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## COMMUTATORS, LITTLE BMO AND WEAK FACTORIZATION

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ABSTRACT. — In this paper, we provide a direct and constructive proof of weak factorization of  $h^1(\mathbb{R} \times \mathbb{R})$  (the predual of little BMO space  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  studied by Cotlar–Sadosky and Ferguson–Sadosky), i.e., for every  $f \in h^1(\mathbb{R} \times \mathbb{R})$  there exist sequences  $\{\alpha_j^k\} \in \ell^1$  and functions  $g_j^k, h_j^k \in L^2(\mathbb{R}^2)$  such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \left( h_j^k H_1 H_2 g_j^k - g_j^k H_1 H_2 h_j^k \right)$$

in the sense of  $h^1(\mathbb{R} \times \mathbb{R})$ , where  $H_1$  and  $H_2$  are the Hilbert transforms on the first and second variable, respectively. Moreover, the norm  $\|f\|_{h^1(\mathbb{R} \times \mathbb{R})}$  is given in terms of  $\|g_j^k\|_{L^2(\mathbb{R}^2)}$  and  $\|h_j^k\|_{L^2(\mathbb{R}^2)}$ . By duality, this directly implies a lower bound on the norm of the commutator  $[b, H_1 H_2]$  in terms of  $\|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})}$ .

Our method bypasses the use of analyticity and the Fourier transform, and hence can be extended to the higher dimension case in an arbitrary  $n$ -parameter setting for the Riesz transforms.

RÉSUMÉ. — Dans ce papier, nous donnons une preuve directe et constructive de la factorisation faible de  $h^1(\mathbb{R} \times \mathbb{R})$  (le prédual de l'espace little BMO  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  étudié par Cotlar–Sadosky et Ferguson–Sadosky), i.e., pour chaque  $f \in h^1(\mathbb{R} \times \mathbb{R})$  il existe des suites  $\{\alpha_j^k\} \in \ell^1$  et des fonctions  $g_j^k, h_j^k \in L^2(\mathbb{R}^2)$  telles que

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \left( h_j^k H_1 H_2 g_j^k - g_j^k H_1 H_2 h_j^k \right)$$

au sens de  $h^1(\mathbb{R} \times \mathbb{R})$ , où  $H_1$  et  $H_2$  sont les transformées de Hilbert dans la première et la seconde variable, respectivement. De plus, la norme  $\|f\|_{h^1(\mathbb{R} \times \mathbb{R})}$  est donnée

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en termes de  $\|g_j^k\|_{L^2(\mathbb{R}^2)}$  et  $\|h_j^k\|_{L^2(\mathbb{R}^2)}$ . Par dualité, ceci implique directement une borne inférieure de la norme du commutateur  $[b, H_1 H_2]$  en termes de  $\|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})}$ .

Notre méthode contourne l'utilisation de l'analyticité et de la transformée de Fourier, et peut donc être étendue en dimension supérieure dans le cadre de  $n$ -paramètres arbitraires, pour les transformées de Riesz.

### 1. Introduction and Statement of Main Results

As motivation for this paper we point to two fundamental results in complex analysis and harmonic analysis. An important result, obtained by Coifman, Rochberg, and Weiss in [3] shows that for the Hilbert transform  $Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{y-x} dy$  and the commutator between multiplication by  $b$  (i. e.,  $M_b f = bf$ ) and the Hilbert transform,  $[b, H] := M_b H - H M_b$ , that:

$$\|[b, H] : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\| \approx \sup_Q \left( \frac{1}{|Q|} \int_Q \left| b(x) - \frac{1}{|Q|} \int_Q b(y) dy \right|^2 dx \right)^{\frac{1}{2}},$$

where the supremum is taken over intervals  $Q$  in  $\mathbb{R}$  and the right-hand side is the well-known  $\text{BMO}(\mathbb{R})$  norm. To obtain this, they used methods of harmonic analysis that were general enough to work for certain Calderón–Zygmund operators, and in particular the Riesz transforms:  $R_j f(x) := c_n \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}} dy$ ,  $1 \leq j \leq n$ , and obtained:

$$\begin{aligned} \max_{1 \leq j \leq n} \|[b, R_j] : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)\| \\ \approx \sup_Q \left( \frac{1}{|Q|} \int_Q \left| b(x) - \frac{1}{|Q|} \int_Q b(y) dy \right|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

where the supremum is taken over cubes  $Q$  in  $\mathbb{R}^n$  and the right-hand side is the well-known  $\text{BMO}(\mathbb{R}^n)$  norm. Commutators play an important role in harmonic analysis, complex analysis, and partial differential equations (see for example [1, 2, 12]) and have a characterization of their boundedness in terms of the symbol  $b$  which is extremely useful.

Nehari studied and characterized the boundedness of Hankel operators in [15]. Recall that  $H^2(\mathbb{R}_+^2)$  is the space of functions that are analytic on the upper half-plane and have boundary values belonging to  $L^2(\mathbb{R})$ . Let  $\mathbb{P}_+ : L^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}_+^2)$  denote the orthogonal projection between these spaces and so we have that  $L^2(\mathbb{R}) = H_+^2(\mathbb{R}_+^2) \oplus H_-^2(\mathbb{R}_+^2)$  where  $H_{\pm}^2(\mathbb{R}_+^2)$  is supported on the positive/negative Fourier frequencies. Then define the Hankel operator  $h_b(f) := \mathbb{P}_-(bf)$  and Nehari's Theorem, stated in modern terminology, is then the relationship:  $\|h_b : H^2(\mathbb{R}_+^2) \rightarrow H_-^2(\mathbb{R}_+^2)\| \approx \|b\|_{\text{BMO}(\mathbb{R}_+^2)}$ .

There is a strong connection between the results of [3] and [15]. To see this recall that we have  $H = i\mathbb{P}_+ - i\mathbb{P}_-$  where  $\mathbb{P}_+$  and  $\mathbb{P}_-$  are the projections onto the positive and negative Fourier supports respectively. It is then a simple computation to show that:  $[b, H] = h_b - h_b^*$ . As the domains and ranges of the operators  $h_b$  and  $h_b^*$  are orthogonal, Nehari's Theorem and the characterization of commutators can then easily be deduced from one another.

Via  $H^1$ -BMO duality and some standard functional analysis it is direct to see that the commutator theorem can be translated to the following statement: For every  $f \in H^1(\mathbb{R})$ , the real Hardy space, there exist functions  $g_j, h_j \in L^2(\mathbb{R})$  so that  $f = \sum_{j=1}^\infty g_j H h_j + h_j H g_j$  in the sense of  $H^1(\mathbb{R})$  and

$$\|f\|_{H^1(\mathbb{R})} \approx \inf \left\{ \sum_{j=1}^\infty \|g_j\|_{L^2(\mathbb{R})} \|h_j\|_{L^2(\mathbb{R})} : f = \sum_{j=1}^\infty g_j H h_j + h_j H g_j \right\},$$

where the infimum is taken over all possible representations of  $f$  as above (for the definition of  $H^1(\mathbb{R})$  see [7]). In fact these factorization results and corresponding commutator results are always equivalent to each other. For more details about the classical Nehari Theorem and background, we refer to the note of Lacey [13] and the references therein.

