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HIGHEST WEIGHT CATEGORIES AND RECOLLEMENTS

by Henning KRAUSE

ABSTRACT. — We provide several equivalent descriptions of a highest weight category using recollements of abelian categories. Also, we explain the connection between sequences of standard and exceptional objects.

RÉSUMÉ. — Nous donnons plusieurs descriptions équivalentes des catégories de plus haut poids au moyen de recollements de catégories abéliennes. En outre, nous expliquons la relation entre suites d'objets standards et exceptionnels.

1. Introduction

Highest weight categories and quasi-hereditary algebras arise naturally in representation theory and were introduced in a series of papers by Cline, Parshall, and Scott [5, 21, 25]; see also the work of Dlab and Ringel [6, 7]. The intimate connection between highest weight categories and recollements of derived categories was noticed right from the beginning. In this note we *characterise* highest weight categories in terms of recollements of abelian categories; see Theorem 3.4.

A highest weight category is determined by its standard objects (usually denoted by Δ_i, where the index i refers to the weight). An efficient way to formulate this for a module category is given by the following result, which is a variation of a result of Dlab and Ringel [7].

THEOREM 1.1. — *Let $\mathcal{A}$ be the category of finitely generated modules over an artin algebra. Then $\mathcal{A}$ is a highest weight category if and only if*

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there are objects $\Delta_1, \ldots, \Delta_n$ *having the following properties:*

(1) $\operatorname{End}_{\mathcal{A}}(\Delta_i)$ *is a division ring for all* i*.*
(2) $\operatorname{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ *for all* $i > j$*.*
(3) $\operatorname{Ext}^1_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ *for all* $i \geqslant j$*.*
(4) *A projective generator of* $\operatorname{Filt}(\Delta_1, \ldots, \Delta_n)$ *is also one for* $\mathcal{A}$*.*

This description of a highest weight category via its sequence of standard objects suggests a close connection with the concept of an exceptional sequence, as introduced in the study of vector bundles [3, 13, 14, 24]. We make this connection precise in Theorem 5.2 and claim that both concepts are basically equivalent, even though their origins are quite different. A special instance of this theorem for vector bundles on rational surfaces is due to Hille and Perling [16]. For further examples of this connection, relating derived categories of Grassmannians and modular representation theory, see [4, 8].

The crucial issue for understanding the concept of a highest weight category is to find out when a recollement of abelian categories extends to a recollement of their derived categories. We address this problem explicitly in an appendix and provide a necessary and sufficient criterion. This is not used in the main part of the paper but serves as an illustration for some of the key arguments and might be of independent interest.

This paper is organised as follows. In §2 we recall definitions and basic facts about recollements of abelian and triangulated categories. The characterisation of highest weight categories via recollements is given in §3. Then we explain in §4 the equivalent concept of a quasi-hereditary ring and provide a method for constructing quasi-hereditary endomorphism rings in abelian categories. The final §5 is devoted to the connection between sequences of standard and exceptional objects.

Polynomial representations of general linear groups provide interesting examples of highest weight categories. In that case it is appropriate to work with k-linear highest weight categories over an arbitrary commutative base ring k, and we refer to [20] for a detailed exposition.

2. Recollements

Recollements of abelian and triangulated categories

We recall the definition of a recollement using the standard notation [2, 1.4]. In fact, any recollement is built from two diagrams involving "localisation" [9] and "colocalisation" [26].

Definition 2.1. — *A* localisation sequence *of abelian (triangulated) categories is a diagram of functors*

$$\mathcal{A}' \underset{i^!}{\overset{i_!}{\rightleftarrows}} \mathcal{A} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{A}'' \tag{2.1}$$

satisfying the following conditions:

(1) $i_!$ *and* j^* *are exact functors of abelian (triangulated) categories.*
(2) $(i_!, i^!)$ *and* (j^*, j_*) *are adjoint pairs.*
(3) $i_!$ *and* j_* *are fully faithful functors.*
(4) *An object in* $\mathcal{A}$ *is annihilated by* j^* *iff it is in the essential image of* $i_!$.

Note that condition (3) admits an equivalent formulation; see [11, I.1.3]. In the presence of (2), the functor $i_!$ is fully faithful iff the unit $\mathrm{id}_{\mathcal{A}'} \to i^! i_!$ is an isomorphism. Also, the functor j_* is fully faithful iff the counit $j^* j_* \to \mathrm{id}_{\mathcal{A}''}$ is an isomorphism.

Definition 2.2. — *A* colocalisation sequence *of abelian (triangulated) categories is a diagram of functors*

$$\mathcal{A}' \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathcal{A} \underset{j^!}{\overset{j_!}{\leftrightarrows}} \mathcal{A}'' \tag{2.2}$$

satisfying the following conditions:

(1) i_* *and* $j^!$ *are exact functors of abelian (triangulated) categories.*
(2) (i^*, i_*) *and* $(j_!, j^!)$ *are adjoint pairs.*
(3) i_* *and* $j_!$ *are fully faithful functors.*
(4) *An object in* $\mathcal{A}$ *is annihilated by* $j^!$ *iff it is in the essential image of* i_*.

Definition 2.3. — *A* recollement *of abelian (triangulated) categories is a diagram of functors*

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "i_*=i_!" description] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.3}$$

such that the subdiagram (2.1) *is a localisation sequence and the subdiagram* (2.2) *is a colocalisation sequence.*

The recollement is called homological *if the functor* i_* *induces for all* $X, Y \in \mathcal{A}'$ *and* $p \geqslant 0$ *isomorphisms*

$$\mathrm{Ext}^p_{\mathcal{A}'}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^p_{\mathcal{A}}(i_*(X), i_*(Y)).$$

The terminology follows that used in [22], *where* i_* *is called* homological embedding.

Given a colocalisation sequence (2.2) and an object X in $\mathcal{A}$, we have the counit $j_!j^!(X) \to X$ and the unit $X \to i_*i^*(X)$. These fit into an exact sequence

$$j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow 0 \qquad (\mathcal{A} \text{ abelian})$$

and an exact triangle

$$j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow \qquad (\mathcal{A} \text{ triangulated}).$$

Often we consider abelian categories having *enough projective objects*, that is, every object X admits an epimorphism $P \to X$ with P projective. We use without mentioning that a left adjoint of an exact functor preserves projectivity.

Recollements of module categories

Let Λ be a ring (associative with identity). We consider the category $\operatorname{Mod}\Lambda$ of right Λ-modules. We write $\operatorname{mod}\Lambda$ for the full subcategory of finitely presented Λ-modules and $\operatorname{proj}\Lambda$ for the full subcategory of finitely generated projective Λ-modules.

The following result summarises some basic facts about subcategories of $\operatorname{Mod}\Lambda$ consisting of modules that are annihilated by a fixed ideal. Note that all ideals in this work are two-sided.

Recall that a full subcategory $\mathcal{C} \subseteq \mathcal{A}$ of an abelian category is a *Serre subcategory* if for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ we have $X \in \mathcal{C}$ iff $X', X'' \in \mathcal{C}$. For example, the objects that are annihilated by an exact functor $\mathcal{A} \to \mathcal{A}'$ form a Serre subcategory.

Proposition 2.4 ([1, Proposition 7.1]). — *Let Λ be a ring. A full subcategory $\mathcal{C}$ of $\operatorname{Mod}\Lambda$ is of the form $\operatorname{Mod}\Lambda/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of Λ if and only if the following holds:*

1. *If $X' \subseteq X$ is a submodule of $X \in \mathcal{C}$, then X' and X/X' are in $\mathcal{C}$.*
2. *If $(X_i)_{i\in I}$ is a family of modules in $\mathcal{C}$, then their product $\prod_{i\in I} X_i$ is in $\mathcal{C}$.*

In this case $\mathfrak{a} = \bigcap_{X\in\mathcal{C}} \operatorname{ann} X$. Moreover, $\mathfrak{a}^2 = \mathfrak{a}$ if and only if $\mathcal{C}$ is a Serre subcategory.

