ANNALES

DE

L'INSTITUT FOURIER

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Tome 67, nº 5 (2017), p. 1905-1961.

<http://aif.cedram.org/item?id=AIF_2017__67_5_1905_0>

Ann. Inst. Fourier, Grenoble
67, 5 (2017) 1905-1961

TROPICAL SKELETONS

by Walter GUBLER, Joseph RABINOFF & Annette WERNER

Abstract. — In this paper, we study the interplay between tropical and analytic geometry for closed subschemes of toric varieties. Let K be a complete non-Archimedean field, and let X be a closed subscheme of a toric variety over K. We define the tropical skeleton of X as the subset of the associated Berkovich space X^{an} which collects all Shilov boundary points in the fibers of the Kajiwara–Payne tropicalization map. We develop polyhedral criteria for limit points to belong to the tropical skeleton, and for the tropical skeleton to be closed. We apply the limit point criteria to the question of continuity of the canonical section of the tropicalization map on the multiplicity-one locus. This map is known to be continuous on all torus orbits; we prove criteria for continuity when crossing torus orbits. When X is schön and defined over a discretely valued field, we show that the tropical skeleton coincides with a skeleton of a strictly semistable pair, and is naturally isomorphic to the parameterizing complex of Helm–Katz.

Résumé. — Nous étudions les relations entre la géométrie tropicale et la géométrie analytique pour les sous-schémas fermés des variétés toriques. Soit K un corps non-archimédien et complet et soit X un sous-schéma fermé d'une variété torique sur K. Nous définissons le squelette tropical de X comme le sous-ensemble de l'espace de Berkovich associé X^{an} qui est composé de tous les points du bord de Shilov dans les fibres du morphisme de tropicalisation de Kajiwara–Payne. Nous développons des critères polyèdraux pour que des points limite appartiennent au squelette tropical, et pour que cet espace soit fermé. Nous appliquons ce critère pour les points limite à la question de la continuité de la section canonique du morphisme de tropicalisation sur le lieu de multiplicité un. On sait que cette section est continue sur chaque orbite du tore; nous donnons des critères de continuité au croisement des orbites. Quand X est schön et défini sur un corps discrètement valué, nous montrons que la squelette tropical coïncide avec le squelette d'une paire strictement semistable, et qu'il est naturellement isomorphe au complexe paramétrisant de Helm–Katz.

1. Introduction

Let K be a field which is complete with respect to a non-Archimedean absolute value, which might be trivial. Tropicalizing a scheme X of finite

Keywords: Tropical geometry, Kajiwara–Payne tropicalization, Berkovich spaces, skeletons.
2010 *Mathematics Subject Classification:* 14G22, 14T05.

type over K means, roughly speaking, applying the valuation map to a set of coordinates on X. This produces a combinatorial shadow of X called the tropical variety of X. Such coordinates are obtained by embedding X (or an open subscheme) into a torus or, more generally, into a toric variety. The tropicalization map extends uniquely to a proper continuous map from the Berkovich space X^{an} to a Kajiwara–Payne compactification of Euclidean space. In fact, by a result of Payne [31] and the generalizations given by Foster, Gross and Payne [19], for any subscheme X of a toric variety, the topological space underlying X^{an} is the inverse limit over all tropicalizations of X with respect to suitable choices of coordinates.

An interesting question is the relationship between the Berkovich space X^{an} and an individual tropicalization. If X is a curve, the problem of finding subgraphs of X^{an} which are isometric to tropical varieties of X was investigated by Baker, Payne and Rabinoff in [2, 3].

In [23], we generalize several of these results to the higher-dimensional setting. Among other results, it is shown in loc. cit. that the tropicalization map for a subvariety of a torus has a canonical continuous section on the locus of all points with tropical multiplicity one. This section is defined by associating to every point ω of tropical multiplicity one the unique Shilov boundary point in the fiber of tropicalization over ω. For an overview of these results, see [37].

