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EQUIVARIANT SCHUBERT CALCULUS AND JEU DE TAQUIN

by Hugh THOMAS & Alexander YONG

ABSTRACT. — We introduce edge labeled Young tableaux. Our main results provide a corresponding analogue of Schützenberger’s theory of *jeu de taquin*. These are applied to the equivariant Schubert calculus of Grassmannians. Reinterpreting, we present new (semi)standard tableaux to study factorial Schur polynomials, after Biedenharn-Louck, Macdonald, Goulden-Greene, and others.

Consequently, we obtain new combinatorial rules for the Schubert structure coefficients, complementing work of Molev-Sagan, Knutson-Tao, Molev, and Kreiman. We also describe a conjectural generalization of one of our rules to the equivariant K -theory of Grassmannians, extending our previous work on non-equivariant K -theory. This conjecture concretely realizes the “positivity” known to exist by a result of Anderson-Griffeth-Miller. It provides an alternative to the conjectural rule of Knutson-Vakil.

RÉSUMÉ. — Nous introduisons le concept de tableaux de Young avec arêtes étiquetées. Nos résultats principaux décrivent un analogue à la théorie du jeu de taquin de Schützenberger, avec applications au calcul de Schubert équivariant des grassmanniennes. Nous présentons de nouveaux tableaux (semi-)standards pour étudier les polynômes de Schur factoriels, d’après Biedenharn-Louck, Macdonald, et Goulden-Greene, entre autres.

Par conséquent, nous obtenons de nouvelles règles combinatoires pour les constantes de structure de Schubert, complémentaires aux travaux de Molev-Sagan, Knutson-Tao, Molev et Kreiman. Nous décrivons également une généralisation conjecturale d’une de nos règles à la K -théorie équivariante des grassmanniennes, étendant nos résultats précédents sur la K -théorie non équivariante. Cette conjecture réalise de façon concrète la positivité déjà connue par un résultat de Anderson-Griffeth-Miller, et offre une alternative à la règle conjecturale de Knutson-Vakil.

1. Introduction

1.1. Overview

The main goal of this paper is to introduce edge labeled Young tableaux, together with a corresponding analogue of the theory of *jeu de taquin*. We

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apply them to the setting of equivariant Schubert calculus of Grassmannians. This paper may also be interpreted as extending (semi)standard tableaux for use with the closely related family of factorial Schur polynomials.

The classical theory of *jeu de taquin*, initiated by M.-P. Schützenberger in [15], has been of significance in combinatorial representation theory. One outcome of this theory is a combinatorial rule for the Littlewood–Richardson coefficients. Perhaps more importantly, it provides a systematic and flexible means to elegantly reconcile a variety of important tableau algorithms. It achieves this using a simple sliding law.

The Littlewood–Richardson coefficients compute Schubert calculus of Grassmannians. More precisely, they are structure coefficients for multiplication with respect to the Schubert basis of the ordinary cohomology ring of Grassmannians. Since a Grassmannian admits the action of the torus T of invertible diagonal matrices, one can instead study the richer T -equivariant cohomology ring and its Schubert calculus. While Littlewood–Richardson rules were already available for this setting [8], further ideas are needed to (provably) extend them to other Lie types or finer cohomology theories. In addition, to date, *jeu de taquin* is the only combinatorial model that admits a root-system uniform rule for Schubert calculus on minuscule G/P 's [16]. These are our principal reasons for seeking new combinatorial models that extend *jeu de taquin*.

1.2. Schubert calculus of Grassmannians

Let $X = Gr(k, \mathbb{C}^n)$ denote the Grassmannian of k -dimensional planes in \mathbb{C}^n . If $\lambda = (n - k \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0)$ is a Young diagram contained in the rectangle $\Lambda := k \times (n - k)$, the associated *Schubert variety* is defined by

$$X_\lambda := \{V \in Gr(k, \mathbb{C}^n) \mid \dim(V \cap F^{n-k+i-\lambda_i}) \geq i, 1 \leq i \leq k\},$$

where $F^d = \text{span}(e_n, e_{n-1}, \dots, e_{n-d+1})$. With this convention, $\text{codim}(X_\lambda) = |\lambda| = \sum_i \lambda_i$.

Let $T \subseteq GL_n$ be the torus of invertible diagonal matrices. Since X_λ is T -stable under the action of T on X , X_λ admits an *equivariant Schubert class* σ_λ in $H_T(X) =$ the T -equivariant cohomology ring of X . Now, $H_T(X)$ is a module over $H_T(\text{pt}) := \mathbb{Z}[t_1, \dots, t_n]$, and these classes form an additive $H_T(\text{pt})$ -basis of $H_T(X)$. The expansion

$$(1.1) \quad \sigma_\lambda \cdot \sigma_\mu = \sum_{\nu} C_{\lambda, \mu}^{\nu} \sigma_{\nu},$$

defines the *equivariant Schubert structure coefficients* $C_{\lambda,\mu}^\nu \in \mathbb{Z}[t_1, \dots, t_n]$. In fact, $C_{\lambda,\mu}^\nu = 0$ unless $|\lambda| + |\mu| \geq |\nu|$. In the case of equality, $C_{\lambda,\mu}^\nu \in \mathbb{N}$ are the *Littlewood–Richardson coefficients*; these compute the number of points of X in $g_1 \cdot X_\lambda \cap g_2 \cdot X_\mu \cap g_3 \cdot X_{\nu^\vee}$, where g_1, g_2, g_3 are generic elements of GL_n and ν^\vee is the 180-degree rotation of the complement of ν inside Λ .

W. Graham [5] proved that the polynomials $C_{\lambda,\mu}^\nu$ have positive coefficients when expressed in the variables $\beta_i := t_i - t_{i+1}$. This positivity is evident in the statement of A. Knutson and T. Tao’s combinatorial *puzzle rule* [8]. Later, alternative tableau rules were given by V. Kreiman [9] and A. Molev [12] (in these rules, the positivity is not hard to prove). See also the work of P. Zinn-Justin [19].

1.3. (Semi)standard tableaux with edge labels

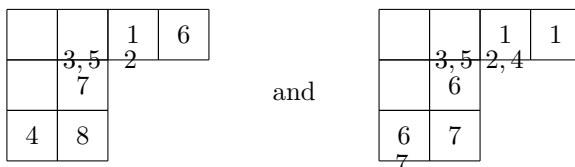
Our work depends on a new kind of Young tableaux. Let \mathbb{Y} denote the set of Young diagrams (drawn in English notation). Given $\lambda, \nu \in \mathbb{Y}$ with λ contained in ν , denote the skew shape by ν/λ . A *horizontal edge* of ν/λ is a horizontal side of a box of Λ which either lies along the upper or lower boundary of ν/λ , or which separates two boxes of ν/λ .

An *equivariant filling* of ν/λ assigns one of the labels $1, 2, \dots, \ell$ to each box of ν/λ and a (possibly empty) subset of $\{1, 2, \dots, \ell\}$ to each horizontal edge of ν/λ . An equivariant filling is *semistandard* if every box label is:

- weakly smaller than the label in the box immediately to its right;
- strictly smaller than any label in its southern edge and the label in the box immediately below it; and
- strictly larger than any label in its northern edge and the label in the box immediately above it.

(No condition is placed on the labels of adjacent edges.) The filling is *standard* if the labels used are $1, 2, \dots, \ell$, and each label is used exactly once.

Let $\text{EqSYT}(\nu/\lambda, \ell)$ and $\text{EqSSYT}(\nu/\lambda, \ell)$ respectively be the set of equivariant standard and semistandard tableaux whose entries come from $\{1, 2, \dots, \ell\}$. For example:



which are in $\text{EqSYT}((4, 2, 2)/(2, 1), 8)$ and $\text{EqSSYT}((4, 2, 2)/(2, 1), 8)$, respectively.

Those $T \in \text{EqSYT}(\nu/\lambda, |\nu/\lambda|)$, where each horizontal edge has no labels, are in obvious bijection with (*ordinary standard Young tableaux*). In this latter case, we also call T an ordinary standard tableau. (We will drop “equivariant” for fillings unless confusion might arise.) Finally, the ordinary standard Young tableau of shape μ that is filled by $1, 2, \dots, \mu_1$ in the first row, $\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2$ in the second row, etc, is called *row superstandard*. Let T_μ denote the row superstandard Young tableau of shape μ .

1.4. Equivariant jeu de taquin (first version)

Our first version of equivariant *jeu de taquin* omits some features (and complexity) of the main construction of Section 2. Nevertheless, this version already suffices to compute the polynomials $C_{\lambda, \mu}^\nu$. Moreover, it suggests generalizations. Specifically, we present a conjectural generalization to equivariant K -theory in Section 4. It also suggests a first step towards an extension to minuscule G/P ’s (further discussion may appear elsewhere), cf. [16].

A box $x \in \lambda$ is an *inner corner* of ν/λ if it is maximally southeast in λ . Given an inner corner x and $T \in \text{EqSYT}(\nu/\lambda, \ell)$, compare the label in the box immediately to the right of x and the smallest label on the southern edge of x , or the label in the box immediately below x , if no label appears on that edge. The smaller of the labels is moved into x , either by vacating a box or moving a label from the southern edge of x . If no labels can be used or if an edge label is moved, the process terminates. Otherwise, some adjacent box has been vacated, and we repeat the above process until termination. Call the result $\text{Eqjdt}_x^<(T)$, the *equivariant jeu de taquin slide* into x . Clearly, $\text{Eqjdt}_x^<(T)$ is also a standard tableau.

Define *the* equivariant rectification of T , denoted $\text{Eqrct}^<(T)$, to be the result of applying the sequence $\text{Eqjdt}_{x(1)}^<, \text{Eqjdt}_{x(2)}^<, \dots, \text{Eqjdt}_{x(|\lambda|)}^<$ starting with T , where we have that $x(1), x(2), \dots, x(|\lambda|)$ are the boxes of λ , read along columns, from bottom to top, and right to left.

Example 1.1. — Let $\nu/\lambda = (4, 3, 1)/(3, 1, 1) \subseteq \Lambda = 3 \times 4$ and

$$T = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & 1 & 2 & \\ \hline & 5 & 6 & \\ \hline & & & \\ \hline & 4 & & \\ \hline \end{array}$$

We use “•” to show the boxes being slid into during the steps of $\text{Eqrct}^<(T)$. The rectification of the third column given by:

(1.2)

		•	3
	1	2	
	5	6	
4			

 \mapsto

		2	3
	1	6	
	5	6	
4			

The rectification of the second column given by:

(1.3)

	•	2	3
	1	6	
	5	6	
4			

 \mapsto

	1	2	3
	5	6	
	5	6	
4			

and finally the rectification of the first column given by:

(1.4)

	1	2	3
	5	6	
•			
4			

 \mapsto

	1	2	3
•	5	6	
4			

 \mapsto

•	1	2	3
4	5	6	

(1.5) $\mapsto \dots \mapsto$

1	2	3	•
4	5	6	

the last tableau being $T_{(3,3)}$. Here the “ $\mapsto \dots \mapsto$ ” refers to slides moving the • right in the first row.

We now define the weight $\text{wt}(T) \in \mathbb{Z}[t_1, \dots, t_n]$ of a standard tableau T . Each box $x \in \Lambda$ is assigned a weight $\beta(x) = t_m - t_{m+1}$ where m is the “Manhattan distance” from the southwest corner (point) of Λ to the northwest corner (point) of x (i.e., the length of any north and east lattice path between the corners); see Example 1.3. We say an edge label l passes through a box x if it occupies x during the equivariant rectification of the column of T in which l begins. Suppose that the boxes passed are x_1, x_2, \dots, x_s . Moreover, once the rectification of a column is complete, suppose the filled boxes strictly to the right of the box x_s are y_1, \dots, y_t . Then set

$$\text{factor}(l) = (\beta(x_1) + \beta(x_2) + \dots + \beta(x_s)) + (\beta(y_1) + \beta(y_2) + \dots + \beta(y_t)).$$

If after rectification of a column, the label l still remains an edge label, $\text{factor}(l)$ is declared to be zero. Otherwise, since the boxes x_1, \dots, x_s ,

y_1, \dots, y_t form a hook inside ν , note that $\mathbf{factor}(i) = t_e - t_f$ with $e < f$. Now define

$$\mathbf{wt}(T) := \prod_{\mathfrak{l}} \mathbf{factor}(\mathfrak{l}),$$

where the product is over all edge labels \mathfrak{l} of T .

THEOREM 1.2. — *The equivariant Schubert structure coefficient is given by the polynomial*

$$C_{\lambda, \mu}^\nu = \sum_T \mathbf{wt}(T),$$

where the sum is over all $T \in \mathbf{EqSYT}(\nu/\lambda, |\mu|)$ such that $\mathbf{Egrect}^<(T) = T_\mu$.

Since each $\mathbf{factor}(\mathfrak{l})$ is a positive sum of the indeterminates $\beta_i = t_i - t_{i+1}$, Theorem 1.2 expresses $C_{\lambda, \mu}^\nu$ as a polynomial with positive coefficients in the β_i 's. It is not hard to see that Theorem 1.2 expresses $C_{\lambda, \mu}^\nu$ as a squarefree polynomial in the “positive root” variables $\alpha_{ij} = \beta_i + \beta_{i+1} + \dots + \beta_{j-1}$, also a feature of the puzzle rule of [8].