Given an idempotent $e \in \Lambda$, the inclusion $i_* \colon \operatorname{Mod}\Lambda/\Lambda e\Lambda \to \operatorname{Mod}\Lambda$ and

$$j^* := \operatorname{Hom}_\Lambda(e\Lambda, -) \cong - \otimes_\Lambda \Lambda e$$

induce a recollement

$$\operatorname{Mod}\Lambda/\Lambda e\Lambda \;\substack{\twoheadleftarrow \\ \xrightarrow{\;i_*\;} \\ \twoheadleftarrow}\; \operatorname{Mod}\Lambda \;\substack{\leftarrowtail \\ \xrightarrow{\;j^*\;} \\ \leftarrowtail}\; \operatorname{Mod} e\Lambda e\,. \tag{2.4}$$

In fact, any recollement of module categories

$$\operatorname{Mod}\Lambda' \;\substack{\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow}\; \operatorname{Mod}\Lambda \;\substack{\leftarrowtail \\ \twoheadrightarrow \\ \leftarrowtail}\; \operatorname{Mod}\Lambda''$$

is up to Morita equivalence of this form. For each Λ-module X there is a natural exact sequence

$$\operatorname{Hom}_\Lambda(e\Lambda, X) \otimes_{e\Lambda e} e\Lambda \xrightarrow{\varepsilon_X} X \longrightarrow X \otimes_\Lambda \Lambda/\Lambda e\Lambda \longrightarrow 0\,. \tag{2.5}$$

If Λ is right artinian, then (2.4) restricts to a colocalisation sequence

$$\operatorname{mod}\Lambda/\Lambda e\Lambda \;\substack{\twoheadleftarrow \\ \rightarrowtail}\; \operatorname{mod}\Lambda \;\substack{\leftarrowtail \\ \twoheadrightarrow}\; \operatorname{mod} e\Lambda e\,.$$

Lemma 2.5. — *Let Λ be a right artinian ring and $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ a Serre subcategory. Then there is an idempotent $e \in \Lambda$ such that $\mathcal{C} = \operatorname{mod}\Lambda/\Lambda e\Lambda$. Moreover, the following holds:*

(1) *The inclusion $\operatorname{mod}\Lambda/\Lambda e\Lambda \to \operatorname{mod}\Lambda$ admits a left and a right adjoint.*
(2) *The functor $\operatorname{Hom}_\Lambda(e\Lambda, -)\colon \operatorname{mod}\Lambda \to \operatorname{mod} e\Lambda e$ admits a left adjoint.*
(3) *The functor $\operatorname{Hom}_\Lambda(e\Lambda, -)\colon \operatorname{mod}\Lambda \to \operatorname{mod} e\Lambda e$ admits a right adjoint provided that $\operatorname{mod}\Lambda$ has enough injective objects.*

Proof. — The annihilator $\mathfrak{a} \subseteq \Lambda$ of the modules in $\mathcal{C}$ is idempotent since $\mathcal{C}$ is closed under forming extension. Thus $\mathfrak{a} = \Lambda e\Lambda$ for some idempotent $e \in \Lambda$.

(1) The right adjoint sends a Λ-module X to the maximal submodule belonging to $\mathcal{C}$. The left adjoint sends X to the maximal factor module belonging to $\mathcal{C}$.

(2) Take $- \otimes_{e\Lambda e} e\Lambda$.

(3) Let E be an injective cogenerator and set $\Gamma = \operatorname{End}_\Lambda(E)^{\mathrm{op}}$. Then we have $(\operatorname{mod}\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{mod}\Gamma$ via $\operatorname{Hom}_\Lambda(-, E)$ and can apply (2). □

Example 2.6. — Consider the right artinian ring $\Lambda = \left[\begin{smallmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{smallmatrix}\right]$ and $e = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then $\operatorname{Hom}_\Lambda(e\Lambda, -)\colon \operatorname{mod}\Lambda \to \operatorname{mod} e\Lambda e$ admits no right adjoint, because it would send $e\Lambda e$ to $\operatorname{Hom}_{e\Lambda e}(\Lambda e, e\Lambda e)$ which is not finitely generated over Λ.

We recall a well known criterion for a recollement of module categories to be homological.

Lemma 2.7. — *Let Λ be a ring and $\mathfrak{a} \subseteq \Lambda$ an ideal. Then the following are equivalent:*

(1) $\Lambda/\mathfrak{a} \otimes_\Lambda \Lambda/\mathfrak{a} \cong \Lambda/\mathfrak{a}$ *and* $\operatorname{Tor}_p^\Lambda(\Lambda/\mathfrak{a}, \Lambda/\mathfrak{a}) = 0$ *for all* $p > 0$.

(2) $\operatorname{Ext}^p_{\Lambda/\mathfrak{a}}(X, Y) \xrightarrow{\sim} \operatorname{Ext}^p_\Lambda(X, Y)$ *for all* $\Lambda/\mathfrak{a}$*-modules* X, Y *and* $p \geqslant 0$.

These conditions are satisfied when $\mathfrak{a}$ is a projective Λ-module.

Proof. — For the first part, see [12, Theorem 4.4]. Now suppose that $\mathfrak{a}$ is projective. This implies $\operatorname{Tor}_*^\Lambda(\mathfrak{a}, \Lambda/\mathfrak{a}) = 0$, and the exact sequence $0 \to \mathfrak{a} \to \Lambda \to \Lambda/\mathfrak{a} \to 0$ induces an isomorphism $\operatorname{Tor}_*^\Lambda(\Lambda/\mathfrak{a}, \Lambda/\mathfrak{a}) \cong \Lambda/\mathfrak{a}$. Thus (1) holds. □

Abelian length categories

Let $\mathcal{A}$ be an abelian *length category*. Thus $\mathcal{A}$ is an abelian category and every object in $\mathcal{A}$ has a finite composition series.

Recall that $\mathcal{A}$ is *Ext-finite* if for every pair of simple objects S and T

$$\dim_{\operatorname{End}_{\mathcal{A}}(T)^{\mathrm{op}}} \operatorname{Ext}^1_{\mathcal{A}}(S, T) < \infty.$$

This property is useful for constructing projective generators.

Proposition 2.8 ([10, 8.2]). — *An abelian length category $\mathcal{A}$ is equivalent to the category* $\operatorname{mod}\Lambda$ *of finitely generated Λ-modules for some right artinian ring Λ if and only if the following holds:*

(1) *$\mathcal{A}$ has only finitely many simple objects.*

(2) *$\mathcal{A}$ is Ext-finite.*

(3) *The supremum of the Loewy lengths of the objects in $\mathcal{A}$ is finite.*

3. Highest weight categories

Highest weight categories were introduced by Cline, Parshall, and Scott in [5] in the context of k-linear categories over a field k. The definition given here uses a slightly different formulation which follows Rouquier [23]. Also, our definition is more general since the endomorphism ring of a standard object can be any division ring. For simplicity, we restrict ourselves to the case that the set of weights is finite and totally ordered.

Let $\Delta_1, \dots, \Delta_n$ be objects in an abelian category $\mathcal{A}$. We denote by $\operatorname{Filt}(\Delta_1, \dots, \Delta_n)$ the full subcategory of objects X in $\mathcal{A}$ that admit a finite filtration $0 = X_0 \subseteq X_1 \subseteq \dots \subseteq X_t = X$ such that each factor X_i/X_{i-1}

is isomorphic to an object of the form Δ_j. Also, let $\operatorname{Thick}(\Delta_1, \dots, \Delta_n)$ denote the smallest thick subcategory of $\mathcal{A}$ containing $\Delta_1, \dots, \Delta_n$. Recall that a full subcategory $\mathcal{C} \subseteq \mathcal{A}$ is *thick* if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ all three of $\{X, Y, Z\}$ belong to $\mathcal{C}$ if two of them are in $\mathcal{C}$.

Recall that a projective object P of an abelian (or exact) category is a *projective generator* if for every object X there is an exact sequence $0 \to N \to P^r \to X \to 0$ for some positive integer r.

Definition 3.1. — *Let $\mathcal{A}$ be an abelian length category having only finitely many isoclasses of simple objects. Then $\mathcal{A}$ is called* highest weight category *if there are finitely many exact sequences*

$$0 \longrightarrow U_i \longrightarrow P_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \leqslant i \leqslant n) \tag{3.1}$$

in $\mathcal{A}$ satisfying the following:

(1) $\operatorname{End}_{\mathcal{A}}(\Delta_i)$ *is a division ring for all i.*
(2) $\operatorname{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ *for all $i > j$.*
(3) U_i *belongs to* $\operatorname{Filt}(\Delta_{i+1}, \dots, \Delta_n)$ *for all i.*
(4) $\bigoplus_{i=1}^n P_i$ *is a projective generator of $\mathcal{A}$.*

The objects $\Delta_1, \dots, \Delta_n$ are called standard objects.