In the present paper we consider higher dimensional subvarieties of toric varieties. By the previous results we can define the locus of multiplicity one and the section map from the tropicalization to the Berkovich analytification of the variety stratum by stratum. Quite surprisingly, it turns out that the section map is in general no longer continuous when passing from one stratum to another. We provide an example where continuity fails in 8.11. A delicate investigation of polyhedra in the tropicalization is necessary to obtain criteria for continuity. We approach this problem from a more general angle by investigating a subset of the Berkovich space X^{an} which we call the tropical skeleton.

Before we will describe this notion and before we explain more general continuity criteria, we formulate our main application which is continuity of the section map in the case of a proper intersection with torus orbits.

Let Y_Δ be the toric variety over K associated to a pointed rational fan Δ. Then Y_Δ can be stratified into torus orbits $O(\sigma)$, where σ runs over the cones in Δ. The Kajiwara–Payne tropicalization of Y_Δ is a natural partial compactification $\overline{N}_{\mathbb{R}}^{\Delta}$ of the real cocharacter space $N_{\mathbb{R}}$ of the dense torus T in Y_Δ. As a set (but not as a topological space) it is the disjoint union of all cocharacter spaces $N_{\mathbb{R}}(\sigma) := N_{\mathbb{R}}/\langle\sigma\rangle$ associated to the torus orbits $O(\sigma)$ in Y_Δ. There is a natural continuous tropicalization map $Y^{\mathrm{an}} \to \overline{N}_{\mathbb{R}}^{\Delta}$. We consider a closed subscheme X of Y and its tropicalization $\mathrm{Trop}(X)$, which is defined as the image of X^{an} under the tropicalization map

$$\mathrm{trop}\colon X^{\mathrm{an}} \hookrightarrow Y_\Delta^{\mathrm{an}} \longrightarrow \overline{N}_{\mathbb{R}}^{\Delta}.$$

The intersection of X with each toric stratum $O(\sigma)$ of $\overline{N}_{\mathbb{R}}^{\Delta}$ is a closed subscheme of a torus, so that its tropical variety, which is simply $\mathrm{Trop}(X)\cap N_{\mathbb{R}}(\sigma)$, can be equipped with the structure of an integral affine polyhedral complex. Hence there is a notion of tropical multiplicity $m_{\mathrm{Trop}}(\omega)$ for any $\omega \in \mathrm{Trop}(X)\cap N_{\mathbb{R}}(\sigma)$ (see §3.5). Using our previous work [23] orbitwise, this defines a canonical section s_X of the tropicalization map on the subset $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ of all points of tropical multiplicity one in $\mathrm{Trop}(X)$.

THEOREM 8.15. — *Let X be a closed subscheme of Y_Δ such that $X\cap T$ is equidimensional and dense in X. Assume additionally that for all $\sigma \in \Delta$, either $X\cap O(\sigma)$ is empty or of dimension $\dim(X)-\dim(\sigma)$. Then $s_X\colon \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \to X^{\mathrm{an}}$ is continuous.*

Under the hypotheses of Theorem 8.15, the map $s_X : \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \to X^{\mathrm{an}}$ induces a homeomorphism onto its image and we may realize the locus $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ as a closed subset of $\mathrm{trop}^{-1}(\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1})$ in X^{an}. Theorem 8.15 follows from Theorem 8.12 mentioned below in the introduction which yields a completely combinatorial criterion for continuity of s_X.

In the higher-dimensional example of the Grassmannian of planes it was shown in [14] that the tropical Grassmannian is homeomorphic to a closed subset of the analytic Grassmannian. This result relies heavily on combinatorial arguments using the interpretation of the tropical Grassmannian of planes as a space of phylogenetic trees. Draisma and Postinghel [15] provide an alternative proof using tropical torus actions.