Extensions of the commutator results and Nehari's Theorem have received lots of attention; in particular we focus on the extensions in the product setting for the little BMO space  $\text{bmo}(\mathbb{R} \times \mathbb{R})$ , introduced and studied by M. Cotlar and Sadosky [5] in connection with weighted norm inequalities for the product Hilbert transform. For this reason, the space  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  was originally defined in terms of the Hilbert transforms, one for each variable. The characterization of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  in terms of mean oscillation on rectangles was given later in [5]. For our purpose here, we take this characterization of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  as our starting point. Note that in [5] and [9], they stated the results on bidisc. Here we state the results on  $\mathbb{R} \times \mathbb{R}$  and study the real analysis approach. More precisely, a function  $b \in L^1_{loc}(\mathbb{R}^2)$  is in  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  if

$$(1.1) \quad \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})} := \sup_{R \subset \mathbb{R} \times \mathbb{R}} \frac{1}{|R|} \iint_R |b(x_1, x_2) - b_R| dx_1 dx_2 < \infty,$$

where

$$b_R := \frac{1}{|R|} \iint_R b(x_1, x_2) dx_1 dx_2$$

is the mean value of  $b$  over the rectangle  $R$ .

It is well known that  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  coincides with the space of integrable functions which are uniformly of bounded mean oscillation in each variable

separately [5]. Moreover, from Ferguson–Sadosky [9], we have the following equivalent characterizations for  $\text{bmo}(\mathbb{R} \times \mathbb{R})$ .

**THEOREM 1.1** ([9]). — *Let  $b \in L^1_{loc}(\mathbb{R}^2)$ . The following conditions are equivalent:*

- (i)  $b \in \text{bmo}(\mathbb{R} \times \mathbb{R})$ ;
- (ii) *The commutators  $[b, H_1]$  and  $[b, H_2]$  are both bounded on  $L^2(\mathbb{R}^2)$ ;*
- (iii) *The commutator  $[b, H_1 H_2]$  is bounded on  $L^2(\mathbb{R}^2)$ .*

We note that  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  can also be equivalently characterized by big Hankel operators and by certain Carleson measures. For the details, we refer to [9].

It was shown in [5] that the predual of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  coincides with  $H^1_{Re}(\mathbb{R}) \otimes L^1(\mathbb{R}) + H^1_{Re}(\mathbb{R}) \otimes L^1(\mathbb{R})$ . Based on the result in [5], Ferguson–Sadosky [9] obtained the weak factorization for  $H^1_{Re}(\mathbb{R}) \otimes L^1(\mathbb{R}) + H^1_{Re}(\mathbb{R}) \otimes L^1(\mathbb{R})$ .

The aim of this paper is to provide a direct and constructive proof for the weak factorization for predual of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$ , which implies the equivalence of (i) and (iii) in Theorem 1.1 directly and our result here bypasses the use of Fourier transform and hence can be extended to the higher dimension case in an arbitrary  $n$ -parameter setting for the Riesz transforms. To get this, we note that in [9], Ferguson–Sadosky also showed that the predual of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  can be characterised in terms of rectangular atoms.

**DEFINITION 1.2** ([9]). — *An atom on  $\mathbb{R} \times \mathbb{R}$  is a function  $a \in L^\infty(\mathbb{R}^2)$  supported on a rectangle  $R \subset \mathbb{R} \times \mathbb{R}$  with  $\|a\|_{L^\infty(\mathbb{R} \times \mathbb{R})} \leq |R|^{-1}$  and satisfying the cancellation property*

$$\int_{\mathbb{R}^2} a(x_1, x_2) dx_1 dx_2 = 0.$$

*Let  $\text{Atom}(\mathbb{R} \times \mathbb{R})$  denote the collection of all such atoms.*

**DEFINITION 1.3** ([9]). — *The atomic Hardy space  $h^1(\mathbb{R} \times \mathbb{R})$  is defined as the set of functions of the form*

$$(1.2) \quad f = \sum_i \alpha_i a_i$$

*with  $\{a_i\}_i \subset \text{Atom}(\mathbb{R} \times \mathbb{R})$ ,  $\{\alpha_i\}_i \subset \mathbb{C}$  and  $\sum_i |\alpha_i| < \infty$ . Moreover,  $h^1(\mathbb{R} \times \mathbb{R})$  is equipped with the norm  $\|f\|_{h^1(\mathbb{R} \times \mathbb{R})} := \inf \sum_i |\alpha_i|$  where the infimum is taken over all possible decompositions of  $f$  in the form (1.2).*

Then we have the following result from [9] on the duality of the atomic Hardy space  $h^1$  and little  $\text{bmo}$ , whose proof will be sketched in Section 2 for the convenience of the reader.

THEOREM 1.4 ([9]). — *A function  $b \in L^1_{loc}(\mathbb{R}^2)$  is in  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  if and only if*

$$\sup_{a \in \text{Atom}(\mathbb{R} \times \mathbb{R})} \int_{\mathbb{R}^2} b(x_1, x_2) a(x_1, x_2) dx_1 dx_2 < \infty.$$

Consequently, the predual of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  is  $h^1(\mathbb{R} \times \mathbb{R})$ .

Our main result of this article is the following.

THEOREM 1.5. — *For every  $f \in h^1(\mathbb{R} \times \mathbb{R})$ , there exist sequences  $\{\alpha_j^k\}_j \in \ell^1$  and functions  $g_j^k, h_j^k \in L^2(\mathbb{R}^2)$  such that*

$$(1.3) \quad f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k)$$

in the sense of  $h^1(\mathbb{R} \times \mathbb{R})$ , where  $\Pi(f, g)$  is the bilinear form defined as

$$(1.4) \quad \Pi(g, h) := hH_1H_2g - gH_1H_2h.$$

Moreover, we have that

$$(1.5) \quad \|f\|_{h^1(\mathbb{R} \times \mathbb{R})} \approx \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}^2)} \|h_j^k\|_{L^2(\mathbb{R}^2)} \right\},$$

where the infimum is taken over all representations of  $f$  in the form (1.3) and the implicit constants are independent of  $f$ .

Remark 1.6. — The functions  $g_j^k$  and  $h_j^k$  in the main theorem above are actually in  $L^\infty(\mathbb{R}^2)$  with compact support.

By duality, we obtain the lower bound of the commutator  $[b, H_1H_2]$ , which was known from the work of Ferguson and Sadosky in [9] (see Theorem 1.1).

COROLLARY 1.7. — *Let  $b \in L^1_{loc}(\mathbb{R}^2)$ . If  $[b, H_1H_2]$  is bounded on  $L^2(\mathbb{R}^2)$ , then we get that  $b \in \text{bmo}(\mathbb{R} \times \mathbb{R})$  and there exists a constant  $C$  so that*

$$(1.6) \quad \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})} \leq C \|[b, H_1H_2] : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)\|.$$

We further remark that in Theorem 1.5 and Corollary 1.7 it is possible to change  $L^2$  to  $L^p$  for  $1 < p < \infty$  and to replace the factorization in terms of  $L^p$  and  $L^q$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . We leave these standard modifications to the reader. Also, as can be seen from the proofs given below, the role of the Hilbert transforms play no substantive role and in fact work for the Riesz transforms just as easily. In the interest of ease of presentation, we have focused on the proof with the Hilbert transforms and leave the direct modifications again to the reader.

We also point out that the results in Corollary 1.7 can be seen as special cases of the work in [16], where Ou et al. first proved the lower bound for commutators with respect to certain BMO spaces (using the ideas from [8, 14]) and then obtained the weak factorization for the predual of their BMO space in the form  $H_{Re}^1(\mathbb{R}^{(d_1, d_2)}) \otimes L^1(\mathbb{R}^{d_3}) + L^1(\mathbb{R}^{d_1}) \otimes H_{Re}^1(\mathbb{R}^{(d_2, d_3)})$  by duality. For more details, we refer to Section 6 in [16].

