

## ANNALES

### DE

# L'INSTITUT FOURIER

Yves FÉLIX, Steve HALPERIN & Jean-Claude THOMAS **On The Growth of the Homology of a Free Loop Space II** Tome 67, nº 6 (2017), p. 2519-2531.

<http://aif.cedram.org/item?id=AIF\_2017\_\_67\_6\_2519\_0>



© Association des Annales de l'institut Fourier, 2017, *Certains droits réservés.* 

Cet article est mis à disposition selon les termes de la licence CREATIVE COMMONS ATTRIBUTION – PAS DE MODIFICATION 3.0 FRANCE. http://creativecommons.org/licenses/by-nd/3.0/fr/

L'accès aux articles de la revue « Annales de l'institut Fourier » (http://aif.cedram.org/), implique l'accord avec les conditions générales d'utilisation (http://aif.cedram.org/legal/).

## cedram

Article mis en ligne dans le cadre du Centre de diffusion des revues académiques de mathématiques http://www.cedram.org/

#### ON THE GROWTH OF THE HOMOLOGY OF A FREE LOOP SPACE II

#### by Yves FÉLIX, Steve HALPERIN & Jean-Claude THOMAS

ABSTRACT. — Controlled exponential growth is a stronger version of exponential growth. We prove that the homology of the free loop space  $\mathcal{L}X$  has controlled exponential growth in two important situations : (1) when X is a connected sum of manifolds whose rational cohomologies are not monogenic, (2) when the rational homotopy Lie algebra  $L_X$  contains an inert element and  $\rho(L_X) < \rho(L_X/[L_X, L_X])$ , where  $\rho(V)$  denotes the radius of convergence of V.

RÉSUMÉ. — La croissance exponentielle controlée est une version forte de la croissance exponentielle. Nous prouvons que les nombres de Betti de l'espace des lacets libres sur un espace X ont une croissance exponentielle controlée dans deux cas: lorsque X est la somme connexe de variétés dont la cohomologie n'est pas monogène, et lorsque l'algèbre de Lie  $L_X$  a une croissance exponentielle strictement plus grande que ses indécomposables.

#### 1. Introduction

In this paper we are concerned with the growth of the homology  $H_*(X^{S^1}; \mathbb{Q})$  of a free loop space on a simply connected space, X.

A graded vector space  $V = V_{\geq 0}$  grows exponentially if there are constants  $1 < C_1 < C_2$  such that for some N,

$$C_1^k \leqslant \sum_{i \leqslant k} \dim V_i \leqslant C_2^k, \qquad k \geqslant N.$$

In particular, if X is a simply connected CW complex of finite type and finite Lusternik–Schnirelmann category then [3] either dim  $\pi_*(X) \otimes \mathbb{Q} < \infty$ (X is rationally elliptic) or  $\pi_*(X) \otimes Q$  grows exponentially (X is rationally hyperbolic). The first examples of elliptic spaces are given by compact homogeneous spaces, but the generic situation is given by hyperbolic spaces.

 $K\!eywords:$  free loop space, exponential growth, inert attachment.

 $<sup>2010\</sup> Mathematics\ Subject\ Classification:\ 55P62.$ 

For instance if the Euler characteristic  $\chi(X) < 0$  then X is hyperbolic (see [4] for other examples of elliptic or hyperbolic spaces)

In [7] Gromov conjectured that  $H_*(X^{S^1}; \mathbb{Q})$  grows exponentially for almost all cases when X is a closed manifold. This would have an important consequence in Riemannian geometry, due to a theorem of Gromov, improved by Ballmann and Ziller:

THEOREM 1.1 ([7], [2]). — Let  $N_g(t)$  denote the number of geometrically distinct closed geodesics of length  $\leq t$  on a simply connected closed Riemannian manifold (M, g). Then, for generic metrics g, there are constants K > 0 and  $\beta > 0$  such that for k sufficiently large,

$$N_g(k) \ge K \cdot \max_{\ell \le \beta k} \dim H_\ell(M^{S^1}; \mathbb{Q}).$$

One of the first applications of Sullivan's minimal models  $(\wedge V, d)$  of a space X was the construction [16] (when X is simply connected) of the minimal model  $(\wedge W, d)$  of  $X^{S^1}$  where  $W^k = V^k \oplus V^{k-1}$ . Since X is elliptic if and only if dim  $V < \infty$  it follows that in that case  $H_*(X^{S^1}; \mathbb{Q})$  grows at most polynomially. In [16] Vigué-Poirrier conjectures that in the hyperbolic case,  $H_*(X^{S^1}; \mathbb{Q})$  should grow exponentially, a conjecture which would give Gromov's conjecture as a special case.

The Vigué-Poirrier conjecture has been proved for a finite wedge of spheres [16], for a non-trivial connected sum of closed manifolds [11] and in the case X is coformal [12].

