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MIXED HODGE STRUCTURES AND SULLIVAN'S MINIMAL MODELS OF SASAKIAN MANIFOLDS

by Hisashi KASUYA (*)

Abstract. — We show that the Malčev Lie algebra of the fundamental group of a compact $2n+1$-dimensional Sasakian manifold with $n \geqslant 2$ admits a quadratic presentation by using Morgan's bigradings of minimal models of mixed-Hodge diagrams. By using bigradings of minimal models, we also simplify the proof of the result of Cappelletti–Montano, De Nicola, Marrero and Yudin on Sasakian nilmanifolds.

Résumé. — Nous montrons, en utilisant les bigraduations de Morgan de modèles minimaux de diagrammes de Hodge, que l'algèbre de Lie de Malčev du groupe fondamental d'une variété sasakienne compacte de dimension $2n+1$ admet une présentation quadratique pour $n \geqslant 2$. A l'aide de bigraduations de modèles minimaux, nous simplifions également la démonstration du résultat de Cappelletti–Montano, De Nicola, Marrero et Yudin sur les nilvariétés sasakiennes.

1. Introduction

Let Γ be a group and $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \dots$ the lower central series (i.e. $\Gamma_i = [\Gamma_{i-1}, \Gamma]$). Consider the tower of nilpotent groups

$$\cdots \to \Gamma/\Gamma_3 \to \Gamma/\Gamma_2 \to \{e\}.$$

Then it is possible to "tensor" these nilpotent groups with $\mathbb{R}$ or $\mathbb{C}$ ([1, 8, 12]) and we obtain the tower of real nilpotent Lie algebras

$$\cdots \to \mathfrak{n}_3 \to \mathfrak{n}_2 \to \{0\}.$$

The inverse limit of this tower is called the Malčev Lie algebra of Γ.

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Let M be a manifold. By Sullivan's de Rham homotopy theory ([16]), the Malčev Lie algebra of the fundamental group $\pi_1(M)$ can be studied by the differential forms on M. The formality of compact Kähler manifolds ([8]) implies that if M is a compact Kähler manifold, then the Malčev Lie algebra of the fundamental group $\pi_1(M)$ admits a quadratic presentation (i.e. is a quotient of a free Lie algebra by an ideal generated in degree two). This fact is very useful to study whether a given finitely generated group can be the fundamental group of a compact Kähler manifold.

In this paper, we consider Sasakian manifolds. Sasakian manifolds constitute an odd-dimensional counterpart of the class of Kähler manifolds. We are interested in whether the Malčev Lie algebras of the fundamental groups of compact Sasakian manifolds admit quadratic presentations.

First, we see important examples. We consider the $2n+1$-dimensional real Heisenberg group H_{2n+1} which is the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & I & {}^t y \\ 0 & 0 & 1 \end{pmatrix}$$

where I is the $n \times n$ unit matrix, $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. For any lattice Γ in H_{2n+1}, the nilmanifold $\Gamma\backslash H_{2n+1}$ admits a Sasakian structure and $\pi_1(\Gamma\backslash H_{2n+1}) \cong \Gamma$. It is known that for any $n \geqslant 2$, the Malčev Lie algebra of Γ admits a quadratic presentation (see [7]). However, if $n = 1$, then the Malčev Lie algebra of Γ cannot admit a quadratic presentation.

In this paper, we extend this observation. We prove the following theorem.

Theorem 1.1. — *Let M be a compact $2n+1$-dimensional Sasakian manifold with $n \geqslant 2$. Then the Malčev Lie algebra of $\pi_1(M)$ admits a quadratic presentation.*

Remark 1.2. — It seems that this fact is known by some experts. However, it is difficult to find an explicit proof in some reference.

We notice that Sasakian manifolds are not formal in general, unlike Kähler manifolds. See [2, 14] for the formality of Sasakian manifolds. But we can apply some Hodge theoretical properties of Sasakian manifolds like algebraic varieties. By using Morgan's techniques of mixed Hodge diagrams [13], we will show the following theorem.

Theorem 1.3. — *Let M be a $2n+1$-dimensional compact Sasakian manifold and $A^*_{\mathbb{C}}(M)$ the de Rham complex of M. Consider the minimal* *model $\mathcal{M}$ (resp 1-minimal model) of $A^*_{\mathbb{C}}(M)$ with a quasi-isomorphism (resp. 1-quasi-isomorphism) $\phi : \mathcal{M} \to A^*_{\mathbb{C}}(M)$. Then we have:*

1. *The real de Rham cohomology $H^*(M, \mathbb{R})$ admits a $\mathbb{R}$-mixed-Hodge structure.*
2. *$\mathcal{M}^*$ admits a bigrading*
$$\mathcal{M}^* = \bigoplus_{p,q \geqslant 0} \mathcal{M}^*_{p,q}$$
such that $\mathcal{M}^*_{0,0} = \mathcal{M}^0 = \mathbb{C}$ and the product and the differential are of type $(0,0)$.
3. *Consider the bigrading $H^*(M, \mathbb{C}) = \bigoplus V_{p,q}$ which will be given in Proposition 2.3 for the $\mathbb{R}$-mixed-Hodge structure. Then the induced map $\phi^* : H^*(\mathcal{M}^*) \to H^*(M, \mathbb{C})$ sends $H^*(\mathcal{M}^*_{p,q})$ to $V_{p,q}$.*

By using this theorem, we prove Theorem 1.1. Moreover, we will give another application of Theorem 1.3. In [6], it is proved that a compact $2n+1$-dimensional nilmanifold admits a Sasakian structure if and only if it is a Heisenberg nilmanifold H_{2n+1}/Γ. By using Theorem 1.3, we can simplify the proof of this result (Section 8).