Example 1.3. — Continuing Example 1.1, the Manhattan distances for $\Lambda = 3 \times 5$ are:

3	4	5	6	7
2	3	4	5	6
1	2	3	4	5

There are three edge labels of T :

- For the edge label 2, we have $\mathbf{factor}(2) = (t_5 - t_6) + (t_6 - t_7) = t_5 - t_7$ since the edge label passes through one box, and after the third column is rectified (1.2), the $\boxed{3}$ lies to its right.
- For the edge label 1, we have $\mathbf{factor}(1) = (t_4 - t_5) + (t_5 - t_6) + (t_6 - t_7) = t_4 - t_7$ since the edge label passes through one box, and after the second column is rectified (1.3), the $\boxed{2 \ 3}$ lies to its right.
- For the edge label 4, we have $\mathbf{factor}(4) = (t_1 - t_2) + (t_2 - t_3) + (t_3 - t_4) + (t_4 - t_5) = t_1 - t_5$ since the edge label passes through two boxes, and after the first column is rectified (1.4) and (1.5), $\boxed{5 \ 6}$ lies to its right.

Therefore, $\mathbf{wt}(T) = (t_5 - t_7)(t_4 - t_7)(t_1 - t_5)$.

In Schützenberger’s *jeu de taquin* theory, one is free to slide at different inner corners. His theory’s “first fundamental theorem” is that rectification does not depend on these choices. The above equivariant *jeu de taquin* avoids this issue altogether by insisting on a specific order of rectification. Even more, the classical theory’s “second fundamental theorem” asserts

the number of tableaux that rectify to a given target tableau is independent of the choice of target tableau. In contrast, we insist on using row superstandard tableaux as our targets.

The above rigid definition of *jeu de taquin* makes nonobvious to us how to directly prove Theorem 1.2. Although one can biject the rule of Theorem 1.2 with earlier rules, our original reason for starting this project was to find a model that could ultimately extend to other equivariant contexts where earlier rules are unavailable.

Therefore, our problem was to find a more flexible version of equivariant *jeu de taquin* possessing features of the fundamental theorems. Our solution is described in Section 2. It has some aspects that are distinctly different than the classical *jeu de taquin* (and our first version of equivariant *jeu de taquin*):

- More than one label can move during a swap.
- Labels can move downwards during a swap.
- Row semistandardness can be violated after a swap (although at most one such violation occurs at any given time, and it is eliminated at the end of a sequence of swaps that defines a slide).

Our main result shows that the order of rectification is independent of the choices, if one rectifies to a “highest weight tableau” and starts with a tableau that is “lattice”. From this, we derive an essentially independent proof of Theorem 1.2.

1.5. Organization

In Section 2, we describe our flexible version of *jeu de taquin* as well as stating and proving our main results. Section 3 uses the results of Section 2 to give two additional formulations of the equivariant Littlewood–Richardson rule. We then deduce Theorem 1.2. In Section 4, we formulate a conjectural formula for equivariant K -theory of Grassmannians. Concluding remarks are given in Section 5.

2. Equivariant *jeu de taquin* (flexible version)

To describe our flexible version of equivariant *jeu de taquin*, it is more convenient to work with semistandard fillings than with standard fillings.

Starting with a semistandard filling T of a skew shape ν/λ , choose an inner corner x and mark it with a \bullet . We now define the equivariant slide

of T into x . As in classical *jeu de taquin*, the slide proceeds by a sequence of swaps, as the \bullet moves through the tableau.

However, the result of a slide is not necessarily a single tableau, but rather a formal sum of tableaux, with coefficients in $\mathbb{Z}[\beta_1, \dots, \beta_{n-1}]$, where $\beta_i = t_i - t_{i+1}$. The way this arises in the course of the sequence of swaps is that sometimes a swap will produce two tableaux. One of them has no \bullet , and it contributes directly to the output (with a coefficient), while the other still has a \bullet , which we continue to swap.

2.1. Definitions of the equivariant swaps

Let $x \in T$ be as above. Suppose y is the box to the immediate right of x , and z is the box immediately below x . Let \mathfrak{b} be the smallest neighbouring label below x (either the smallest one on the lower edge of x or the one in the box z) and let τ be the label in y . Define $\mathcal{N}_{x, \mathfrak{l}}^T$ to be the number of occurrences of a label \mathfrak{l} in columns weakly to the right of the box x in T .

There are four kinds of swaps (I)–(IV) that we use:

- (I) “vertical swap”: $\mathfrak{b} \leq \tau$ (or there is no τ) and \mathfrak{b} is a box label of z : T' is obtained by exchanging \bullet and \mathfrak{b} , i.e.,

$$T = \begin{array}{|c|c|} \hline \bullet & \tau \\ \hline \mathfrak{b} & \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline \mathfrak{b} & \tau \\ \hline \bullet & \\ \hline \end{array} = T'.$$

Output: T' .

- (II) “expansion swap”: $\mathfrak{b} \leq \tau$ and \mathfrak{b} is a label of the lower edge of x : T' is obtained by moving \mathfrak{b} into x ; the \bullet is eliminated. T'' is obtained by moving \mathfrak{b} to the top edge of x (and \bullet remains in place). In this case,

$$\begin{array}{|c|} \hline \bullet \\ \hline \mathfrak{b} \\ \hline \end{array} \mapsto \beta(x) \cdot \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \end{array} + \begin{array}{|c|} \hline \mathfrak{b} \\ \hline \bullet \\ \hline \end{array} = \beta(x) \cdot T' + T''.$$

Output: $\beta(x) \cdot T' + T''$.

- (III) “resuscitation swap”: $\mathfrak{b} > \tau$ (or there is no \mathfrak{b}), and the largest label u on the upper edge of x satisfies $u = \tau$: In this case, T' is obtained by having $u = \tau$ replace the \bullet in x , replace τ by \bullet in y , and placing τ on the lower edge of y . This move locally looks like:

$$T = \begin{array}{|c|c|} \hline \tau & \tau \\ \hline \bullet & \tau \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline \tau & \bullet \\ \hline \tau & \tau \\ \hline \end{array} = T'.$$

Output: T' .

(IV) “horizontal swap”: $\mathfrak{b} > \mathfrak{r}$ (or there is no \mathfrak{b}), and (III) does not apply: Define Z to be the set of consecutive integers $\{\mathfrak{r}, \mathfrak{r} + 1, \dots, \mathfrak{m}\}$ where \mathfrak{m} is chosen largest so that:

- (i) $\mathfrak{m} < \mathfrak{b}$ and \mathfrak{m} is at least as large as the entry in the box to the left of \mathfrak{x} ;
- (ii) $\mathcal{N}_{y,l}^T = \mathcal{N}_{y,\mathfrak{r}}^T$ for all $\mathfrak{r} \leq l \leq \mathfrak{m}$.
- (iii) $\{\mathfrak{r} + 1, \dots, \mathfrak{m}\}$ are labels on the lower edge of y .

Set $Z' = Z \setminus \{\mathfrak{m}\}$, $W = U \cup Z'$ and $Y' = Y \setminus Z$. Then locally the swap is:

$$(2.1) \quad T = \begin{array}{|c|c|} \hline U & \\ \hline \bullet & \mathfrak{r} \\ \hline \mathfrak{b} & Y \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline W & \\ \hline \mathfrak{m} & \bullet \\ \hline \mathfrak{b} & Y' \\ \hline \end{array} = T'.$$

That is, T' is the result of moving \mathfrak{m} into \mathfrak{x} and putting the smaller entries of Z in the upper edge of \mathfrak{x} . Conclude by placing \bullet into y . Output: T' .

Example 2.1 (of swap (IV)). — We have

$$\begin{array}{|c|c|} \hline \bullet & 1 \\ \hline 3 & 2, 3 \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline 1 & \\ \hline 2 & \bullet \\ \hline 3 & 3 \\ \hline \end{array}$$

where $Z = \{1, 2\}$.

On the other hand:

$$\begin{array}{|c|c|c|} \hline & & 1 \\ \hline \bullet & 1 & \\ \hline & 2 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \bullet \\ \hline & 2 & \\ \hline \end{array}$$

where the edge label “2” is not in Z because of (IV) (ii).

In addition, the following swap (IV) is valid, even though it “breaks” row semistandardness in the “obvious” sense:

$$\begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & 1 & 1 \\ \hline & 2 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & \bullet & 1 \\ \hline 2 & & 2 \\ \hline \end{array}$$

Note that the next swap will also be of type (IV), “fixing” the broken semistandardness in the second row. Claims 2.8 and 2.9 below explain how this example generalizes.

We now describe $\text{Eqjdt}_x(T)$ (as opposed to $\text{Eqjdt}_x^{\leq}(U)$ for U an equivariant standard tableau, described in Section 1). Begin by replacing T by the result of swapping at \mathfrak{x} . The result is a formal sum of terms of the form $\omega \cdot S$ where $\omega \in \mathbb{Z}[\beta_1, \dots, \beta_{n-1}]$, and S is a tableau. If a tableau U in this formal sum either has no \bullet , or the \bullet has no neighbouring labels southeast,

then do nothing. Otherwise, let x' be the box containing the \bullet of U and replace U by swapping at x' . Repeat until no more tableaux need replacement. Now erase all any \bullet 's from the tableaux in the formal sum. We need to show (under assumptions) that $\text{Eqjdt}_x(T)$ is a well-defined algorithm.

Call a tableau T with at most a single \bullet *really good* if:

- (a) it is semistandard, once one ignores the \bullet (i.e., the rows are weakly increasing and the columns are strictly increasing);
- (b) the label of the box directly left of the box with the \bullet is weakly less than the smallest label on the edge below the \bullet (if the latter label exists), i.e.,

$$\begin{array}{|c|c|} \hline \ell & \bullet \\ \hline & b \\ \hline \end{array} \quad (\ell \leq b);$$

- (c) the label of the box directly right of the box with the bullet is weakly larger than the largest label on the edge above the \bullet (if the latter exists), i.e.,

$$\begin{array}{|c|c|} \hline u & \\ \hline \bullet & r \\ \hline \end{array} \quad (u \leq r).$$

(Note that the latter two conditions would be automatic if the \bullet were a numerical label.) Call T *nearly bad* if (b) and (c) above hold, and (a) holds except that the label to the immediate left of the \bullet may be larger than the label to the immediate right of \bullet . We will say T is *good* if it is either really good or nearly bad; otherwise T is *bad*.

The third swap in Example 2.1 demonstrates that swap (IV) can turn a really good tableau to a nearly bad one. In fact, in Section 2.4 we see only swap (IV) can cause near badness.

2.2. Statement of the main results

An equivariant filling T is *lattice* if for a given column c and label l (that may not be in column c), the number of occurrences of l in columns weakly to the right of column c weakly exceeds the occurrences of $l + 1$ in that region.

The appropriate class of tableaux to apply our Eqjdt swaps to are the lattice and semistandard tableaux, in the sense that Eqjdt preserves this class:

PROPOSITION 2.2. — *Suppose T is semistandard and lattice, and that x is an inner corner. Then $\text{Eqjdt}_x(T)$ is well-defined as an algorithm: it*

terminates in a finite number of steps, and outputs a formal sum of semistandard and lattice tableaux. Each intermediate tableau in the calculation of $\text{Eqjdt}_x(T)$ is good and lattice.

Assuming this proposition (the proof being delayed until Section 2.3), we define (an) *equivariant rectification*. Given T , pick an inner corner x and replace T by the formal sum $\text{Eqjdt}_x(T)$. Now, for each U appearing in $\text{Eqjdt}_x(T)$, which has an inner corner x' , replace U by $\text{Eqjdt}_{x'}(U)$. Repeat until no such U exists. Let $\text{Eqrect}(T)$ be the resulting formal sum of equivariant semistandard tableaux. We will call the choices of x and of each x' the *rectification order*.

Call a straight shape tableau *regular* if does not have any edge labels; it is *irregular* otherwise. The regular tableau S_μ whose i -th row uses only the labels i is called a *highest weight tableau*. The *content* of a tableau T is $\mu = (\mu_1, \mu_2, \dots)$ if T has μ_1 1's, μ_2 2's, etc. For T of content μ , $\text{Eqrect}(T)$ is μ -*highest weight* if S_μ is the only regular tableau that appears. (We allow the possibility that no regular tableau appears at all.)

Let us also define the *a priori weight* of a good and lattice tableau T , denoted by $\text{apwt}(T)$. Declare $\text{apwt}(T) = 0$ if:

- (i) there is an edge label i weakly above the upper edge of the box x in row i (in its column), and it is not possible to apply a resuscitation swap (III) to T such that i moves into x ; or
- (ii) there is a box label i located strictly higher than row i .

We will say that a label satisfying (i) or (ii) is *too high*. It will also be convenient to say that a label i is *nearly too high* if it lies on the upper edge of a box in row i but is not too high (i.e., a resuscitation swap (III) applies to T and moves i into x).

Now suppose neither (i) nor (ii) holds. Given an edge label i , suppose it lies on the lower edge of a box x in row r . (If i is on a top edge of Λ then $r = 0$.) Define $\text{apfactor}(i)$ as follows:

$$(2.2) \quad \text{apfactor}(i) = t_{\text{Man}(x)} - t_{\text{Man}(x)+r-i+1+\# \text{ of } i\text{'s strictly to the right of } x}$$

where $\text{Man}(x)$ is the Manhattan distance as defined in Section 1.

Finally, let

$$\text{apwt}(T) = \prod_{i \text{ is an edge label of } T} \text{apfactor}(i).$$

We are now ready to state our main result, a partial analogue of the fundamental theorems of *jeu de taquin*.

THEOREM 2.3. — *Let T be a lattice semistandard tableau of content μ . Then:*

- (1) $\mathbf{Eqrect}(T)$ is μ -highest weight for any choice of rectification order.
- (2) The coefficient of S_μ in $\mathbf{Eqrect}(T)$ is invariant under these choices.
- (3) The coefficient in (2) is $\mathbf{apwt}(T)$.

Theorem 2.3 is the key needed to formulate our second version of the equivariant Littlewood–Richardson rule (Theorem 3.1).

