 1985)*, Advanced Studies in Pure Mathematics, vol. 10, North-Holland, 1987, p. 283-360.
- [13] S. KEEL, “Basepoint freeness for nef and big line bundles in positive characteristic”, *Ann. Math.* **149** (1999), no. 1, p. 253-286.
- [14] D. S. KEELER, “Ample filters of invertible sheaves”, *J. Algebra* **259** (2003), no. 1, p. 243-283.
- [15] S. L. KLEIMAN, “Toward a numerical theory of ampleness”, *Ann. Math.* **84** (1966), p. 293-344.
- [16] J. KOLLÁR, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, 2013, With a collaboration of Sándor Kovács, x+370 pages.
- [17] J. KOLLÁR & S. KOVÁCS, “Birational geometry of log surfaces”, preprint available at <https://sites.math.washington.edu/~kovacs/pdf/BiratLogSurf.pdf>.
- [18] J. KOLLÁR & S. MORI, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original, viii+254 pages.
- [19] R. LAZARSFELD, *Positivity in algebraic geometry. I: Classical setting: Line bundles and linear series*, Springer, 2004, xviii+387 pages.
- [20] ———, *Positivity in algebraic geometry. II: Positivity for vector bundles, and multiplier ideals*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge.*, vol. 49, Springer, 2004, xvii+385 pages.
- [21] J. LIPMAN, “Desingularization of two-dimensional schemes”, *Ann. Math.* **107** (1978), no. 1, p. 151-207.
- [22] Q. LIU, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, 2002, Translated from the French by Reinie Ern e, xvi+576 pages.
- [23] H. MATSUMURA, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, 1989, Translated from the Japanese by M. Reid, xiv+320 pages.
- [24] M. MIYANISHI, *Noncomplete algebraic surfaces*, Lecture Notes in Mathematics, vol. 857, Springer, 1981, xviii+244 pages.
- [25] F. SAKAI, “Classification of normal surfaces”, in *Algebraic geometry, Bowdoin 1985 (Brunswick, 1985)*, Proceedings of Symposia in Pure Mathematics, vol. 46, American Mathematical Society, 1987, p. 451-465.
- [26] K. SCHWEDE, “ F -adjunction”, *Algebra Number Theory* **3** (2009), no. 8, p. 907-950.
- [27] K. SCHWEDE & K. TUCKER, “On the behavior of test ideals under finite morphisms”, *J. Algebr. Geom.* **23** (2014), no. 3, p. 399-443.
- [28] A. SEIDENBERG, “The hyperplane sections of normal varieties”, *Trans. Am. Math. Soc.* **69** (1950), p. 357-386.
- [29] I. R. SHAFAREVICH, *Lectures on minimal models and birational transformations of two dimensional schemes*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Mathematics, vol. 37, Tata Institute of Fundamental Research, 1966, iv+175 pages.
- [30] H. TANAKA, “Minimal models and abundance for positive characteristic log surfaces”, *Nagoya Math. J.* **216** (2014), p. 1-70.
- [31] ———, “The X-method for klt surfaces in positive characteristic”, *J. Algebr. Geom.* **24** (2015), no. 4, p. 605-628.
- [32] ———, “Behavior of canonical divisors under purely inseparable base changes”, <https://arxiv.org/abs/1502.01381v4>, to appear in *J. Reine Angew. Math.*, 2016.

- [33] ———, “Pathologies on Mori fibre spaces in positive characteristic”, <https://arxiv.org/abs/1609.00574v2>, 2016.

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