2. Mixed Hodge structures

Definition 2.1. — *An* $\mathbb{R}$-Hodge structure *of weight n on a $\mathbb{R}$-vector space V is a finite bigrading*
$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V_{p,q}$$
on the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that
$$\overline{V_{p,q}} = V_{q,p}\,,$$
or equivalently, a finite decreasing filtration F^* on $V_{\mathbb{C}}$ such that
$$F^p(V_{\mathbb{C}}) \oplus \overline{F^{n+1-p}(V_{\mathbb{C}})}$$
for each p.

Definition 2.2. — *An* $\mathbb{R}$-mixed-Hodge structure *on an $\mathbb{R}$-vector space V is a pair (W_*, F^*) such that:*

1. *W_* is an increasing filtration which is bounded below,*
2. *F^* is a decreasing filtration on $V_{\mathbb{C}}$ called such that the filtration on $Gr^W_n V_{\mathbb{C}}$ induced by F^* is an $\mathbb{R}$-Hodge structure of weight n.*

We call W_* the weight filtration and F^* the Hodge filtration.

Proposition 2.3 ([13, Proposition 1.9]). — *Let* (W_*, F^*) *be an* $\mathbb{R}$-*mixed-Hodge structure on an* $\mathbb{R}$-*vector space* V. *Define* $V_{p,q} = R_{p,q} \cap L_{p,q}$ *where* $R_{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap F^p(V_{\mathbb{C}})$ *and* $L_{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})} + \sum_{i\geqslant 2} W_{p+q-i}(V_{\mathbb{C}}) \cap \overline{F^{q-i+1}(V_{\mathbb{C}})}$. *Then we have the bigrading* $V_{\mathbb{C}} = \bigoplus V_{p,q}$ *such that:* $W_i(V_{\mathbb{C}}) = \bigoplus_{p+q\leqslant i} V_{p,q}$ *and* $F^i(V_{\mathbb{C}}) = \bigoplus_{p\geqslant i} V_{p,q}$.

Proposition 2.4 ([13, Proposition 1.11]). — *Let* V *be an* $\mathbb{R}$-*vector space. We suppose that we have a bigrading* $V_{\mathbb{C}} = \bigoplus V^{p,q}$ *such that* $\overline{V_{p,q}} = V_{q,p}$ *modulo* $\bigoplus_{r+s<p+q} V_{r,s}$ *and the grading* $\mathrm{Tot}^* V_{*,*}$ *is bounded below where* $\mathrm{Tot}^r V_{*,*} = \bigoplus_{p+q=r} V_{p,q}$. *Then the filtrations* W *and* F *such that* $W_i(V_{\mathbb{C}}) = \bigoplus_{p+q\leqslant i} V_{p,q}$ *and* $F^i(V_{\mathbb{C}}) = \bigoplus_{p\geqslant i} V_{p,q}$ *give an* $\mathbb{R}$-*mixed Hodge structure on* V.

3. Sullivan's minimal models

Let k be a field of characteristic 0. We recall Sullivan's minimal model of a differential graded algebra (shortly DGA).

Definition 3.1. — *Let* A^* *and* B^* *be* k-*DGAs.*

- *A morphism* $\phi : A^* \to B^*$ *is a quasi-isomorphism (resp.* 1-*quasi-isomorphism) if* ϕ *induces a cohomology isomorphism (resp. isomorphisms on the* 0-*th and first cohomologies and an injection on the second cohomology).*
- A^* *and* B^* *are* quasi-isomorphic *(resp.* 1-quasi-isomorphic*) if there is a finite diagram of* k-*DGAs*

$$A \leftarrow C_1 \to C_2 \leftarrow \cdots \leftarrow C_n \to B$$

such that all the morphisms are quasi-isomorphisms (resp. 1-*quasi-isomorphisms).*

Definition 3.2. — *A* k-*DGA* A^* *with a differential* d *is* minimal *if the following conditions hold.*

- A^* *is a free graded algebra* $\bigwedge \mathcal{V}^*$ *generated by a graded* k-*vector space* $\mathcal{V}^*$ *with* $* \geqslant 1$.
- $\mathcal{V}^*$ *admits a basis* $\{x_i\}$ *which is indexed by a well-ordered set such that* $dx_i \in \bigwedge\langle\{x_j | j < i\}\rangle$
- $d(\mathcal{V}^*) \subset \bigwedge^2 \mathcal{V}^*$.

A minimal k-*DGA* A^* *is* 1-minimal *if* A^* *is generated by elements of degree* 1.