Remark 2.4. — In the classical theory, T rectifies to S_μ if and only if T is lattice and has content μ . However, in our setting, analogues of these two conditions are no longer equivalent. Specifically, it is possible for a non-lattice tableau to become lattice using the equivariant swaps. For example, the starting tableau T below is not lattice, but $\mathbf{Eqjdt}_x(T)$ is:

$$T = \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & 1 & \\ \hline \bullet & 2 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & 1 & \\ \hline 2 & \bullet & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & 1 & \\ \hline 2 & 2 & \bullet \\ \hline \end{array} = \mathbf{Eqjdt}_x(T)$$

Therefore, we proceed to develop an equivariant Littlewood–Richardson rule using the second of the two classically equivalent conditions.

In order to develop a rule using an analogue of the first condition, one needs swapping rules with the property that non-lattice fillings stay non-lattice after a swap. It seems to us that such rules would be more complicated than our current rules.

Example 2.5. — In the following rectification (inside $\Lambda = 2 \times 2$), we suppress the computations concerning tableaux with labels that are too high (i.e., will rectify to a irregular tableau). For each tableau, the \bullet indicates the box where the next swap will be applied.

$$T = \begin{array}{|c|c|} \hline & \bullet \\ \hline & 1 \\ \hline 1 & \\ \hline \end{array} \mapsto \beta_3 \left(\begin{array}{|c|c|} \hline & 1 \\ \hline \bullet & \\ \hline 1 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & 1 \\ \hline & \bullet \\ \hline 1 & \\ \hline \end{array} \right) \\ \mapsto \beta_3 \left(\beta_1 \left(\begin{array}{|c|c|} \hline \bullet & 1 \\ \hline 1 & \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & 1 \\ \hline 1 & \\ \hline & \\ \hline \end{array} \right) + \dots \right)$$

$$\begin{aligned}
 &\mapsto \beta_1\beta_3 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \bullet & \\ \hline \end{array} + \beta_3 \left(\beta_2 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \bullet \\ \hline & 1 \\ \hline \end{array} \right) + \dots \\
 &\mapsto (\beta_1\beta_3 + \beta_2\beta_3) \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \\ \hline \end{array} + \beta_3 \begin{array}{|c|c|} \hline 1 & \bullet \\ \hline & 1 \\ \hline & \\ \hline \end{array} + \dots \\
 &\mapsto (\beta_1\beta_3 + \beta_2\beta_3) \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \\ \hline \end{array} + \beta_3 \left(\beta_3 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \bullet \\ \hline & 1 \\ \hline \end{array} \right) + \dots \\
 &\mapsto (\beta_1\beta_3 + \beta_2\beta_3 + \beta_3^2) \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \\ \hline \end{array} + \dots = \mathbf{Eqrect}(T)
 \end{aligned}$$

Hence $\mathbf{Eqrect}(T)$ is (2)-highest weight. Now, $\mathbf{apwt}(T) = (\beta_1 + \beta_2 + \beta_3)\beta_3$ which equals the coefficient of $S_{(2)}$ in $\mathbf{Eqrect}(T)$. These two facts agree with parts (1) and (3) of Theorem 2.3, respectively.

2.3. Proof of Proposition 2.2

Suppose we start the computation of $\mathbf{Eqjdt}_x(T)$, giving rise to a sequence of swaps of tableaux:

$$T = T^{(0)} \mapsto T^{(1)} \mapsto \dots \mapsto T^{(i)}.$$

(If we use swap (II) a “branching” occurs in the computation. The above sequence represents one of the paths of the computation.)

We argue by induction that each successive tableau is good and lattice; the base case is the hypothesis on T . If $T^{(i)}$ either has no \bullet or no labels southeast of \bullet then this is one of the tableaux appearing in $\mathbf{Eqjdt}_x(T)$. Otherwise, we must show that we can apply exactly one of the swaps (I)–(IV) to obtain $S' = T^{(i+1)}$ which is good and lattice.

There are two cases, depending on whether $S = T^{(i)}$ is really good or nearly bad.

Case 1: S is really good. — We break our argument into several claims.

CLAIM 2.6. — *If it is possible to apply one of the swaps (I)–(IV) to S then the result is good.*

Proof. — Suppose the vertical swap (I) is applied. Thus, S locally looks like

$$S = \begin{array}{|c|c|c|} \hline d & \bullet & e \\ \hline f & g & h \\ \hline \end{array}$$

where $g \leq e$ (and there is no label on the edge above the g). Thus we obtain

$$S' = \begin{array}{|c|c|c|} \hline d & g & e \\ \hline f & \bullet & h \\ \hline \end{array}.$$

To check that S' is really good, one only needs $d \leq g$ (if d exists). If d exists, so must f and $d < f \leq g$ (since S is good), as needed.

Next, suppose the expansion swap (II) is applied, thus

$$S = \begin{array}{|c|c|c|} \hline d & \bullet & e \\ \hline f & y & h \\ \hline \end{array}$$

where y is the smallest label on its edge and $y \leq e$. If S' is the result of having the y jump to the top edge, then S' is really good since S is really good and, as we have assumed, $y \leq e$. Also, we know $d \leq y$ (again since S is good) and hence if S' is the result of replacing \bullet by y , then S' is really good.

If a resuscitation swap (III) is used, we would have:

$$S = \begin{array}{|c|c|c|} \hline p & q & t \\ \hline m & \tau & \tau \\ \hline s & f & h \\ \hline \end{array} \mapsto S' = \begin{array}{|c|c|c|} \hline p & q & t \\ \hline m & \tau & \bullet \\ \hline s & f & h \\ \hline \end{array}$$

where $u = \tau$ is the largest label on its edge. Since S is really good, $m \leq \tau$. Hence S' is really good.

Finally, suppose we use a horizontal swap (IV) to arrive at S' . Thus:

$$S = \begin{array}{|c|c|c|c|} \hline d & \bullet & \tau & w \\ \hline s & f & h & v \\ \hline \end{array} \mapsto S' = \begin{array}{|c|c|c|c|} \hline d & \mathbf{m} & \bullet & w \\ \hline s & f & h & v \\ \hline \end{array}$$

Recall $Z = \{\tau, \tau + 1, \dots, \mathbf{m}\}$ is the set of labels that move from the third column to the second (relative to our local picture). Removal of these labels

clearly keeps the third column of S' semistandard since the third column of S is assumed to be semistandard. By the really goodness of S and the assumption that (III) does not apply, it follows that the maximal element immediately above the \bullet in S is strictly less than τ . These considerations, and condition (IV) (i), imply the semistandardness of the second column of S' . Now, (IV) (i) allows, at worst, the possibility that S' is nearly bad, i.e., that $w < m$. However, even in that case, S' is good (by definition). \square

CLAIM 2.7. — *Exactly one of the swaps (I)–(IV) is applicable.*

Proof. — In the case $b \leq \tau$ (or τ does not exist), one can apply either a vertical or expansion swap but not both. Thus suppose $b > \tau$ (or there is no b). Locally, we have

$$S = \begin{array}{|c|c|c|} \hline & u & \\ \hline a & \bullet & \tau \\ \hline & b & Y \\ \hline \end{array}.$$

(The argument is the same if b is the label of the box below the \bullet .) Since S is good we have $u \leq \tau$. If $u = \tau$ then one can apply a resuscitation move (III) (and, by definition, not a horizontal swap (IV)).

Hence we may assume $u < \tau < b$. Now, (IV) is always possible since the set Z in the definition of (IV) is nonempty by the given inequalities and the assumption $a \leq \tau$ (since S is really good). \square

We also need to show S' is lattice. This will be argued after Case 2 since the proof only assumes S is good.

Case 2: S is nearly bad. — In Case 1 we proved a really good tableau can become nearly bad only after using swap (IV) on some tableau S^- . Suppose then that S was obtained using swap (IV) from some tableau S^- , where S^- may be nearly bad. Let the local pictures of these tableaux be

$$S^- = \begin{array}{|c|c|c|c|} \hline p & q & t & x \\ \hline d & \overline{U} & \overline{U} & \\ \hline & \bullet & \tau & \overline{\tau} \\ \hline & b & Y & \overline{Y} \\ \hline \end{array} \qquad S = \begin{array}{|c|c|c|c|} \hline p & q & t & x \\ \hline d & \overline{W} & \overline{U} & \\ \hline & m & \bullet & \overline{\tau} \\ \hline & b & Y' & \overline{Y} \\ \hline \end{array}$$

where S being nearly bad means $m > \overline{\tau}$. We now construct the next swap $S \mapsto S'$.

CLAIM 2.8. — *No swap of type (I), (II), or (III) is applicable to S .*

Proof. — If $Y' \neq \emptyset$ then let $b' = \min Y'$. Then $b' > m > \overline{\tau}$ (by column semistandardness of S^- and the assumption S is nearly bad). Therefore (II) cannot be applied. If $Y' = \emptyset$ then a similar argument shows that (I) cannot be applied either. Also, if $\overline{u} = \max \overline{U}$ exists, then $\overline{u} < \tau \leq \overline{\tau}$. Hence a resuscitation swap (III) cannot be applied either. \square

CLAIM 2.9. — Swap (IV) is applicable to S .

Proof. — Since S^- is good, we have $\max \bar{U} < \mathfrak{r} \leq \bar{\mathfrak{r}}$. Also, by the definition of (IV) we have $\min Y' > \mathfrak{m} > \bar{\mathfrak{r}}$. Therefore, $\bar{\mathfrak{r}}$ can be placed in the edge of \bar{U} in S and maintain the vertical semistandardness in that column. Thus, let $\bar{\mathfrak{m}}$ be the largest label from \bar{Y} with this property such that the consecutive sequence $\bar{Z} = \{\bar{\mathfrak{r}}, \bar{\mathfrak{r}} + 1, \dots, \bar{\mathfrak{m}}\}$ could form the sequence of labels that move left in the swap (IV) starting from S . That is, they satisfy (IV) (i)–(iii) provided $\bar{\mathfrak{m}} \geq \mathfrak{m}$. In this case, the swap (IV) $S \mapsto S'$ would result in a good tableau:

$$S' = \begin{array}{|c|c|c|c|} \hline p & q & t & x \\ \hline d & \bar{\mathfrak{m}} & \bar{\mathfrak{m}} & \bullet \\ \hline & \mathfrak{b} & Y' & \bar{Y}' \\ \hline \end{array} .$$

In order to reach a contradiction, suppose $\bar{\mathfrak{m}} < \mathfrak{m}$.

S^- is lattice (by induction). Condition(IV) (ii) gives

$$(2.3) \quad \mathcal{N}_{\text{col } 3, \mathfrak{r}}^{S^-} = \mathcal{N}_{\text{col } 3, \mathfrak{r}+1}^{S^-} = \dots = \mathcal{N}_{\text{col } 3, \mathfrak{m}}^{S^-} .$$

Since the labels $\mathfrak{r}, \mathfrak{r} + 1, \dots, \mathfrak{m}$ appear in column 3 of S^- , (2.3) implies

$$(2.4) \quad \mathcal{N}_{\text{col } 4, \mathfrak{r}}^S = \mathcal{N}_{\text{col } 4, \mathfrak{r}+1}^S = \dots = \mathcal{N}_{\text{col } 4, \mathfrak{m}}^S .$$

We also know that

$$(2.5) \quad \mathfrak{r} \leq \bar{\mathfrak{r}} \leq \bar{\mathfrak{m}} < \mathfrak{m} .$$

The first inequality is the induction hypothesis: S^- is row semistandard (to the right of the \bullet). The second inequality is the vertical semistandardness in the fourth column of S combined with the fact $\bar{\mathfrak{m}} \in \bar{Y}$. The third inequality is our assumption to be contradicted.

Suppose $\bar{\mathfrak{m}} + 1 (\leq \mathfrak{m})$ appears in column 4 of S . If $\bar{\mathfrak{m}} + 1$ were in the box below the edge of \bar{Y} in S^- then since S^- is good, the box to its immediate left must be filled with $\mathfrak{q} \leq \bar{\mathfrak{m}} + 1$. But $\mathfrak{m} \in Y$ and $\mathfrak{m} \geq \bar{\mathfrak{m}} + 1 \geq \mathfrak{q}$ implying this filling is impossible. Hence we may assume $\bar{\mathfrak{m}} + 1 \in \bar{Y}$. Then this, together with (2.4) and (2.5), imply $\bar{\mathfrak{m}} + 1 (\leq \mathfrak{m} < \min Y')$ should have been included in \bar{Z} , contradicting the definition of $\bar{\mathfrak{m}}$.

Therefore $\bar{\mathfrak{m}} + 1$ does not appear in column 4 of S^- . Then let X be the subtableau of S^- using the boxes in columns weakly to the right of column 4 of S^- . Then X has the labels $\mathfrak{r}, \dots, \bar{\mathfrak{m}} + 1$ in equal numbers, is lattice in those labels, and does not have $\bar{\mathfrak{m}} + 1$ in its leftmost column. This is impossible, another contradiction. Hence, in fact, $\mathfrak{m} \leq \bar{\mathfrak{m}}$ as desired. This means S' is at worst nearly bad and therefore good, as desired. \square

Summarizing, if S is nearly bad then it was obtained by a horizontal swap (IV) from either a really good S^- or a nearly bad S^- whose near badness occurs in the same row but one step to the right.

To complete both Cases 1 and 2, it remains to prove:

CLAIM 2.10. — *Any swap $T \mapsto T'$ starting from a good and lattice T , results in T' being lattice.*

Proof. — None of the swaps (I), (II) nor (III) can turn a lattice tableau into a non-lattice tableau, since in each case the set of labels in each column stays the same. Therefore, suppose that a horizontal swap (IV) destroys latticeness.

Consider the local diagram (2.1). The labels that move from the second column to the first column (with respect to our local diagram) are $Z = \{\mathfrak{r}, \mathfrak{r} + 1, \dots, \mathfrak{m}\}$.