Now fix a highest weight category $\mathcal{A}$ with standard objects $\Delta_1, \dots, \Delta_n$. Let P denote a projective generator and set $\Lambda = \operatorname{End}_{\mathcal{A}}(P)$. We identify $\mathcal{A} = \operatorname{mod}\Lambda$ via $\operatorname{Hom}_{\mathcal{A}}(P, -)$. Set $\Gamma = \operatorname{End}_{\Lambda}(\Delta_n)$ and note that Δ_n is projective. For each Λ-module X there is a natural exact sequence

$$\operatorname{Hom}_{\Lambda}(\Delta_n, X) \otimes_{\Gamma} \Delta_n \xrightarrow{\varepsilon_X} X \longrightarrow \bar{X} \longrightarrow 0. \tag{3.2}$$

Note that $\operatorname{Ker}\varepsilon_X$ and $\bar{X}$ are annihilated by $\operatorname{Hom}_{\Lambda}(\Delta_n, -)$ since the map $\operatorname{Hom}_{\Lambda}(\Delta_n, \varepsilon_X)$ is invertible. The homomorphism $\Lambda \to \bar{\Lambda}$ identifies via restriction of scalars

$$\operatorname{mod}\bar{\Lambda} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_{\Lambda}(\Delta_n, X) = 0\}$$

and $\bar{X} \cong X \otimes_{\Lambda} \bar{\Lambda}$ for all $X \in \operatorname{mod}\Lambda$.

Lemma 3.2.

(1) *The counit ε_X is a monomorphism for X in* $\operatorname{Filt}(\Delta_1, \dots, \Delta_n)$.
(2) *The assignment $X \mapsto \bar{X}$ provides an exact left adjoint of the inclusion* $\operatorname{Filt}(\Delta_1, \dots, \Delta_{n-1}) \to \operatorname{Filt}(\Delta_1, \dots, \Delta_n)$.

Proof. — An induction on the length of a filtration of an object X in $\operatorname{Filt}(\Delta_1, \dots, \Delta_n)$ yields some $r \geqslant 0$ and an exact sequence $0 \to \Delta_n^r \to X \to X' \to 0$ with X' in $\operatorname{Filt}(\Delta_1, \dots, \Delta_{n-1})$. Then we have $\operatorname{Hom}_{\Lambda}(\Delta_n, X) \otimes_{\Gamma}$

$\Delta_n \cong \Delta_n^r$ and $\bar{X} \cong X'$. The exactness follows from the snake lemma since $\operatorname{Hom}_\Lambda(\Delta_n, -) \otimes_\Gamma \Delta_n$ is exact. ☐

Lemma 3.3. — *Let $\mathcal{A}$ be a highest weight category with standard objects $\Delta_1, \dots, \Delta_n$. For the full subcategory $\bar{\mathcal{A}} = \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(\Delta_n, X) = 0\}$ the following holds:*

(1) *$\bar{\mathcal{A}}$ is a highest weight category with standard objects $\Delta_1, \dots, \Delta_{n-1}$.*
(2) *The inclusion $\bar{\mathcal{A}} \to \mathcal{A}$ induces bijections $\operatorname{Ext}^p_{\bar{\mathcal{A}}}(X, Y) \xrightarrow{\sim} \operatorname{Ext}^p_{\mathcal{A}}(X, Y)$ for all X, Y in $\bar{\mathcal{A}}$ and $p \geqslant 0$.*

Proof. — (1) Applying the assignment $X \mapsto \bar{X}$ to (3.1) yields exact sequences

$$0 \longrightarrow \bar{U}_i \longrightarrow \bar{P}_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \leqslant i \leqslant n-1) \tag{3.3}$$

with $\bar{U}_i$ in $\operatorname{Filt}(\Delta_{i+1}, \dots, \Delta_{n-1})$ by Lemma 3.2. It remains to observe that $\bigoplus_{i=1}^{n-1} \bar{P}_i$ is a projective generator of $\bar{\mathcal{A}}$.

(2) Let $\mathfrak{a}$ denote the kernel of $\Lambda \to \bar{\Lambda}$. This is a projective Λ-module because it is a direct sum of copies of Δ_n by Lemma 3.2. Thus the assertion follows from Lemma 2.7. Alternatively, use Proposition A.1. ☐

The following result establishes the precise connection between highest weight categories and recollements of abelian categories with semisimple factors. For a similar result involving recollements of derived categories, see [21, Theorem 5.13].

Theorem 3.4. — *Let $\mathcal{A}$ be an abelian length category with finitely many simple objects. Suppose that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are Ext-finite. Then the following are equivalent:*

(1) *The category $\mathcal{A}$ is a highest weight category.*
(2) *There is a finite chain of full subcategories*

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_n = \mathcal{A}$$

and a sequence of division rings $\Gamma_1, \dots, \Gamma_n$ such that each inclusion $\mathcal{A}_{i-1} \to \mathcal{A}_i$ induces a homological recollement of abelian categories

$$\begin{tikzcd} \mathcal{A}_{i-1} \arrow[r, tail] & \mathcal{A}_i \arrow[l, two heads, shift right=2ex] \arrow[l, two heads, shift left=2ex] \arrow[r, two heads] & \operatorname{mod}\Gamma_i \arrow[l, tail, shift right=2ex] \arrow[l, tail, shift left=2ex] \end{tikzcd}. \tag{3.4}$$

In that case the standard objects $\Delta_1, \dots, \Delta_n$ in $\mathcal{A}$ are obtained by applying the left adjoint $\operatorname{mod}\Gamma_i \to \mathcal{A}_i$ to Γ_i for $1 \leqslant i \leqslant n$. Conversely, the subcategories $\mathcal{A}_i \subseteq \mathcal{A}$ are obtained by setting $\mathcal{A}_i = \operatorname{Thick}(\Delta_1, \dots, \Delta_i)$ or recursively $\mathcal{A}_{i-1} = \{X \in \mathcal{A}_i \mid \operatorname{Hom}_{\mathcal{A}}(\Delta_i, X) = 0\}$.

Remarks 3.5.

(1) The assertion of Theorem 3.4 remains true if one requires each Γ_i to be a semisimple ring.

(2) The number n in Theorem 3.4 equals the number of pairwise non-isomorphic simple objects in $\mathcal{A}$ and the Γ_i are their endomorphism rings.

(3) Each recollement (3.4) restricts to a diagram of exact functors

$$\begin{tikzcd} \operatorname{Filt}(\Delta_1,\dots,\Delta_{i-1}) \arrow[r, tail, shift right] & \operatorname{Filt}(\Delta_1,\dots,\Delta_i) \arrow[l, two heads, shift right] \arrow[r, two heads, shift right] & \operatorname{Filt}(\Delta_i)\,. \arrow[l, tail, shift right] \end{tikzcd}$$

(4) Each recollement (3.4) induces for the corresponding bounded derived categories a recollement of triangulated categories

$$\begin{tikzcd}[column sep=large] \mathbf{D}^b(\mathcal{A}_{i-1}) \arrow[r, tail] & \mathbf{D}^b(\mathcal{A}_i) \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads] & \mathbf{D}^b(\operatorname{mod}\Gamma_i)\,. \arrow[l, tail, shift right=2] \arrow[l, tail, shift left=2] \end{tikzcd}$$

This follows, for example, from [17, Lemme 2.1.3].

Proof. — (1) $\Rightarrow$ (2): Suppose $\mathcal{A}$ is a highest weight category with standard objects $\Delta_1,\dots,\Delta_n$. Observe that $\mathcal{A}$ has enough injective objects by Proposition 2.8, since $\mathcal{A}^{\mathrm{op}}$ is Ext-finite. We give a recursive construction of a chain

$$0=\mathcal{A}_0\subseteq\mathcal{A}_1\subseteq\ldots\subseteq\mathcal{A}_n=\mathcal{A}$$

of full subcategories satisfying the conditions in (2). Let $\mathcal{A}_{n-1}$ denote the full subcategory of objects X in $\mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(\Delta_n,X)=0$ and set $\Gamma_n=\operatorname{End}_{\mathcal{A}}(\Delta_n)$. The object Δ_n is projective and $\operatorname{Hom}_{\mathcal{A}}(\Delta_n,-)$ induces a recollement

$$\begin{tikzcd}[column sep=large] \mathcal{A}_{n-1} \arrow[r, tail] & \mathcal{A} \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads] & \operatorname{mod}\Gamma_n \arrow[l, tail, shift right=2] \arrow[l, tail, shift left=2] \end{tikzcd}$$

by Lemma 2.5. In Lemma 3.3 it is shown that the recollement is homological and that $\mathcal{A}_{n-1}$ is a highest weight category.