Another interesting case from the point of view of moduli spaces is discussed in [11, Theorem 3.14], where it is shown that the tropicalization of a suitable Hassett space can be identified with its Berkovich skeleton.

We will now explain the other results in this paper and how they lead to Theorem 8.15. In §§2–3 we provide some background material on tropical and analytic geometry. Working with several torus orbits at once forces us to consider reducible subschemes of tori. We work out some fundamental properties in this situation which we did not find in the literature. Note that our ground field may be an arbitrary field endowed with the trivial absolute value. This often requires a careful study of the behaviour of our objects under non-Archimedean field extensions.

Let X be any closed subscheme of the toric variety Y_Δ over a K and let $O(\sigma)$ be the orbit associated to the cone σ in the fan Δ. For every $\omega \in \mathrm{Trop}(X)$ the fiber $\mathrm{trop}^{-1}(\{\omega\})$ of the tropicalization map over ω is an affinoid domain in $(X \cap O(\sigma))^{\mathrm{an}}$. Therefore it contains a finite subset of points B, the Shilov boundary, such that every element in the associated affinoid algebra achieves its maximum absolute value on B. Then we define the tropical skeleton of X in §4 as the subset

$$S_{\mathrm{Trop}}(X) = \left\{\xi \in X^{\mathrm{an}} \mid \xi \text{ is a Shilov boundary point of } \mathrm{trop}^{-1}(\mathrm{trop}(\xi))\right\}$$

of X^{an}. The tropical skeleton does not change by passing to the induced reduced subscheme and it is compatible with valued field extensions. Moreover, we show that it is locally closed in X^{an}, and that the tropical skeleton of X is the union of the tropical skeletons of the irreducible components. In Example 4.9 we discuss a concrete hypersurface in affine 3-space such that its tropical skeleton is not closed. Motivated by this example, we define the local dimension $d(\omega)$ of a point $\omega \in \mathrm{Trop}(X)$ as the dimension of the local cone at ω of the tropicalization of X. This coincides with the dimension of the initial degeneration of X at ω. Then a polyhedron in the tropicalization of $X \cap O(\sigma)$ containing ω is called d-maximal at ω if its dimension coincides with $d(\omega)$. Note that the local cone of $\mathrm{Trop}(X)$ at ω can be identified with the tropicalization of the initial degeneration at ω over the residue field of K endowed with the trivial absolute value.

In Theorem 6.1, we prove a very general criterion for a limit point of a sequence $\xi_i \in S_{\mathrm{Trop}}(X)$ to stay inside the tropical skeleton. For simplicity, we assume here that the sequence ξ_i is in the dense torus T as well, and that $X \cap T$ is of pure dimension d. As a consequence of Theorem 6.1, we obtain the following result:

Theorem 6.3. — *Let X be a closed subscheme of the toric variety Y_Δ such that $X \cap O(\sigma)$ is equidimensional of dimension d_σ for any $\sigma \in \Delta$. We suppose that for all faces $\tau \prec \sigma$ of Δ and any d_τ-dimensional polyhedron P in $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\tau)$ such that its closure meets $N_{\mathbb{R}}(\sigma)$, the natural projection of P to $N_{\mathbb{R}}(\sigma)$ has dimension d_σ. Then $S_{\mathrm{Trop}}(X)$ is closed.*

The situation is particularly nice if X meets every torus orbit not at all or properly, which means that either $X \cap O(\sigma) = \emptyset$ or that its dimension is equal to $\dim(X) - \dim(\sigma)$. We investigate this situation in §7. In particular, we show in Corollary 7.7 that for such X the tropical skeleton $S_{\mathrm{Trop}}(X)$ is closed in X^{an}.

