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MCSHANE-TYPE IDENTITIES FOR AFFINE DEFORMATIONS

by Virginie CHARETTE & William M. GOLDMAN (*)

Dedicated to the memory of Maryam Mirzakhani

Abstract. — We derive an identity for Margulis invariants of affine deformations of a complete orientable one-ended hyperbolic surface following the identities of McShane, Mirzakhani and Tan–Wong–Zhang. As a corollary, a deformation of the surface which infinitesimally lengthens all interior simple closed curves must infinitesimally lengthen the boundary.

Résumé. — À partir des identités de McShane, de Mirzakhani et de Tan–Wong–Zhang, nous obtenons une identité pour les invariants de Margulis associés à une déformation affine d'une surface hyperbolique complète, orientable, à un trou. Il en découle le corollaire suivant : une déformation de la surface, dont les courbes simples fermées intérieures s'allongent infinitésimalement, doit également allonger le bord de manière infinitésimale.

Introduction

Beginning with McShane [26], sums of geometric quantities over the simple closed curves on a surface Σ have been shown to exhibit remarkable properties, such as being constant over the deformation space of geometric structures on Σ. Let $\mathfrak{F}(\Sigma)$ denote the *Fricke space*, consisting of isotopy classes of marked complete hyperbolic structures on Σ. If γ is a closed curve

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on a marked hyperbolic surface X, then the function

$$\begin{aligned}\mathfrak{F}(\Sigma) &\longrightarrow \mathbb{R}\\ X &\longmapsto \ell_X(\gamma)\end{aligned}$$

assigning to X the length of the closed geodesic on X (or 0 if γ is homotopic to a cusp) is a basic quantity, depending on the homotopy class of γ. See Abikoff [2] and Wolpert [40, §3] for background on hyperbolic Riemann surfaces, where $\mathfrak{F}(\Sigma)$ idenfifies with the *Teichmüller space* of Σ by the uniformization theorem.

Mirzakhani [28] and Tan–Wong–Zhang [36] express the length of a boundary component as an infinite series of functions of the lengths of interior curves. (Compare also Hu–Tan–Zhang [19] and Wolpert [40, pp 86–87]) This led Mirzakhani to new calculations of the Weil–Petersson volume of the moduli space of curves.

In another direction, Labourie–McShane [23] found identities involving quantities generalizing geodesic lengths for *Hitchin representations* of $\pi_1(\Sigma)$ into $\mathsf{SL}(n,\mathbb{R})$. For $n = 3$, these quantities are the geodesic length functions for the Hilbert metric on convex $\mathbb{R}\mathsf{P}^2$-structures on Σ. In contrast, our identities are the first such identities for a *non-reductive* Lie group.

Closely related to hyperbolic structures on sufraces are *flat Lorentzian structures* on 3-manifolds, that is, geometric structures modeled on Minkowski space $\mathsf{E}^{2,1}$ and groups of (Lorentzian) isometries. A *Margulis spacetime* M^3 is a quotient $\mathsf{E}^{2,1}/\Gamma$, where $\Gamma \subset \mathsf{Isom}(\mathsf{E}^{2,1})$ is a discrete group of Lorentzian isometries which acts properly and freely on M. We assume Γ is not solvable, as the solvable cases are easily classified (Fried–Goldman [12]). Then M^3 is an geodesically complete flat Lorentzian manifold with free fundamental group Γ. It is necessarily orientable ([7]). The linearization $\mathsf{Isom}^+(\mathsf{E}^{2,1}) \longrightarrow \mathsf{SO}(2,1)$ maps Γ isomorphically onto a discrete subgroup of $\mathsf{SO}(2,1) \cong \mathsf{Isom}(\mathsf{H}^2)$ ([12]). Therefore, associated to M^3 is a complete hyperbolic surface $X = \mathsf{H}^2/\Gamma_0$. Just as essential closed curves in X are canonically homotopic to unique closed geodesics (having length $\ell_X(\gamma)$), nonparabolic closed curves in M are canonically homotopic to closed geodesics which are necessarily spacelike. These geodesics have a natural Lorentzian length $\alpha(\gamma)$, called the *Margulis invariant*, as this quantity was discovered and developed by Margulis [24, 25]. Such a geodesic is the quotient of a line $\mathsf{Axis}(\gamma)$ upon which γ acts by translation of Lorentzian displacement $\alpha(\gamma)$. This is similar to how $\ell(\gamma)$ measures the minimal displacement of the linear part of γ acting isometrically on H^2.