Definition 3.3. — *Let A^* be a k-DGA with $H^0(A^*) = k$. A minimal (resp.* 1*-minimal) k-DGA $\mathcal{M}$ is the* minimal model *(resp.* 1-minimal model*) of A^* if there is a quasi-isomorphism (resp.* 1*-quasi-isomorphism) $\mathcal{M} \to A^*$.*

Theorem 3.4 ([16]). — *For a k-DGA A^* with $H^0(A^*) = k$, the minimal model (resp.* 1*-minimal model) of A^* exists and it is unique up to k-DGA isomorphism.*

Consequently, if A is quasi-isomorphic (resp. 1-quasi-isomorphic) to a k-DGA B, then A^* and B^* have the same minimal model (resp. 1-minimal model).

Definition 3.5. — *Let A^* be a k-DGA. We say that A^* is* formal *(resp.* 1-formal*) if A^* is quasi-isomorphic (resp.* 1*-quasi-isomorphic) to the k-DGA $H^*(A^*)$ with the trivial differential.*

Theorem 3.6 ([8]). — *Let M be a compact Kähler manifold. Then the de Rham complex $A^*(M)$ is formal.*

Example 3.7. — Let N be a real simply connected nilpotent Lie group with a lattice (i.e. cocompact discrete subgroup) Γ. Then the compact quotient $\Gamma\backslash N$ is called nilmanifold. Let $\mathfrak{n}$ be the Lie algebra of N. Consider the cochain complex $\bigwedge \mathfrak{n}^*$ with the differential d which is the dual to the Lie bracket. Then, by the nilpotency, we can easily check that $\bigwedge \mathfrak{n}^*$ is a minimal DGA. We regard $\bigwedge \mathfrak{n}^*$ as the space of left-invariant differential forms on $\Gamma\backslash N$. Let $A^*(\Gamma\backslash N)$ be the de Rham complex of $\Gamma\backslash N$. In [15], Nomizu proved that the canonical inclusion $\bigwedge \mathfrak{n}^* \hookrightarrow A^*(\Gamma\backslash N)$ induces a cohomology isomorphism. Thus the DGA $\bigwedge \mathfrak{n}^*$ is the minimal model of the de Rham complex $A^*(\Gamma\backslash N)$. Since $\bigwedge \mathfrak{n}^*$ is generated by elements of degree 1, $\bigwedge \mathfrak{n}^*$ is 1-minimal and hence it is also the 1-minimal model of $A^*(\Gamma\backslash N)$. It is proven in [11] that the DGA $\bigwedge \mathfrak{n}^*$ is formal if and only if the Lie algebra $\mathfrak{n}$ is Abelian. Hence, $A^*(\Gamma\backslash N)$ is formal if and only if the nilmanifold $\Gamma\backslash N$ is a torus. Consequently, Kähler nilmanifolds are only tori.

Remark 3.8. — In [9], Fernández and Muñoz define the "s-formality" ([9, Definition 2.4]). We notice that the 1-formality in the sense of Fernández–Muñoz is essentially different from the 1-formality as in Definition 3.5. In fact, for a nilpotent Lie algebra $\mathfrak{g}$, the DGA $\bigwedge \mathfrak{g}^*$ is 1-formal in the sense of Fernández–Muñoz if and only if $\mathfrak{g}$ is Abelian. On the other hand, there exist non-Abelian nilpotent Lie algebras $\mathfrak{g}$ such that $\bigwedge \mathfrak{g}^*$ is 1-formal as in Definition 3.5.

4. 1-minimal model and 1-formality

We see the construction of 1-minimal model of the de Rham complex in detail. The article [1, Chapter 3] is a good reference. Let A^* be a k-DGA and $\mathcal{M}$ the 1-minimal model of A^* with a 1-quasi-isomorphism $\phi : \mathcal{M}^* \to A^*$. Then we have the canonical sequence of DGAs

$$\mathcal{M}^*(1) \subset \mathcal{M}^*(2) \subset \dots$$

such that:

(1) $\mathcal{M}^* = \bigcup_{k=1}^{\infty} \mathcal{M}^*(k)$.
(2) $\mathcal{M}^*(1) = \bigwedge \mathcal{V}_1$ with the trivial differential such that ϕ induces an isomorphism $\mathcal{V}_1 \to H^1(A^*)$.
(3) Consider the map $\phi_1 : \mathcal{V}_1 \wedge \mathcal{V}_1 \to H^2(A^*)$ induced by ϕ. We have $\mathcal{M}^*(2) = \mathcal{M}^*(1) \otimes \bigwedge \mathcal{V}_2$ such that the differential d induces an isomorphism $\mathcal{V}_2 \to \operatorname{Ker} \phi_1$.
(4) Consider the map $\phi_n : H^2(\mathcal{M}^*(n)) \to H^2(A^*)$ induced by ϕ for each integer n. We have $\mathcal{M}^*(n+1) = \mathcal{M}^*(n) \otimes \bigwedge \mathcal{V}_{n+1}$ such that the differential $d : \mathcal{V}_{n+1} \to \mathcal{M}^2(n)$ induces an isomorphism $\mathcal{V}_{n+1} \to \operatorname{Ker} \phi_n$.