For simplicity we write H(X) and  $H^*(X)$  respectively for the rational homology and cohomology of a space X, and denote the free loop space of maps  $S^1 \to X$  by  $\mathcal{L}X$ . If X is simply connected and dim  $\pi_*(X) \otimes \mathbb{Q} < \infty$ then it is immediate from Sullivan's model of  $\mathcal{L}X$  [15] that  $H(\mathcal{L}X)$  grows at most polynomially. However, even in the case when X is a rationally hyperbolic finite simply connected complex it is not known if  $H(\mathcal{L}X)$  grows exponentially.

Next, for a graded vector space V denote by

$$V(z) := \sum_{k \ge 0} \dim V_k z^k$$

the formal Hilbert series of V and denote by  $\rho_V$  or  $\rho(V)$  the radius of convergence of V(z). If X is a topological space we denote by X(z) and by  $\rho_X$  or by  $\rho(X)$  the Hilbert series of H(X) and its radius of convergence.

In [5] we introduced a much stronger version of exponential growth: V has controlled exponential growth if  $0 < \rho_V < 1$  and for each  $\lambda > 1$  there

is an infinite sequence  $n_1 < n_2 < \cdots$  such that  $n_{i+1} < \lambda n_i$ ,  $i \ge 1$ , and

$$\lim_{i} \frac{\log \dim V_{n_i}}{n_i} = -\log(\rho_V) \,.$$

As usual,  $\Omega X$  denotes the (based) loop space on a space X. We recall [14] or [4] that if X is simply connected, then  $H(\Omega X)$  is the universal enveloping algebra of the graded Lie algebra  $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ ;  $L_X$  is called the homotopy Lie algebra of X. According to [5, Lemma 4],

(1.1) 
$$\rho_{\Omega X} = \rho(L_X).$$

If X has rational homology of finite type and infinite dimensional rational homotopy, then Sullivan's model for  $\mathcal{L}X$  gives

(1.2) 
$$\rho_{\mathcal{L}X} \leqslant \rho_{\Omega X}$$
.

Our objective here is to establish new classes of spaces X (Theorems 1.3 and 1.4 below) for which  $H(\mathcal{L}X)$  has controlled exponential growth and

$$\rho_{\mathcal{L}X} = \rho_{\Omega X} \,.$$

Our approach is by constructing maps

$$F \to X \xrightarrow{p} Y$$

in which F is the homotopy fibre of p.

THEOREM 1.2. — With the above notations if F is rationally a wedge of spheres, and if  $0 < \rho_{\Omega F} < \rho_{\Omega Y}$  then  $H(\mathcal{L}X)$  has controlled exponential growth and  $\rho_{\mathcal{L}X} = \rho_{\Omega X}$ .

*Proof.* — This follows from [5, formula (4)], together with Theorems 1.2 and 1.4.  $\hfill \Box$ 

One method for constructing other maps  $p: X \to Y$  is via inert elements  $\alpha \in L_X$ , where  $L_X$  is the homotopy Lie algebra of X. Any  $\alpha \in (L_X)_k$  corresponds up to a scalar multiple to a map  $\sigma: S^{k+1} \to X$  and  $\alpha$  is called *inert* if the map

$$p: X \to X \cup_{\sigma} D^{k+2}$$

is surjective in rational homotopy. In Lemma 2.2 we recall the proof that if  $\alpha$  is inert then the homotopy fibre of p is a wedge of spheres with homology isomorphic to  $H(\Omega(X \cup_{\sigma} D^{k+2})) \otimes \mathbb{Q}\alpha$ . For instance the attaching map of the top cell in a simply connected manifold whose cohomology is not monogenic is inert [8]. (Recall that a graded algebra  $A = \mathbb{Q} \oplus A^{\geq 1}$  is monogenic if it is generated by a single element  $a \in A^{\geq 1}$ ). Also, every nonzero element  $\alpha$  in a free Lie algebra generated by elements of even degrees is inert ([8]).

A key condition in our theorems is the hypothesis

(1.3) 
$$\Omega X(\rho_{\Omega X}) := \lim_{z \to \rho_{\Omega X}} \Omega X(z) = \infty$$

There are no examples where this is known to fail if X is a rationally hyperbolic, finite, simply connected CW complex. In fact (Proposition 2.1) this follows from the condition

$$\rho(L_X) < \rho\left(\frac{L_X}{[L_X, L_X]}\right),$$

which is not known to fail for such X. When dim  $L_X/[L_X, L_X] < \infty$ , Proposition 2.1 follows from a result of Anick [1].

With this preamble we can state our two theorems:

THEOREM 1.3. — Suppose X is a simply connected CW complex with rational homology of finite type. If  $L_X$  contains an inert element  $\gamma$  and if  $\rho(L_X) < \rho(L_X/[L_X, L_X])$  then  $H(\mathcal{L}X)$  has controlled exponential growth and  $\rho_{\mathcal{L}X} = \rho_{\Omega X}$ .

THEOREM 1.4. — Suppose M # N is the connected sum of two closed simply connected *n*-manifolds with  $H^*(N)$  not monogenic and M not rationally a sphere. If  $\rho_{\Omega N} \leq \rho_{\Omega M}$  and if  $\Omega N(\rho_{\Omega N}) = \infty$  then  $H(\mathcal{L}(M \# N))$ has controlled exponential growth and  $\rho_{\mathcal{L}(M \# N)} = \rho_{\Omega(M \# N)}$ .