(Compare the survey articles Abels [1], Charette–Drumm [6], Drumm [11] and Guéritaud [18] for background on these geometric structures.)

This paper develops an identity for Margulis spacetimes for the Margulis invariants $\alpha(\gamma)$ analogous to McShane's identity for hyperbolic surfaces.

The Lie group of interest here is the *tangent bundle* Lie group TG (see below) of the isometry group $G = \mathsf{Isom}(\mathsf{H}^2)$ of the hyperbolic plane H^2. The Lie algebra $\mathfrak{g}$ of G naturally identifies with the Lorentzian vector space $\mathbb{R}^{2,1}$, and Ad identifies with the isometric action of $G \cong \mathsf{SO}(2,1)$ on $\mathbb{R}^{2,1}$. The affine space $\mathsf{E}^{2,1}$ identifies with the quotient $TG/\mathbf{0}_G$ by the zero-section.

Given any Lie group G with Lie algebra $\mathfrak{g}$, the tangent bundle of G enjoys a Lie group structure as the semidirect product

$$TG := G \rtimes_{\mathsf{Ad}} \mathfrak{g} \tag{0.1}$$

where $G \xrightarrow{\mathsf{Ad}} \mathsf{Aut}(\mathfrak{g})$ is the adjoint representation. Then $\mathfrak{g}$ is an abelian normal subgroup, and the zero-section

$$G \xrightarrow{\mathbf{0}_G} TG$$

embeds G as a subgroup of TG which is *not* normal.

This can be elegantly described in terms of base change $\mathbb{R} \hookrightarrow \mathbb{R}[\epsilon]$, where $\mathbb{R}[\epsilon]$ is the $\mathbb{R}$-algebra of *dual numbers.* (Here ϵ is an indeterminate with $\epsilon^2 = 0$). Suppose G is the group $\mathbf{G}(\mathbb{R})$ of $\mathbb{R}$-points of an algebraic group scheme $\mathbf{G}$. Then $TG = \mathbf{G}(\mathbb{R}[\epsilon])$ and

$$\mathsf{Hom}(\Gamma, TG) = \mathsf{Hom}(\Gamma, \mathbf{G}(\mathbb{R}[\epsilon])) = \big(\mathsf{Hom}(\Gamma, \mathbf{G})\big)(\mathbb{R}[\epsilon]) = T\mathsf{Hom}(\Gamma, G)$$

is the Zariski tangent bundle of the representation variety $\mathsf{Hom}(\Gamma, G)$. In particular homomorphisms $\Gamma \longrightarrow TG$ correspond to a homomorphism $\Gamma \xrightarrow{\rho} G$ and an *infinitesimal deformation* of ρ, that is, a tangent vector in $T_\rho\big(\mathsf{Hom}(\Gamma, G)\big)$.

We obtain an identity for affine deformations Γ of a Fuchsian group Γ_0 by differentiating the basic length identity for Γ_0. To simplify notation, define two functions $H(u,v)$ and $K(u,v)$:

$$\begin{aligned} H(u,v) &:= \frac{1}{1+e^{(u+v)/2}} + \frac{1}{1+e^{(u-v)/2}} \\ K(u,v) &:= \frac{1}{1+e^{(u+v)/2}} - \frac{1}{1+e^{(u-v)/2}} = \frac{-\sinh(v/2)}{\cosh(u/2)+\cosh(v/2)}. \end{aligned}$$

Theorem 0.1. — *Let $X = \mathsf{H}^2/\Gamma_0$ be a complete one-ended orientable hyperbolic surface. For any affine deformation of Γ_0, the Margulis invariants* *satisfy:*

$$\Big(1 - \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} H\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\Big)\alpha(\partial) = \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} K\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\big(\alpha(\gamma_1)+\alpha(\gamma_2)\big).$$

Here $\mathcal{C}$ denotes the set of unordered pairs of isotopy classes of simple closed curves such that γ_1, γ_2 and ∂X cobound an embedded 3-holed sphere in X.

Theorem 0.1 implies the following result in hyperbolic geometry:

Corollary 0.2. — *Suppose that X is a complete one-ended orientable hyperbolic surface. Let X_t be a deformation of hyperbolic surfaces such that the derivative of the length satisfies:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell_{X_t}(\gamma) > 0$$

for all simple closed curves $\gamma \subset \mathsf{interior}(X)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell_{X_t}(\partial) > 0.$$

Yair Minsky pointed out that this corollary can also be deduced from Bestvina–Bromberg–Fujiwara–Souto [3], using the convexity of length functions with respect to shear deformations. François Guéritaud observed that this corollary also follows from the theory of strip deformations developed in his collaboration [9, 10, 18] with Jeffrey Danciger and Fanny Kassel. In particular, he described deformations where all the interior simple loops lengthen, and all but one boundary component lengthens.