The violation of latticeness must occur in the second column (and nowhere else in T'), since it is the only column such that the multiset of entries weakly to its right has changed. An offending label $\mathfrak{l} + 1$ (i.e., one such that $\mathcal{N}_{y, \mathfrak{l}+1}^{T'} > \mathcal{N}_{y, \mathfrak{l}}^{T'}$) is not weakly less than \mathfrak{n} (the neighboring label of \bullet to the north) since none of those labels moved. Also, $\mathfrak{l} + 1 \notin Z$ since they do not appear in the second column of T' . Moreover $\mathfrak{l} + 1 \leq \mathfrak{m} + 1$ since the labels $\mathfrak{m} + 1$ and larger have not moved. Thus the offending label must be $\mathfrak{l} + 1 = \mathfrak{m} + 1$, i.e., $\mathcal{N}_{y, \mathfrak{m}+1}^{T'} > \mathcal{N}_{y, \mathfrak{m}}^{T'}$. Hence, there must be a $\mathfrak{m} + 1$ in the second column.

We cannot have $\mathfrak{m} + 1$ as a box label in the box immediately below y , because then the box label in the neighbor to the left would also be $\mathfrak{m} + 1$ (other values would violate the prerequisite (IV)(i) or that T is good). Since \mathfrak{m} does not already occur in the first column of T , the assumption that T' is not lattice implies T fails the lattice condition for the label $\mathfrak{m} + 1$, at the first column, contrary to our assumption that T is lattice.

Therefore, $\mathfrak{m} + 1$ is on the lower edge of y . Why does it not lie in the set Z ? The reason must be failure of the prerequisite (IV)(i) or (IV)(ii). If it is condition (IV)(i), then there must already be a $\mathfrak{m} + 1$ in the first column, and again we conclude T is not lattice, contrary to our assumption. If it violates condition (IV)(ii), then that means $\mathcal{N}_{y, \mathfrak{m}+1}^T < \mathcal{N}_{y, \mathfrak{r}}^T$ (since T is lattice). However, since swap (IV) was used, by (IV)(ii) we see

$$\mathcal{N}_{y, \mathfrak{m}}^T = \mathcal{N}_{y, \mathfrak{r}}^T > \mathcal{N}_{y, \mathfrak{m}+1}^T.$$

Since $\mathcal{N}_{y, \mathfrak{m}}^{T'} = \mathcal{N}_{y, \mathfrak{m}}^T - 1$, $\mathcal{N}_{y, \mathfrak{m}+1}^{T'} = \mathcal{N}_{y, \mathfrak{m}+1}^T$ and all the numbers involved are integers, we have $\mathcal{N}_{y, \mathfrak{m}}^{T'} \geq \mathcal{N}_{y, \mathfrak{m}+1}^{T'}$ and so $\mathfrak{m} + 1$ satisfies the lattice condition in T' after all, a contradiction. □

Concluding, we have shown that after each swap we obtain a good and lattice tableau. Moreover, given such a tableau, exactly one of the swaps (I)–(IV) is applicable. These swaps have the property of either eliminating the \bullet , moving the \bullet strictly east or south, or strictly decreasing the number of labels southeast of the \bullet . Hence after a finite number of steps, each tableau will have either no \bullet or a single \bullet on an outer corner (which can then be erased). Hence the `Eqjdt` algorithm is well-defined and terminates as desired. \square

2.4. Proof of Theorem 2.3

Having established the well-definedness of `Eqjdt` in Proposition 2.2, the next proposition is the remaining main step in our proof of Theorem 2.3.

PROPOSITION 2.11. — *Let T be a good and lattice tableau arising in the process of computing `Eqjdt` starting from a semistandard and lattice tableau. If $T \mapsto T'$ is the result of one of the swaps (I), (III) or (IV) then $\mathbf{apwt}(T) = \mathbf{apwt}(T')$. In the case of the expansion swap (II), if $T \mapsto \beta(x)T' + T''$ then we have $\mathbf{apwt}(T) = \beta(x)\mathbf{apwt}(T') + \mathbf{apwt}(T'')$.*

Proof. — We analyze each of the swaps (I)–(IV) in turn:

Vertical swap (I). — Only the box label \mathfrak{b} moves (up by one square). Hence if any label was too high in T , it will also be too high in T' . So we may assume no label is too high in T . In addition, since we use (I), no labels of T are even nearly too high. Hence no labels in T' other than perhaps \mathfrak{b} can be even nearly too high. Thus, if \mathfrak{b} is not too high in T' , then the computation of each `apfactor` will be the same in T and T' .

Suppose \mathfrak{b} becomes too high in T' . Since the swap does not destroy the lattice or goodness properties, there must be some $\mathfrak{b} - 1$ to the right of the \mathfrak{b} , which must therefore be strictly higher than the new position of the \mathfrak{b} . But this implies that the $\mathfrak{b} - 1$ was too high in T , contrary to our assumption.

Since the edge labels are in the same positions in T and T' , it now follows that $\mathbf{apwt}(T) = \mathbf{apwt}(T')$, as desired.

Expansion swap (II). — Recall T' is the tableau obtained by moving \mathfrak{b} into the box x , “emitting the weight” $\beta(x)$, whereas T'' is the tableau obtained by \mathfrak{b} “jumping over” x . Thus, if T has any labels that are too high, this will be true of both T' and T'' , in which case

$$0 = \mathbf{apwt}(T) = \beta(x)\mathbf{apwt}(T') + \mathbf{apwt}(T'') = \beta(x) \cdot 0 + 0,$$

as desired. Hence we may assume no labels of T are too high.

Case 1: The \mathfrak{b} in T'' is not too high. — Note that the \mathfrak{b} in T' is also not too high: we could only have \mathfrak{b} too high in T' if \mathfrak{b} was at the top edge of the box in row \mathfrak{b} in T . However, since it was not resuscitated, it would have been too high in T , a contradiction. Thus the highest \mathfrak{b} can be in T' is row \mathfrak{b} . The **apfactor** of all edge labels other than the \mathfrak{b} are the same in T, T' and T'' . (No label could become nearly too high in T'' except possibly \mathfrak{b} .) So, it remains to prove that

$$(2.6) \quad \mathbf{apfactor}_T(\mathfrak{b}) = \beta(x) + \mathbf{apfactor}_{T''}(\mathfrak{b}).$$

Since the box above \mathfrak{b} in T'' has Manhattan distance $\mathbf{Man}(x) + 1$, we have by (2.2) that

$$\begin{aligned} \mathbf{apfactor}_{T''}(\mathfrak{b}) &= t_{\mathbf{Man}(x)+1} \\ &\quad - t_{\mathbf{Man}(x)+1+(r-1)-\mathfrak{b}+1+\# \text{ of } \mathfrak{b}'\text{s strictly to the right of } x \text{ in } T''}. \end{aligned}$$

But the number of \mathfrak{b} 's strictly to the right of our \mathfrak{b} in T'' equals the number of \mathfrak{b} 's strictly to the right of \mathfrak{b} in T . Thus, since $\beta(x) = t_{\mathbf{Man}(x)} - t_{\mathbf{Man}(x)+1}$, (2.6) follows immediately.

Case 2: The \mathfrak{b} in T'' is too high. — Since \mathfrak{b} is not too high in T , \mathfrak{b} in T'' is on the upper edge of a box in row \mathfrak{b} . This label is too high because it cannot be resuscitated. Consider the box y to the immediate right of x in T (or T''). For us to have done an expansion step $T \rightarrow T''$, if there is a label \mathfrak{r} in y of T , it must satisfy $\mathfrak{r} \geq \mathfrak{b}$. However, if $\mathfrak{r} > \mathfrak{b}$ then since y is in row \mathfrak{b} we can conclude \mathfrak{r} is too high in T , a contradiction of our assumption about T .

Thus y either has no box label (explaining why we can't do a resuscitation) or y contains \mathfrak{b} . If the former is true then $\mathbf{apfactor}_T(\mathfrak{b}) = \beta(x)$ since there can be no \mathfrak{b} 's strictly to the right of x (by the goodness and highness assumptions). So we are done in this situation. Hence assume y contains \mathfrak{b} . Thus the resuscitation swap (III) was possible after all in T'' , contradicting our assumption that the \mathfrak{b} in T'' is too high.

Resuscitation swap (III). — Only two labels move, namely $u = \mathfrak{r}$ and \mathfrak{r} go downwards. Suppose a label n on the top edge of box y in T is too high but becomes only nearly too high in T' . Hence y must be in row n . But since swap (III) was applied, by semistandardness, $n < u = \mathfrak{r}$. Hence $u = \mathfrak{r}$ must be too high in both T and T' and thus $\mathbf{apwt}(T) = \mathbf{apwt}(T') = 0$. Thus we may assume this does not happen. It is therefore clear that no other labels, except possibly $u = \mathfrak{r}$ and \mathfrak{r} can be too high in T and become

not too high in T' . Hence we assume all labels of T except possibly $u = \tau$ and τ are not too high.

If $u = \tau$ is on the upper edge of a box in row u then it must be only nearly too high in T , since by assumption we can apply a resuscitation swap (III) to bring that label downwards. If $u = \tau$ were any higher in T (and thus too high), it would be still too high in T' . Thus we may assume it was not too high in T . Thus τ must be not too high. Summarizing, we can assume that no label of T is too high.

Since $u = \tau$ and τ move down when $T \rightarrow T'$, no labels of T' are too high. Since the set of labels in each column is the same, it follows that $\text{apwt}(T) = \text{apwt}(T')$ provided that

$$(2.7) \quad \text{apfactor}_T(u) = \text{apfactor}_{T'}(\tau).$$

(Recall we argued above that no labels other than u can be nearly too high in T or T' .)

There is a box above x , say w . By (2.2) we have

$$\text{apfactor}_T(u) = t_{\text{Man}(w)} - t_{\text{Man}(w)+\text{row}(w)-u+1+\Delta_{x,u}^T},$$

where $\Delta_{x,u}^T$ is the number of u 's strictly to the right of x in T .

Also by (2.2)

$$\text{apfactor}_{T'}(\tau) = t_{\text{Man}(y)} - t_{\text{Man}(y)+\text{row}(y)-\tau+1+\Delta_{y,\tau}^{T'}}.$$

where $\Delta_{y,\tau}^{T'}$ is the number of τ 's strictly to the right of y in T' .

Noting that

$$\begin{aligned} \text{Man}(w) &= \text{Man}(y) \\ \text{row}(w) &= \text{row}(y) - 1 \\ \Delta_{y,\tau}^{T'} &= \Delta_{x,u=\tau}^T - 1 \end{aligned}$$

we conclude (2.7) is true.

Horizontal swap (IV). — If any label of T is too high then since labels are moving weakly upwards, that label will also be too high in T' . Thus, we may assume that no label of T is too high. We did not resuscitate $u = \max U$, nor labels on the upper edge of y . Hence labels on these edges are not even nearly too high in T .

Recall $Z = \{\tau, \tau + 1, \dots, m\}$ are the labels that moved during the swap (IV). If $m = \tau$, then τ is the only label that moves, and moreover it simply moves directly to the left from box y to box x . So if τ was not too high in T , nor is it too high in T' . Next suppose $m > \tau$. Now τ moves into the upper edge of x . Since $\tau + 1 \in Y$ and $\tau + 1$ is not too high in T ,

we see x and y are in row $R \geq r + 1$. Similarly, in fact x and y must be in row $R \geq m$, since otherwise m would be too high. Consequently, in T' , all of the labels in Z are still not too high.

We also need to rule out the possibility that an edge-label which is too high in T could become nearly too high in T' (because the next step after T' would be a swap (III) resuscitating it). Suppose locally the picture looks like

$$(2.8) \quad T = \begin{array}{|c|c|c|} \hline U & \bar{r} & \\ \hline \bullet & r & \bar{r} \\ \hline Y & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline W & \bar{r} & \\ \hline m & \bullet & \bar{r} \\ \hline Y' & & \\ \hline \end{array} = T'$$

If T' is really good then $m \leq \bar{r}$, but $m > \bar{r}$ by the vertical semistandardness of T , a contradiction. Otherwise if T' is nearly bad, then the next swap is (IV) not (III), by Claim 2.8. Thus, again T' cannot be of the form in (2.8).

Consider any edge label i that did not change in the swap $T \rightarrow T'$. Note $\text{apfactor}(i)$ and $\text{apfactor}_{T'}(i)$ could only differ if the number of i 's strictly to the right of the given i changes as we compare T and T' . However, there could not be a nonzero change, by the definition of the swap (IV).

We now establish a weight-preserving correspondence between the edge labels of T which moved and the edge labels of T' which resulted from the move; specifically,

$$\text{apfactor}_{T'}(l) = \text{apfactor}_T(l + 1)$$

for $l = r, r + 1, \dots, m - 1$, using (2.2). To see this, first note that in each case l is in a one higher row in T' than in T . Therefore it remains to show

$$(2.9) \quad \mathcal{N}_{y,l+1}^T = \mathcal{N}_{x,l}^{T'}$$

Now by (IV) (ii) we have

$$(2.10) \quad \mathcal{N}_{y,r}^T = \mathcal{N}_{y,r+1}^T = \dots = \mathcal{N}_{y,m}^T$$

Finally, by (IV) (i), we know there were no l 's in the column of x in T , so

$$(2.11) \quad \mathcal{N}_{x,l}^{T'} = \mathcal{N}_{y,l}^T$$

Now (2.10) and (2.11) combined immediately gives (2.9). □

Conclusion of the proof of Theorem 2.3. — By Proposition 2.2, any tableau in $\text{Eqrrect}(T)$ (under any rectification order) is semistandard and lattice. The only regular, semistandard, lattice tableaux of straight shape are the highest weight tableaux. Since T is lattice then $\text{Eqrrect}(T)$ (with respect to any order) will be a sum of tableaux that are lattice and which have the same multiset of labels as T . Hence the only regular tableau that can appear is S_μ . Any irregular U that appears in $\text{Eqrrect}(T)$ has

$\text{apwt}(U) = 0$. Hence, by Proposition 2.11, the coefficient of S_μ in $\text{Eqrect}(T)$ is $\text{apwt}(T)$ and the theorem holds. \square

3. Equivariant jeu de taquin computes Schubert calculus

Let

$$D_{\lambda,\mu}^\nu := \sum_T [S_\mu] \text{Eqrect}(T) = \sum_T \text{apwt}(T)$$

where the sums are over all lattice and semistandard tableaux T of shape ν/λ and content μ such that $\text{Eqrect}(T)$ is μ -highest weight. (By the arguments of Section 2, the last condition can be replaced by $\text{apwt}(T) \neq 0$.) Also, here $[S_\mu] \text{Eqrect}(T)$ means the coefficient of S_μ under some (or, as we proved in Theorem 2.3, any) rectification order. (The second equality is Theorem 2.3(3).)