Section 8 deals with continuous sections of the tropicalization map. Let X be a closed subscheme of Y_Δ and consider a point ω in $\mathrm{Trop}(X \cap O(\sigma))$ of tropical multiplicity one. We show in Proposition 8.3 that in this case there exists a unique irreducible component C of $X \cap O(\sigma)$ of (maximal possible) dimension $d(\omega)$ such that ω lies in $\mathrm{Trop}(C)$ and such that $\mathrm{trop}^{-1}(\omega) \cap C^{\mathrm{an}}$ has a unique Shilov boundary point. Hence for every point ω of tropical multiplicity one we can single out a Shilov boundary point in the fiber of the tropicalization map over ω. This defines a section s_X of the tropicalization map on the subset $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ of all points of tropical multiplicity one of $\mathrm{Trop}(X)$. The question of continuity of s_X is closely related to the question of the tropical skeleton being closed. We will deduce the following theorem from the results of §§6–7:

Theorem 8.12. — *Let Δ be a pointed rational fan in $N_{\mathbb{R}}$ and let $X \subset Y_\Delta$ be a closed subscheme. Let $\{\omega_i\}_{i=1}^{\infty}$ be a sequence in $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}$ converging to a point $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}(\sigma)$ for $\sigma \in \Delta$, $\sigma \neq \{0\}$. Suppose that there exists a polyhedron $P \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}$ which is d-maximal at each ω_i. If the natural projection of P to $N_{\mathbb{R}}(\sigma)$ is d-maximal at ω, then $s_X(\omega_i) \to s_X(\omega)$.*

We assume now that the intersection of X with the dense torus T in Y_Δ is dense in X. We can apply Theorem 8.12 to the case that the intersection $X \cap O(\sigma)$ of X with all torus orbits is equidimensional. We deduce in Theorem 8.14 that if $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$ can be covered by finitely many maximal-dimensional polyhedra P which project to polyhedra of maximal dimension in all boundary strata which are met by the closure of P, then $s_X \colon \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \to X^{\mathrm{an}}$ is continuous. As an immediate consequence, we get the neat criterion in Theorem 8.15 which we highlighted before.

In §9 we specialize to the case of a so-called schön subvariety X of a torus T, defined over a discretely valued subfield $K_0 \subset K$. In this situation, X admits a compactification $\mathscr{X}$ in a toric scheme over the valuation ring $K^\circ \subset K$, such that the pair of $\mathscr{X}$ with the boundary divisor on the generic fiber H form a strictly semistable pair in the sense of [23]. This allows us to use the results of loc. cit.to define a skeleton $S(\mathscr{X}, H) \subset X^{\mathrm{an}}$ associated to the pair $(\mathscr{X}, H)$. In Theorem 9.12 we show that $S(\mathscr{X}, H)$ coincides with the tropical skeleton $S_{\mathrm{Trop}}(X)$ (as subsets of X^{an}), and that both are naturally isomorphic to the parameterizing complex Γ_X defined by Helm–Katz [24]. As a consequence, the parameterizing complex is a deformation retract of X^{an}, so the canonical isomorphism $H^r(\Gamma_X, \mathbb{Q}_\ell) \cong W_0 H^r_{\mathrm{\acute{e}t}}(X_{\overline{K}_0}, \mathbb{Q}_\ell)$ of [24] follows, at least over local fields, from a very general comparison result of Berkovich; see Remark 9.14. Here, ℓ is assumed to be different from the residue characteristic of K, and the isomorphism relates the singular cohomology to the weight zero part of the étale cohomology of the base change of X to the algebraic closure $\overline{K}_0$.

The interplay between tropical and analytic geometry has been intensely studied during the last years. It plays an important role in the investigation of tropical moduli spaces [1] and also for applications of tropical geometry to arithmetic problems [26]. We hope that the general conceptual picture of the relationship between analytic and tropical subschemes of toric varieties which we develop in this paper will prove useful for further developments in this area.

1.1. Acknowledgments

Part of the work on this project took place during the Simons symposium on *Non-archimedean and Tropical Geometry* in 2015. The authors are very grateful to the Simons Foundation for financial support of this meeting and to the organizers, Matt Baker and Sam Payne. The first author was partly supported by the collaborative research center SFB 1085 funded by the Deutsche Forschungsgemeinschaft. The second author was sponsored by the National Security Agency Young Investigator Grant number H98230-15-1-0105. We also thank the referee for his helpful comments.