Let M be a manifold and $A^*(M)$ the de Rham complex. Suppose $A^* = A^*(M)$. Dualizing the sequence

$$\mathcal{M}^*(1) \subset \mathcal{M}^*(2) \subset \dots ,$$

we obtain the tower of real nilpotent Lie algebras

$$\dots \to L_2 \to L_1 \to 0 .$$

Sullivan showed that this tower gives the Malčev Lie algebra of the fundamental group $\pi_1(M)$ (see [8]).

Proposition 4.1 ([1, Chapter 3]). — *Let M be a manifold. Then the following conditions are equivalent:*

(1) *$A^*(M)$ is 1-formal.*
(2) *The Malčev Lie algebra of the fundamental group $\pi_1(M)$ admits a quadratic presentation.*
(3) *Consider the 1-minimal model $\mathcal{M}^*$ of $A^*(M)$ and the canonical sequence of DGAs*

$$\mathcal{M}^*(1) \subset \mathcal{M}^*(2) \subset \dots$$

as above. Then the map $H^2(\mathcal{M}^*(1)) \to H^2(\mathcal{M}^*)$ is surjective.

By the result in [8], for a compact Kähler manifold M, the Malčev Lie algebra of the fundamental group $\pi_1(M)$ admits a quadratic presentation. For example, for a compact surface S with the genus $g \geqslant 2$, the Malčev Lie algebra of the fundamental group $\pi_1(S)$ is the free Lie algebra on $X_1, \dots X_g, Y_1, \dots, Y_g$ modulo the relation $\sum_{i=1}^{g}[X_i, Y_i] = 0$ (see [16, Section 12]).

Example 4.2. — Let N be a real simply connected nilpotent Lie group with a lattice Γ and $\mathfrak{n}$ be the Lie algebra of N. Then, as in Example 3.7, the DGA $\bigwedge \mathfrak{n}^*$ is the 1-minimal model of $A^*(\Gamma\backslash N)$. Since the fundamental group of the nilmanifold $\Gamma\backslash N$ is Γ, the Malčev Lie algebra of Γ is $\mathfrak{n}$. Suppose $N = H_{2n+1}$ where H_{2n+1} is the $2n+1$-dimensional real Heisenberg group. Then the DGA $\bigwedge \mathfrak{n}^*$ is given by $\mathfrak{n}^* = \langle x_1, \dots, x_{2n}, y\rangle$ so that $dx_i = 0$ and

$$dy = x_1 \wedge x_2 + \dots + x_{2n-1} \wedge x_{2n}\,.$$

As in the above argument, we have $\mathcal{V}_1 = \langle x_1, \dots x_{2n}\rangle$ and $\mathcal{V}_2 = \langle y\rangle$. Suppose $n \geqslant 2$. Then we have $H^2(\mathfrak{n}) = \mathcal{V}_1 \wedge \mathcal{V}_1/\langle dy\rangle$ and hence the map $H^2(\mathcal{M}^*(1)) \to H^2(\mathcal{M}^*)$ is surjective. By Proposition 4.1, the Malčev Lie algebra of Γ admits a quadratic presentation. On the other hand, if $n = 1$, then we can check that Γ does not admit a quadratic presentation (see [1, Example 3.31]). Consequently, a lattice in the 3-dimensional real Heisenberg group can not be the fundamental group of any compact Kähler manifold.

See [7] for more examples of nilpotent groups admitting quadratic presentations.

5. Morgan's mixed Hodge diagrams

Definition 5.1 ([13, Definition 3.5]). — *An* $\mathbb{R}$-mixed-Hodge diagram *is a pair of filtered $\mathbb{R}$-DGA (A^*, W_*) and bifiltered $\mathbb{C}$-DGA (E^*, W_*, F^*) and filtered DGA map $\phi : (A^*_{\mathbb{C}}, W_*) \to (E^*, W_*)$ such that:*

1. *ϕ induces an isomorphism $\phi^* : {}_WE_1^{*,*}(A^*_{\mathbb{C}}) \to {}_WE_1^{*,*}(E^*)$ where ${}_WE_*^{*,*}(\cdot)$ is the spectral sequence for the decreasing filtration $W^* = W_{-*}$.*
2. *The differential d_0 on ${}_WE_0^{*,*}(E^*)$ is strictly compatible with the filtration induced by F.*
3. *The filtration on ${}_WE_1^{p,q}(E^*)$ induced by F is a $\mathbb{R}$-Hodge structure of weight q on $\phi^*({}_WE_1^{*,*}(A^*))$.*

Theorem 5.2 ([13, Theorem 4.3]). — *Let* $\{(A^*, W_*), (E^*, W_*, F^*), \phi\}$ *be an* $\mathbb{R}$*-mixed-Hodge diagram. Define the filtration* W'_* *on* $H^r(A^*)$ *(resp* $H^r(E^*)$*) as* $W'_i H^r(A^*) = W_{i-r}(H^r(A^*))$ *(resp.* $W'_i H^r(E^*) = W_{i-r}(H^r(E^*))$*). Then the filtrations* W'_* *and* F^* *on* $H^r(E^*)$ *give an* $\mathbb{R}$*-mixed-Hodge on* $\phi^*(H^r(A^*))$*.*