## 2. Weak factorization of the product Hardy space

$$h^1(\mathbb{R} \times \mathbb{R})$$

In this section we will first sketch the proof of Theorem 1.4, and then we provide the proof of the weak factorization for the predual of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  characterised by rectangular atoms (as in Definition 1.3). We adapt the idea from [17] (see also a recent refinement of the idea in [6]) to our current product setting for atoms and for the  $\text{bmo}$  defined via rectangles. The main approach here is to approximate each  $h^1(\mathbb{R} \times \mathbb{R})$  atom  $a$  by a related bilinear form  $\Pi(f, g)$  with two  $L^2(\mathbb{R}^2)$  functions  $f$  and  $g$  constructed with respect to  $a$ .

*Sketch of the proof of Theorem 1.4.* — We first consider the definition of  $h^1(\mathbb{R} \times \mathbb{R})$  via  $q$ -atoms. Suppose  $q \in (1, \infty]$ . A  $q$ -atom on  $\mathbb{R} \times \mathbb{R}$  is a function  $a \in L^q(\mathbb{R}^2)$  supported on a rectangle  $R \subset \mathbb{R} \times \mathbb{R}$  with  $\|a\|_{L^q(\mathbb{R}^2)} \leq |R|^{\frac{1}{q}-1}$  and satisfying the cancellation property

$$\int_{\mathbb{R} \times \mathbb{R}} a(x_1, x_2) dx_1 dx_2 = 0.$$

Let  $\text{Atom}_q(\mathbb{R} \times \mathbb{R})$  denote the collection of all such atoms. The atomic Hardy space  $h^{1,q}(\mathbb{R} \times \mathbb{R})$  is defined as the set of functions of the form

$$(2.1) \quad f = \sum_i \alpha_i a_i$$

with  $\{a_i\}_i \subset \text{Atom}_q(\mathbb{R} \times \mathbb{R})$ ,  $\{\alpha_i\}_i \subset \mathbb{C}$  and  $\sum_i |\alpha_i| < \infty$ . Moreover,  $h^{1,q}(\mathbb{R} \times \mathbb{R})$  is equipped with the norm  $\|f\|_{h^{1,q}(\mathbb{R} \times \mathbb{R})} := \inf \sum_i |\alpha_i|$  where the infimum is taken over all possible decompositions of  $f$  in the form (2.1).

Next, it suffices to prove that for  $q \in (1, \infty)$ , the spaces  $h^{1,q}(\mathbb{R} \times \mathbb{R})$  and  $h^{1,\infty}(\mathbb{R} \times \mathbb{R})$  coincide with equivalent norms. Assuming that this is true at the moment, then to prove the duality of  $h^1(\mathbb{R} \times \mathbb{R})$  with  $\text{bmo}(\mathbb{R} \times \mathbb{R})$ , we just need to show the dual space of  $h^{1,2}(\mathbb{R} \times \mathbb{R})$  is  $\text{bmo}(\mathbb{R} \times \mathbb{R})$ . This follows from a standard argument, see for example [4], also [11, Section II, Chapter 3].

Concerning the equivalence of the spaces  $h^{1,q}(\mathbb{R} \times \mathbb{R})$  and  $h^{1,\infty}(\mathbb{R} \times \mathbb{R})$ , we first point out that the inclusion  $h^{1,\infty}(\mathbb{R} \times \mathbb{R}) \subset h^{1,q}(\mathbb{R} \times \mathbb{R})$  for  $q \in (1, \infty)$  is obvious, since an  $\infty$ -atom must be a  $q$ -atom for all  $q \in (1, \infty)$ . Thus, we only need to establish the converse. We do so by showing that any  $(1, q)$ -atom  $a$  with  $\text{supp}(a) \subset R_0$ ,  $b := |R_0|a$  has an atomic decomposition  $b = \sum_{i=0}^{\infty} \alpha_i b_i$ , where each  $b_i$ ,  $i \in \mathbb{Z}_+$ , is a  $(1, \infty)$ -atom and  $\sum_{i=0}^{\infty} |\alpha_i| \lesssim 1$ . Actually, this follows from a standard induction argument (see e.g. [4]) using the Whitney covering lemma and a variant of the argument in [4, Lemma (3.9)].  $\square$

**THEOREM 2.1.** — *Let  $\epsilon$  be an arbitrary positive number. Let  $a(x_1, x_2)$  be an atom as defined in Definition 1.2. Then there exist  $f, g \in L^2(\mathbb{R}^2)$  and a constant  $C(\epsilon)$  depending only on  $\epsilon$  such that*

$$\|a - \Pi(f, g)\|_{h^1(\mathbb{R} \times \mathbb{R})} < \epsilon,$$

where  $\|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)} \leq C(\epsilon)$ .

To prove Theorem 2.1, we first provide a technical lemma as follows.

**LEMMA 2.2.** — *Let  $R = I \times J$  be a rectangle in  $\mathbb{R} \times \mathbb{R}$  with center  $(x_I, x_J)$ . For every  $\epsilon > 0$ , we choose  $M$  such that*

$$(2.2) \quad \frac{\log M}{M} < \epsilon.$$

Then define  $\tilde{R} = \tilde{I} \times \tilde{J}$  as another rectangle in  $\mathbb{R} \times \mathbb{R}$  centered at  $(x_{\tilde{I}}, x_{\tilde{J}})$  and satisfying:  $\ell(\tilde{I}) = \ell(I)$ ,  $\ell(\tilde{J}) = \ell(J)$  and  $|x_I - x_{\tilde{I}}| = M\ell(I)$ ,  $|x_J - x_{\tilde{J}}| = M\ell(J)$ .

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  and assume that  $\text{supp } f \subseteq R \cup \tilde{R}$ . Further, assume that

$$|f(x_1, x_2)| \lesssim \frac{1}{M|R|} \left( \chi_R(x_1, x_2) + \chi_{\tilde{R}}(x_1, x_2) \right)$$

and that  $f$  has mean zero property:

$$(2.3) \quad \int_{\mathbb{R} \times \mathbb{R}} f(x_1, x_2) \, dx_1 dx_2 = 0.$$

Then  $\|f\|_{h^1(\mathbb{R} \times \mathbb{R})} \lesssim \epsilon$ , where the implicit constant is independent of  $f$ ,  $\epsilon$  and  $M$ .

*Proof.* — Suppose  $f$  satisfies the conditions as stated in the lemma above. We will show that  $f$  has an atomic decomposition as the form in Definition 1.3. To see this, we first define two functions  $f_1(x)$  and  $f_2(x)$  by

$$f_1(x_1, x_2) = f(x_1, x_2), (x_1, x_2) \in R; \quad f_1(x_1, x_2) = 0, (x_1, x_2) \in \mathbb{R}^2 \setminus R, \text{ and}$$

$$f_2(x_1, x_2) = f(x_1, x_2), (x_1, x_2) \in \tilde{R}; \quad f_2(x_1, x_2) = 0, (x_1, x_2) \in \mathbb{R}^2 \setminus \tilde{R}.$$



Then we have  $f = f_1 + f_2$  and

$$|f_1(x_1, x_2)| \lesssim \frac{1}{M|R|} \chi_R(x_1, x_2) \quad \text{and} \quad |f_2(x_1, x_2)| \lesssim \frac{1}{M|R|} \chi_{\widetilde{R}}(x_1, x_2).$$

Define

$$\begin{aligned} g_1^1(x_1, x_2) &:= \frac{\chi_{2R}(x_1, x_2)}{|2R|} \iint_R f_1(y_1, y_2) dy_1 dy_2, \\ f_1^1(x_1, x_2) &:= f_1(x_1, x_2) - g_1^1(x_1, x_2), \\ \alpha_1^1 &:= \|f_1^1\|_{L^\infty(\mathbb{R}^2)} |2R|. \end{aligned}$$