Remarks 1.5.

(1) Theorem 1.3 is proved in [5] under the considerably stronger hypothesis that

$$\dim L_X/[L_X, L_X] < \infty \,.$$

- (2) If  $H^*(M)$  and  $H^*(N)$  are monogenic, but of dimension > 2 then M # N is elliptic and so  $H(\mathcal{L}(M \# N))$  grows at most polynomially.
- (3) Theorem 1.4 strengthens a result of Lambrechts [10], which asserts that H(L(M#N)) grows exponentially unless both H\*(M) and H\*(N) are monogenic.

#### 2. Proposition 2.1 and Theorem 1.3

Suppose  $A = \mathbb{Q}1 \oplus A_{\geq 1}$  is a finitely generated graded algebra satisfying  $\rho_A < 1$ . Then it follows from a result of Anick [1] that

$$A(\rho_A) = \infty$$
.

We generalize this with

ANNALES DE L'INSTITUT FOURIER

2522

PROPOSITION 2.1. — Let  $L = L_{\geq 1}$  be a graded Lie algebra of finite type such that  $0 < \rho_{UL} < 1$ . If L is generated by a subspace V with  $\rho_{UL} < \rho_V$ then  $UL(\rho_{UL}) = \infty$ .

*Proof.* — We assume  $UL(\rho_{UL}) < \infty$ , and deduce a contradiction. By Anick's result we have dim  $V = \infty$ . Choose some  $\sigma$  with  $\rho_{UL} < \sigma < \rho_V$ . Then  $V(\sigma) < \infty$  and so  $V_{\geq r}(\sigma) \to 0$  as  $r \to \infty$ . In particular, we may choose r so that

 $UL(\rho_{UL}) \cdot V_{\geq r}(\sigma) < 1.$ 

Now let *E* be the sub Lie algebra generated by  $V_{<r}$  and note that by Anick's result,  $E \neq L$ . In particular,  $UE(\rho_{UL}) < UL(\rho_{UL})$ . Clearly  $\rho_{UE} \ge \rho_{UL}$ . If  $\rho_{UE} = \rho_{UL}$ , then  $0 < \rho_{UE} < 1$ . Then by Anick's result  $UE(\rho_{UE}) = \infty$ , and  $UL(\rho_{UL}) = \infty$ . It follows that  $\rho_{UE} > \rho_{UL}$ . Thus for some  $\tau$  with  $\rho_{UL} < \tau < \rho_{UE}$  we have  $UE(\tau) < UL(\rho_{UL})$ .

Choose  $\rho$  so that  $\rho_{UL} < \rho < \tau$  and  $\rho < \sigma$ . Then

$$UE(\rho) \cdot V_{\geqslant r}(\rho) < UE(\tau) \cdot V_{\geqslant r}(\sigma) < UL(\rho_{UL}) \cdot V_{\geqslant r}(\sigma) < 1$$

Now let  $W = UE \circ V_{\geq r}$  where "o" denotes the adjoint action and note that  $W(\rho) < 1$ . Then, let *I* be the sub Lie algebra generated by *W*. The inclusion of *W* in *I* extends to a surjection  $TW \to UI$ . Since  $(TW)(\rho) = \frac{1}{1-W(\rho)} < \infty$ , it follows that

$$\rho_{UI} \geqslant \rho_{TW} \geqslant \rho > \rho_{UL} \,.$$

On the other hand, since  $W \supset V_{\geqslant r}$  and  $[E, W] \subset W$ , it follows that I is an ideal in L. The surjection  $L \to L/I$  kills  $V_{\geqslant r}$ , and so it restricts to a surjection  $E \to L/I$ . Thus  $\rho_{U(L/I)} \ge \rho_{UE} > \rho_{UL}$ . But as graded vector spaces  $UL \cong UI \otimes U(L/I)$  and so

$$\rho_{UL} = \min\{\rho_{UI}, \rho_{U(L/I)}\}.$$

This is the desired contradiction because  $\rho_{UL} < \rho_{UI}$  and  $\rho_{UL} < \rho_{U(L/I)}$ .

We also require the following lemma announced in the Introduction, and which is essentially proved, if not stated, in [8].

LEMMA 2.2. — Let X be a simply connected CW complex that is not rationally a sphere. If  $\alpha \in (L_X)_k$  is an inert element corresponding to  $\sigma: S^{k+1} \to X$ , then

- (1) The homotopy fibre  $i: F \to X$  of  $p: X \to X \cup_{\sigma} D^{k+2} = Y$  is rationally a wedge of spheres.
- (2)  $H(\Omega i)$  restricts to an isomorphism  $L_F \xrightarrow{\cong} I$ , where  $I \subset L_X$  is the ideal generated by  $\alpha$ .

Yves FÉLIX, Steve HALPERIN & Jean-Claude THOMAS

- (3) I is a free Lie algebra and  $I/[I, I] \cong U(L_X/I) \otimes \mathbb{Q}\alpha$ .
- (4)  $H_*(\Omega p)$  induces an isomorphism  $U(L_X/I) \xrightarrow{\cong} H_*(\Omega Y)$ .