We now connect these polynomials to the Schubert structure coefficients:

THEOREM 3.1. — $D_{\lambda,\mu}^\nu = C_{\lambda,\mu}^\nu$.

The **Eqrect** method of computing $D_{\lambda,\mu}^\nu$ generates each monomial of this polynomial (as expressed in the variables β_i) separately. This is somewhat different than other rules for these polynomials, which express the answer (as the **apwt** computation does) by combining many of these monomials into one.

Our proof follows the same general strategy used in [8]. However the technical details are, naturally, significantly different. Although we can state the rule $C_{\lambda,\mu}^\nu = \sum_T \text{apwt}(T)$ without development of **Eqjdt**, our proof relies on this construction.

PROPOSITION 3.2. — $D_{\lambda,\mu}^\lambda = C_{\lambda,\mu}^\lambda$.

We delay the proof of the above proposition until after the proof of Theorem 3.1.

For completeness, we restate and prove the following recurrence from [13, Proposition 3.4] and also observed by A. Okounkov; see also [8, Proposition 2].

LEMMA 3.3. — *We have*

$$(3.1) \quad \sum_{\lambda^+} C_{\lambda^+,\mu}^\nu = C_{\lambda,\mu}^\nu \cdot \text{wt}(\nu/\lambda) + \sum_{\nu^-} C_{\lambda,\mu}^{\nu^-}$$

where

- λ^+ is obtained by adding an outer corner to λ ;

- ν^- is obtained by removing an outer corner of ν ; and
- $\mathbf{wt}(\nu/\lambda) = \sum_{x \in \nu/\lambda} \beta(x)$.

Proof. — The equivariant Pieri rule states

$$(3.2) \quad \sigma_{(1)} \cdot \sigma_\lambda = \sum_{\lambda^+} \sigma_{\lambda^+} + \mathbf{wt}(\lambda)\sigma_\lambda \in H_T(X).$$

Equation (3.2) is proved in [8, Proposition 2]. To repeat the argument, it follows from the classical Pieri rule combined with the localization computation $C_{\lambda,(1)}^\lambda = \mathbf{wt}(\lambda)$; this localization computation is easily recovered from the earlier results discussed in Section 3.1. Hence

$$\begin{aligned} \sigma_{(1)} \cdot (\sigma_\lambda \cdot \sigma_\mu) &= \sigma_{(1)} \cdot \left(\sum_\nu C_{\lambda,\mu}^\nu \sigma_\nu \right) = \sum_\nu C_{\lambda,\mu}^\nu \sigma_{(1)} \cdot \sigma_\nu \\ &= \sum_\nu C_{\lambda,\mu}^\nu \mathbf{wt}(\nu)\sigma_\nu + \sum_\nu C_{\lambda,\mu}^\nu \sum_{\nu^+} \sigma_{\nu^+}. \end{aligned}$$

Also,

$$\begin{aligned} (\sigma_{(1)} \cdot \sigma_\lambda) \cdot \sigma_\mu &= \left(\mathbf{wt}(\lambda)\sigma_\lambda + \sum_{\lambda^+} \sigma_{\lambda^+} \right) \cdot \sigma_\mu \\ (3.3) \quad &= \mathbf{wt}(\lambda)\sigma_\lambda \cdot \sigma_\mu + \sum_{\lambda^+} \sigma_{\lambda^+} \cdot \sigma_\mu \\ &= \mathbf{wt}(\lambda) \sum_\nu C_{\lambda,\mu}^\nu \sigma_\nu + \sum_{\lambda^+} \sum_\nu C_{\lambda^+,\mu}^\nu \sigma_\nu. \end{aligned}$$

Now, $\sigma_{(1)} \cdot (\sigma_\lambda \cdot \sigma_\mu) = (\sigma_{(1)} \cdot \sigma_\lambda) \cdot \sigma_\mu$ since H_T is an associative ring. Thus taking the coefficient of σ_ν on both sides of this identity gives the conclusion. □

Proof of Theorem 3.1. — Suppose that $\{D_{\lambda,\mu}^\nu\}$ satisfies

$$(3.4) \quad \sum_{\lambda^+} D_{\lambda^+,\mu}^\nu = D_{\lambda,\mu}^\nu \cdot \mathbf{wt}(\nu/\lambda) + \sum_{\nu^-} D_{\lambda,\mu}^{\nu^-}$$

and we have established Proposition 3.2 (as done in Section 3.1). Then, by induction on $|\nu| - |\lambda| \geq 0$, the recurrence (3.4) together with the initial condition $D_{\lambda,\mu}^\lambda = C_{\lambda,\mu}^\lambda$ uniquely determine $D_{\lambda,\mu}^\nu$; cf. [8, Corollary 1]. Hence, by Lemma 3.3 it follows that $D_{\lambda,\mu}^\nu = C_{\lambda,\mu}^\nu$. This would complete the proof of the theorem.

Hence it remains to show that the polynomials $\{D_{\lambda,\mu}^\nu\}$ satisfy (3.4). Let $\mathcal{D}_{\lambda,\mu}^\nu$ denote the set of witnessing lattice and semistandard tableaux that rectify to S_μ . Fix λ^+ and consider $T \in \mathcal{D}_{\lambda^+,\mu}^\nu$. Let $x = \lambda^+/\lambda$ and consider the tableaux $\{S : [S] \mathbf{Eqjdt}_x(T) \neq 0\}$. Among these S , exactly one is of shape ν^-/λ (for some ν^-). For this S we have $\omega_S = 1$ and $S \in \mathcal{D}_{\lambda,\mu}^{\nu^-}$. The

other S appearing in the formal sum arise from an expansion of an edge label into a box y in ν/λ and $\omega_S = \beta(y)$; also $S \in \mathcal{D}'_{\lambda,\mu}$. By construction, no other kinds of tableaux can appear. (In this paragraph, we have tacitly used Proposition 2.2.)

It remains to show that:

- (a) Given $W \in \mathcal{D}'_{\lambda,\mu}$ there is a unique λ^+ and a unique $T \in \mathcal{D}'_{\lambda^+,\mu}$ such that

$$[W] \text{Eqjdt}_x(T) = 1.$$

- (b) Given $W \in \mathcal{D}'_{\lambda,\mu}$ and a box $\mathbf{b} \in \nu/\lambda$ there is a unique λ^+ and a unique $T \in \mathcal{D}'_{\lambda^+,\mu}$ such that

$$[W] \text{Eqjdt}_x(T) = \beta(\mathbf{b}).$$

In order to prove (a) and (b), we need to develop a notion of reverse Eqjdt . In (a), we wish to argue that from W and the box $\mathbf{b} = \nu/\nu^-$ there is a unique sequence of tableaux

$$(3.5) \quad T = U^{(-N)} \mapsto \dots \mapsto U^{(-1)} \mapsto U^{(0)} = W,$$

(for some N) where each $U^{(-j)}$ is a good and lattice tableau. Moreover, $U^{(-j)} \mapsto U^{(-j+1)}$ means $U^{(-j+1)}$ is obtained from $U^{(-j)}$ by one of the swaps (I)–(IV) into the box of $U^{(-j)}$ containing the \bullet . In (b) we wish to make the same argument, except that $U^{(0)}$ is obtained from W by moving the label in \mathbf{b} to the lower edge of \mathbf{b} , and a \bullet is placed in \mathbf{b} .

Now, (a) and (b) follow from three claims.

CLAIM 3.4. — Suppose $U = U^{(-i)}$ is a really good and lattice tableau with \bullet in box \mathbf{b} and locally near \mathbf{b} we label the boxes as

$$U = \begin{array}{|c|c|} \hline \cdots & \mathbf{a} \\ \hline \mathbf{c} & \mathbf{b} \\ \hline \end{array}.$$

If box \mathbf{a} or box \mathbf{c} has a label, or if the upper edge of \mathbf{b} has a label, then there exists a unique good and lattice tableau V with \bullet in box $\mathbf{d} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ such that $V \rightarrow U$, using one of the swaps (I)–(IV).

Proof of Claim 3.4. — There are two main cases, depending on whether the upper edge of \mathbf{b} is empty or not.

Case 1: Locally U looks like

$$\begin{array}{|c|c|c|} \hline z & y & w \\ \hline x & \bullet & q \\ \hline \end{array},$$

where the upper edge of the box b containing \bullet is empty, but other edges are possibly nonempty. —

Subcase 1a: $x \leq y$ or x does not exist. — Since U is good, we have $z < x \leq y \leq w < q$. If

$$V = \begin{array}{|c|c|c|} \hline z & \bullet & w \\ \hline x & y & q \\ \hline \end{array}$$

then V is (really) good and also lattice since U is lattice. Moreover, since $y \leq w$ then we can apply the vertical swap (I) to give U . Hence it remains to show that there are no other possible choices of V .

Clearly a expansion swap (II) could not result in U since we assume the edge immediately above the \bullet in U is empty. Also, swaps (III) and (IV) are not possible if x does not exist. Thus, we assume x exists.

If resuscitation (III) results in U then the box with x in U had a \bullet in V , and the $u = x$ is on the top edge of this box in V . But $y \geq x$ implies V is not semistandard in the second column.

Finally, if a horizontal swap (IV) resulted in U , then

$$V = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline \bullet & \tau & q \\ \hline \end{array}$$

Y

where $x \in \{\tau\} \cup Y$. However, since $x \leq y$, we have a violation of vertical semistandardness in the second column of V . Hence, (IV) could not have used either.

Subcase 1b: $x > y$, or y does not exist. — If y does not exist then clearly the vertical swap (I) did not result in U . If y exists then the same is true since we would have

$$V = \begin{array}{|c|c|c|} \hline z & \bullet & w \\ \hline x & y & q \\ \hline \end{array}$$

but since $x > y$ then we obtain a violation of semistandardness in the second row.

As in subcase 1a, the expansion swap (II) cannot produce U since we have assumed that the edge directly above the \bullet in U is empty.

Resuscitation (III) can happen if

$$V = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline x & x & q \\ \hline \bullet & & \end{array}$$

and x is the least label in the edge below the \bullet in U . Note V is good and lattice since U has these properties. Clearly, there is at most one way to reverse using (III).

On the other hand, if a reversal using (III) is not possible, then we aim to construct a horizontal swap $V' \mapsto U$ where

$$V' = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline \bullet & \tau & q \\ \hline & A & \\ \hline \end{array},$$

$x \in \{\tau\} \cup A$, and in the notation of swap (IV) we have $\mathbf{m} = x$. More precisely, suppose one can find a set of labels $\tau = x - d, x - d + 1, \dots, x - 1, x$ (for some $d \geq 0$) where $x - d, x - d + 1, \dots, x - 1$ are labels in the edge above the box containing x in U and

$$\mathcal{N}_{\text{col } 1, x-i}^U = \mathcal{N}_{\text{col } 1, x}^U$$

for $1 \leq i \leq d$. Further suppose if those labels are moved where A is (and combined with labels already on that edge in U) then V' is good. In this case, take d to be maximal among all choices satisfying these conditions and define A and thus V' in this manner.

SUBCLAIM 3.5. — *If $d \geq 0$ exists then V' is lattice.*

Proof. — U is lattice (by the induction hypothesis) and only two columns of U change to construct V' . Thus, if V' is not lattice, the failure of latticeness can be blamed on one of these two columns. It cannot be the first column of the local picture of V' since we moved labels rightward and thus $\mathcal{N}_{\text{col } 1, t}^U = \mathcal{N}_{\text{col } 1, t}^{V'}$ for any label t . If there is a problem in the second column, it would have to be that $\mathcal{N}_{\text{col } 2, x-d}^{V'} > \mathcal{N}_{\text{col } 2, x-d-1}^{V'}$, so assume this holds. Since U is lattice we have $\mathcal{N}_{\text{col } 1, x-d-1}^U \geq \mathcal{N}_{\text{col } 1, x-d}^U$. In combination with our assumption, it must be that $\mathcal{N}_{\text{col } 1, x-d-1}^U = \mathcal{N}_{\text{col } 1, x-d}^U$, and $x - d - 1$ appears in column 1 of U . It must appear either in the edge above x or in the box above it. We also note that $x - d - 1$ cannot appear in column 2 of U , since if it did, we would have $\mathcal{N}_{\text{col } 3, x-d-1}^U < \mathcal{N}_{\text{col } 3, x-d}^U$, contrary to the assumption that U is lattice.

Suppose first that $x - d - 1$ appears in the first column of U in the box above x . That is to say, using our labelling of entries of U defined above, that $z = x - d - 1$. Now consider the value y . Since we have assumed that V' is good, we must have $y < x - d$, and semistandardness requires $y \geq z = x - d - 1$. So $y = x - d - 1$, but that contradicts our argument above that $x - d - 1$ does not appear in column 2 of U .