Then we claim that  $a_1^1 := (\alpha_1^1)^{-1} f_1^1$  is a rectangle atom as in Definition 1.2. First, it is direct that  $a_1^1$  is supported in  $2R$ . Moreover, we have that

$$\begin{aligned} &\int_{\mathbb{R}^2} a_1^1(x_1, x_2) dx_1 dx_2 \\ &= (\alpha_1^1)^{-1} \int_{\mathbb{R}^2} (f_1(x_1, x_2) - g_1^1(x_1, x_2)) dx_1 dx_2 \\ &= (\alpha_1^1)^{-1} \left( \int_{\mathbb{R}^2} f_1(x_1, x_2) dx_1 dx_2 - \int_{\mathbb{R}^2} f_1(x_1, x_2) dx_1 dx_2 \right) \\ &= 0 \end{aligned}$$

and that

$$\|a_1^1\|_{L^\infty(\mathbb{R}^2)} \leq |(\alpha_1^1)^{-1}| \|f_1^1\|_{L^\infty(\mathbb{R}^2)} = \frac{1}{|2R|}.$$

Thus,  $a_1^1$  is an atom as in Definition 1.2. Moreover, we have

$$|\alpha_1^1| = \|f_1^1\|_{L^\infty(\mathbb{R}^2)} |2R| \leq \|f_1\|_{L^\infty(\mathbb{R}^2)} |2R| \lesssim \frac{1}{M|R|} \cdot |2R| \lesssim \frac{1}{M}.$$

And

$$f_1(x_1, x_2) = f_1^1(x_1, x_2) + g_1^1(x_1, x_2) = \alpha_1^1 a_1^1 + g_1^1(x_1, x_2).$$

For  $g_1^1(x_1, x_2)$ , we further write it as

$$g_1^1(x_1, x_2) = g_1^1(x_1, x_2) - g_1^2(x_1, x_2) + g_1^2(x_1, x_2) =: f_1^2(x_1, x_2) + g_1^2(x_1, x_2)$$

with

$$g_1^2(x_1, x_2) := \frac{\chi_{4R}(x_1, x_2)}{|4R|} \iint_R f_1(y_1, y_2) dy_1 dy_2.$$

Again, we define

$$\alpha_1^2 := \|f_1^2\|_{L^\infty(\mathbb{R}^2)} |4R| \quad \text{and} \quad a_1^2 := (\alpha_1^2)^{-1} f_1^2,$$

and following similar estimates as for  $a_1^1$ , we see that  $a_1^2$  is an atom as in Definition 1.2 with

$$\|a_1^2\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{|4R|} \quad \text{and} \quad |\alpha_1^2| \lesssim \frac{1}{M}.$$

Then we have

$$f_1(x_1, x_2) = \sum_{i=1}^2 \alpha_1^i a_1^i + g_1^2(x_1, x_2).$$

Continuing in this fashion we see that for  $i \in \{1, 2, \dots, i_0\}$ ,

$$f_1(x_1, x_2) = \sum_{i=1}^{i_0} \alpha_1^i a_1^i + g_1^{i_0}(x_1, x_2),$$

where for  $i \in \{2, \dots, i_0\}$ ,

$$\begin{aligned} g_1^i(x_1, x_2) &:= \frac{\chi_{2^i R}(x_1, x_2)}{|2^i R|} \iint_R f_1(y_1, y_2) dy_1 dy_2, \\ f_1^i(x_1, x_2) &:= g_1^{i-1}(x_1, x_2) - g_1^i(x_1, x_2), \\ \alpha_1^i &:= \|f_1^i\|_{L^\infty(\mathbb{R}^2)} |2^i R| \quad \text{and} \\ a_1^i &:= (\alpha_1^i)^{-1} f_1^i. \end{aligned}$$

Here we choose  $i_0$  to be the smallest positive integer such that  $\tilde{R} \subset 2^{i_0} R$ . Then from the definition of  $\tilde{R}$ , we obtain that

$$i_0 \approx \log_2 M.$$

Moreover, for  $i \in \{1, 2, \dots, i_0\}$ , we have

$$\begin{aligned} |\alpha_1^i| &\leq \|f_1^i\|_{L^\infty(\mathbb{R}^2)} |2^i R| \leq |2^i R| (\|g_1^{i-1}\|_{L^\infty(\mathbb{R}^2)} + \|g_1^i\|_{L^\infty(\mathbb{R}^2)}) \\ &\leq |2^i R| \left( \frac{1}{|2^{i-1} R|} \iint_R |f_1(y_1, y_2)| dy_1 dy_2 + \frac{1}{|2^i R|} \iint_R |f_1(y_1, y_2)| dy_1 dy_2 \right) \\ &\lesssim |2^i R| \frac{1}{|2^{i-1} R|} \|f_1\|_{L^\infty(\mathbb{R}^2)} |R| \\ &\lesssim |R| \frac{1}{M|R|} \\ &= \frac{1}{M}. \end{aligned}$$

Following the same steps, we also obtain that for  $i \in \{1, 2, \dots, i_0\}$ ,

$$f_2(x_1, x_2) = \sum_{i=1}^{i_0} \alpha_2^i a_2^i + g_2^{i_0}(x_1, x_2),$$

where for  $i \in \{2, \dots, i_0\}$ ,

$$\begin{aligned} g_2^i(x_1, x_2) &:= \frac{\chi_{2^i R}(x_1, x_2)}{|2^i R|} \iint_{\tilde{R}} f_2(y_1, y_2) dy_1 dy_2, \\ f_2^i(x_1, x_2) &:= g_2^{i-1}(x_1, x_2) - g_2^i(x_1, x_2), \\ \alpha_2^i &:= \|f_2^i\|_{L^\infty(\mathbb{R}^2)} |2^i R| \quad \text{and} \\ a_2^i &:= (\alpha_2^i)^{-1} f_2^i. \end{aligned}$$

Similarly, for  $i \in \{1, 2, \dots, i_0\}$ , we have

$$|\alpha_2^i| \lesssim \frac{1}{M}.$$

Combining the decompositions above, we obtain that

$$f(x_1, x_2) = \sum_{j=1}^2 \sum_{i=1}^{i_0} \alpha_j^i a_j^i + g_j^{i_0}(x_1, x_2).$$

We now consider the tail  $g_1^{i_0}(x_1, x_2) + g_2^{i_0}(x_1, x_2)$ . To handle that, consider the rectangle  $\bar{R}$  centered at the point

$$\left( \frac{x_I + x_{\tilde{I}}}{2}, \frac{x_J + x_{\tilde{J}}}{2} \right)$$

with sidelength  $2^{i_0+1}\ell(I)$  and  $2^{i_0+1}\ell(J)$ . Then, it is clear that  $R \cup \tilde{R} \subset \bar{R}$ , and that  $2^{i_0}R, 2^{i_0}\tilde{R} \subset \bar{R}$ . Thus, we get that

$$\frac{\chi_{\bar{R}}(x_1, x_2)}{|\bar{R}|} \iint_{\bar{R}} f_1(y_1, y_2) dy_1 dy_2 + \frac{\chi_{\bar{R}}(x_1, x_2)}{|\bar{R}|} \iint_{\bar{R}} f_2(y_1, y_2) dy_1 dy_2 = 0.$$