*Proof.* — Since  $\alpha$  is inert  $\pi_*(p) \otimes \mathbb{Q}$  is surjective. Thus  $\pi_*(\Omega p) \otimes \mathbb{Q}$  is surjective and

 $\pi_*(\Omega i)\otimes\mathbb{Q}: L_F=\pi_*(\Omega F)\otimes\mathbb{Q}\xrightarrow{\cong} \ker \pi_*(\Omega p)\otimes\mathbb{Q}.$ 

Moreover, it follows from [8, Theorem 1.1], that  $L_F = I$ , and so  $H_*(\Omega p) \otimes \mathbb{Q}$ induces an isomorphism  $U(L_X/I) \xrightarrow{\cong} H_*(\Omega Y)$ . Theorem 1.1 of [8] also asserts that I is a free Lie algebra, and that

$$I/[I,I] \cong U(L_X/I) \otimes \mathbb{Q}\alpha$$
.

It remains to show that F is rationally a wedge of spheres. Let  $\sigma_i : S^{n_i} \to F$  corresponding to elements  $\alpha_i \in L_F$  which represent a basis of I/[I, I]. Then the map

$$\varphi = \vee_i \sigma_i : \vee S^{n_i} \to F$$

induces a map  $\Omega \varphi : \Omega(\vee S^{n_i}) \to \Omega F$  and  $\pi_*(\Omega \varphi) \otimes \mathbb{Q}$  is a morphism between free Lie algebras inducing an isomorphism  $I/[I,I] \cong L_F/[L_F,L_F]$ . Thus  $\pi_*(\Omega \varphi) \otimes \mathbb{Q}$  is an isomorphism and  $\varphi$  is a rational homotopy equivalence.  $\Box$ 

Proof of Theorem 1.3. — Denote  $L_X$  simply by L, let  $\alpha \in L_k$  be the inert element corresponding to  $\sigma: S^{k+1} \to X$ , and let  $p: X \to X \cup_{\sigma} D^{k+2}$  be the map considered in Lemma 1. Then by Lemma 1, with I the ideal generated by  $\alpha$  and V = I/[I, I], we have isomorphisms

$$H_*(\Omega F) \cong UI \cong TV$$
 and  $H(\Omega(X \cup_{\sigma} D^{k+2})) \cong U(L/I)$ .

Thus, as observed in the Introduction, Theorem 1.3 will be established once we prove

(2.1) 
$$\rho_{UI} < \rho_{U(L/I)} \,.$$

Clearly  $\rho_{UL} \leq \rho_{U(L/I)}$  and if  $\rho_{UL} < \rho_{U(L/I)}$  then  $\rho_{UI} < \rho_{U(L/I)}$  since  $UL \cong UI \otimes U(L/I)$ . It remains to consider the case that  $\rho_{UL} = \rho_{U(L/I)}$ . Since  $UI \cong TV$  and since dim  $V \ge 2$  it follows that  $\rho_{UL} \le \rho_{UI} < 1$ . Since L/[L, L] maps surjectively to (L/I)/[L/I, L/I], we obtain

$$\rho_{U(L/I)} = \rho_{UL} < \rho_{L/[L,L]} \leq \rho_{(L/I/[L/I,L/I])}$$

Thus by Proposition 2.1,

$$U(L/I)(\rho_{U(L/I)}) = \infty$$

On the other hand,  $UI \cong TV$  with  $V \cong U(L/I) \otimes \mathbb{Q}\alpha$ . Thus

$$UI(z) = \frac{1}{1 - z^k U(L/I)(z)}.$$

ANNALES DE L'INSTITUT FOURIER

2524

Since  $\lim_{z\to\rho(U(L/I))} U(L/I)(z) = \infty$ , it follows that  $r^k U(L/I)(r) = 1$  for some  $r < \rho_{U(L/I)}$ . But then  $r = \rho_{UI}$  and so again  $\rho_{UI} < \rho(U(L/I))$ .

#### 3. Connected sums

The objective of this section is to prove Theorem 1.4, and we shall frequently rely on the acyclic closure [6] of a cdga, (A, d) in which  $A^0 = \mathbb{Q}$ and  $H^1(A) = 0$ . This is a cdga of the form  $(A \otimes \wedge U, d)$  containing (A, d)as a sub cdga, where the quotient  $(\wedge U, \overline{d})$  is a minimal Sullivan algebra, and such that  $H(A \otimes \wedge U, d) = \mathbb{Q}$ . The acyclic closure is determined up to isomorphism ([6, Theorem 3.2]).

For the proof of Theorem 1.4 we establish a preliminary proposition to deal with the case that  $H^*(M)$  is monogenic and  $H^*(N)$  is not. Recall that a model for a space X is a connected commutative graded differential algebra whose minimal Sullivan model is also a minimal Sullivan model for the rational polynomial differential forms on X ([15], [4]).