Theorem 5.3 ([13, Section 6]). — *Let* $\{(A^*, W_*), (E^*, W_*, F^*), \phi\}$ *be an* $\mathbb{R}$*-mixed-Hodge diagram. Then the minimal model (resp. 1-minimal model)* $\mathcal{M}^*$ *of the DGA* E^* *with a quasi-isomorphism (resp. 1-quasi-isomorphism)* $\phi : \mathcal{M}^* \to E^*$ *satisfies the following conditions:*

- $\mathcal{M}^*$ *admits a bigrading*
$$\mathcal{M}^* = \bigoplus_{p,q\geqslant 0} \mathcal{M}^*_{p,q}$$
such that $\mathcal{M}^*_{0,0} = \mathcal{M}^0 = \mathbb{C}$ *and the product and the differential are of type* $(0, 0)$*.*
- *Consider the bigrading* $H^*(E^*) = \bigoplus V_{p,q}$ *as in Proposition 2.3 for the* $\mathbb{R}$*-mixed-Hodge structure as in Theorem 5.2. Then* $\phi^* : H^*(\mathcal{M}^*) \to H^*(E^*)$ *sends* $H^*(\mathcal{M}^*_{p,q})$ *to* $V_{p,q}$*.*

We explain Morgan's construction for 1-minimal models. For an $\mathbb{R}$-mixed-Hodge diagram $\{(A^*, W_*), (E^*, W_*, F^*), \phi\}$, let $\mathcal{M}^*$ be the 1-minimal model of E^* with a 1-quasi-isomorphism $\phi : \mathcal{M}^* \to E^*$. Take the canonical sequence of DGAs
$$\mathcal{M}^*(1) \subset \mathcal{M}^*(2) \subset \dots$$
as in Section 4. Then the bigrading as in Theorem 5.3 constructed by the following inductive way. For each n, we let $\mathcal{M}^*(n)$ have a bigrading
$$\mathcal{M}^*(n) = \bigoplus_{p,q\geqslant 0} \mathcal{M}^*_{p,q}(n)$$
such that:

- $\mathcal{M}^*_{0,0}(n) = \mathcal{M}^0(n) = \mathbb{C}$ and the product and the differential are of type $(0, 0)$.
- Consider the bigrading $H^*(E^*) = \bigoplus V_{p,q}$ as in Proposition 2.3 for the $\mathbb{R}$-mixed-Hodge structure as in Theorem 5.2. Then $\phi^* : H^*(\mathcal{M}^*(n)) \to H^*(E^*)$ sends $H^*(\mathcal{M}^*_{p,q}(n))$ to $V_{p,q}$.
- For $\mathcal{M}^*(n+1) = \mathcal{M}^*(n) \otimes \bigwedge \mathcal{V}_{n+1}$, $\mathcal{V}_{n+1}$ has a bigrading such that the differential $d : \mathcal{V}_{n+1} \to \mathcal{M}^2(n)$ is compatible with bigradings and the bigrading on $\mathcal{M}^*(n + 1)$ is the multiplicative extension of the bigradings on $\mathcal{V}_{n+1}$ and $\mathcal{M}^*(n)$.

6. Mixed Hodge diagrams of Sasakian manifolds

Let M be a compact $(2n+1)$-dimensional Sasakian manifold with a Sasakian metric g and η the contact structure associated with the Sasakian structure. Take ξ the Reeb vector field. ξ gives the 1-dimensional foliation $\mathcal{F}_\xi$. Let $A^*(M)$ be the de Rham complex of M. A differential form $\alpha \in A^*(M)$ is basic if $\iota_\xi \alpha = 0$ and $\iota_\xi d\alpha = 0$. Denote by $A^*_B(M)$ the differential graded algebra of the basic differential forms on M and denote by $H^*_B(M,\mathbb{R})$ (resp. $H^*_B(M,\mathbb{C})$) the cohomology of $A^*_B(M)$ (resp. $A^*_B(M)\otimes\mathbb{C}$). For a closed basic form α, we denote by $[\alpha]_B$ the cohomology class of α in $H^*_B(M,\mathbb{R})$ (resp. $H^*_B(M,\mathbb{C})$).

We have the transverse complex structure on $\operatorname{Ker}\iota_\xi \subset \bigwedge TM^*$ and we obtain the bigrading $A^r_B(M)\otimes\mathbb{C} = \bigoplus_{p+q=r} A^{p,q}_B(M)$ with the bidifferential $d = \partial_B + \bar\partial_B$. We have the $\partial_B\bar\partial_B$-Lemma:

Proposition 6.1 ([17, Proposition 3.7]). — *For each r, in $A^r_B(M)\otimes\mathbb{C}$, we have*

$$\operatorname{Ker}\partial_B \cap \operatorname{Ker}\bar\partial_B \cap \operatorname{Im} d = \operatorname{Im}\partial_B\bar\partial_B\,.$$

We define the DGA $A^* = H^*_B(M,\mathbb{R})\otimes \bigwedge\langle y\rangle$ such that y is an element of degree 1 and $dy = [d\eta]_B$. By using Proposition 6.1, we obtain the following theorem.