Hence, we write

$$\begin{aligned} &g_1^{i_0}(x_1, x_2) + g_2^{i_0}(x_1, x_2) \\ &= \left( g_1^{i_0}(x_1, x_2) - \frac{\chi_{\bar{R}}(x_1, x_2)}{|\bar{R}|} \iint_{\bar{R}} f_1(y_1, y_2) dy_1 dy_2 \right) \\ &\quad + \left( g_2^{i_0}(x_1, x_2) - \frac{\chi_{\bar{R}}(x_1, x_2)}{|\bar{R}|} \iint_{\bar{R}} f_2(y_1, y_2) dy_1 dy_2 \right) \\ &=: f_1^{i_0+1} + f_2^{i_0+1}. \end{aligned}$$

For  $j = 1, 2$ , we now define

$$\begin{aligned} \alpha_j^{i_0+1} &:= \|f_j^{i_0+1}\|_{L^\infty(\mathbb{R}^2)} |2^{i_0+1}R| \quad \text{and} \\ a_j^{i_0+1} &:= (\alpha_j^{i_0+1})^{-1} f_j^{i_0+1}. \end{aligned}$$

Again we can verify that for  $j = 1, 2$ ,  $a_j^{i_0+1}$  is an atom as in Definition 1.2 with

$$\|a_j^{i_0+1}\|_{L^\infty(\mathbb{R}^2)} = \frac{1}{|2^{i_0+1}R|}.$$

Moreover, we also have

$$|\alpha_j^{i_0+1}| \lesssim \frac{1}{M}.$$

Thus, we obtain that

$$f(x_1, x_2) = \sum_{j=1}^2 \sum_{i=1}^{i_0+1} \alpha_j^i a_j^i,$$

which implies that  $f \in h^1(\mathbb{R} \times \mathbb{R})$  and

$$\|f\|_{h^1(\mathbb{R} \times \mathbb{R})} \leq \sum_{j=1}^2 \sum_{i=1}^{i_0+1} |\alpha_j^i| \lesssim \sum_{j=1}^2 \sum_{i=1}^{i_0+1} \frac{1}{M} \lesssim \frac{\log M}{M} < \epsilon.$$

Therefore, we finish the proof of Lemma 2.2. □

*Proof of Theorem 2.1.* — Suppose  $a$  is an atom of  $h^1(\mathbb{R} \times \mathbb{R})$  supported in a rectangle  $R$  centered at  $(x_I, x_J)$ , as in Definition 1.2. For every fixed  $\epsilon > 0$ , we now let  $M, \tilde{R}$  be the same as in Lemma 2.2.

We define the two functions

$$f(x_1, x_2) := \mathbb{1}_{\tilde{R}}(x_1, x_2) \quad \text{and} \quad g(x_1, x_2) := \frac{a(x_1, x_2)}{H_1 H_2 f(x_I, x_J)}.$$

Then by definition, we have

$$\|f\|_{L^2(\mathbb{R}^2)} = |\tilde{R}|^{\frac{1}{2}} = |R|^{\frac{1}{2}}$$

and

$$\|g\|_{L^2(\mathbb{R}^2)} = \frac{1}{|H_1 H_2 f(x_I, x_J)|} \|a\|_{L^2(\mathbb{R}^2)} \leq \frac{|R|^{-\frac{1}{2}}}{|H_1 H_2 f(x_I, x_J)|}.$$

Observe that

$$|H_1 H_2 f(x_I, x_J)| = \left| \int_{\tilde{R}} \frac{1}{x_I - y_1} \frac{1}{x_J - y_2} dy_1 dy_2 \right| \approx \frac{1}{M^2}.$$

Thus, we have that

$$\|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)} \leq CM^2$$

with the positive constant  $C$  independent of  $a(x_1, x_2)$  and  $M$ . We take  $C(\epsilon)$  as

$$(2.4) \quad C(\epsilon) := CM^2,$$

then it is easy to see that  $C(\epsilon)$  depends only on  $\epsilon$  as  $M$  only depends on  $\epsilon$ . Now, write

$$a - \Pi(f, g) = (a - gH_1H_2f) + fH_1H_2g =: w_1 + w_2.$$

First, consider  $w_1$ . Observe that  $\text{supp } w_1 \subseteq R$  and

$$|w_1(x_1, x_2)| = \frac{|a(x_1, x_2)|}{|H_1H_2f(x_I, x_J)|} |H_1H_2f(x_I, x_J) - H_1H_2f(x_1, x_2)|.$$

Then as  $x \in R$ , we can estimate

$$\begin{aligned} & |H_1H_2f(x_I, x_J) - H_1H_2f(x_1, x_2)| \\ &= \left| \int_{\tilde{R}} \frac{1}{(y_1 - x_I)(y_2 - x_J)} - \frac{1}{(y_1 - x_1)(y_2 - x_2)} dy_1 dy_2 \right| \\ &\leq \int_{\tilde{R}} \frac{|x_1 - x_I|}{|y_1 - x_I||y_1 - x_1||y_2 - x_J|} + \frac{|x_2 - x_J|}{|y_1 - x_1||y_2 - x_2||y_2 - x_J|} dy_1 dy_2 \\ &\leq \int_{\tilde{R}} \frac{\ell(I)}{M^2 \ell(I)^2 M \ell(J)} + \frac{\ell(J)}{M \ell(I) M^2 \ell(J)^2} dy_1 dy_2 \\ &\lesssim \frac{1}{M^3}. \end{aligned}$$

Combining this with the definition of  $w_1$  immediately gives:

$$|w_1(x_1, x_2)| \lesssim \frac{1}{M} |a(x_1, x_2)|,$$

which implies that

$$\|w_1\|_{L^2(\mathbb{R}^2)} \lesssim \frac{1}{M} \|a\|_{L^2(\mathbb{R}^2)} \lesssim \frac{1}{M|R|^{\frac{1}{2}}}.$$

Now, consider  $w_2(x_1, x_2)$ . Note that

$$\begin{aligned} w_2(x_1, x_2) &= f(x_1, x_2)H_1H_2g(x_1, x_2) \\ &= \frac{1}{H_1H_2f(x_I, x_J)} \mathbf{1}_{\tilde{R}}(x_1, x_2)H_1H_2a(x_1, x_2). \end{aligned}$$

Clearly,  $\text{supp } w_2 \subseteq \tilde{R}$ . Furthermore, using the mean zero property of  $a(x_1, x_2)$ , we have:

$$\begin{aligned} & H_1H_2a(x_1, x_2) \\ &= \int_R \frac{a(y_1, y_2)}{(y_1 - x_1)(y_2 - x_2)} dy_1 dy_2 \\ &= \int_R \left( \frac{1}{(y_1 - x_1)(y_2 - x_2)} - \frac{1}{(x_I - x_1)(x_J - x_2)} \right) a(y_1, y_2) dy_1 dy_2. \end{aligned}$$

It is immediate that

$$|\mathbb{1}_{\tilde{R}}(x_1, x_2)H_1H_2a(x_1, x_2)| \lesssim \mathbb{1}_{\tilde{R}}(x_1, x_2)\frac{1}{M^3}\|a\|_{L^\infty(\mathbb{R}^2)}.$$

Thus, we can conclude that

$$|w_2(x_1, x_2)| \lesssim \mathbb{1}_{\tilde{R}}(x_1, x_2)\frac{1}{M}\|a\|_{L^\infty(\mathbb{R}^2)},$$

which implies that

$$\|w_2\|_{L^2(\mathbb{R}^2)} \lesssim \frac{1}{M|R|^{\frac{1}{2}}}.$$

Combining the estimates of  $w_1$  and  $w_2$ , we can conclude that  $a - \Pi(f, g)$  has support contained in

$$R \cup \tilde{R}$$

and satisfies

$$\|a - \Pi(f, g)\|_{L^2(\mathbb{R}^2)} \lesssim \frac{1}{M|R|^{\frac{1}{2}}}.$$

Moreover, from the definition of the bilinear form, we obtain that

$$\int_{\mathbb{R}^2} (a(x_1, x_2) - \Pi(f, g)(x_1, x_2)) dx_1 dx_2 = 0.$$

Then, the fact that  $\|a - \Pi(f, g)\|_{h^1(\mathbb{R} \times \mathbb{R})} \lesssim \epsilon$  now immediately follows from Lemma 2.2. □

*Remark 2.3.* — From the proof of Theorem 2.1 as above, we observe that the functions  $f$  and  $g$  that we constructed are actually in  $L^\infty(\mathbb{R}^2)$  with compact support.