Now suppose that $x - d - 1$ appears on the edge above x . Since we know that $x - d - 1$ does not appear in column 2, we could have chosen $\tau = x - d - 1$ rather than $\tau = x - d$, which contradicts the fact that d was chosen to be maximal.

We have found a contradiction based on our assumption that V' was not lattice, so it must be that V' is lattice. □

SUBCLAIM 3.6. — *Suppose \tilde{V}' is good, lattice and $\tilde{V}' \mapsto U$ is obtained by swap (IV). Then \tilde{V}' is unique (and hence $\tilde{V}' = V'$ as just constructed above).*

Proof. — The only question is whether in our given construction of V' we can instead use $0 \leq d' < d$ in place of d . That is, we construct \tilde{V}' by moving fewer labels right than we could have, i.e., we move $\tau = x - d', x - d' + 1, \dots, x - 1, x = z$. If we do this then note that \tilde{V}' is not lattice since

$$\mathcal{N}_{\text{col } 2, x-d'}^{\tilde{V}'} = \mathcal{N}_{\text{col } 1, x-d'}^U = \mathcal{N}_{\text{col } 1, x-d'-1}^U = \mathcal{N}_{\text{col } 2, x-d'-1}^{\tilde{V}'} + 1.$$

(The first equality holds since there is no $x - d'$ in column 2 of U .) Hence we find $\mathcal{N}_{\text{col } 2, x-d'}^{\tilde{V}'} > \mathcal{N}_{\text{col } 2, x-d'-1}^{\tilde{V}'}$, so \tilde{V}' is not lattice. □

SUBCLAIM 3.7. — *If V and V' are good and lattice then they cannot both result in U , using swaps (III) and (IV) respectively.*

Proof. — If (III) could be applied to V to give U then

$$U = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline x & \bullet & q \\ \hline & x & \\ \hline \end{array}$$

where the edge label x is the least label on its edge. However, then V' is ruled out since we must have two x 's in the second column of V' , a contradiction. □

SUBCLAIM 3.8. — *One can actually reverse from U using either (III) or (IV).*

Proof. — Let γ be the smallest label on the edge directly below the \bullet in U . It satisfies $x \leq \gamma$ (since U is good). If $\gamma = x$ we saw (III) is applicable: $V \mapsto U$ where V is good and lattice. If $\gamma > x (> y)$ then since $x \leq q$ (since U is really good) the construction of V' can be achieved, and we saw V' is good and lattice, as desired. □

Case 2: Suppose

$$U = \begin{array}{|c|c|c|} \hline d & e & f \\ \hline x & y & t \\ \hline & \bullet & \\ \hline \end{array}$$

where y is the largest label in its edge. —

Subcase 2a: $x \leq y$. — Clearly a vertical swap (I) could not have produced U . If a resuscitation swap (III) produced U then V looks locally like

$$V = \begin{array}{|c|c|} \hline x & y \\ \hline \bullet & x \\ \hline \end{array}$$

where semistandardness requires $y < x$. This contradicts the assumption of this subcase. On the other hand, if a horizontal swap (IV) produced U then

$$V = \begin{array}{|c|c|c|} \hline d & e & f \\ \hline \bullet & \tau & t \\ \hline & A & \\ \hline \end{array}$$

where $x \in \{\tau\} \cup A$, which by vertical semistandardness implies that $x > y$, which is again a contradiction.

Finally, consider

$$(3.6) \quad V = \begin{array}{|c|c|c|} \hline d & e & f \\ \hline x & \bullet & t \\ \hline & y & \\ \hline \end{array}$$

where y is the least label on its edge (the other labels being those on the same edge of U .) Clearly V is good and lattice (since we assume $x \leq y$ and U is good and lattice) and an expansion swap (II) produces U .

Subcase 2b: $x > y$. — Clearly U did not arise from a vertical swap (I). Next, suppose an expansion swap (II) produced U . Then V is of the form (3.6), where y is the least element on its edge. But $y < x$, so V is not good.

A resuscitation swap (III) can produce U if

$$V = \begin{array}{|c|c|c|} \hline d & e & f \\ \hline x & y & \\ \hline \bullet & x & t \\ \hline \end{array} \mapsto U = \begin{array}{|c|c|c|} \hline d & e & f \\ \hline x & y & \\ \hline x & \bullet & t \\ \hline & x & \\ \hline \end{array}$$

Suppose the resuscitation swap (III) is not possible starting with V . We need to construct a unique

$$V' = \begin{array}{|c|c|c|} \hline d & e & f \\ \hline \bullet & y & \\ \hline & x & t \\ \hline & Y & \\ \hline \end{array}$$

such that V' is good and lattice, and $V' \mapsto U$ using (IV). The arguments are exactly the same as in subcase 1b.

We have now completed our proof of Claim 3.4. □

CLAIM 3.9. — *In the process of reversing from W , if we arrive at a tableau $U = U^{(-i)}$ that is nearly bad, then the forward step $U \mapsto U^* = U^{(-i+1)}$ was a horizontal swap.*

Proof. — By assumption, locally we have

$$U = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline x & \bullet & q \\ \hline j & k & m \\ \hline \end{array},$$

where $x > q$. We show that $U \mapsto U^*$ could not be swaps (I), (II) and (III).

Suppose $U \mapsto U^*$ is swap (I). Then $k \leq q$. But then U^* is not good since $x > k$ and x and k are adjacent in U^* ; this is a contradiction. Similarly, we could not have used swap (II). Finally, if swap (III) was used, then $q = u$ where u is the largest label in the upper edge of the box in U with the \bullet . But $x > q = u$ means that, again, U^* would not be good. \square

Claim 3.4 tells us how to reverse from W until we arrive at a nearly bad tableau U . Claim 3.9 says that we can only arrive at a nearly bad tableau by (reversing) a horizontal swap (IV). The remaining claim below explains how to reverse from a nearly bad tableau:

CLAIM 3.10. — *Suppose we are in the process (3.5) of reversing from W and we arrive at a nearly bad $U = U^{(-i)}$. Then there is a good and lattice tableau V such that $V \mapsto U$ is a swap (IV). If V is nearly bad, the defect occurs in the same row as the defect of U , but one square to the left.*

Proof. — By Claim 3.9 we may suppose $U \mapsto U^* = U^{(-i+1)}$, where the local pictures are

$$U = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline A & T & \\ \hline x & \bullet & q \\ \hline B & C & Y \\ \hline \end{array} \qquad U^* = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline A & W & \\ \hline x & f & \bullet \\ \hline B & C & Y' \\ \hline \end{array}$$

and $x > q$ (since U is nearly bad).

We need to show that we can take some of the labels of A and move them right so as to construct

$$V = \begin{array}{|c|c|c|} \hline z & y & w \\ \hline A' & T & \\ \hline \bullet & \tau & q \\ \hline B & C' & Y \\ \hline \end{array}$$

where all the conditions on being good (but possibly nearly bad) are met, and $V \mapsto U$ using (IV).

We have $x \leq f < \min C$ (since U^* is good). Also, $x > q$ and $U \mapsto U^*$ occurs, so $x > q > \max T$. Hence x can be placed into C 's edge and maintain vertical semistandardness in that column. Note that $\mathfrak{r} = x$ is not possible since then V is bad. Let $A = \{a_k < a_{k-1} < \dots < a_1\}$ where $a_1 < x$ (by column semistandardness). We need to show there exists $j \geq 1$ satisfying the following conditions:

- $a_j, a_{j-1}, \dots, a_1, x$ forms an interval,
- $\mathcal{N}_{\text{col } 1, a_j}^U = \mathcal{N}_{\text{col } 1, x}^U$,
- a_j is strictly larger than the maximum entry of T (or y , if T is empty).

Then choose j to be maximal subject to those conditions. We want to establish that $a_j \leq q$ so that we can set $\mathfrak{r} = a_j$ and $C' = C \cup \{a_{j-1}, \dots, a_1, x\}$, and have V be good.

Now, since $U \mapsto U^*$ using swap (IV) we know $q + 1, q + 2, \dots, f - 1, f \in Y$. Moreover, by the prerequisite (IV) (ii) we have $\mathcal{N}_{\text{col } 2, i}^{U^*} = \mathcal{N}_{\text{col } 2, q}^{U^*}$ for $q \leq i \leq f$. Using this, together with the fact that $q < x \leq f$, and the fact that the first column of U and U^* are the same, we deduce that there exists an $x - 1$ in column 1 of U : otherwise we find that $\mathcal{N}_{\text{col } 1, x-1}^{U^*} < \mathcal{N}_{\text{col } 1, x}^{U^*}$ so that U^* is not lattice (contradicting our induction hypothesis). If $x - 1 \notin A$ it must be z . But then $y \geq x - 1$ which contradicts that $U \mapsto U^*$ is possible. Hence $x - 1 \in A$. Continuing this same reasoning implies $x - 2, x - 3, \dots, q + 1, q \in A$. It then follows that $a_j \leq q$, so V is good.

We now check that $V \mapsto U$. The only concern is if $x + 1 \in C'$, so that $x + 1$ might also move left when we apply the horizontal swap (IV), so that we do not arrive at U after all. However, if this were true then $\mathcal{N}_{\text{col } 2, x}^V = \mathcal{N}_{\text{col } 2, x+1}^V$. This would imply that $\mathcal{N}_{\text{col } 2, x}^U < \mathcal{N}_{\text{col } 2, x+1}^U$, violating the lattice property of U .

It remains to check that V is lattice. Recall U is lattice (by the induction hypothesis) and V and U agree except in two columns. Since we are moving labels to the right from column 1 of U into column 2, if V is not lattice we have $\mathcal{N}_{\text{col } 2, a_j}^V > \mathcal{N}_{\text{col } 2, a_j-1}^V$.

In order for this to happen, we must have an $a_j - 1$ in column 1 of U . Further, there must be no $a_j - 1$ in column 2 of U , since otherwise $\mathcal{N}_{\text{col } 2, a_j-1}^U > \mathcal{N}_{\text{col } 2, a_j}^U$, and U is not lattice, contrary to our assumption.

Hence, it must be true that $\mathcal{N}_{\text{col } 1, a_j}^U = \mathcal{N}_{\text{col } 1, a_j-1}^U$. Moreover, in fact $a_j - 1 \in A$: Otherwise in U , $z = a_j - 1$. Since $y \neq a_j - 1$, by U 's goodness, $y \geq a_j$ implying $V \mapsto U$ is impossible, and thus violating the definition of

a_j . Therefore we should also have moved $a_j - 1 (= a_{j+1})$ in our construction of V . This contradicts the maximality of j .

Summarizing, V is good, but possibly nearly bad: It might be that \mathfrak{r} is strictly smaller than the first numerical label to its left (if it exists). However, in this case, the near badness has moved one square left, as claimed. □

Conclusion of the proof of the Theorem 3.1. — First suppose we are considering the case (b) and our initial tableau $U^{(0)}$ that we are reversing from is obtained from W by pushing the label in box \mathfrak{b} to its lower edge. Then $U^{(0)}$ is really good and lattice. So we are in the situation of Claim 3.4 and can take a first step in the reversal process (3.5). If this reversal results after some steps in a nearly bad tableau, then we can utilize Claim 3.9 and Claim 3.10. At each step we obtain a good tableau with strictly fewer labels northwest of the \bullet . Thus, by induction, we eventually arrive at the situation that the \bullet has no labels northwest of it. This happens when \bullet arrives at an outer corner of λ . Call the final tableau T of shape λ^+ . Then T is good (thus semistandard) and lattice. Moreover, the final position of \bullet and T itself was uniquely determined from $U^{(0)}$. This completes the proof for (b). The argument for (a) is the same, except we start with $U^{(0)} = W$. □

3.1. Proof of Proposition 3.2

We now show that $C_{\lambda,\mu}^\lambda = D_{\lambda,\mu}^\lambda$, a fact we needed in the above proof of Theorem 3.1.

For $\lambda \subseteq \Lambda = k \times (n - k)$, the Grassmannian permutation associated to λ is the permutation $\pi(\lambda) \in S_n$ uniquely defined by $\pi(\lambda)_i = i + \lambda_{k-i+1}$ for $1 \leq i \leq k$ and which has at most one descent, which (if it exists) appears at position k .

Let $w', v' \in S_n$ be the Grassmannian permutations for the conjugate shapes $\lambda', \mu' \subseteq (n - k) \times k$. The following identity relates $C_{\lambda,\mu}^\lambda$ to the localization at e_μ of the class σ_λ , as expressed in terms of a specialization of the double Schubert polynomial. It is well known to experts; it can be proved (in the conventions we use) by, e.g., combining [8, Lemma 4] and [18, Theorem 4.5]:

$$C_{\lambda,\mu}^\lambda(Gr_k(\mathbb{C}^n)) = \overline{\mathfrak{S}_{v'}(t_{w'(1)}, \dots, t_{w'(n)}; t_1, \dots, t_n)}.$$

Here $\overline{p(t_1, \dots, t_n)}$ is the polynomial obtained from $p(t_1, \dots, t_n)$ under the substitution $t_j \mapsto t_{n-j+1}$. We refer the reader to [11] for background about

Schubert polynomials; however, we will only use a subset of the theory, which we describe now.

Since v' is Grassmannian, we have

$$\mathfrak{S}_{v'}(X; Y) = \sum_T \text{SSYTwt}(T)$$

where the sum is over all (ordinary) semistandard Young tableau T of shape μ' with entries bounded above by $n - k$. Here

$$\text{SSYTwt}(T) = \prod_{b \in \mu'} (x_{\text{val}(b)} - y_{\text{val}(b)+j(b)})$$

where $j(b) = \text{col}(b) - \text{row}(b)$. This formula is well-known (see, e.g., a more general form in [7, Theorem 5.8]).