Theorem 6.2 ([17, Section 4.3]). — *The DGAs A^* and $A^*(M)$ are quasi-isomorphic.*

Define the basic Bott–Chern cohomology $H^{*,*}_B(M)$ as

$$H^{*,*}_B(M) = \frac{\operatorname{Ker}\partial_B \cap \operatorname{Ker}\bar\partial_B}{\operatorname{Im}\partial_B\bar\partial_B}\,.$$

Then we have $\overline{H^{p,q}_B(M)} = H^{q,p}_B(M)$ and the natural map

$$\operatorname{Tot}^* H^{*,*}_B(M) \to H^*_B(M,\mathbb{C})\,.$$

By $\partial_B\bar\partial_B$-Lemma, this natural map is an isomorphism (see [8, Remark 5.16]). Hence we have the $\mathbb{R}$-Hodge structure of weight r on $H^r_B(M,\mathbb{R})$ as

$$H^r_B(M,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_B(M)\,.$$

We consider the bigrading $A^*_{\mathbb{C}} = \bigoplus A^*_{p,q}$ such that

$$A^*_{p,q} = H^{p,q}_B(M) \oplus H^{p-1,q-1}_B(M)\wedge y\,.$$

Definition 6.3.

(1) *We define the increasing filtration W_* on the DGA A^* such that $W_{-1}A^* = 0$, $W_0A^* = H_B^*(M, \mathbb{R})$ and $W_1A^* = A^*$.*

(2) *We define the decreasing filtration F^* on the DGA $A_{\mathbb{C}}^*$ such that*

$$F^k(A_{\mathbb{C}}^*) = \bigoplus_{p \geqslant k} A_{p,q}^* .$$

Theorem 6.4. — *The pair (A^*, W_*), $(A_{\mathbb{C}}^*, W_*, F^*)$ and $\mathrm{id} : A_{\mathbb{C}}^* \to A_{\mathbb{C}}^*$ is a $\mathbb{R}$-mixed-Hodge diagram.*

Proof. — We can easily check that the differential d_0 on ${}_WE_0^{p,q}(A_{\mathbb{C}}^*)$ is zero and we obtain

$$\begin{aligned} {}_WE_1^{0,q}(A_{\mathbb{C}}^*) &\cong H_B^q(M, \mathbb{C}), \\ {}_WE_1^{-1,q}(A_{\mathbb{C}}^*) &\cong H_B^{q-2}(M, \mathbb{C}) \wedge y \end{aligned}$$

and other ${}_WE_1^{p,q}(A_{\mathbb{C}}^*)$ is trivial. Hence, by $H_B^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H_B^{p,q}(M)$ and $\overline{H_B^{p,q}(M)} = H_B^{q,p}(M)$, the filtration F induces $\mathbb{R}$-Hodge structures of weight q on ${}_WE_1^{0,q}(A_{\mathbb{C}}^*)$ and ${}_WE_1^{-1,q}(A_{\mathbb{C}}^*)$ such that ${}_WE_1^{-1,q}(A_{\mathbb{C}}^*) = \bigoplus_{s+t=q} V_{s,t}$ with $V_{s,t} = H_B^{s-1,t-1}(M) \wedge y$. Hence the theorem follows. □

Hence, by Theorem 5.3, we obtain the following theorem.

Theorem 6.5. — *Let M be a $2n+1$-dimensional compact Sasakian manifold and $A_{\mathbb{C}}^*(M)$ the de Rham complex of M. Consider the minimal model $\mathcal{M}$ (resp 1-minimal model) of $A_{\mathbb{C}}^*(M)$ with a quasi-isomorphism (resp. 1-quasi-isomorphism) $\phi : \mathcal{M} \to A_{\mathbb{C}}^*(M)$. Then we have:*

(1) *The real de Rham cohomology $H^*(M, \mathbb{R})$ admits a $\mathbb{R}$-mixed-Hodge structure.*

(2) *$\mathcal{M}^*$ admits a bigrading*

$$\mathcal{M}^* = \bigoplus_{p,q \geqslant 0} \mathcal{M}_{p,q}^*$$

such that $\mathcal{M}_{0,0}^* = \mathcal{M}^0 = \mathbb{C}$ and the product and the differential are of type $(0,0)$.

(3) *Consider the bigrading $H^*(M, \mathbb{C}) = \bigoplus V_{p,q}$ as in Proposition 2.3 for the $\mathbb{R}$-mixed-Hodge structure. Then the induced map $\phi^* : H^*(\mathcal{M}^*) \to H^*(M, \mathbb{C})$ sends $H^*(\mathcal{M}_{p,q}^*)$ to $V_{p,q}$.*

By this theorem, we have a canonical mixed Hodge structure on the cohomology $H^*(M, \mathbb{R})$. In more detail the mixed Hodge structure is given by the bigrading $A_{\mathbb{C}}^* = \bigoplus A_{p,q}^*$ such that

$$A_{p,q}^* = H_B^{p,q}(M) \oplus H_B^{p-1,q-1}(M) \wedge y .$$

We can easily show the following proposition.