Now we provide the proof of the main result in this paper. To begin with, we need the following two auxiliary lemmas.

LEMMA 2.4. — *Suppose  $b \in \text{bmo}(\mathbb{R} \times \mathbb{R})$ . Then we have*

$$(2.5) \quad \|[b, H_1H_2]\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \lesssim \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})},$$

where the implicit constant is independent of  $b$ .

*Proof.* — We point out that the proof of upper bound of  $[b, H_1H_2]$  follows directly from the property of  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  and the  $L^2$  boundedness of the Hilbert transforms  $H_1$  and  $H_2$ .

Suppose that  $b \in \text{bmo}(\mathbb{R} \times \mathbb{R})$ . Then we know that for any fixed  $x_2 \in \mathbb{R}$ ,  $b(x_1, x_2)$  as a function of  $x_1$  is in the standard one-parameter  $\text{BMO}(\mathbb{R})$ , symmetric result holds for the roles of  $x_1$  and  $x_2$  interchanged. Moreover, we further have that

$$(2.6) \quad \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})} \approx \sup_{x_1 \in \mathbb{R}} \|b(x_1, \cdot)\|_{\text{BMO}(\mathbb{R})} + \sup_{x_2 \in \mathbb{R}} \|b(\cdot, x_2)\|_{\text{BMO}(\mathbb{R})},$$

where the implicit constants are independent of the function  $b$ .

Next, we point out that

$$[b, H_1 H_2] = H_1 [b, H_2] + [b, H_1] H_2.$$

Then based on (2.6) and the result of Coifman–Rochberg–Weiss [3], we know that

$$\begin{aligned} & \| [b, H_2] \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} + \| [b, H_1] \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \\ & \lesssim \sup_{x_1 \in \mathbb{R}} \| b(x_1, \cdot) \|_{\text{BMO}(\mathbb{R})} + \sup_{x_2 \in \mathbb{R}} \| b(\cdot, x_2) \|_{\text{BMO}(\mathbb{R})} \\ & \lesssim \| b \|_{\text{bmo}(\mathbb{R} \times \mathbb{R})}. \end{aligned}$$

Then, denote by  $\text{Id}_1$  and  $\text{Id}_2$  the identity operator on  $L^2(\mathbb{R})$  for the first and second variable, respectively. We further have

$$[b, H_1 H_2] = (H_1 \otimes \text{Id}_2) \circ [b, H_2] + [b, H_1] \circ (\text{Id}_1 \otimes H_2),$$

where we use  $T_1 \circ T_2$  to denote the composition of two operators  $T_1$  and  $T_2$ . Thus, we obtain that

$$\begin{aligned} & \| [b, H_1 H_2] \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \\ & = \| (H_1 \otimes \text{Id}_2) \circ [b, H_2] + [b, H_1] \circ (\text{Id}_1 \otimes H_2) \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \\ & \leq \| (H_1 \otimes \text{Id}_2) \circ [b, H_2] \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \\ & \quad + \| [b, H_1] \circ (\text{Id}_1 \otimes H_2) \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \\ & \leq \| H_1 \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \| [b, H_2] \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \\ & \quad + \| [b, H_1] \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \| H_2 \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \\ & \lesssim \| b \|_{\text{bmo}(\mathbb{R} \times \mathbb{R})}, \end{aligned}$$

which shows that (2.5) holds.  $\square$

LEMMA 2.5. — Suppose  $f, g \in L^\infty(\mathbb{R}^2)$  with compact supports. Then the bilinear form  $\Pi(f, g)$  defined as in (1.4) is in  $h^1(\mathbb{R} \times \mathbb{R})$  with the norm satisfying

$$(2.7) \quad \| \Pi(f, g) \|_{h^1(\mathbb{R} \times \mathbb{R})} \lesssim \| f \|_{L^2(\mathbb{R}^2)} \| g \|_{L^2(\mathbb{R}^2)},$$

where the implicit constant is independent of  $f$  and  $g$ .

*Proof.* — We first note that for every  $b \in \text{bmo}(\mathbb{R} \times \mathbb{R})$ ,  $b$  is in  $L^q_{loc}(\mathbb{R}^2)$  for  $q \in (1, \infty)$ . In fact, for any compact set  $\Omega$  in  $\mathbb{R} \times \mathbb{R}$ , there exist two closed intervals  $I, J \in \mathbb{R}$ , such that  $\Omega \subset I \times J$ . For any  $x_1 \in I$ , we have  $b(x_1, x_2)$  as a function of  $x_2$  is in  $\text{BMO}(\mathbb{R})$ . Hence,  $b(x_1, x_2)$  as a function

of  $x_2$  is in  $L^q(J)$ . Again, for any  $x_2 \in J$ ,  $b(x_1, x_2)$  as a function of  $x_1$  is in  $L^q(I)$ . As a consequence, we have that for any  $q \in (1, \infty)$ ,

$$\begin{aligned} \int_{\Omega} |b(x_1, x_2)|^q dx_1 dx_2 &\leq \int_I \int_J |b(x_1, x_2)|^q dx_2 dx_1 \leq \int_I \|b(x_1, \cdot)\|_{\text{BMO}(\mathbb{R})}^q dx_1 \\ &\leq \sup_{x_1 \in I} \|b(x_1, \cdot)\|_{\text{BMO}(\mathbb{R})}^q |I| \\ &\leq C \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})}^q |I|, \end{aligned}$$

which shows that  $b$  is in  $L^q_{loc}(\mathbb{R}^2)$  for  $q \in (1, \infty)$  with

$$(2.8) \quad \int_{\Omega} |b(x_1, x_2)|^q dx_1 dx_2 \leq C_{\Omega} \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})}^q$$

for any compact set  $\Omega \in \mathbb{R} \times \mathbb{R}$ .

We now consider the property of the bilinear form  $\Pi(f, g)$  defined as in (1.4). For each  $f, g \in L^{\infty}(\mathbb{R}^2)$  with compact support, we have that  $\Pi(f, g) = gH_1H_2f - fH_1H_2g$  is in  $L^2(\mathbb{R}^2)$  with compact support. In fact, since  $f$  is in  $L^{\infty}(\mathbb{R}^2)$  with compact support, we get that  $f$  is in  $L^2(\mathbb{R}^2)$  with compact support, which implies that  $H_1H_2f$  is in  $L^2(\mathbb{R}^2)$ , and hence  $gH_1H_2f$  is in  $L^2(\mathbb{R}^2)$  with compact support. Similar argument holds for  $fH_1H_2g$ . Also note that from (2.8), for each  $b \in \text{bmo}(\mathbb{R} \times \mathbb{R})$ ,  $b$  is in  $L^2_{loc}(\mathbb{R}^2)$ . We have that

$$\begin{aligned} \left| \langle b, \Pi(f, g) \rangle_{L^2(\mathbb{R}^2)} \right| &= \left| \int_{\mathbb{R} \times \mathbb{R}} b(x_1, x_2) \Pi(f, g)(x_1, x_2) dx_1 dx_2 \right| \\ &\leq C \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})} < \infty, \end{aligned}$$

where the constant  $C$  depends on the support of  $f$  and  $g$ . Hence  $\langle b, \Pi(f, g) \rangle_{L^2(\mathbb{R}^2)}$  is well-defined.