(2) $\Rightarrow$ (1): Fix a chain of full subcategories $\mathcal{A}_i\subseteq\mathcal{A}$ satisfying the conditions in (2). We show by induction on n that $\mathcal{A}$ is a highest weight category. Let Δ_n denote the image of Γ_n under the left adjoint $\operatorname{mod}\Gamma_n\to\mathcal{A}$. Clearly, $\operatorname{End}_{\mathcal{A}}(\Delta_n)\cong\Gamma_n$ and Δ_n is a projective object. The induction hypothesis for $\mathcal{A}_{n-1}$ yields a collection of exact sequences (3.3). We modify them as follows to obtain exact sequences (3.1).

Fix $1\leqslant t<n$. Observe that $\Delta_n/\operatorname{rad}\Delta_n$ is a simple object and that

$$\operatorname{Ext}^1_{\mathcal{A}}(\bar{P}_t,\Delta_n)\xrightarrow{\sim}\operatorname{Ext}^1_{\mathcal{A}}(\bar{P}_t,\Delta_n/\operatorname{rad}\Delta_n)$$

since $\operatorname{rad}\Delta_n$ belongs to $\mathcal{A}_{n-1}$. Using the Ext-finiteness of $\mathcal{A}$, we can form the universal extension

$$0\longrightarrow\Delta_n^r\longrightarrow P_t\longrightarrow\bar{P}_t\longrightarrow 0 \tag{3.5}$$

in $\mathcal{A}$, that is, the induced map $\operatorname{Hom}_{\mathcal{A}}(\Delta_n^r, \Delta_n) \to \operatorname{Ext}^1_{\mathcal{A}}(\bar{P}_t, \Delta_n)$ is surjective. This implies $\operatorname{Ext}^1_{\mathcal{A}}(P_t, \Delta_n) = 0$.

We claim that P_t is a projective object. First observe that for any object X in $\mathcal{A}$, the recollement (3.4) yields an exact sequence

$$0 \longrightarrow \operatorname{Ker} \varepsilon_X \longrightarrow j_! j^!(X) \xrightarrow{\varepsilon_X} X \longrightarrow i_* i^*(X) \longrightarrow 0 \tag{3.6}$$

with $\operatorname{Ker} \varepsilon_X$ in the image of i_*, since $j^!(\varepsilon_X)$ is invertible (using the notation of (2.3)). The functor $\operatorname{Ext}^p_{\mathcal{A}}(P_t, -)$ vanishes for all $p > 0$ on the image of i_* because the recollement is homological, and $\operatorname{Ext}^1_{\mathcal{A}}(P_t, -)$ vanishes on the image of $j_!$ since $\operatorname{Ext}^1_{\mathcal{A}}(P_t, \Delta_n) = 0$. Now one applies the sequence (3.6) by writing it as composite of two exact sequences

$$0 \to \operatorname{Ker} \varepsilon_X \to j_! j^!(X) \to X' \to 0 \quad \text{and} \quad 0 \to X' \to X \to i_* i^*(X) \longrightarrow 0.$$

From the first sequence one gets $\operatorname{Ext}^1_{\mathcal{A}}(P_t, X') = 0$, and then the second sequence gives $\operatorname{Ext}^1_{\mathcal{A}}(P_t, X) = 0$.

Combining the exact sequences (3.3) and (3.5) gives for each t the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Delta_n^r \arrow[r, equal] \arrow[d] & \Delta_n^r \arrow[d] & & \\
0 \arrow[r] & U_t \arrow[r] \arrow[d] & P_t \arrow[r] \arrow[d] & \Delta_t \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bar{U}_t \arrow[r] \arrow[d] & \bar{P}_t \arrow[r] \arrow[d] & \Delta_t \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

This yields exact sequences (3.1) with U_t in $\operatorname{Filt}(\Delta_{t+1}, \ldots, \Delta_n)$, where $P_n := \Delta_n$ and $U_n := 0$. We observe that $\bigoplus_t P_t$ is a projective generator of $\mathcal{A}$.

It remains to show that $\mathcal{A}_t = \operatorname{Thick}(\Delta_1, \ldots, \Delta_t)$ for $1 \leqslant t \leqslant n$. We prove this by induction on t, using the recollement (3.4) and the induced sequence (3.6). For $X \in \mathcal{A}_t$ we have $j_! j^!(X) = \Delta_t^r$ for some $r \geqslant 0$. Also, $\operatorname{Ker} \varepsilon_X$ and $i_* i^*(X)$ are in $\mathcal{A}_{t-1} = \operatorname{Thick}(\Delta_1, \ldots, \Delta_{t-1})$. Thus $X \in \operatorname{Thick}(\Delta_1, \ldots, \Delta_t)$. The other inclusion is clear. □

4. Quasi-hereditary rings

Quasi-hereditary rings provide an alternative concept for describing a highest weight category. Quasi-hereditary algebras over a field were introduced by Scott [25]; the definition given here for semiprimary rings is due to Dlab and Ringel [6].

Recall that a ring Λ is *semiprimary* if its Jacobson radical $J(\Lambda)$ is nilpotent and $\Lambda/J(\Lambda)$ is semisimple. For example, the endomorphism ring of an object having finite composition length is semiprimary.

Definition 4.1. — *An ideal $\mathfrak{a} \subseteq \Lambda$ of a semiprimary ring Λ is an* heredity ideal *if $\mathfrak{a}$ is idempotent, $\mathfrak{a}$ is a projective Λ-module, and $\mathfrak{a}J(\Lambda)\mathfrak{a} = 0$.*

Note that an ideal $\mathfrak{a}$ of a semiprimary ring Λ is idempotent iff there exists an idempotent $e \in \Lambda$ such that $\mathfrak{a} = \Lambda e \Lambda$; see [6, Statement 6]. In that case $\mathfrak{a}J(\Lambda)\mathfrak{a} = 0$ iff the ring $e\Lambda e$ is semisimple.

Definition 4.2. — *A semiprimary ring Λ is* quasi-hereditary *if there is a finite sequence of surjective ring homomorphisms*

$$\Lambda = \Lambda_n \to \Lambda_{n-1} \to \cdots \to \Lambda_1 \to \Lambda_0 = 0 \tag{4.1}$$

such that for each $1 \leqslant i \leqslant n$ the kernel of $\Lambda_i \to \Lambda_{i-1}$ is an heredity ideal. Clearly, such a sequence is equivalent to a finite chain of ideals

$$0 = \mathfrak{a}_n \subseteq \ldots \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_0 = \Lambda$$

such that $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ is an heredity ideal in $\Lambda/\mathfrak{a}_i$ for all i.

For k-linear highest weight categories over a field k, the following result is due to Cline, Parshall, and Scott [5].

Theorem 4.3. — *Let $\mathcal{A}$ be an abelian length category $\mathcal{A}$ having only finitely many isoclasses of simple objects. Then $\mathcal{A}$ is a highest weight category if and only there is a quasi-hereditary ring Λ such that $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}\Lambda$.*

Proof. — The proof is by induction on the number of simple objects in $\mathcal{A}$ and yields an explicit correspondence between the standard objects in $\mathcal{A}$ and the chain of ideals in Λ.

Suppose that $\mathcal{A}$ is a highest weight category and fix the standard objects $\Delta_1, \ldots, \Delta_n$. Then we have $\mathcal{A} = \operatorname{mod}\Lambda$ for a ring Λ and there is a surjective homomorphism $f\colon \Lambda \to \bar{\Lambda}$ such that $\operatorname{mod}\bar{\Lambda} = \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(\Delta_n, X) = 0\}$ is a highest weight category, by Lemma 3.3. The induction hypothesis implies that $\bar{\Lambda}$ is quasi-hereditary, and we need to show that $\mathfrak{a} := \operatorname{Ker} f$ is an heredity ideal. Observe first that $\Delta_n \cong e\Lambda$ for some idempotent $e \in \Lambda$, and therefore $\mathfrak{a} = \Lambda e \Lambda$. We have $eJ(\Lambda)e = 0$ since $e\Lambda e \cong \operatorname{End}_\Lambda(\Delta_n)$ is a division ring. Moreover, $\mathfrak{a}$ is a direct sum of copies of Δ_n, since the counit ε_Λ in (3.2) is a monomorphism by Lemma 3.2. Thus $\mathfrak{a}$ is a projective Λ-module.