McShane [27] also discusses differentiated length identities and rearrangements of absolutely convergent series. Papadopoulos–Théret [31] explains a construction of Thurston's which shortens (or lengthens) all closed geodesics; see also [9, 10, 15, 18].

A direct application of the technique of Labourie–McShane [23] fails, since the cross-ratios of [23] will not apply directly to the non-reductive group $\mathsf{Isom}^+(\mathsf{E}^{2,1})$. However, a generalization of these cross-ratios which includes infinitesimal information may give an alternate proof of our results.

As noted above, McShane's original identity [26] concerned once-punctured tori. The generalized McShane identity for a bordered *orientable* hyperbolic surface X is due, independently, to Tan–Wong–Zhang [36] and Mirzakhani [28]. Norbury [30] extended the identity to include non-orientable surfaces. In subsequent work we consider identities for affine deformations when X is possibly nonorientable and may have more than one end.

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1. Deformations and the Margulis invariant

The Fricke space $\mathfrak{F}(\Sigma)$ embeds in the quotient space $\mathsf{Hom}(\Gamma, G)/G$, where G acts on the $\mathbb{R}$-algebraic set $\mathsf{Hom}(\Gamma, G)$ by left-composition with inner automorphisms of G. Here $G = \mathsf{Isom}(\mathsf{H}^2) \cong \mathsf{SO}(2,1)$, and indeed $\mathfrak{F}(\Sigma)$ embeds as a connected component of $\mathsf{Hom}(\Gamma, G)/G$. Openness of $\mathfrak{F}(\Sigma)$ in $\mathsf{Hom}(\Gamma, G)/G$ was first proved by Weil [38]. (See also Raghunathan [32, Thm. 6.19]). It is a special case of the openness of the holonomy map, which was first noticed by Thurston; compare the discussion in Kapovich [20, Thm. 7.2], Goldman [13], Canary–Epstein–Green [5] and Koszul [22, Ch. IV, §3, Thm. 3].

Closedness is originally due to Chuckrow [8] in this particular context, but it follows from Kazhdan–Margulis uniform discreteness [21]; see [32, Ch. VIII], Kapovich [20, §4.12], or Thurston [37, §4.1]. (See also Goldman–Millson [17].) The Fenchel–Nielsen coordinates imply that $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6+3b}$ is connected, completing the proof that $\mathfrak{F}(\Sigma)$ is a connected component of $\mathsf{Hom}(\Gamma, G)/G$. (See, for example, Buser [4], Abikoff [2], Thurston [37, §4.6], or Wolpert [40] for details.)

Tangent vectors to $\mathfrak{F}(\Sigma)$ identify with affine deformations of Fuchsian representations of $\pi_1(\Sigma) \cong \Gamma_0 \subset G$. An *affine deformation* of Γ_0 consists of a lift of $\Gamma_0 \subset \mathsf{SO}(2,1)$ to:

$$\begin{tikzcd} & \mathsf{Isom}^+(\mathsf{E}^{2,1}) \arrow[d, "L"] \\ \Gamma_0 \arrow[ur, dashed, "\rho"] \arrow[r, hook] & \mathsf{SO}(2,1) \end{tikzcd}$$

where $\mathsf{E}^{2,1}$ is 3-dimensional *Minkowski space*, that is, a simply connected geodesically complete flat Lorentzian 3-manifold, and L is projection onto the linear part. Affine deformations of Fuchsian groups arise as holonomy groups of *Margulis spacetimes*, complete flat Lorentz 3-manifolds with free fundamental group.

Affine deformations correspond to infinitesimal deformations of the hyperbolic structure on the quotient surface $\Sigma_0 := \mathsf{H}^2/\Gamma_0$, using the identification of $\mathsf{Isom}^+(\mathsf{E}^{2,1})$ with the *tangent bundle Lie group* as in (0.1). The infinitesimal deformation theory was developed by Weil [39] (see also Raghunathan [32, §6]), whereby the tangent space to $\mathfrak{F}(\Sigma)$ identifies with the cohomology group of $\pi_1(\Sigma)$ with coefficients in the composition of the holonomy representation with the adjoint representation $\mathsf{SO}(2,1) \longrightarrow \mathsf{Aut}(\mathfrak{so}(2,1))$.