Proposition 6.6. — *Consider the bigrading $H^*(M,\mathbb{C}) = \bigoplus V_{p,q}$ as in Proposition 2.3.*

(1) $H^1(M,\mathbb{C}) = V_{1,0} \oplus V_{0,1}$.
(2) $V_{n+1,n+1} = H^{2n+1}(M,\mathbb{C})$.
(3) *If $n \geqslant 2$, then $H^2(M,\mathbb{C}) = V_{2,0} \oplus V_{1,1} \oplus V_{0,2}$.*

Proof. — The first and second assertions are easy.

Suppose $n \geqslant 2$. Then it is known that the map $[d\eta]\wedge : H^1_B(M,\mathbb{C}) \to H^3_B(M,\mathbb{C})$ is injective (see [5, Section 7.2]). Hence we have $\operatorname{Ker} d_{|A^2_{\mathbb{C}}} \subset H^2_B(M,\mathbb{C})$. Thus the third assertion follows. □

Example 6.7. — Let $N = H_{2n+1}$ and Γ be a lattice in N. Consider the DGA $\bigwedge \mathfrak{n}^* = \bigwedge\langle x_1,\dots,x_{2n},y\rangle$ as in Example 4.2. Then the nilmanifold $\Gamma\backslash N$ is a regular Sasakian manifold with the left-invariant contact structure y. In this case, the foliation induced by Reeb vector field is a principal S^1 bundle over the n-dimensional Abelian variety T associated with the Kähler class on T. We have $H^*_B(M,\mathbb{R}) \cong H^*(T,\mathbb{R}) \cong \bigwedge\langle x_1,\dots,x_{2n}\rangle$. Thus the DGA $A^* = H^*_B(M,\mathbb{R}) \otimes \bigwedge\langle y\rangle$ is identified with $\bigwedge \mathfrak{n}^*$ and hence it is the minimal model of the de Rham complex $A^*(\Gamma\backslash N)$. For $\mathcal{V} = \langle x_1,\dots,x_{2n}\rangle$, take $\mathcal{V}\otimes\mathbb{C} = \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}$ associated with the complex structure on T. Take the bigrading on $\bigwedge \mathfrak{n}^*_{\mathbb{C}} = \bigwedge(\mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}) \otimes \bigwedge\langle y\rangle$ so that y is considered as an element of type $(1,1)$. Then we can check that this bigrading is a bigrading as in Theorem 6.5 (see the explanation after Theorem 5.3). As in Proposition 6.6, if $n \geqslant 2$ we can easily check that $H^2(\Gamma\backslash N,\mathbb{C}) = V_{2,0} \oplus V_{1,1} \oplus V_{0,2}$. But if $n = 1$, we have $H^2(\Gamma\backslash N,\mathbb{C}) = V_{2,1} \oplus V_{1,2}$

7. 1-formality of Sasakian manifolds

Theorem 7.1. — *Let M be a compact $2n+1$-dimensional Sasakian manifold with $n \geqslant 2$. Then the DGA $A^*_{\mathbb{C}}(M)$ is 1-formal equivalently the Malčev Lie algebra of $\pi_1(M)$ admits a quadratic presentation.*

Proof. — Consider the 1-minimal model $\mathcal{M}^*$ of the de Rham complex $A^*_{\mathbb{C}}(M)$ with a 1-quasi-isomorphism $\phi : \mathcal{M}^* \to A^*_{\mathbb{C}}(M)$. We take a bigrading

$$\mathcal{M}^* = \bigoplus_{p,q\geqslant 0} \mathcal{M}^*_{p,q}$$

as Theorem 6.5. Since the induced injection $\phi^2 : H^*(\mathcal{M}^*) \to H^2(M,\mathbb{C})$ sends $H^2(\mathcal{M}^*_{p,q})$ to $V_{p,q}$, by Proposition 6.6, we have $H^2(\mathcal{M}^*) = H^2(\mathcal{M}^*_{2,0}\oplus$

$\mathcal{M}^*_{1,1}\oplus\mathcal{M}^*_{0,2})$. We also consider the canonical sequence of DGAs

$$\mathcal{M}^*(1)\subset\mathcal{M}^*(2)\subset\dots.$$

as Section 4. By Proposition 6.6, $\mathcal{V}_1$ is spanned by elements of type $(1,0)$ and $(0,1)$. Since we take $d\mathcal{V}_2\subset\mathcal{V}_1\wedge\mathcal{V}_1$, we can see that $\mathcal{V}_2$ is spanned by elements of type $(2,0)$, $(1,1)$ and $(0,2)$. For $n\geqslant 3$, since we take $d\mathcal{V}_n\subset\bigoplus_{i,j<0,\,i+j>2}\mathcal{V}_i\wedge\mathcal{V}_j$, $\mathcal{V}_n$ is spanned by elements of type (p,q) with $p+q\geqslant 3$. Hence we have

$$H^2(\mathcal{M}^*)=H^2(\mathcal{M}^*_{2,0}\oplus\mathcal{M}^*_{1,1}\oplus\mathcal{M}^*_{0,2})=\mathcal{V}_1\wedge\mathcal{V}_1/d(\mathcal{V}_2).$$

Hence the theorem follows from Proposition 4.1. □

Let H_3 be the 3-dimensional real Heisenberg group and Γ a lattice in H_3. Then the nilmanifold $\Gamma\backslash H_3$ is Sasakian. In [17], Tievsky observed that for any compact even dimensional simply connected manifold M with $\dim M>0$, the product $\Gamma\backslash H_3\times M$ is not Sasakian. We can extend this observation.