Now suppose that Λ is a quasi-hereditary ring with $\mathcal{A} = \operatorname{mod}\Lambda$. Thus there is a sequence of surjective ring homomorphisms (4.1) such that the kernel of each $\Lambda_i \to \Lambda_{i-1}$ is an heredity ideal. We may assume that n is maximal. Set $\bar{\Lambda} = \Lambda_{n-1}$. Then the induction hypothesis implies that $\operatorname{mod}\bar{\Lambda}$ is a highest weight category, say with standard objects $\Delta_1, \dots, \Delta_{n-1}$, and we view this as full subcategory of $\operatorname{mod}\Lambda$ via restriction along $f\colon \Lambda \to \bar{\Lambda}$. There is an idempotent $e \in \Lambda$ such that $\operatorname{Ker} f = \Lambda e \Lambda$ and we set $\Delta_n = e\Lambda$. Then $\operatorname{End}_\Lambda(\Delta_n) \cong e\Lambda e$ is semisimple since $eJ(\Lambda)e = 0$. In fact, it is a divison ring because of the maximality of n. The induction hypothesis yields a collection of exact sequences (3.3) in $\operatorname{mod}\bar{\Lambda}$. We modify them exactly as in the second part of the proof of Theorem 3.4 to obtain exact sequences (3.1). For this construction one uses that $\mathcal{A}$ is Ext-finite (holds by Proposition 2.8) and that $\operatorname{Ext}^p_{\bar{\Lambda}}(-,-) \xrightarrow{\sim} \operatorname{Ext}^p_\Lambda(-,-)$ for all $p \geqslant 0$ (holds by Lemma 2.7). Thus $\operatorname{mod}\Lambda$ is a highest weight category. □

We continue with a reformulation of the definition of a quasi-hereditary ring which makes the concept accessible for interesting constructions. The basic idea is to extend the definition of an heredity ideal to the context of additive categories.

Let $\mathcal{C}$ be an additive category. An *ideal* $\mathfrak{I}$ in $\mathcal{C}$ is given by subgroups

$$\mathfrak{I}(X,Y) \subseteq \operatorname{Hom}_\mathcal{C}(X,Y) \qquad (X,Y \in \mathcal{C})$$

such that any composition $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ of morphisms in $\mathcal{C}$ belongs to $\mathfrak{I}$ if ϕ or ψ belongs to $\mathfrak{I}$.

For a full additive subcategory $\mathcal{B} \subseteq \mathcal{C}$ let $\mathfrak{I}_\mathcal{C}$ denote the ideal in $\mathcal{C}$ given by

$$\mathfrak{I}_\mathcal{C}(X,Y) = \{\phi \in \operatorname{Hom}_\mathcal{C}(X,Y) \mid \phi \text{ factors through some } B \in \mathcal{B}\}.$$

We denote by $\mathcal{C}/\mathcal{B}$ the additive category having the same objects as $\mathcal{C}$ while the morphisms for objects $X, Y \in \mathcal{C}$ are defined by the quotient

$$\operatorname{Hom}_{\mathcal{C}/\mathcal{B}}(X,Y) = \operatorname{Hom}_\mathcal{C}(X,Y)/\mathfrak{I}_\mathcal{B}(X,Y).$$

The *Jacobson radical* $J(\mathcal{C})$ of an additive category $\mathcal{C}$ is by definition the unique ideal in $\mathcal{C}$ such that $J(\mathcal{C})(X,X)$ equals the Jacobson radical of the endomorphism ring $\operatorname{End}_\mathcal{C}(X)$ for every object X in $\mathcal{C}$.

DEFINITION 4.4. — *A full additive subcategory $\mathcal{B} \subseteq \mathcal{C}$ of an additive category $\mathcal{C}$ is called* heredity subcategory *if $J(\mathcal{B}) = 0$ and the inclusion admits a right adjoint $p\colon \mathcal{C} \to \mathcal{B}$ such that for each X in $\mathcal{C}$ the counit $p(X) \to X$ is a monomorphism.*

For a semiprimary ring Λ there is a bijective correspondence between idempotent ideals of Λ and certain additive subcategories of $\operatorname{proj}\Lambda$. Next we show that this restricts to a correspondence between heredity ideals and heredity subcategories.

For an object X of an additive category let $\operatorname{add} X$ denote the full subcategory of direct summands of finite direct sums of copies of X.

LEMMA 4.5. — *Let Λ be a semiprimary ring and set $\mathcal{C} = \operatorname{proj}\Lambda$. The assignments*

$$\Lambda \supseteq \mathfrak{a} \longmapsto \{X \in \mathcal{C} \mid \operatorname{Hom}_\Lambda(X, \Lambda/\mathfrak{a}) = 0\} \subseteq \mathcal{C}$$

and

$$\mathcal{C} \supseteq \mathcal{B} \longmapsto \mathfrak{I}_{\mathcal{B}}(\Lambda, \Lambda) \subseteq \Lambda$$

give mutually inverse and incluson preserving bijections between the sets of

(1) *idempotent ideals of Λ, and*
(2) *strictly full and idempotent complete additive subcategories of* $\operatorname{proj}\Lambda$.

These bijections restrict to a correspondence between heredity ideals and heredity subcategories.

Proof. — For an idempotent ideal $\mathfrak{a} = \Lambda e\Lambda$, an analysis of the recollement (2.4) shows that $\operatorname{add} e\Lambda = \{X \in \mathcal{C} \mid \operatorname{Hom}_\Lambda(X, \Lambda/\mathfrak{a}) = 0\}$. Conversely, any strictly full and idempotent complete additive subcategory $\mathcal{B} \subseteq \mathcal{C}$ is of the form $\mathcal{B} = \operatorname{add} e\Lambda$ for some idempotent $e \in \Lambda$, because the ring Λ is semiperfect. Then $\mathfrak{I}_{\mathcal{B}}(\Lambda, \Lambda) = \Lambda e\Lambda$.

Now fix an ideal $\mathfrak{a} = \Lambda e\Lambda$ and a subcategory $\mathcal{B} = \operatorname{add} e\Lambda$ that correspond to each other. Then $\mathfrak{a}J(\Lambda)\mathfrak{a} = 0$ if and only if $J(\mathcal{B}) = 0$. Assume this property, which means that $e\Lambda e$ is semisimple. The assignment $X \mapsto \operatorname{Hom}_\Lambda(e\Lambda, X) \otimes_{e\Lambda e} e\Lambda$ provides a right adjoint for the inclusion $\mathcal{B} \to \mathcal{C}$. We claim that $\mathfrak{a}$ is a projective Λ-module if and only if the counit ε_X in (2.5) is a monomorphism for all X in $\mathcal{C}$. For this it suffices to consider ε_Λ, using that its image equals $\mathfrak{a}$. If $\mathfrak{a}$ is projective, then ε_Λ is a monomorphism since $\operatorname{Hom}_\Lambda(e\Lambda, \operatorname{Ker}\varepsilon_\Lambda) = 0$. Conversely, if ε_Λ is a monomorphism, then $\mathfrak{a}$ belongs $\mathcal{B}$ and is therefore projective. We conclude that $\mathfrak{a}$ is an heredity ideal if and only if $\mathcal{B}$ is an heredity subcategory. □

Proposition 4.6. — *Let Λ be a semiprimary ring and set $\mathcal{C} = \operatorname{proj}\Lambda$. Then Λ is quasi-hereditary if and only if there is a finite chain of full additive subcategories*

$$0 = \mathcal{C}_n \subseteq \ldots \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_0 = \mathcal{C} \tag{4.2}$$

such that $\mathcal{C}_{i-1}/\mathcal{C}_i$ is an heredity subcategory of $\mathcal{C}/\mathcal{C}_i$ for $1 \leqslant i \leqslant n$.

Proof. — Apply Lemma 4.5. □

We provide a sufficient criterion for the existence of a chain of full additive subcategories (4.2).

Proposition 4.7. — *A semiprimary ring Λ is quasi-hereditary via a chain of full additive subcategories* (4.2) *if the following holds for $1 \leqslant i \leqslant n$:*

(1) *$J(\mathcal{C}_{i-1})(X,Y) \subseteq \mathfrak{I}_{\mathcal{C}_i}(X,Y)$ for all $X, Y \in \mathcal{C}_{i-1}$.*
(2) *The inclusion $\mathcal{C}_i \to \mathcal{C}_{i-1}$ admits a right adjoint p_i such that the counit $p_i(X) \to X$ is a monomorphism in $\mathcal{C}$ for all $X \in \mathcal{C}_{i-1}$.*

The proof is based on the following elementary lemma.