The tangent space to the Fricke space at X identifies with $H^1(\Gamma, \mathfrak{g})$, where Γ is the image of the holonomy representation for X and $\mathfrak{g}$ is the Lie algebra of the group of isometries of the hyperbolic plane. Since $\mathfrak{g} \cong \mathbb{R}^{2,1}$, a tangent vector V corresponds to an *affine deformation* of Γ, that is, a group of affine isometries of the Minkowski spacetime $\mathbb{R}^{2,1}$. Given $\gamma \in \Gamma$, the affine deformation of γ acts by translation on a unique invariant spacelike line $\mathsf{Axis}(\gamma)$. The signed Lorentzian distance of this translation is called the *Margulis invariant* $\alpha(\gamma)$ of the affine deformation.

A *deformation* of hyperbolic surfaces X_t is a one-parameter family of marked hyperbolic surfaces of fixed topology, varying smoothly in $t \in [a, b]$, where $[a, b] \in \mathbb{R}$ is a closed interval. The space of equivalence classes of marked hyperbolic surfaces forms a smooth manifold, corresponding to a smooth submanifold of the space of representations $\pi_1(X) \longrightarrow \mathsf{SO}(2,1)$. Thus a deformation X_t corresponds to a smooth path of holonomy representations ρ_t in the top-dimensional stratum of $\mathsf{Hom}\big(\pi_1(X), \mathsf{SO}(2,1)\big)$. *Infinitesimal deformations* of X correspond to tangent vectors in this space, and also to affine deformations defined above. The Margulis invariant equals the derivative of the geodesic length function $\ell_{X_t}(\gamma)$ at $t = 0$ (Goldman–Margulis [16]).

By [14], the $\mathbb{R}$-valued function $\gamma \mapsto \alpha(\gamma)/\ell(\gamma)$ on Γ_0 extends to a continuous function on the compact set of *geodesic currents* when Γ_0 is convex cocompact, and is therefore bounded. Thus:

$$|\alpha(\gamma)| \leqslant \kappa\ell(\gamma) \tag{1.1}$$

for some $0 < \kappa$ and all $\gamma \in \Gamma$. Inequality (1.1) also holds when Γ_0 is only assumed to finitely generated; for the elementary argument (which uses trace identities in $\mathsf{SL}(2)$), see [15].

2. Length identities for simple geodesics

Let X be a complete hyperbolic surface with one end, which we denote ∂. The length of a closed geodesic γ in X will be written $\ell_X(\gamma)$, or simply $\ell(\gamma)$, when the context is clear. When γ is peripheral, then defining $\ell_X(\gamma)$ to be zero makes the function $X \longmapsto \ell_X(\gamma)$ continuous on the deformation space.

The end of X lies in various three-holed spheres such that the other two ends correspond to simple closed geodesics $\gamma_1, \gamma_2 \subset X$. (These curves coincide when $g = 1$.) The set $\mathcal{C}$ of isotopy classes of such subsurfaces identifies with the set of subsets $\{\gamma_1, \gamma_2\}$ of isotopy classes of simple closed curves such that $\partial, \gamma_j, \gamma_k$ cobound a three-holed sphere in X. Set:

$$\mathsf{O}(X_t) := \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} \mathcal{D}\big(\ell_{X_t}(\partial), \ell_{X_t}(\gamma_1), \ell_{X_t}(\gamma_2)\big) \tag{2.1}$$

where $\mathcal{D}$ is the continuous function:

$$\mathcal{D}(x,y,z) := 2\log\left(\frac{e^{x/2} + e^{(y+z)/2}}{e^{-x/2} + e^{(y+z)/2}}\right) \tag{2.2}$$

The generalized McShane identity for one-ended orientable X is:

$$\ell_X(\partial) = \mathsf{O}(X).$$

We briefly discuss the convergence of this series, as the ideas will be crucially used when we differentiate it. The key idea is that while the number of simple closed geodesics grows polynomially, the summands in these series decay exponentially.

Break $\mathcal{C}$ into a countable disjoint union of finite sets $\mathcal{C}_N$ as follows. For every nonnegative integer N,

$$\mathcal{C}_N := \big\{\{\gamma_1,\gamma_2\} \in \mathcal{C} \,\big|\, N \leqslant \big(\ell(\gamma_1) + \ell(\gamma_2)\big) < N+1\big\}$$

is finite with cardinality:

$$|\mathcal{C}_N| \leqslant m(N+1)^{6g-4} \tag{2.3}$$

(Mirzakhani [29, Prop. 3.1 (3.15)] and Rivin [33, (1)]). Compare also Rivin [34] and Wolpert [40] for further discussion.