Proposition 7.2. — *For any compact even dimensional manifold M (not necessarily simply connected) with $\dim M>0$, the product $\Gamma\backslash H_3\times M$ is not Sasakian.*

Proof. — The nilmanifold $\Gamma\backslash H_3$ has a non-trivial Massey triple product on the first cohomology and it is an obstruction of 1-formality. The product $\Gamma\backslash H_3\times M$ also has a non-trivial Massey triple product on the first cohomology. Hence $\Gamma\backslash H_3$ is not Sasakian. □

It is known that for a compact contact manifold X and compact surface S, the product $X\times S$ admits a contact structure (see [3, 4]). Hence we can construct many non-Sasakian contact manifolds.

Corollary 7.3. — *Let $S_1,\dots,S_n$ be compact surfaces. Then the product $\Gamma\backslash H_3\times S_1\times\cdots\times S_n$ is a non-Sasakian contact manifold.*

8. Sasakian nilmanifolds

First, we show the following proposition inspired by Hain's paper [10].

Proposition 8.1. — *Let $\mathfrak{n}$ be a real $2n+1$-dimensional nilpotent Lie algebra. Suppose that the DGA $\bigwedge\mathfrak{n}^*_{\mathbb{C}}$ admits a bigrading $\bigwedge\mathfrak{n}^*_{\mathbb{C}}=\bigoplus\mathcal{M}^*_{p,q}$ such that $\mathcal{M}^*_{0,0}=\bigwedge^0\mathfrak{n}^*_{\mathbb{C}}=\mathbb{C}$ and the product and the differential are of type $(0,0)$. We suppose that $H^1(\mathfrak{n},\mathbb{C})=\operatorname{Ker} d_{|\mathfrak{n}^*_{\mathbb{C}}}=\mathcal{M}^*_{1,0}\oplus\mathcal{M}^*_{0,1}$ and $H^{2n+1}(\mathfrak{n},\mathbb{C})=\bigwedge^{2n+1}\mathfrak{n}^*_{\mathbb{C}}=\mathcal{M}^*_{n+1,n+1}$. Then we have $\dim H^1(\mathfrak{n},\mathbb{R})=2n$.*

Proof. — Let $w_r = \sum_{p+q=r} \dim \mathcal{M}^1_{p,q}$. Then, by the assumption, we have $\sum_r w_r = 2n+1$ and $\sum_r rw_r = 2n+2$. Hence $\sum_r (r-1)w_r = 1$ and this implies $w_2 = 1$, $w_r = 0$ for $r \geqslant 3$ and so $w_1 = 2n$. By the assumption, we have $w_1 = \dim H^1(\mathfrak{n}, \mathbb{C})$. Hence the proposition follows. □

THEOREM 8.2 ([6]). — *Let N be a real $(2n+1)$-dimensional simply connected nilpotent Lie group with a lattice Γ. If the nilmanifold $\Gamma\backslash N$ admits a Sasakian structure, then N is the real $(2n+1)$-dimensional Heisenberg group.*

Proof. — Let $\mathfrak{n}$ be the Lie algebra of N. Then we can say that $\bigwedge \mathfrak{n}^*$ is the minimal model of $A^*(\Gamma\backslash N)$ (see Example 3.7). By Theorem 6.5 and Proposition 6.6, the assumptions of Proposition 8.1 hold and hence we can say that $\bigwedge^1 \mathfrak{n}^* = V_1 \oplus \langle v_2 \rangle$ such that $H^1(\mathfrak{n}, \mathbb{R}) = V_1$, $\dim V_1 = 2n$ and $dv_2 \in \bigwedge^2 V_1$. By Theorem 6.2, we have a morphism $\phi : \bigwedge \mathfrak{n}^* \to A^* = H^*_B(M, \mathbb{R}) \otimes \bigwedge \langle y \rangle$ which induces a cohomology isomorphism. By $H^1(A^*) = H^1_B(M, \mathbb{R})$, we have $\phi(V_1) = H^1_B(M, \mathbb{R})$. By $H^{2n+1}(A^*) = H^{2n}_B(M, \mathbb{R}) \wedge y$, we have $\phi(\bigwedge^{2n+1} \mathfrak{n}^*) = H^{2n}_B(M, \mathbb{R}) \wedge y$. Hence we can obtain $\phi(v_2) = a + cy$ such that $a \in H^1_B(M, \mathbb{R})$ and $c \in \mathbb{R}$ with $c \neq 0$. This implies that $\phi(dv_2) \in \langle [d\eta] \rangle$. It is known that $[d\eta]^{2n} \neq 0$ (see [5, Section 7.2]). Hence dv_2 is non-degenerate in $\bigwedge^2 V_1$ and so the Theorem follows. □

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