Lemma 4.8. — *Let $\mathcal{C}$ be an additive category and $\mathcal{B} \subseteq \mathcal{C}' \subseteq \mathcal{C}$ full additive subcategories. If the inclusion $\mathcal{C}' \to \mathcal{C}$ admits a right adjoint p such that the counit $p(X) \to X$ is a monomorphism in $\mathcal{C}$ for all $X \in \mathcal{C}$, then the inclusion $\mathcal{C}'/\mathcal{B} \to \mathcal{C}/\mathcal{B}$ admits a right adjoint $\bar{p}$ such that the counit $\bar{p}(X) \to X$ is a monomorphism in $\mathcal{C}/\mathcal{B}$ for all $X \in \mathcal{C}/\mathcal{B}$.*

Proof of Proposition 4.7. — Set $\mathcal{C} = \mathcal{C}_0$. We need to check that $\mathcal{C}_{i-1}/\mathcal{C}_i$ is an heredity subcategory of $\mathcal{C}/\mathcal{C}_i$ for $1 \leqslant i \leqslant n$; see Proposition 4.6. Clearly, (1) implies that $J(\mathcal{C}_{i-1}/\mathcal{C}_i) = 0$. The composite $p_{i-1} \ldots p_2 p_1$ provides a right adjoint for the inclusion $\mathcal{C}_{i-1} \to \mathcal{C}$ and the counit is a monomorphism in $\mathcal{C}$. Now apply Lemma 4.8. □

The following result provides a natural construction of quasi-hereditary rings which is due to Iyama [18].

Corollary 4.9. — *Let $\mathcal{A}$ be an abelian category and suppose that every object in $\mathcal{A}$ has a semiprimary endomorphism ring. Fix an object $X = X_0$ and set $X_{t+1} = \mathfrak{r}X_t$ for $t \geqslant 0$, where $\mathfrak{r}Y = \sum_{\phi \in J(\operatorname{End}_{\mathcal{A}}(Y))} \operatorname{Im}\phi$ for any object Y in $\mathcal{A}$. Then the endomorphism ring of $\bigoplus_{t \geqslant 0} X_t$ is quasi-hereditary.*

Proof. — In [18] the result is stated for modules over artin algebras. The same proof works in our more general setting.

We apply the criterion of Proposition 4.7. Set $\mathcal{C}_i = \operatorname{add}(\bigoplus_{t \geqslant i} X_t)$ for $i \geqslant 0$ and $\Lambda = \operatorname{End}_{\mathcal{A}}(\bigoplus_{t \geqslant 0} X_t)$. Thus we can identify $\operatorname{proj}\Lambda = \mathcal{C}_0$. Note that

$\mathcal{C}_i = 0$ for $i \gg 0$ since $J(\mathrm{End}_{\mathcal{A}}(X))$ is nilpotent. The inclusion $\mathcal{C}_{i+1} \to \mathcal{C}_i$ admits a right adjoint p_i given by $p_i(X_t) = X_t$ for $t > i$ and $p_i(X_i) = X_{i+1}$. The counit $p_i(Y) \to Y$ is a monomorphism for all Y; it is for $Y = X_t$ the identity when $t > i$ and the inclusion $\mathfrak{r}X_i \to X_i$ when $t = i$. This follows from the fact that $\mathfrak{r}X_i \to X_i$ induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X_t, \mathfrak{r}X_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X_t, X_i)$$

for all $t > i$. It remains to observe that $J(\mathcal{C}_i)(X,Y) \subseteq \mathfrak{I}_{\mathcal{C}_{i+1}}(X,Y)$ for all $X, Y \in \mathcal{C}_i$ by construction. □

5. Exceptional sequences

In Theorem 1.1 we have seen a description of highest weight categories via standard objects that suggests a close connection with the concept of an exceptional sequence, as introduced in the study of vector bundles [3, 13, 14, 24]. We make this connection precise, and this involves the use of derived categories.

For an exact category $\mathcal{A}$ let $\mathbf{D}^b(\mathcal{A})$ denote its bounded derived category [19].

Definition 5.1. — *Let $\mathcal{A}$ be an abelian category. An object E in $\mathcal{A}$ is* exceptional *if $\mathrm{End}_{\mathcal{A}}(E)$ is a division ring and $\mathrm{Ext}^p_{\mathcal{A}}(E,E) = 0$ for all $p > 0$. A sequence of objects $(E_1, \dots, E_n)$ in $\mathcal{A}$ is called* exceptional *if each E_i is exceptional and $\mathrm{Ext}^p_{\mathcal{A}}(E_i, E_j) = 0$ for all $i > j$ and $p \geqslant 0$. The sequence is* full *if the objects $E_1, \dots, E_n$ generate $\mathbf{D}^b(\mathcal{A})$ as a triangulated category, and we say that the sequence is* strictly full *if the inclusion $\mathrm{Filt}(E_1, \dots, E_n) \to \mathcal{A}$ induces a triangle equivalence*

$$\mathbf{D}^b(\mathrm{Filt}(E_1, \dots, E_n)) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}).$$

Note that a full exceptional sequence need not be strictly full; see Examples 5.9 and 5.10.

Theorem 5.2. — *Let k be a commutative artinian ring and $\mathcal{A}$ a k-linear abelian category such that $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ and $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$ are finitely generated over k for all X, Y in $\mathcal{A}$. For a sequence $(E_1, \dots, E_n)$ of objects in $\mathcal{A}$ the following are equivalent:*

1. *The sequence $(E_1, \dots, E_n)$ is a strictly full exceptional sequence.*
2. *There is a highest weight category $\mathcal{A}'$ and a triangle equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}')$ that maps $(E_1, \dots, E_n)$ to the sequence of standard objects in $\mathcal{A}'$.*

A special instance of this theorem for vecor bundles on rational surfaces is due to Hille and Perling [16]. For further examples of this connection, relating derived categories of Grassmannians and modular representation theory, see [4, 8].

I am grateful to Lutz Hille for pointing out the following criterion for an exceptional sequence to be strictly full; it is an immediate consequence of the proof of Theorem 5.2.

Remark 5.3 (Hille). — An exceptional sequence $(E_1,\dots,E_n)$ in $\mathcal{A}$ is strictly full if any tilting object in $\operatorname{Filt}(E_1,\dots,E_n)$ is also a tilting object in $\mathcal{A}$.

Recall that an object T of an exact category $\mathcal{A}$ is a *tilting object* if $\operatorname{Ext}^p_{\mathcal{A}}(T,T)=0$ for all $p>0$ and T generates $\mathbf{D}^b(\mathcal{A})$ as a triangulated category.

We need some preparations for the proof Theorem 5.2 and we begin with the following well known fact [27, III.2.4].

Lemma 5.4. — *Let $\mathcal{A}$ be an abelian (or exact) category with projective generator P and set $\Lambda=\operatorname{End}_{\mathcal{A}}(P)$. The inclusion $\operatorname{proj}\Lambda\to\mathcal{A}$ induces a triangle equivalence $\mathbf{D}^b(\operatorname{proj}\Lambda)\xrightarrow{\sim}\mathbf{D}^b(\mathcal{A})$ if all objects of $\mathcal{A}$ have finite projective dimension.*

Lemma 5.5. — *Let $\mathcal{A}$ be a highest weight category with standard objects $\Delta_1,\dots,\Delta_n$. Then the inclusion $\operatorname{Filt}(\Delta_1,\dots,\Delta_n)\to\mathcal{A}$ induces a triangle equivalence*

$$\mathbf{D}^b(\operatorname{Filt}(\Delta_1,\dots,\Delta_n))\xrightarrow{\sim}\mathbf{D}^b(\mathcal{A}).$$

Proof. — This follows from Lemma 5.4 once we have shown that every object in $\mathcal{A}$ has finite projective dimension, keeping in mind that every object in $\operatorname{Filt}(\Delta_1,\dots,\Delta_n)$ admits a projective resolution in $\mathcal{A}$ that belongs to $\operatorname{Filt}(\Delta_1,\dots,\Delta_n)$.

The fact that every object in $\mathcal{A}$ has finite projective dimension is shown by induction on n. Consider $\bar{\mathcal{A}}=\{X\in\mathcal{A}\mid\operatorname{Hom}_{\mathcal{A}}(\Delta_n,X)=0\}$ and for each X in $\mathcal{A}$ the exact sequence (3.2). Then $\operatorname{Ker}\varepsilon_X$ and $\bar{X}$ have finite projective dimension in $\bar{\mathcal{A}}$ since $\bar{\mathcal{A}}$ is a highest weight category with $n-1$ standard objects, by Lemma 3.3. Every projective object from $\bar{\mathcal{A}}$ has projective dimension at most one in $\mathcal{A}$ since it is of the form $\bar{P}$ for some projective P in $\mathcal{A}$ and ε_P is a monomorphism. Thus $\operatorname{Ker}\varepsilon_X$ and $\bar{X}$ have finite projective dimension in $\mathcal{A}$. It follows that X has finite projective dimension. $\square$

Remark 5.6. — The proof of Lemma 5.5 shows that $\operatorname{Ext}^{2n-1}_{\mathcal{A}}(-,-) = 0$ for a highest weight category $\mathcal{A}$ with n standard objects. This bound is well known [6].