Next we claim that for each  $f, g \in L^{\infty}(\mathbb{R}^2)$  with compact support,

$$(2.9) \quad \langle b, \Pi(f, g) \rangle_{L^2(\mathbb{R}^2)} = \langle [b, H_1H_2] f, g \rangle_{L^2(\mathbb{R}^2)}.$$

To see this, note that by definition of  $\Pi(f, g)$ ,

$$\langle b, \Pi(f, g) \rangle_{L^2(\mathbb{R}^2)} = \langle b, gH_1H_2f - fH_1H_2g \rangle_{L^2(\mathbb{R}^2)}.$$

Next, since  $f, g \in L^{\infty}(\mathbb{R}^2)$  with compact support and  $b \in L^2_{loc}(\mathbb{R}^2)$ , it is direct that

$$\langle b, gH_1H_2f \rangle_{L^2(\mathbb{R}^2)} = \langle g, bH_1H_2f \rangle_{L^2(\mathbb{R}^2)}$$



and that

$$\begin{aligned} \langle b, fH_1H_2g \rangle_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R} \times \mathbb{R}} b(x_1, x_2) f(x_1, x_2) H_1 H_2 g(x_1, x_2) dx_1 dx_2 \\ &= \int_{\mathbb{R} \times \mathbb{R}} H_1 H_2 (b \cdot f)(x_1, x_2) g(x_1, x_2) dx_1 dx_2 \\ &= \langle H_1 H_2 (b \cdot f), g \rangle_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Combining these two equalities, we get that the claim (2.9) holds.

From the claim (2.9) and the upper bound as in (2.5), we obtain that

$$(2.10) \quad \left| \langle b, \Pi(f, g) \rangle_{L^2(\mathbb{R}^2)} \right| = \left| \langle [b, H_1 H_2] f, g \rangle_{L^2(\mathbb{R}^2)} \right| \lesssim \|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})} \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)},$$

where the implicit constant is independent of  $f$  and  $g$ .

Now for any fixed  $f, g \in L^\infty(\mathbb{R}^2)$  with compact support, we claim that  $\Pi(f, g)$  is in  $h^1(\mathbb{R} \times \mathbb{R})$ .

To see this, we now show that  $\Pi(f, g)$  is the product of a constant and a 2-atom of  $h^1(\mathbb{R} \times \mathbb{R})$ . In fact, from the definition of the bilinear form, we obtain that

$$\int_{\mathbb{R}^2} \Pi(f, g)(x_1, x_2) dx_1 dx_2 = 0.$$

Next, since both  $f$  and  $g$  are in  $L^\infty(\mathbb{R}^2)$  with compact support, we get that  $\Pi(f, g) \in L^2(\mathbb{R} \times \mathbb{R})$  with compact support, denoted it by a rectangle  $R \subset \mathbb{R} \times \mathbb{R}$ . And we further have  $\|\Pi(f, g)\|_{L^2(\mathbb{R}^2)} \leq C_{f,g} \|g\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^\infty(\mathbb{R}^2)}$ , where the constant  $C_{f,g}$  depends on the compact supports of  $f$  and  $g$ . Moreover, we assume that  $\|\Pi(f, g)\|_{L^2(\mathbb{R}^2)} \neq 0$  since otherwise  $\Pi(f, g) = 0$  almost everywhere and hence it is in  $h^1(\mathbb{R} \times \mathbb{R})$ .

Now we can write

$$\Pi(f, g)(x_1, x_2) =: a(x_1, x_2) \cdot \|\Pi(f, g)\|_{L^2(\mathbb{R}^2)} |R|^{\frac{1}{2}},$$

where

$$a(x_1, x_2) := \frac{\Pi(f, g)(x_1, x_2)}{\|\Pi(f, g)\|_{L^2(\mathbb{R}^2)} |R|^{\frac{1}{2}}}.$$

Then it is direct that  $a(x_1, x_2)$  is supported in  $R$ ,  $\int_{\mathbb{R} \times \mathbb{R}} a(x_1, x_2) dx_1 dx_2 = 0$  and that  $\|a\|_{L^2(\mathbb{R}^2)} \leq |R|^{-\frac{1}{2}}$ . Hence  $a(x_1, x_2)$  is a 2-atom of  $h^1(\mathbb{R} \times \mathbb{R})$ , which implies that  $\Pi(f, g)$  is in  $h^1(\mathbb{R} \times \mathbb{R})$ , i.e., the claim holds.

Note that  $\Pi(f, g)$  is in  $h^1(\mathbb{R} \times \mathbb{R})$ , we then further have

$$\|h\|_{h^1(\mathbb{R} \times \mathbb{R})} \approx \sup_{\|b\|_{\text{bmo}(\mathbb{R} \times \mathbb{R})} \leq 1} |\langle b, h \rangle|,$$

which follows from the fundamental fact as in 1.4.12(b) in [10].

This, together with (2.10), immediately implies that (2.7) holds.  $\square$

We now provide the proof of our main result.

*Proof of Theorem 1.5.* — We first point out from Remark 1.6, the functions  $g_j^k$  and  $h_j^k$  in the representation (1.3) are actually in  $L^\infty(\mathbb{R}^2)$  with compact support. Hence, from (2.7), for every  $f \in h^1(\mathbb{R} \times \mathbb{R})$  having the representation (1.3) with

$$\sum_{k=1}^\infty \sum_{j=1}^\infty |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}^2)} \|h_j^k\|_{L^2(\mathbb{R}^2)} < \infty,$$

it follows that

$$\|f\|_{h^1(\mathbb{R} \times \mathbb{R})} \lesssim \inf \left\{ \sum_{k=1}^\infty \sum_{j=1}^\infty |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}^2)} \|h_j^k\|_{L^2(\mathbb{R}^2)} : f = \sum_{k=1}^\infty \sum_{j=1}^\infty \alpha_j^k \Pi(g_j^k, h_j^k) \right\}.$$

It remains to show that for each  $f \in h^1(\mathbb{R} \times \mathbb{R})$ ,  $f$  has a representation as in (1.3) with

$$(2.11) \quad \inf \left\{ \sum_{k=1}^\infty \sum_{j=1}^\infty |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}^2)} \|h_j^k\|_{L^2(\mathbb{R}^2)} : f = \sum_{k=1}^\infty \sum_{j=1}^\infty \alpha_j^k \Pi(g_j^k, h_j^k) \right\} \lesssim \|f\|_{h^1(\mathbb{R} \times \mathbb{R})}.$$

To this end, assume that  $f$  has the following atomic representation  $f = \sum_{j=1}^\infty \alpha_j^1 a_j^1$  with  $\sum_{j=1}^\infty |\alpha_j^1| \leq C_0 \|f\|_{h^1(\mathbb{R} \times \mathbb{R})}$  for certain absolute constant  $C_0 \in (1, \infty)$ . We show that for every  $\epsilon \in (0, C_0^{-1})$  and every  $K \in \mathbb{N}$ ,  $f$  has the following representation

$$(2.12) \quad f = \sum_{k=1}^K \sum_{j=1}^\infty \alpha_j^k \Pi(g_j^k, h_j^k) + E_K,$$

where

$$(2.13) \quad \sum_{j=1}^\infty |\alpha_j^k| \leq \epsilon^{k-1} C_0^k \|f\|_{h^1(\mathbb{R} \times \mathbb{R})},$$

and  $E_K \in h^1(\mathbb{R} \times \mathbb{R})$  with

$$(2.14) \quad \|E_K\|_{h^1(\mathbb{R} \times \mathbb{R})} \leq (\epsilon C_0)^K \|f\|_{h^1(\mathbb{R} \times \mathbb{R})},$$

and  $g_j^k \in L^2(\mathbb{R}^2)$ ,  $h_j^k \in L^2(\mathbb{R}^2)$  for each  $k$  and  $j$ ,  $\{\alpha_j^k\}_j \in \ell^1$  for each  $k$  satisfying that

$$(2.15) \quad \|g_j^k\|_{L^2(\mathbb{R}^2)} \|h_j^k\|_{L^2(\mathbb{R}^2)} \lesssim C(\epsilon)$$

with the absolute constant  $C(\epsilon)$  defined as in (2.4).