Let (A, d) and (B, d) be finite dimensional models for the closed *n*manifolds M and N of Theorem 1.4. We may suppose  $A^0 = B^0 = \mathbb{Q}$ ,  $A^1 = B^1 = 0, A^{>n} = B^{>n} = 0, A^n = \mathbb{Q}\alpha$  and  $B^n = \mathbb{Q}\beta$ .

LEMMA 3.1. — A model for the connected sum M # N is given the cdga

$$((A \oplus_{\mathbb{Q}} B) \oplus \mathbb{Q}w, d)$$

with  $dw = \alpha - \beta$  and  $w \cdot A^+ = w \cdot B^+ = 0$ .

Proof. — By [4, §12], the cdga  $A \oplus_{\mathbb{Q}} B$  is a model for the wedge  $M \vee N$ . Denote by  $p : M \# N \to M \vee N$  the pinch map and  $(\wedge X, d)$  a Sullivan minimal model for  $M \vee N$ . Since  $H^{<n}(p)$  is an isomorphism and  $H^n(p)$ simply identifies the classes  $\alpha$  and  $\beta$ , a model of p is given by the inclusion  $(\wedge X, d) \to (\wedge X \otimes \wedge u \otimes \wedge Z, d)$  where  $du = \alpha - \beta$  with  $[\alpha]$  and  $[\beta]$  the fundamental classes of M and N, and where  $Z = Z^{\leq n-1}$  is introduced to kill recursively all new cohomology classes. We then have clearly a commutative diagram, where the vertical maps are quasi-isomorphisms

Now consider the case that  $H^*(M)$  is monogenic. Then  $H^*(M) = \wedge a/a^{n+1}$ , where deg a = 2p, n = 2pk, and  $k \ge 2$  because M is not rationally

TOME 67 (2017), FASCICULE 6

a sphere. In this case  $(\wedge a/a^{n+1}, 0)$  is a model for M and we choose as model (B, d) for N a quotient of the minimal Sullivan model such that  $B^{>n} = 0$ and  $B^n = \mathbb{Q}\beta$ . Then a represents a cohomology class in  $H^{2p}(M \# N)$  and hence determines a map  $p: M \# N \to K(2p, \mathbb{Q})$  with homotopy fibre F.

**PROPOSITION 3.2.** — The homotopy fibre F has a model of the form

$$(C,d) = (B/\beta,d) \oplus (B^{\geq 1},d) \otimes \mathbb{Q}\overline{a}$$

where deg  $\overline{a} = 2p - 1$ ,  $(B/\beta, d)$  is the quotient cdga of (B, d) acting by multiplication on the left on  $(B^{\ge 1}, d) \otimes \mathbb{Q}\overline{a}$ , and  $(B^{\ge 1} \otimes \mathbb{Q}\overline{a}) \cdot (B^{\ge 1} \otimes \mathbb{Q}\overline{a}) = 0$ .

Proof. — As observed above, a model for M # N is given by  $((\wedge a/a^{k+1} \times_{\mathbb{Q}} B) \oplus \mathbb{Q}w, d)$  with  $dw = a^k - \beta$ . Now a quasi-isomorphism

$$((\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B, d) \xrightarrow{\simeq} (\wedge a/a^{k+1} \times_{\mathbb{Q}} B) \oplus \mathbb{Q}w$$

is given by dividing by the elements  $a^q$  and  $a^r w$ ,  $q \ge k+1$  and  $r \ge 1$ ; here on the left  $dw = a^k - \beta$ . (This follows by filtering by the degree in B.)

Thus it follows from Theorem 15.3 in [4] or Theorem 5.1 in [6] that the Sullivan fibre of the morphism  $\wedge a \to ((\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B))$  is a model for F. Let  $(\wedge a \otimes \wedge \overline{a}, d\overline{a} = a)$  be the acyclic closure of  $(\wedge a, 0)$ . Then this Sullivan fibre is given by  $((\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B) \otimes_{\wedge a} (\wedge a \otimes \wedge \overline{a})$ . Hence

$$(\wedge a \otimes \wedge w \otimes \wedge \overline{a}) \oplus (B^{\geq 1} \otimes \wedge \overline{a}) = (\wedge a \otimes \wedge w \otimes \wedge \overline{a}) \times_{\wedge \overline{a}} (B \otimes \wedge \overline{a})$$
$$= [(\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B] \otimes_{\wedge a} (\wedge a \otimes \wedge \overline{a})$$

is also a model for F.

Next note that  $I = (\wedge^{\geq 2} a \oplus \wedge^{\geq 1} a \cdot \overline{a}) \otimes \wedge w \subset (\wedge a \otimes \wedge w \otimes \wedge \overline{a}) \oplus (B^{\geq 1} \otimes \wedge \overline{a})$ is an ideal preserved by d, and that H(I, d) = 0. Thus division by I produces another model for F, given explicitly by

$$(\mathbb{Q}(1 \oplus a \oplus \overline{a}) \otimes \wedge w) \oplus (B^{\geq 1} \otimes \wedge \overline{a})$$

with  $a^2 = a\overline{a} = \overline{a}^2 = 0$ ,  $d\overline{a} = a$  and, since  $k \ge 2$ ,  $dw = -\beta$ . In this cdga,  $d(\overline{a}w) = aw + \overline{a}\beta$ . Moreover, the subspace spanned by  $\overline{a}w$  and  $aw + \overline{a}\beta$  is an ideal. Thus a quasi-isomorphism

$$(\mathbb{Q}(1 \oplus a \oplus \overline{a}) \otimes \wedge w) \oplus (B^{\geqslant 1} \otimes \wedge \overline{a}) \to \mathbb{Q}(1 \oplus a \oplus \overline{a} \oplus w) \oplus (B^{\geqslant 1} \otimes \wedge \overline{a})$$

is given by  $\overline{a}w \mapsto 0$  and  $aw \mapsto -\overline{a}\beta$ .