Proposition 5.7. — *The standard objects of a highest weight category form a strictly full exceptional sequence.*

Proof. — Let $\Delta_1, \dots, \Delta_n$ be the exceptional objects. It follows from Lemma 3.3 by induction on n that the sequence $(\Delta_1, \dots, \Delta_n)$ is exceptional. The sequence is strictly full by Lemma 5.5. □

The following lemma is the key for relating exceptional sequences and highest weight categories; it is a variation of the "standardisation" which Dlab and Ringel introduced in [7].

Lemma 5.8. — *Let $\mathcal{A}$ be an abelian category and $(E_1, \dots, E_n)$ a sequence of objects satisfying the following:*

(1) $\operatorname{Ext}^1_{\mathcal{A}}(E_i, E_j) = 0$ *for all* $i \geqslant j$.
(2) $\operatorname{Ext}^1_{\mathcal{A}}(X, E_j)$ *is finitely generated over* $\operatorname{End}_{\mathcal{A}}(E_j)^{\mathrm{op}}$ *for all* $X \in \mathcal{A}$.

Then there are exact sequences

$$0 \longrightarrow U_i \longrightarrow P_i \longrightarrow E_i \longrightarrow 0 \qquad (1 \leqslant i \leqslant n) \tag{5.1}$$

in $\mathcal{A}$ such that U_i belongs to $\operatorname{Filt}(E_{i+1}, \dots, E_n)$ *for all i and $\bigoplus_{i=1}^n P_i$ is a projective generator of* $\operatorname{Filt}(E_1, \dots, E_n)$.

Proof. — We use induction on n. The induction hypothesis yields a collection of exact sequences

$$0 \longrightarrow \bar{U}_i \longrightarrow \bar{P}_i \longrightarrow E_i \longrightarrow 0 \qquad (1 \leqslant i < n)$$

in $\operatorname{Filt}(E_{i+1}, \dots, E_{n-1})$. We modify them as follows. Using Ext-finiteness we can form the universal extension

$$0 \longrightarrow E_n^r \longrightarrow P_i \longrightarrow \bar{P}_i \longrightarrow 0$$

in $\mathcal{A}$, that is, the induced map $\operatorname{Hom}_{\mathcal{A}}(E_n^r, E_n) \to \operatorname{Ext}^1_{\mathcal{A}}(\bar{P}_i, E_n)$ is surjective. This implies $\operatorname{Ext}^1_{\mathcal{A}}(P_i, E_n) = 0$. Also, $\operatorname{Ext}^1_{\mathcal{A}}(P_i, -)$ vanishes on $\operatorname{Filt}(E_1, \dots, E_{n-1})$.

We claim that P_i is a projective object in $\operatorname{Filt}(E_1, \dots, E_n)$. First observe that each object X in $\operatorname{Filt}(E_1, \dots, E_n)$ fits into an exact sequence $0 \to E_n^s \to X \to \bar{X} \to 0$ for some $s \geqslant 0$ with $\bar{X}$ in $\operatorname{Filt}(E_1, \dots, E_{n-1})$, since E_n is projective. Now apply $\operatorname{Ext}^1_{\mathcal{A}}(P_i, -)$ to this sequence.

We obtain the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E_n^r \arrow[r, equal] \arrow[d] & E_n^r \arrow[d] & & \\
0 \arrow[r] & U_i \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bar{U}_i \arrow[r] \arrow[d] & \bar{P}_i \arrow[r] \arrow[d] & E_i \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and get exact sequences (5.1) with U_i in $\operatorname{Filt}(E_{i+1},\dots,E_n)$, where $P_n := E_n$ and $U_n := 0$. It remains to observe that $\bigoplus_i P_i$ is a projective generator of $\operatorname{Filt}(E_1,\dots,E_n)$. $\square$

Proof of Theorem 5.2. — (1) $\Rightarrow$ (2): Let $(E_1,\dots,E_n)$ be an exceptional sequence in $\mathcal{A}$. Then it follows from Lemma 5.8 that $\operatorname{Filt}(E_1,\dots,E_n)$ admits a projective generator, say P. Set $\Lambda = \operatorname{End}_{\mathcal{A}}(P)$ and $\Delta_i = \operatorname{Hom}_{\mathcal{A}}(P,E_i)$ for $1 \leqslant i \leqslant n$. Then $\operatorname{Hom}_{\mathcal{A}}(P,-)$ induces a fully faithful and exact functor $\operatorname{Filt}(E_1,\dots,E_n) \to \operatorname{mod}\Lambda$, and $\operatorname{mod}\Lambda$ is a highest weight category with standard objects $\Delta_1,\dots,\Delta_n$ because of the sequences (5.1). If $(E_1,\dots,E_n)$ is strictly full, then $\operatorname{Hom}_{\mathcal{A}}(P,-)$ extends to a triangle equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$ by Lemma 5.5.

(2) $\Rightarrow$ (1): Let $F\colon \mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}')$ be a triangle equivalence that identifies $(E_1,\dots,E_n)$ with the sequence of standard objects $(\Delta_1,\dots,\Delta_n)$ in $\mathcal{A}'$. Then the sequence $(E_1,\dots,E_n)$ is exceptional, because $(\Delta_1,\dots,\Delta_n)$ is exceptional by Proposition 5.7. An induction on n shows that F induces an equivalence

$$\operatorname{Filt}(E_1,\dots,E_n) \xrightarrow{\ \sim\ } \operatorname{Filt}(\Delta_1,\dots,\Delta_n)\,.$$

Here we use the fact that for each object X in $\operatorname{Filt}(E_1,\dots,E_n)$ there is some $r \geqslant 0$ and an exact sequence $0 \to E_n^r \to X \to X' \to 0$ with X' in $\operatorname{Filt}(E_1,\dots,E_{n-1})$. This equivalence extends to a triangle equivalence

$$\mathbf{D}^b(\operatorname{Filt}(E_1,\dots,E_n)) \xrightarrow{\ \sim\ } \mathbf{D}^b(\operatorname{Filt}(\Delta_1,\dots,\Delta_n))$$

making the following square of exact functors commutative

$$\begin{tikzcd} \mathbf{D}^b(\mathrm{Filt}(E_1,\dots,E_n)) \arrow[r, "\sim"] \arrow[d] & \mathbf{D}^b(\mathrm{Filt}(\Delta_1,\dots,\Delta_n)) \arrow[d] \\ \mathbf{D}^b(\mathcal{A}) \arrow[r, "F"'] & \mathbf{D}^b(\mathcal{A}') \end{tikzcd}$$

where the vertical functors are induced by the inclusions $\mathrm{Filt}(E_1,\dots,E_n) \to \mathcal{A}$ and $\mathrm{Filt}(\Delta_1,\dots,\Delta_n) \to \mathcal{A}'$ respectively. The vertical functor on the right is an equivalence by Lemma 5.5, and it follows that the vertical functor on the left is an equivalence. Thus the sequence $(E_1,\dots,E_n)$ is strictly full. $\square$

I am grateful to Martin Kalck for providing the following example of a full exceptional sequence that is not strictly full.

Example 5.9 (Kalck). — Fix a field k and consider the finite dimensional k-algebra Λ given by the following quiver with relations.

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] & \\ 2 \arrow[rr, "\beta"'] & & 3 \arrow[ul, "\gamma"'] \end{tikzcd} \qquad \begin{aligned} \gamma\beta &= 0 \\ \alpha\gamma &= 0 \end{aligned}$$

For each vertex i let S_i denote the corresponding simple Λ-module and P_i its projective cover. Then (S_1, P_2, P_3) is an exceptional sequence in $\mathcal{A} = \mathrm{mod}\,\Lambda$ which generates $\mathbf{D}^b(\mathcal{A})$ as a triangulated category. Set $\mathcal{B} = \mathrm{Filt}(S_1, P_2, P_3)$. Then we have $\mathcal{B} = \mathrm{add}(S_1 \oplus P_2 \oplus P_3)$ but $\mathrm{Ext}^2_\Lambda(S_1, P_2) \neq 0$. Thus the canonical functor $\mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ is not full.