The constant $m = m(g, Q)$ depends only on the genus g and a quantity Q defined as follows. Choose $\epsilon > 0$ sufficiently small so that closed geodesics shorter than ϵ do not intersect, such as:

$$\epsilon < 2\sinh^{-1}(1) = 2\log(1+\sqrt{2});$$

compare Buser [4, §4]. The Margulis lemma and the collar lemma imply only finitely many such geodesics exist. Explicitly, any set of disjoint closed geodesics extends to a pants decomposition. Let b denote the number of ends of X. Since the number of pants equals $-\chi$ where $\chi = 2-2g-b$ is the Euler characteristic, at most $3g-3+b$ closed geodesics are shorter than ϵ.

Let Q be the product of the inverses of the lengths of these geodesics. By taking t in an interval $[a,b]$ containing 0, the continuity of Q implies we can choose m so that inequality (2.3) holds for all $t \in [a,b]$ although the finite subsets $\mathcal{C}_N$ may vary with t.

To prove that the series (2.1) converges absolutely, rearrange its terms as follows:

$$f(t) := \mathsf{O}(X_t) = \sum_{N=0}^{\infty} f_N(t)$$

where $f_N(t)$ denotes the finite sum of positive terms:

$$f_N(t) := \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}_N} \mathcal{D}\big(\ell_{X_t}(\partial), \ell_{X_t}(\gamma_1), \ell_{X_t}(\gamma_2)\big). \tag{2.4}$$

Lemma 2.1.

$$|\mathcal{D}(x,y,z)| \leqslant 4\sinh(x/2)\exp\big(-(y+z)/2\big). \tag{2.5}$$

Proof. — Let $X = e^{x/2} > 0$ and $Y = e^{(y+z)/2} > 0$. Write:

$$\mathcal{D}(x,y,z) = 2\log\left(\frac{X+Y}{X^{-1}+Y}\right)$$

where:

$$\frac{X+Y}{X^{-1}+Y} = 1 + \frac{X-X^{-1}}{X^{-1}+Y}.$$

Using the estimate $0 \leqslant \log(1+u) \leqslant u$ for $u \geqslant 0$, and taking $u = \big(X - X^{-1}\big)/\big(X^{-1}+Y\big)$,

$$\begin{aligned}\mathcal{D}(x,y,z) = 2\log\left(\frac{X+Y}{X^{-1}+Y}\right) &\leqslant 2\,\frac{X-X^{-1}}{X^{-1}+Y}\\ &< 2\big(X-X^{-1}\big)Y^{-1}\\ &= 4\sinh(x/2)e^{-(y+z)/2}.\end{aligned}$$ □

Combine (2.3) with (2.5), and the definition (2.4) of f_N, obtaining:

$$|f_N(t)| \leqslant A_N := 4m\sinh\big(\ell_X(\partial)/2\big)(N+1)^{6g-4}e^{-N/2}.$$

Since $\sum A_N < \infty$, the series (2.1) for the generalized McShane identity converges uniformly and absolutely.

3. Differentiating length identities

Now differentiate the generalized McShane identity to establish uniform absolute convergence of $\sum f_N(t)$. The partial derivatives:

$$\begin{aligned}\frac{\partial \mathcal{D}(x,y,z)}{\partial x} &= H(y+z,x)\,,\\ \frac{\partial \mathcal{D}(x,y,z)}{\partial y} = \frac{\partial \mathcal{D}(x,y,z)}{\partial z} &= K(-y-z,x)\end{aligned} \tag{3.1}$$

decay exponentially as well:

$$\left|\frac{\partial \mathcal{D}}{\partial x}\right|, \left|\frac{\partial \mathcal{D}}{\partial y}\right| = \left|\frac{\partial \mathcal{D}}{\partial z}\right| < 2\cosh(|x|/2)\exp\big(-(y+z)/2\big)\,. \tag{3.2}$$

Lemma 3.1. — *The derivative $f_N'(t)$ is the sum over $\{\gamma_1,\gamma_2\} \in \mathcal{C}_N$ of terms:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\mathcal{D}\big(\ell_{X_t}(\partial),\ell_{X_t}(\gamma_1),\ell_{X_t}(\gamma_2)\big)\\ &= H\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\alpha(\partial)\\ &\qquad + K\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\big(\alpha(\gamma_1)+\alpha(\gamma_2)\big)\,.\end{aligned}$$