Now the inclusion  $\mathbb{Q} \oplus \mathbb{Q} w \oplus (B^{\geq 1} \otimes \wedge \overline{a})$  in  $\mathbb{Q}(1 \oplus a \oplus \overline{a} \oplus w) \oplus (B^{\geq 1} \otimes \wedge \overline{a})$  is clearly a quasi-isomorphism. Since  $dw = -\beta$ , division by w and  $\beta$  then gives a quasi-isomorphism

$$\mathbb{Q} \oplus \mathbb{Q} w \oplus (B^{\geq 1} \otimes \wedge \overline{a}) \xrightarrow{\simeq} B/\beta \oplus (B^{\geq 1} \otimes \mathbb{Q} \overline{a}).$$

ANNALES DE L'INSTITUT FOURIER

2526

(Note that in the left hand cdga  $\beta \otimes \overline{a}$  is not the product of  $\beta$  and  $\overline{a}$ , since  $\overline{a}$  is not an element in the cdga!).

Proof. — We consider separately the cases that  $H^*(M)$  is monogenic and  $H^*(N)$  is not, and that neither  $H^*(M)$  nor  $H^*(N)$  are monogenic. Note that since M and N are simply connected, and N is not a rational sphere,  $n \ge 4$ .

Case 1:  $H^*(M)$  is monogenic. — We adopt the notation of Proposition 3.2, and for simplicity denote  $-\otimes \mathbb{Q}\overline{a}$  simply by  $-\otimes \overline{a}$ . It is immediate from Theorem 3 and (4) in [5] that it is sufficient to prove that  $H(\mathcal{L}F)$ has controlled exponential growth and that  $\rho_{\mathcal{L}F} = \rho_{\Omega F}$ . Let  $(\wedge W, d) \rightarrow$  $(B/\beta, d)$  be a minimal Sullivan model, and extend this to a Sullivan model  $(\wedge W \otimes \wedge Z, d) \xrightarrow{\cong} (C, d)$ . By Proposition 3.2,  $(\wedge W \otimes \wedge Z, d)$  is a Sullivan model for F. Now, letting  $(\wedge W \otimes \wedge U, d)$  be the acyclic closure of  $(\wedge W, d)$ , we have for the Sullivan fibre  $(\wedge Z, \overline{d})$  that

$$\begin{split} (\wedge Z, \overline{d}) &\simeq (\wedge W \otimes \wedge Z \otimes_{\wedge W} \wedge W \otimes \wedge U, d) = (\wedge W \otimes \wedge Z \otimes \wedge U, d) \\ & \stackrel{\sim}{\to} \left( B/\beta \oplus (B^{\geqslant 1} \otimes \overline{a}) \otimes \wedge U, d \right) \\ & \stackrel{\sim}{\to} \mathbb{Q} \oplus (B^{\geqslant 1} \otimes \overline{a} \otimes \wedge U, d) \,. \end{split}$$

Since products in  $(B^{\geq 1} \otimes \overline{a})$  are zero it follows that  $(\wedge Z, \overline{d})$  is the minimal Sullivan model of a wedge of spheres with cohomology  $\mathbb{Q} \oplus H(B^{\geq 1} \otimes \overline{a} \otimes \wedge U, d)$ .

Thus in this case Theorem 1.4 will follow from the Sullivan model version of Theorem 3 and (4) in [5] once we show that the Sullivan acyclic closure  $(\wedge Z \otimes \wedge S, \overline{d})$  of  $(\wedge Z, \overline{d})$  satisfies

$$(3.1) \qquad \qquad \rho_{\wedge S} < \rho_{\wedge U} \,.$$

Denote  $H(B^{\geq 1} \otimes \overline{a} \otimes \wedge U)$  simply by H. Since  $(\wedge Z, \overline{d})$  is the model of a wedge of spheres, it follows that  $\wedge S$  is the dual of a tensor algebra TE with  $E_i \simeq H^{i+1}$ . Thus

(3.2) 
$$\wedge S(z) = \frac{1}{1 - E(z)} = \frac{1}{1 - \frac{1}{z}H(z)}$$

It remains to estimate H(z).