The following geometric example is more involved, and I am grateful to Nathan Broomhead for allowing me to include this.

Example 5.10 (Broomhead). — Let $\mathbb{X}$ be the blow up of $\mathbb{P}^3$ at a torus invariant point. We consider this as a toric variety given by a fan with rays:

$$\{(1,0,0), (0,1,0), (0,0,1), (-1,-1,-1), (1,1,1)\}.$$

Label the corresponding divisors D_1, D_2, D_3, D, E. Note that D and E form a basis of $\mathrm{Pic}\,\mathbb{X}$, where $D_i \sim D - E$ for $i = 1, 2, 3$. An explicit calculation shows that

$$X = (\mathcal{O}(-3D+2E), \mathcal{O}(-2D+E), \mathcal{O}(-D), \mathcal{O}(-2D+2E), \mathcal{O}(-D+E), \mathcal{O})$$

is a full strong exceptional sequence in $\mathcal{A} = \mathrm{coh}\,\mathbb{X}$. Mutating this sequence, we obtain a new full exceptional sequence

$$X' = (\mathcal{O}(-2D+E), \mathcal{O}(-D), \mathcal{O}(-E), \mathcal{O}(-2D+2E), \mathcal{O}(-D+E), \mathcal{O})$$

which is not strictly full. Set $\mathcal{B} = \mathrm{Filt}(X')$. Then

$$\mathcal{B} = \mathrm{add}(\mathcal{O}(-2D+E)\oplus\mathcal{O}(-D)\oplus\mathcal{O}(-E)\oplus\mathcal{O}(-2D+2E)\oplus\mathcal{O}(-D+E)\oplus\mathcal{O})$$

but

$$\mathrm{Ext}^2_{\mathbb{X}}(\mathcal{O}(-E), \mathcal{O}(-2D+2E)) \neq 0\,.$$

Thus the canonical functor $\mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ is not full.

We end this note by giving the proof of Theorem 1.1 from the introduction.

Proof of Theorem 1.1. — Let $\mathcal{A} = \mathrm{mod}\,\Lambda$ for some artin algebra. Suppose first that $\mathcal{A}$ is a highest weight category and fix the standard objects $\Delta_1, \dots, \Delta_n$. Then all but one of the conditions (1)–(4) hold by definition, while (3) follows by induction on n from Lemma 3.3. The converse is an immediate consequence of Lemma 5.8. □

Appendix A. Homological recollements

There are well known criteria for an inclusion of abelian categories $\mathcal{A}' \to \mathcal{A}$ to extend to a fully faithful functor between their derived categories [15, 17, 19], and closely related is the question when the inclusion induces isomorphisms

$$\mathrm{Ext}^p_{\mathcal{A}'}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^p_{\mathcal{A}}(X, Y)$$

for all $X, Y \in \mathcal{A}'$ and $p \geqslant 0$.

The following proposition provides a necessary and sufficient criterion for a colocalisation sequence of abelian categories

$$\text{(A.1)} \qquad \mathcal{A}' \underset{i_*}{\overset{i^*}{\leftleftarrows\!\rightarrowtail}} \mathcal{A} \underset{j^!}{\overset{j_!}{\leftarrowtail\!\twoheadrightarrow}} \mathcal{A}''$$

to extend to a colocalisation sequence of derived categories.

Proposition A.1. — *Suppose that $\mathcal{A}$ has enough projective objects and that $j^!$ preserves projectivity. Then the following are equivalent:*

(1) *The counit $j_! j^!(X) \to X$ is a monomorphism for every projective $X \in \mathcal{A}$.*

(2) *There is an induced colocalisation sequence of triangulated categories*

$$\text{(A.2)} \qquad \mathbf{D}^-(\mathcal{A}') \underset{i_*}{\overset{i^*}{\leftleftarrows\!\rightarrowtail}} \mathbf{D}^-(\mathcal{A}) \underset{j^!}{\overset{j_!}{\leftarrowtail\!\twoheadrightarrow}} \mathbf{D}^-(\mathcal{A}'')\,.$$

Proof. — (1) $\Rightarrow$ (2): Let $\mathcal{P}$ denote the full subcategory of projective objects in $\mathcal{A}$; the categories $\mathcal{P}'$ and $\mathcal{P}''$ are defined analogously. We view $\mathcal{A}'$ and $\mathcal{A}''$ as full subcategories of $\mathcal{A}$ via i_* and $j_!$, respectively, and write $\mathrm{Filt}(\mathcal{P}',\mathcal{P}'')$ for the smallest extension closed subcategory of $\mathcal{A}$ containing $\mathcal{P}'$ and $\mathcal{P}''$. This contains $\mathcal{P}$ since each projective object X fits into an exact sequence

$$0 \longrightarrow j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow 0.$$

Note that the diagram (A.1) restricts to

$$\text{(A.3)} \qquad \mathcal{P}' \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathrm{Filt}(\mathcal{P}',\mathcal{P}'') \underset{j^!}{\overset{j_!}{\leftrightarrows}} \mathcal{P}''$$

and all functors in this diagram are exact. The only functor for which this is not obvious is i^*. In that case exactness follows from the snake lemma because the counit $j_!j^!(X) \to X$ is a monomorphism for every X in $\mathrm{Filt}(\mathcal{P}',\mathcal{P}'')$. Thus the diagram (A.3) induces a colocalisation sequence

$$\text{(A.4)} \qquad \mathbf{D}^-(\mathcal{P}') \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathbf{D}^-(\mathrm{Filt}(\mathcal{P}',\mathcal{P}'')) \underset{j^!}{\overset{j_!}{\leftrightarrows}} \mathbf{D}^-(\mathcal{P}'').$$

We claim that the diagrams (A.2) and (A.4) are equivalent via triangle equivalences induced by the inclusions

$$f'\colon \mathcal{P}' \to \mathcal{A}' \qquad f''\colon \mathcal{P}'' \to \mathcal{A}'' \qquad f\colon \mathrm{Filt}(\mathcal{P}',\mathcal{P}'') \to \mathcal{A}.$$

This is clear for f' and f'', since $\mathcal{A}'$ and $\mathcal{A}''$ have enough projective objects. For f it suffices to note that the inclusion $\mathcal{P} \to \mathrm{Filt}(\mathcal{P}',\mathcal{P}'')$ yields a triangle equivalence $\mathbf{D}^-(\mathcal{P}) \xrightarrow{\sim} \mathbf{D}^-(\mathrm{Filt}(\mathcal{P}',\mathcal{P}''))$, since $\mathcal{P}$ equals the full subcategory of projective objects of the exact category $\mathrm{Filt}(\mathcal{P}',\mathcal{P}'')$.

(2) $\Rightarrow$ (1): Suppose there is a colocalisation sequence (A.2). Given a projective object X in $\mathcal{A}$, we have an exact triangle

$$j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow$$

in $\mathbf{D}^-(\mathcal{A})$. This uses the fact that for complexes of projectives the derived functors of i^* and $j_!$ are defined degreewise via i^* and $j_!$, respectively. Taking cohomology, we obtain an exact sequence

$$\cdots \longrightarrow 0 \longrightarrow j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow 0 \longrightarrow \cdots$$

in $\mathcal{A}$. It follows that the counit $j_!j^!(X) \to X$ is a monomorphism. $\square$

Remark A.2. — There is a dual version of Proposition A.1 for localisation sequences of abelian categories with enough injective objects. This situation arises frequently, a typical example being a Grothendieck abelian category $\mathcal{A}$ with localising subcategory $\mathcal{A}'$.

Remark A.3. — Proposition A.1 covers a couple of known criteria for an inclusion of abelian categories $\mathcal{A}' \to \mathcal{A}$ to extend to a fully faithful functor between their derived categories. Consider a colocalisation sequence (A.1) and suppose that $\mathcal{A}$ has enough projective objects.

(1) The criterion in [15, Proposition 4.8] requires that every object Y in $\mathcal{A}'$ admits an epimorphism $X \to Y$ in $\mathcal{A}'$ with X projective in $\mathcal{A}$. This implies easily that $j^!$ preserves projectivity and that for each projective X in $\mathcal{A}$ the counit $j_! j^!(X) \to X$ is a *split* monomorphism.

(2) The criterion in [19, §12] requires for every epimorphism $X \to Y$ in $\mathcal{A}$ with Y in $\mathcal{A}'$ the existence of an epimorphism $X' \to Y$ in $\mathcal{A}'$ that factors through $X \to Y$. Given our assumptions, this condition is equivalent to the one in [15, Proposition 4.8].

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