Proof. — Apply (3.1) and the chain rule. □

By (3.2), the coefficient $H\big(\ell_{X_t}(\gamma_1)+\ell_{X_t}(\gamma_2),\ell_{X_t}(\partial)\big)$ of $\alpha(\partial)$ and the coefficient $K\big(\ell_{X_t}(\gamma_1)+\ell_{X_t}(\gamma_2),\ell_{X_t}(\partial)\big)$ of $\alpha(\gamma_1)+\alpha(\gamma_2)$ above are each bounded by:

$$2\cosh\big(\ell_{X_t}(\partial)/2\big)\exp(-(\ell_X(\gamma_1)+\ell_X(\gamma_2))/2)\,.$$

The contributions from the first term for $\{\gamma_1,\gamma_2\}\in\mathcal{C}_N$ are bounded by:

$$2\cosh\big(\ell_{X_t}(\partial)/2\big)|\alpha(\partial)|\ e^{-N/2}$$

and the contributions from the second terms are bounded by:

$$4\cosh\big(\ell_{X_t}(\partial)/2\big)|\alpha(\gamma_1)+\alpha(\gamma_2)|e^{-N/2} \leqslant 8\cosh\big(\ell_{X_t}(\partial)/2\big)\kappa\ Ne^{-N/2}$$

using the linear bound for the Margulis invariant (1.1). Adding these contributions, $|f_N'(t)|$ is bounded by:

$$M_N := 4m\cosh\big(\ell_{X_t}(\partial)/2\big)\ \big(2\kappa N + |\alpha(\partial)|\big)(N+1)^{6g-4}e^{-N/2}\,.$$

Proposition 3.2. — *Let X_t be a smooth path of marked complete orientable one-ended hyperbolic surfaces tangent to an affine deformation with Margulis invariant α. Then:*

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} \Big(H\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\ \alpha(\partial) \\ + K\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\big(\alpha(\gamma_1)+\alpha(\gamma_2)\big)\Big)$$

converges absolutely to $\alpha(\partial)$.

Proof. — Evidently $\sum M_N < \infty$, and $|f'_N(t)| < M_N$. Then $\sum f'_N(t)$ converges uniformly and absolutely to $f'(t)$ (see, for example, Rudin [35]). □

Theorem 0.1 follows from Proposition 3.2. Corollary 0.2 now follows from Proposition 3.2 and the following lemma:

Lemma 3.3 (Mirzakhani).

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} H\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big) > 1\,.$$

Proof. — Differentiate $H(x,y)$ with respect to y, obtaining, for $x,y>0$:

$$\frac{\partial H(x,y)}{\partial y} = \frac{\exp\big((x-y)/2\big)(e^x-1)(e^y-1)}{\big\{1+\exp\big((x-y)/2\big)\big\}^2\big\{1+\exp\big((x+y)/2\big)\big\}^2} > 0\,. \tag{3.3}$$

For fixed $x,y>0$ set:

$$f(L) := LH(x+y,L) - \mathcal{D}(L,x,y)$$

where $L>0$. Now $f(0)=0$, and:

$$f(L) = \int_0^L f'(u)\,\mathrm{d}u = \int_0^L u\frac{\partial H(x+y,u)}{\partial u}\,\mathrm{d}u > 0\,,$$

that is, $\mathcal{D}(x,y,L)/L < H(x+y,L)$.

Let $L=\ell_X(\partial)$. Then:

$$\begin{aligned} 1 &= \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} \mathcal{D}\big(L\ell_X(\gamma_1),\ell_X(\gamma_2)\big)/L \\ &< \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} H\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),L)\big) \end{aligned}$$

as desired. □

Proof of Corollary 0.2. — Apply Proposition 3.2, obtaining:

$$\Big(1 - \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} H\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\Big)\alpha(\partial) = \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{C}} K\big(\ell_X(\gamma_1)+\ell_X(\gamma_2),\ell_X(\partial)\big)\big(\alpha(\gamma_1)+\alpha(\gamma_2)\big).$$

By Lemma 3.3, the coefficient on the left-hand side is negative. By (3.1), each coefficient on the right-hand side is negative also. If all $\alpha(\gamma) > 0$ for interior curves, then $\alpha(\partial) > 0$ as desired. □

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