For this recall that the morphism  $B \to B/\beta$  corresponds to the inclusion

 $N - D^n \to (N - D^n) \cup_{S^{n-1}} D^n,$ 

where  $S^{n-1}$  is the boundary of a small disk  $D^n \subset N$ . Since H(N) is not monogenic Theorem 5.1 of [8] asserts that the sphere  $S^{n-1}$  corresponds

TOME 67 (2017), FASCICULE 6

to an inert element in the homotopy Lie algebra of  $N - D^n$ . Thus by [8, Theorem 1.1],

$$H(\Omega(N-D^n)) \cong TV \otimes H(\Omega N)$$

where  $V \cong H(\Omega N) \otimes v$  and  $\deg v = n - 2$ . Since  $V(z) = z^{n-2}\Omega N(z)$  it follows that  $\rho_V = \rho_{\Omega N}$  and that  $V(\rho_V) = \infty$ . Since

$$TV(z) = \frac{1}{1 - V(z)}$$

it follows that  $\rho_{TV} < \rho_V$  and that  $TV(\rho_{TV}) = \infty$ .

Moreover, the minimal Sullivan model  $(\wedge W, d)$  of  $B/\beta$  has the form  $(\wedge W_N \otimes \wedge P, d)$  in which  $\wedge W_N$  is the minimal Sullivan model of N. Thus the acyclic closure  $(\wedge W \otimes \wedge U, d)$  has the form

$$(\wedge W_N \otimes \wedge U_N \otimes \wedge P \otimes \wedge U_P, d)$$

in which  $(\wedge W_N \otimes \wedge U_N, d)$  is the acyclic closure of  $(\wedge W_N, d)$ . In particular,  $\wedge U \cong \wedge U_N \otimes \wedge U_P$ , and there are linear isomorphisms

(3.3) 
$$\wedge U_N \cong H^*(\Omega N) \quad \text{and} \quad \wedge U_P \cong TV^{\#},$$

 $V^{\#}$  denoting the dual of V. Thus

$$\rho_{\wedge U_P} = \rho_{TV} < \rho_V = \rho_{\wedge U_N} \,.$$

Since  $\wedge U = \wedge U_N \otimes \wedge U_P$ , it follows that

$$\rho_{\wedge U} = \rho(\wedge U_N \otimes \wedge U_P) = \rho_{\wedge U_P} \,,$$

and that  $\wedge U(\rho_{\wedge U}) = \infty$ .

Now consider the short exact sequence

$$0 \to (B^{\geq 1} \otimes \overline{a} \otimes \wedge U, d) \to (B \otimes \overline{a} \otimes \wedge U, d) \to (\overline{a} \otimes \wedge U, 0) \to 0.$$

Since  $(B \otimes \overline{a} \otimes \wedge U, d) = (B \otimes \overline{a} \otimes \wedge U_N \otimes \wedge U_P, d)$  it follows that

$$H = H(B \otimes \overline{a} \otimes \wedge U, d) \cong \overline{a} \otimes \wedge U_P$$
.

It follows that  $H(B^{\geq 1} \otimes \overline{a} \otimes \wedge U, d)$  contains a subspace T with

$$T^{i+\deg \overline{a}+1} \cong (\wedge^{\geq 1} U_N \otimes \wedge U_P)^i.$$

In particular, with  $\gg$  denoting coefficient-wise inequality, we have

$$E(z) \gg z^{\deg,\overline{a}} \cdot \left(\wedge^{\geq 1} U_N\right)(z) \cdot \left(\wedge U_P\right)(z)$$

Thus  $\rho_E \leq \rho_{\wedge U}$  and if  $\rho_E = \rho_{\wedge U}$ , then  $E(\rho_E) = \infty$ . Since

$$\wedge S(z) = \frac{1}{1 - E(z)}$$

it follows in either case that  $\rho_{\wedge S} < \rho_{\wedge U}$ , which completes the proof of Theorem 1.4 in this case.

ANNALES DE L'INSTITUT FOURIER

Case 2: Neither H(M) nor H(N) is monogenic. — In this case Theorem 5.4 of [8] asserts that the collar sphere  $S^{n-1}$  joining  $M - \{pt\}$  to  $N - \{pt\}$  represents an inert element in  $L_{M\#N}$ . Attaching a disk to this sphere gives  $M \vee N$  and thus by Theorem 1.1 in [8] the homotopy fibre Fof the map  $p: M\#N \to M \vee N$  is rationally a wedge of spheres with

$$H_i(F) \cong H_{i-n+2}(\Omega(M \vee N))$$

Thus

$$H(\Omega F) = TV$$
 and  $V_i \cong H_{i-n+2}(\Omega(M \lor N))$ ,

and so

$$\Omega F(z) = \frac{1}{1 - z^{n-2} (\Omega(M \lor N))(z)}.$$

On the other hand it is a classical fact that the homotopy fibre G of the map  $q: M \vee N \to M \times N$  is the join  $\Omega M * \Omega N$ , (we sketch the proof in Lemma 3.3 below). Thus G is the suspension of  $\Omega M \wedge \Omega N$  and therefore rationally a wedge of spheres. Since  $\pi_*(q)$  is trivially surjective. It follows that

$$H(\Omega G) = TW$$
 with  $W_i \cong H_{i-1}(\Omega M * \Omega N)$ .