The Schubert polynomial $\mathfrak{S}_{v'}$ for a Grassmannian permutation v' can also be identified as the *factorial Schur function* $s_{\mu'}$ (cf. [2, 4, 10]): One has (see, e.g., [9, Section 2]), after (re)conjugating the shapes, that if we take $\lambda, \mu \subseteq \Lambda$ then $s_\lambda \cdot s_\mu = \sum_{\nu \subseteq \Lambda} C_{\lambda, \mu}^\nu(t_j \mapsto -y_j)s_\nu$. We will not need this identification.

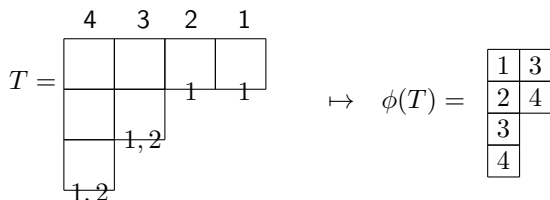
Let $\text{SSYTeqwt}(T)$ be the result of the substitution $x_j \mapsto t_{w'(j)}, y_j \mapsto t_j$. Define \mathcal{A} to be the set of semistandard and lattice tableaux T of shape λ/λ and content μ such that $\text{apwt}(T) \neq 0$. Define \mathcal{B} to be the set of semistandard tableaux U of shape μ' where $\text{SSYTeqwt}(U) \neq 0$.

It remains to prove the following:

CLAIM 3.11. — *There is a weight-preserving bijection $\phi : \mathcal{A} \rightarrow \mathcal{B}$ where if $T \in \mathcal{A}$ then $\text{apwt}(T) = \overline{\text{SSYTeqwt}(\phi(T))}$.*

Proof. — Define ϕ as follows. Label the columns of $\Lambda = k \times (n - k)$ by $(n - k), (n - k) - 1, \dots, 3, 2, 1$ from left to right. Given T , let $\text{col}(T)$ be the word $c_1 c_2 \dots c_{|\mu|}$ obtained by recording the column indices of the 1's (from left to right), 2's (from left to right) etc. Now let $\phi(T)$ be obtained by placing this word into the boxes of shape μ' from bottom to top along columns, and from left to right (noting there are μ_i labels i in T for each i). We have a candidate inverse map $\phi^{-1} : \mathcal{B} \rightarrow \mathcal{A}$ obtained by reading $U \in \mathcal{B}$ in the same way and placing edge labels on the bottom edge of λ/λ : the placement of the i 's is determined by the labels in column i of U .

Example 3.12. — Let $n = 7, k = 3, \lambda = (4, 2, 1)$ and $\mu = (4, 2)$. Then T , together with the column labels $1, \dots, 4$ and $\phi(T)$ are depicted below:



Here we had $\text{col}(T) = 432143$.

We compute

$$\text{apwt}(T) = (t_1 - t_7)(t_3 - t_7)(t_5 - t_7)(t_6 - t_7)(t_1 - t_4)(t_3 - t_4),$$

where the first four factors correspond to the labels 1 of T read from left to right and the last two factors correspond to the labels 2 of T read from left to right. Now,

$$\text{SSYTwt}(\phi(T)) = (x_4 - y_1)(x_3 - y_1)(x_2 - y_1)(x_1 - y_1)(x_4 - y_4)(x_3 - y_4),$$

where the factors correspond to the entries of $\phi(T)$ as read up columns from left to right (i.e., consistent with the order of factors of $\text{apwt}(T)$ above).

Since $\lambda' = (3, 2, 1, 1)$ and $\mu' = (2, 2, 1, 1)$ we have $w' = 2357146$ and $v' = 2356147$ (one line notation). So substituting, we get

$$\text{SSYTEqwt}(\phi(T)) = (t_7 - t_1)(t_5 - t_1)(t_3 - t_1)(t_2 - t_1)(t_7 - t_4)(t_5 - t_4).$$

Finally, the reader can check $\overline{\text{SSYTEqwt}(T)} = \text{apwt}(T)$, in agreement with the Claim.

ϕ^{-1} is well-defined and is weight-preserving. — Let $U \in \mathcal{B}$. Since $\phi^{-1}(U)$ is of shape λ/λ , it is vacuously standard. The fact that U is semistandard easily implies that $\phi^{-1}(U)$ is lattice.

We check that the weight assigned to a label ℓ in box \mathbf{b} and column $c = \text{col}(\mathbf{b})$ of U is the same as the apfactor assigned to the corresponding label c in $\phi^{-1}(U)$. The label ℓ gets assigned the weight $\text{SSYTEqfactor} = t_{\lambda'_{(n-k)-\ell+1} + \ell} - t_{\ell+j(\mathbf{b})}$. Hence we must show the equality of these two quantities:

$$\overline{\text{SSYTEqfactor}(\ell)} = t_{n - (\lambda'_{(n-k)-\ell+1} + \ell) + 1} - t_{n - (\ell+j(\mathbf{b})) + 1}, \quad \text{and}$$

$$\text{apfactor}(c) = t_{\text{Man}(x)}$$

$$- t_{\text{Man}(x) + r - c + 1 + \# \text{ of } c\text{'s strictly to the right of } x},$$

where here x is the bottom edge of λ in column ℓ from the right edge of Λ and $r = \lambda'_{(n-k)-\ell+1}$.

Now, counting the number of columns and rows which separate x from the bottom-left corner of Λ , we have

$$\text{Man}(x) = ((n - k) - \ell) + (k - \lambda'_{(n-k)-\ell+1} + 1) = n - (\lambda'_{(n-k)-\ell+1} + \ell) + 1.$$

Thus, the first term of $\overline{\text{SSYTeqfactor}}(\ell)$ and $\text{apfactor}(c)$ agree. To compare the second terms note that

$$\begin{aligned} \text{Man}(x) + r - c + 1 + \# \text{ of } c\text{'s strictly to the right of } x &= [n - (\lambda'_{(n-k)-\ell+1} + \ell) + 1] + \lambda'_{(n-k)-\ell+1} - c + 1 \\ &\quad + \# \text{ of } c\text{'s strictly to the right of } x \\ &= n - \ell + 1 - c + 1 + \# \text{ of } c\text{'s strictly to the right of } x. \end{aligned}$$

Hence it suffices to show

$$-j(\mathbf{b}) = -c + 1 + \# \text{ of } c\text{'s strictly to the right of } x,$$

or equivalently,

$$\text{row}(\mathbf{b}) - 1 = \# \text{ of } c\text{'s strictly to the right of } x.$$

However, this final equality is clear by the definition of ϕ^{-1} .

Thus $0 \neq \overline{\text{SSYTeqwt}}(U) = \text{apwt}(\phi^{-1}(U))$ and we are done.

ϕ is well-defined and weight-preserving. — Let $T \in \mathcal{A}$. By construction, $\phi(T)$ is strictly increasing along columns.

Now suppose $\phi(T)$ is not weakly increasing along rows. Thus there is a violation between columns $c + 1$ and c . We may suppose $c + 1$ is the leftmost column of Λ , recalling the reverse labelling of columns; the general argument is similar. Now suppose the violation occurs M rows from the top. Hence in T , the M -th label 1 (counting from the right) is in a column strictly to the left of the label M -th label 2. Then it must be true that T is not lattice.

Hence $\phi(T)$ is a semistandard tableau of shape μ' . The same computations which show that ϕ^{-1} is weight preserving also show that $0 \neq \text{apwt}(T) = \overline{\text{SSYTeqwt}}(\phi(T))$ and so the desired conclusions hold. \square

3.2. Proof of Theorem 1.2

Let \mathcal{C} be the set of lattice semistandard tableaux S of shape ν/λ whose content is μ and $\text{apwt}(S) \neq 0$. Also, let \mathcal{D} be the set of tableaux from Theorem 1.2. Define a map $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ as follows: given $S \in \mathcal{C}$ relabel the μ_1 labels 1 that appear by $1, 2, \dots, \mu_1$, from left to right; then relabel the

μ_2 (original) labels 2 by $\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2$, etc. This map is clearly reversible. Theorem 1.2 follows from:

PROPOSITION 3.13. — $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ is a weight preserving bijection: $\text{apwt}(S) = \text{wt}(\Phi(S))$.

Proof.

Φ is well-defined. — Since $S \in \mathcal{C}$ is semistandard, clearly $T = \Phi(S)$ is standard.

Let $T_\mu[i]$ be the set of labels in row i of T_μ . By construction, the labels of $T_\mu[i]$ form a horizontal strip in T . The following is an easy induction using the definition of $\text{Eqjdt}^<$:

CLAIM 3.14. — The labels of $T_\mu[i]$ form a horizontal strip in each tableau arising in the process of column rectifying T .

Translating the assumption that S is lattice, for any column c of T , the number of labels from $T_\mu[i]$ appearing in columns weakly to the right of column c weakly exceeds the number from $T_\mu[i + 1]$ in the same region, for any $i \geq 1$. Mildly abusing terminology, we say that T is also lattice.

CLAIM 3.15. — Each tableau appearing in the column rectification of T is lattice.

Proof. — Suppose that in the process of column rectification we arrive at a tableau U (which may have a \bullet in the middle of it) which is lattice and the next swap $U \mapsto U'$ breaks latticeness. Then this swap must locally look like

$$U = \begin{array}{|c|c|} \hline a & b \\ \hline \bullet & c \\ \hline d & e \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline a & b \\ \hline c & \bullet \\ \hline d & e \\ \hline \end{array} = U'$$

where $c \in T_\mu[i]$ moving left causes more labels of $T_\mu[i + 1]$ than of $T_\mu[i]$ to appear weakly right of column 2 of U' . So there must be a label ℓ of $T_\mu[i + 1]$ in column 2 of U (and of U'), since otherwise U is not lattice, a contradiction.

Suppose e does not exist. Then since U is standard, ℓ cannot exist, a contradiction. Hence we assume e and thus d exists. By standardness of U and Claim 3.14, $e(= \ell) \in T_\mu[i + 1]$. Notice that no label of column 1 of U can be in $T_\mu[i]$ since we would contradict Claim 3.14 (applied to U'). Now $d > c$ (since otherwise the swap would not have been used). So by standardness and Claim 3.14 (applied to U), $d \in T_\mu[i + 1]$. But then U was not lattice in column 1 to begin with. This is our final contradiction. \square

Write $T^{(k)}$ for the tableau that consists of the k rightmost columns of the column rectification of the k rightmost columns of T .

CLAIM 3.16. — *The i -th row of $T^{(k)}$ is a consecutive sequence of integers from $T_\mu[i]$, ending with $\mu_1 + \dots + \mu_i (= \max T_\mu[i])$.*

Proof. — The argument is by induction on $k \geq 0$. The base case $k = 0$ is trivial. Suppose after rectifying the $k - 1$ rightmost columns of T , $T^{(k-1)}$ has the claimed form. Now we are rectifying column k (from the right). Suppose we are $\text{Eqjdt}^<$ sliding into a square x in row R and the slide $\text{Eqjdt}_x^<$ is a horizontal one (i.e., a label moves left). Observe that in this case, the \bullet must only move right in the same row until the slide completes: otherwise, by the form of $T^{(k-1)}$, it must be that the rows R and $R + 1$ of $T^{(k)}$ are of the same length, and the rightmost label of row $R + 1$ moves up into row R ; however this contradicts Claim 3.15.

Suppose the labels in the column we are presently rectifying are $\ell_1 < \ell_2 < \dots < \ell_t$. Now $\ell_m \in T_\mu[i_m]$ where $i_1 < i_2 < \dots < i_t$. By the form of $T^{(k-1)}$, it is easy to see ℓ_m completes at row i_m . Now, by Claim 3.14 it follows that ℓ_m is the largest label of $T_\mu[i_m]$ that does not appear in $T^{(k-1)}$. This completes the induction step. □

Claim 3.16 immediately shows $\text{Eqrect}^<(T) = T_\mu$, as desired.

Φ^{-1} is well-defined. — Let $T \in \mathcal{D}$. Let $S = \Phi^{-1}(T)$; proving well definedness means we need to show S is semistandard, lattice and $\text{apwt}(S) \neq 0$ (the content of S being μ is by construction).

CLAIM 3.17. — *The labels $T_\mu[i]$ form a horizontal strip in T , as well as in each tableau T' in the column rectification of T .*

Proof. — Suppose j and $j + 1$ appear in the same row of T_μ . Then we claim that $j + 1$ is strictly east (and, by standardness of T , thus weakly north) of j in T (respectively, T'). Otherwise, if this is false, it remains false after each $\text{Eqjdt}^<$ step. This implies $\text{Eqrect}^<(T) \neq T_\mu$, a contradiction. □

Given Claim 3.17, the semistandardness of S is clear.

Next we argue that S is lattice. Otherwise, there is a column c and label i such that $\mathcal{N}_{\text{col } c, i+1}^S > \mathcal{N}_{\text{col } c, i}^S$. We may assume c is rightmost with this property. Hence T is not lattice.

CLAIM 3.18. — *Assuming (for the sake of contradiction) that T is not lattice, it follows that after every swap in the process that column rectifies T to T_μ , the resulting tableau is also not lattice.*

Proof. — Without loss of generality, it suffices to argue about the first swap applied to T . If the result T° is lattice then there is a label $\ell \in T_\mu[i+1]$ in column c of T that moved to the column $c - 1$. Locally, the swap looks like

$$\begin{array}{|c|c|} \hline a & b \\ \hline \bullet & \ell \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline a & b \\ \hline \ell & \bullet \\ \hline \end{array}.$$

By Claim 3.17, the labels of $T_\mu[i + 1]$ form a horizontal strip in T . Hence $a, b \notin T_\mu[i + 1]$. Also, no label in column c is in $T_\mu[i]$ since otherwise there is a violation of latticeness strictly to the right of column c that is not fixed by this swap. Now, some label m in column $c - 1$ is in $T_\mu[i]$ (since we have fixed non-latticeness by the swap). This m cannot be below the \bullet since $\ell > m$ so m would move into the \bullet instead of ℓ . Hence $a = m$. Now what about b ? We have excluded the possibility that $b \in T_\mu[i] \cup T_\mu[i + 1]$. However, by standardness of T , there are no other possibilities for b . This is a contradiction and T° is not lattice. \square

Thus, by Claim 3.18, T_μ is not lattice, a contradiction. Hence S is lattice.