In fact, for given  $\epsilon$  and each  $a_j^1$ , by Theorem 2.1 we obtain that there exist  $g_j^1 \in L^2(\mathbb{R}^2)$  and  $h_j^1 \in L^2(\mathbb{R}^2)$  with

$$\|g_j^1\|_{L^2(\mathbb{R}^2)} \|h_j^1\|_{L^2(\mathbb{R}^2)} \lesssim C(\epsilon)$$

and

$$\|a_j^1 - \Pi(g_j^1, h_j^1)\|_{h^1(\mathbb{R} \times \mathbb{R})} < \epsilon.$$

Actually, from Remark 2.3, these two functions  $g_j^1$  and  $h_j^1$  are in  $L^\infty(\mathbb{R}^2)$  with compact supports.

Now we write

$$\begin{aligned} f &= \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 = \sum_{j=1}^{\infty} \alpha_j^1 \Pi(g_j^1, h_j^1) + \sum_{j=1}^{\infty} \alpha_j^1 [a_j^1 - \Pi(g_j^1, h_j^1)] \\ &=: M_1 + E_1. \end{aligned}$$

Observe that

$$\|E_1\|_{h^1(\mathbb{R} \times \mathbb{R})} \leq \sum_{j=1}^{\infty} |\alpha_j^1| \|a_j^1 - \Pi(g_j^1, h_j^1)\|_{h^1(\mathbb{R} \times \mathbb{R})} \leq \epsilon C_0 \|f\|_{h^1(\mathbb{R} \times \mathbb{R})}.$$

Since  $E_1 \in h^1(\mathbb{R} \times \mathbb{R})$ , for the given  $C_0$ , there exists a sequence of atoms  $\{a_j^2\}_j$  and numbers  $\{\alpha_j^2\}_j$  such that  $E_1 = \sum_{j=1}^{\infty} \alpha_j^2 a_j^2$  and

$$\sum_{j=1}^{\infty} |\alpha_j^2| \leq C_0 \|E_1\|_{h^1(\mathbb{R} \times \mathbb{R})} \leq \epsilon C_0^2 \|f\|_{h^1(\mathbb{R} \times \mathbb{R})}.$$

Again, we have that for given  $\epsilon$ , there exists a representation of  $E_1$  such that

$$\begin{aligned} E_1 &= \sum_{j=1}^{\infty} \alpha_j^2 \Pi(g_j^2, h_j^2) + \sum_{j=1}^{\infty} \alpha_j^2 [a_j^2 - \Pi(g_j^2, h_j^2)] \\ &=: M_2 + E_2, \end{aligned}$$

and

$$\|g_j^2\|_{L^2(\mathbb{R}^2)} \|h_j^2\|_{L^2(\mathbb{R}^2)} \lesssim C(\epsilon) \quad \text{and} \quad \|a_j^2 - \Pi(g_j^2, h_j^2)\|_{h^1(\mathbb{R} \times \mathbb{R})} < \frac{\epsilon}{2}.$$

Moreover,

$$\begin{aligned}
 (2.16) \quad \|E_2\|_{h^1(\mathbb{R} \times \mathbb{R})} &\leq \sum_{j=1}^{\infty} |\alpha_j^2| \|a_j^2 - \Pi(g_j^2, h_j^2)\|_{h^1(\mathbb{R} \times \mathbb{R})} \\
 &\leq (\epsilon C_0)^2 \|f\|_{h^1(\mathbb{R} \times \mathbb{R})}.
 \end{aligned}$$

Now we conclude that

$$(2.17) \quad f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 = \sum_{k=1}^2 \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) + E_2,$$

Again, from Remark 2.3, all these functions  $g_j^k$  and  $h_j^k$  are in  $L^\infty(\mathbb{R}^2)$  with compact supports.

Continuing in this way, we deduce that for every  $K \in \mathbb{N}$ ,  $f$  has the representation (2.12) satisfying (2.15), (2.13), and (2.14). Thus letting  $K \rightarrow \infty$ , we see that (1.3) holds. Moreover, since  $\epsilon C_0 < 1$ , we have that

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \leq \sum_{k=1}^{\infty} \epsilon^{-1} (\epsilon C_0)^k \|f\|_{h^1(\mathbb{R} \times \mathbb{R})} \lesssim \|f\|_{h^1(\mathbb{R} \times \mathbb{R})},$$

which implies (2.11) and hence, completes the proof of Theorem 1.5.  $\square$

Next, by duality, we provide the proof of our second main result in this paper.

*Proof of Corollary 1.7.* — Suppose that  $b \in \cup_{q>1} L^q_{loc}(\mathbb{R}^2)$ . Assume that  $[b, H_1 H_2]$  is bounded on  $L^2(\mathbb{R}^2)$  and  $f \in h^1(\mathbb{R} \times \mathbb{R})$  and  $f$  has compact support. From Theorem 1.5, we deduce that

$$\begin{aligned}
 \langle b, f \rangle_{L^2(\mathbb{R}^2)} &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \langle b, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}^2)} \\
 &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \langle g_j^k, [b, H_1 H_2] h_j^k \rangle_{L^2(\mathbb{R}^2)},
 \end{aligned}$$

where in the second equality we have applied the fact that

$$\langle b, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}^2)} = \langle g_j^k, [b, H_1 H_2] h_j^k \rangle_{L^2(\mathbb{R}^2)},$$

which follows from (2.9) since the functions  $g_j^k, h_j^k$  here are constructed as in  $L^\infty(\mathbb{R}^2)$  with compact support (see Remark 2.3).

This implies that

$$\begin{aligned}
 & \left| \langle b, f \rangle_{L^2(\mathbb{R}^2)} \right| \\
 & \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}^2)} \|[b, H_1 H_2] h_j^k\|_{L^2(\mathbb{R}^2)} \\
 & \leq \|[b, H_1 H_2] : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)\| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}^2)} \|h_j^k\|_{L^2(\mathbb{R}^2)} \\
 & \lesssim \|[b, H_1 H_2] : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)\| \|f\|_{h^1(\mathbb{R} \times \mathbb{R})}.
 \end{aligned}$$

Then by the fact that  $\{f \in h^1(\mathbb{R} \times \mathbb{R}) : f \text{ has compact support}\}$  is dense in  $h^1(\mathbb{R} \times \mathbb{R})$ , and the duality between  $h^1(\mathbb{R} \times \mathbb{R})$  and  $\text{bmo}(\mathbb{R} \times \mathbb{R})$  (see [9]), we finish the proof of Corollary 1.7.  $\square$

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