By hypothesis,  $\rho_{\Omega N} \leq \rho_{\Omega M}$  and  $\rho_{\Omega N}(\Omega N) = \infty$ . In particular,  $W(\rho_{\Omega N}) = \infty$  and, since  $\Omega G(z) = \frac{1}{1 - W(z)}$ , it follows that the radius of convergence,  $\rho$ , of  $\Omega G(z)$  satisfies

 $\rho < \rho_{\Omega N} \leqslant \rho_{\Omega M}$  and  $W(\rho) = 1$ .

Moreover, since  $\pi_*(q)$  is surjective,

$$H(\Omega(M \lor N)) = H(\Omega G) \otimes H(\Omega M) \otimes H(\Omega N)$$

and so  $\rho$  is also the radius of convergence of  $\Omega(M \vee N)(z)$  and

$$\Omega(M \vee N)(\rho) = \infty.$$

Finally, since

$$\Omega F(z) = \frac{1}{1 - z^{n-2} \Omega(M \vee N)(z)}$$

it follows that  $\rho_{\Omega F} < \rho = \rho_{\Omega(M \vee N)}$  and Theorem 1.4 follows from Theorem 1, Theorem 3 and (4) in [5].

LEMMA 3.3. — The homotopy fiber G of the injection  $q: M \vee N \rightarrow M \times N$  has the homotopy type of  $\Omega M * \Omega N$ .

*Proof.* — Recall the Cube Lemma ([13]): In a homotopy commutative cube, if the vertical faces are homotopy pullbacks and the lower face an homotopy push-out, then the upper face is also an homotopy push-out.

TOME 67 (2017), FASCICULE 6

Let  $j: G \to M \lor N$  be the homotopy fibre of the inclusion q. Then we form the following cube by taking the pullbacks of j along the injections  $M \to M \lor N$  and  $N \to M \lor N$ .



This shows that  $G \cong \Omega M * \Omega N$ .

#### BIBLIOGRAPHY

- D. J. ANICK, "The smallest singularity of a Hilbert series", Math. Scand 51 (1982), p. 35-44.
- [2] W. BALLMANN & W. ZILLER, "On the number of closed geodesics on a compact riemannian manifold", Duke Math. J. 49 (1982), p. 629-632.
- [3] Y. FÉLIX, S. HALPERIN & T. JEAN-CLAUDE, "The homotopy Lie algebra for finite complexes", Publ. Math., Inst. Hautes Étud. Sci. 56 (1983), p. 387-410.
- [4] ——, Rational Homotopy Theory, Graduate Texts in Mathematics, vol. 205, Springer, 2001, xxxii+535 pages.
- [5] \_\_\_\_\_, "On the growth of the homology of a free loop space", Pure Appl. Math. Q. 9 (2013), no. 1, p. 167-187.
- [6] , Rational Homotopy II, World Scientific, 2015, xxxvi+412 pages.
- M. GROMOV, "Homotopical effects of dilatations", J. Differ. Geom. 13 (1978), p. 303-310.
- [8] S. HALPERIN & L. JEAN-MICHEL, "Suites inertes dans les algèbres de Lie graduées", Math. Scand. 61 (1987), p. 39-67.
- [9] S. HALPERIN & G. LEVIN, "High skeleta of CW complexes", in Algebra, Algebraic Topology and their Interactions (Stockholm 1983), Lecture Notes in Math., vol. 1183, Springer, 1986, p. 211-217.
- [10] P. LAMBRECHTS, "Analytic properties of Poincaré series of spaces", Topology 37 (1998), p. 1363-1370.
- [11] ——, "The Betti numbers of the free loop space of a connected sum", J. Lond. Math. Soc. 64 (2001), p. 205-228.
- [12] ——, "On the Betti numbers of the free loop space of a coformal space", J. Pure Appl. Algebra 161 (2001), p. 177-192.
- [13] M. MATHER, "Pull-backs in homotopy theory", Can. J. Math. 28 (1976), p. 225-263.
- [14] J. W. MILNOR & J. MOORE, "On the structure of Hopf algebras", Ann. Math. 81 (1965), p. 211-264.

ANNALES DE L'INSTITUT FOURIER

 $\square$ 

#### on the growth of the homology of a free loop space II 2531

- [15] D. SULLIVAN, "Infinitesimal Computations in Topology", Publ. Math., Inst. Hautes Étud. Sci. 47 (1977), p. 269-331.
- [16] M. VIGUÉ-POIRRIER, "Homotopie rationnelle et nombre de géodésiques fermées", Ann. Sci. Éc. Norm. Supér. 17 (1984), p. 413-431.

Manuscrit reçu le 16 avril 2015, révisé le 27 mai 2017, accepté le 7 août 2017.

Yves FÉLIX Université Catholique de Louvain, Institut de Mathématique, 2, Chemin du cyclotron, 1348 Louvain-La-Neuve (Belgium) vves.felix@uclouvain.be

Steve HALPERIN University of Maryland, Department of Mathematics, Mathematics Building, College Park, MD 20742 (USA) shalper@umd.edu

Jean-Claude THOMAS Université d'Angers, LAREMA, 2 Bd Lavoisier, 49045 Angers Cedex (France) jean-claude.thomas@univ-angers.fr