Finally, in the weight preservation argument below, we see $\text{apwt}(S) = \text{wt}(T)$. Thus we have $\text{apwt}(S) \neq 0$ since by construction $\text{wt}(T) \neq 0$.

Φ and Φ^{-1} are weight preserving. — Suppose $T \in \mathcal{D}$ and we consider a label ℓ in that column which finishes in row i . Claim 3.16 (and its proof) shows that the labels to the right (and in the same row) of ℓ (once it completed rectifying in its column) are precisely those to its right in T_μ , and moreover than any edge label rises exactly to its row in T_μ (although it may move left in that row in subsequent column rectifications). Hence by the definition of apfactor , if ℓ' is the corresponding label in $S = \Phi^{-1}(T)$ then $\text{factor}(\ell) = \text{apfactor}(\ell')$. So $\text{wt}(T) = \text{apfactor}(\Phi^{-1}(T))$. Thus Φ^{-1} is weight-preserving. Reversing the argument shows Φ is weight preserving. \square

4. Conjectural extension to equivariant K -theory

The ring $K_T(\text{Gr}(k, \mathbb{C}^n))$ has a $K_T(\text{pt})$ -basis of equivariant K -theory classes σ_λ^K indexed by $\lambda \subseteq \Lambda$. Here $K_T(\text{pt}) := \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$ is the Laurent polynomial ring in t_1, \dots, t_n . Consequently, the equivariant K -theory Schubert structure coefficients are defined by the expansion

$$(4.1) \quad \sigma_\lambda^K \cdot \sigma_\mu^K = \sum_\nu K_{\lambda, \mu}^\nu \sigma_\nu^K,$$

where $K_{\lambda, \mu}^\nu \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$.

Earlier, a puzzle conjecture for these Laurent polynomials was given by A. Knutson and R. Vakil and reported in [3]. One aspect of their conjecture is that it does not specialize to K -theory puzzle rules (compare Sections 3 and 5 of [3] and see specifically the remarks of the fourth paragraph of the latter section). In contrast, our conjecture transparently recovers the *jeu de taquin* rules for K -theory, T -equivariant cohomology and ordinary cohomology, by “turning off” parts of our construction.

Recently, A. Knutson [6] obtained a puzzle rule for an equivariant K -theory problem different than the one considered here (or in the Knutson–Vakil puzzle conjecture).

4.1. Statement of the equivariant K -theory rule

To state our conjectural generalization of Theorem 1.2, we need to broaden the class of equivariant tableaux. The ideas contained below also generalize the notions concerning *increasing tableau* that we gave in our earlier paper [17], where a *jeu de taquin* rule for K -theory of Grassmannians was proved.

An *equivariant increasing tableau* is an equivariant filling of ν/λ by the labels $1, 2, \dots, \ell$ such that each label in a box is:

- strictly smaller than the label in the box immediately to its right;
- strictly smaller than the label in its southern edge, and the label in the box immediately below it; and
- strictly larger than the label in the northern edge.

Moreover, any subset of the boxes of ν/λ may be marked by a “ \star ”, subject to:

- if the labels i and $i + 1$ appear as box labels in the same row of T , then only the box containing $i + 1$ may be marked by a “ \star ”.

Let $\text{EqInc}(\nu/\lambda, \ell)$ denote the set of all equivariant increasing tableaux.

Example 4.1. — If $\nu/\lambda = (3, 2)/(2)$ and $\ell = 3$ the first two tableaux below are in $\text{EqInc}(\nu/\lambda, \ell)$ while the third is not:

		1					
1 \star	3 \star						
2							

		2					
1	2 \star						
2							

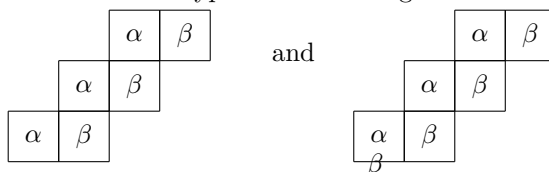
		2					
1 \star	2 \star						
2							

We also need an extension of the algorithms $\text{Eqjdt}^<$ and $\text{Eqrect}^<$ defined in Section 1.

A *short ribbon* R is a connected skew shape that does not contain a 2×2 subshape and where each row and column contains at most two boxes. An *alternating ribbon* is a filling of R by two symbols, say α and β such that

- adjacent boxes are filled differently;
- all edges except the southwest-most edge are empty; and
- if the southwest-most edge is filled, it is filled with a different symbol than the symbol in the box above it.

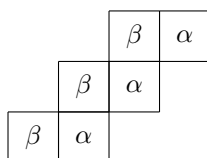
Example 4.2. — The two types of alternating ribbons are of the form:



(where in the tableau on the right, the edge label β is the smallest label on that edge).

We define $\text{switch}(R)$ to be the alternating ribbon of the same shape but where each box is instead filled with the other symbol. If the southmost edge was filled by one of these symbols, that symbol is deleted. If R is a ribbon consisting of a single box with only one symbol used, then switch does nothing to it. We also define switch to act on a skew shape consisting of multiple connected components, each of which is a alternating ribbon, by acting on each separately.

Example 4.3. — Applying switch to either of the alternating ribbons above gives



Given $T \in \text{EqInc}(\nu/\lambda, \ell)$, consider an *inner corner* $x \in \lambda$ which we label with a \bullet . Erase all \star 's appearing in T . Consider the alternating ribbon made of \bullet and 1. (It is allowed for the southmost edge of R_1 in T consists of the label 1 and other labels as well.) Apply switch to R_1 . Now let R_2 be the union of ribbons consisting of \bullet and 2, and proceed as before. Repeat this process until the \bullet 's have been switched past all the numerical labels in T ; the final placement of these labels gives $\text{KEqjdt}_x^<(T)$. Finally, define $\text{KEqrect}^<(T)$ by successively applying $\text{KEqjdt}^<$ in the column rectification order.

Example 4.4. — Erasing the \star 's in

$$T = \begin{array}{|c|c|c|} \hline & & 2 \\ \hline 1\star & 1 & \\ \hline 3 & 4 & \\ \hline \end{array} \quad \text{gives} \quad \begin{array}{|c|c|c|} \hline & & 2 \\ \hline 1 & 1 & \\ \hline 3 & 4 & \\ \hline \end{array} .$$

There is nothing to do to rectify the third column. Rectifying the second column is achieved in one step:

$$\begin{array}{|c|c|c|} \hline & \bullet & 2 \\ \hline & 1 & \\ \hline 1 & 4 & \\ \hline 3 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline & & \\ \hline 1 & 4 & \\ \hline 3 & & \\ \hline \end{array}$$

while rectifying the first column demands three steps:

$$\begin{array}{|c|c|c|} \hline \bullet & 1 & 2 \\ \hline 1 & 4 & \\ \hline 3 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & \bullet & 2 \\ \hline \bullet & 4 & \\ \hline 3 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 2 & \bullet \\ \hline \bullet & 4 & \\ \hline 3 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 2 & \bullet \\ \hline 3 & 4 & \\ \hline & & \\ \hline \end{array}$$

which gives the final tableau $T_{(2,2)}$.

While the definition of $\text{KEqrect}^<$ above does not depend on the markings of boxes of T by \star , these markings play a role in our modification of the equivariant weight $\text{wt}(T)$ defined in Section 1.3. We say that a label $\mathfrak{s} \in T$ is a *special label* if it is either

- an edge label; or
- lies in a box that has been marked by a \star .

To each special label \mathfrak{s} we associate a Laurent binomial $\text{factor}_K(\mathfrak{s})$: given a box x define a weight $\hat{\beta}(x) = t_m/t_{m+1}$ where m is the “Manhattan distance” as defined in Section 1. Note that at most one of the labels “ \mathfrak{s} ” or “ $\mathfrak{s}\star$ ” can appear in a column. Moreover, each step of the rectification moves an \mathfrak{s} at most one step north (and it remains in the same column). Therefore one can precisely say a special label \mathfrak{s} *passes* through a box x if it occupies it during the K -equivariant rectification of the column that \mathfrak{s} initially occupies *and* if \mathfrak{s} did not initially begin in x . (This notion of “pass” reduces to our original notion in Section 1.4 if \mathfrak{s} is an edge label.) Now, let x_1, \dots, x_s be the boxes passed through by \mathfrak{s} and y_1, \dots, y_t be the numerically labelled boxes in the same row as x_s and strictly to its right. Set

$$\text{factor}_K(\mathfrak{s}) = 1 - \prod_{i=1}^s \hat{\beta}(x_i) \prod_{j=1}^t \hat{\beta}(y_j).$$

We apply the convention that if any special label \mathfrak{s} does not move during the rectification of the column that it initially sits in, then $\mathbf{factor}_K(\mathfrak{s}) = 0$. Now set

$$\mathbf{wt}_K(T) = \prod_{\mathfrak{s}} \mathbf{factor}_K(\mathfrak{s}),$$

where the product is over all special labels \mathfrak{s} .

Example 4.5. — Assume that in Example 4.4, we are working inside $Gr(2, \mathbb{C}^5)$. There are three special labels:

- The edge label “1” in the second column gives $\mathbf{factor}(1) = 1 - \frac{t_3}{t_4} \cdot \frac{t_4}{t_5} = 1 - \frac{t_3}{t_5}$ since it passes through one box during the rectification of column 2, and ends in a row with a single labelled box $\boxed{2}$ to its right.
- The marked label “1 \star ” gives $\mathbf{factor}(1\star) = 1 - \frac{t_2}{t_3} \cdot \frac{t_3}{t_4} = 1 - \frac{t_2}{t_4}$ since it passes through one box and has one box $\boxed{2}$ to its right after rectifying column 1.
- The edge label “3” gives $\mathbf{factor}(3) = 1 - \frac{t_1}{t_2} \cdot \frac{t_2}{t_3} = 1 - \frac{t_1}{t_3}$ since it passes through one box and has one box $\boxed{4}$ to its right when the rectification of column 1 is complete.

Hence $\mathbf{wt}_K(T) = \left(1 - \frac{t_3}{t_5}\right) \left(1 - \frac{t_2}{t_4}\right) \left(1 - \frac{t_1}{t_3}\right)$.

Lastly, given T we define

$$\begin{aligned} \mathbf{sgn}(T) &= (-1)^{\#\star\text{'s in } T + \#\text{labels in } T - |\mu|} \\ (4.2) \quad &= (-1)^{\#\star\text{'s in } T + \#\text{edge labels in } T + |\nu| - |\lambda| - |\mu|} \end{aligned}$$

Example 4.6. — Continuing Example 4.4 we calculate that $\mathbf{sgn}(T) = (-1)^{1+2+5-2-4} = (-1)^2 = 1$.

CONJECTURE 4.7. — *The equivariant K -theory Schubert structure coefficient is*

$$K_{\lambda, \mu}^{\nu} = \sum_T \mathbf{sgn}(T) \cdot \mathbf{wt}_K(T)$$

where the sum is over all $T \in \mathbf{EqInc}(\nu/\lambda, |\mu|)$ such that $\mathbf{KEqrect}^{<}(T) = T_{\mu}$.

Conjecture 4.7 manifests the equivariant K -theory positivity proved (for all generalized flag varieties G/P) by [1]. Let $z_i := \frac{t_i}{t_{i+1}} - 1$. Note that for $j > i$,

$$(4.3) \quad 1 - \frac{t_i}{t_j} = -(z_i + 1) \dots (z_{j-1} + 1) + 1.$$

Thus

$$\begin{aligned} \text{sgn}(T) \cdot \text{wt}_K(T) &= (-1)^{|\nu| - |\lambda| - |\mu|} \\ &\quad \left((-1)^{\#\star\text{'s in } T + \#\text{edge labels in } T} \cdot \text{wt}_K(T) \right). \end{aligned}$$

Notice $\text{wt}_K(T)$ is a product of ($\#\star\text{'s in } T + \#\text{edge labels in } T$)-many factors of the form (4.3) and also $(z_i + 1) \dots (z_{j-1} + 1) - 1$ is manifestly positive in the variables $\{z_i\}$. Hence Conjecture 4.7 expresses $K_{\lambda, \mu}^\nu \cdot (-1)^{|\nu| - |\lambda| - |\mu|}$ as a manifestly positive polynomial in the variables $\{z_i\}$; this is the positivity of [1], after the substitution $z_i \mapsto e^{\beta_i} - 1$.

We have computer verified this conjecture for all $Gr(k, \mathbb{C}^n)$ for $n \leq 5$ as well as a number of cases for larger n .

5. Final remarks

We are attempting to extend ideas in this paper to prove Conjecture 4.7. Specifically, we desire an analogue of the results of Section 2. This would specialize to a “semistandard” version of the results of [17].

One can reformulate Theorem 1.2 to avoid edge labels. In this version, a bullet \bullet in a box can either be replaced by a label using a classical *jeu de taquin* slide or it can be replaced by a label not already present in the tableau, at the cost of the weight associated with the box containing the \bullet .

We mentioned that the equivariant cohomology of Grassmannians is controlled by multiplication of factorial Schur polynomials. A generalization of this was introduced by A. Molev and B. Sagan [13], geometrically relevant to “triple Schubert calculus”, see [8]. The ideas of Section 2 also generalize to provide a *jeu de taquin* rule for the Molev–Sagan coefficients.

Note added in proof. Conjecture 4.7 has since been proved by O. Pechenik and A. Yong in [14]

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