2. Analytic spaces and their reductions

In this section we present some technical facts about analytic spaces, mostly concerning reductions and Shilov boundaries. Since in our investigation of torus orbits we are forced to consider reducible varieties and since we want to include the case of an arbitrary non-archimedean ground field endowed possibly with the trivial absolute value, we have to provide some proofs which we could not locate in the literature in the required generality.

We assume that the reader is familiar with the terminology used in Berkovich's book [4].

2.1. General notation

This paper uses standard notations from the fields of non-Archimedean analytic geometry and toric geometry. Appendix A contains a list of notations.

If X is an object (scheme, analytic space, formal scheme, algebra, arrow) over a ring K and $K \to K'$ is a ring homomorphism, the extension of scalars of X to K' will be denoted $X_{K'}$ when convenient.

2.2. Non-Archimedean fields

By a *non-Archimedean field* we mean a field K which is complete with respect to a (potentially trivial) non-Archimedean valuation $v\colon K \to \mathbb{R} \cup \{\infty\}$. If K is a non-Archimedean field then we write K° for its ring of integers, $K^{\circ\circ}$ for the maximal ideal in K°, and $\widetilde{K} = K^\circ/K^{\circ\circ}$ for the residue field. We also write $\Gamma = \Gamma_K = v(K^\times)$ for the value group of K and $\sqrt{\Gamma}$ for its saturation in $\mathbb{R}$. Let $|\cdot| = \exp(-v(\cdot))$ be a corresponding absolute value on K.

Throughout this paper, K will denote a non-Archimedean field. By a *valued extension field* of K we mean a non-Archimedean field K' containing K such that the valuation on K' restricts to the valuation on K.

For $r_1, \ldots, r_n \in \mathbb{R}_{>0}$, we denote the *generalized Tate algebra* by

$$K\langle r_1^{-1}x_1, \ldots, r_n^{-1}x_n\rangle = \left\{ \sum_{I \in \mathbb{Z}^n_{\geqslant 0}} a_I x^I \mid |a_I| r^I \to 0 \text{ as } |I| \to \infty \right\}.$$

2.3. Analytic spaces

We will generally use calligraphic letters to denote K-affinoid algebras. The Berkovich spectrum of a (strictly) K-affinoid algebra $\mathcal{A}$ is denoted $\mathscr{M}(\mathcal{A})$. These are the building blocks of a Berkovich (strictly) K-analytic space X, see [6]. For $x \in X$ we let $\mathscr{H}(x)$ denote the completed residue field at x. This is a valued extension field of K.

Of major importance for us are *good (strictly) K-analytic spaces* which means that every point has a neighborhood of the form $\mathscr{M}(\mathcal{A})$, where $\mathcal{A}$ is a (strictly) K-affinoid algebra. Note that only good K-analytic spaces are considered in [4].

For any K-scheme X locally of finite type, we let X^{an} denote its analytification, as constructed in [4, §3.4–3.5]. This is a good strictly K-analytic space.

2.4. Dimension theory and irreducible decomposition

The basic dimension theory of K-analytic spaces is developed in [4, §2.3]. The *dimension* $\dim(X)$ of a strictly K-affinoid space $X = \mathscr{M}(\mathcal{A})$ is by definition the Krull dimension of $\mathcal{A}$. The dimension of a general K-affinoid space X is the dimension of $X_{K'}$ for K'/K a valued field extension such that $X_{K'}$ is strictly K'-affinoid. The *dimension* $\dim(X)$ of a K-analytic space X is the maximum dimension of a K-affinoid domain in X. If X is strictly K-analytic then this is equal to the maximal Krull dimension of the stalk $\mathscr{O}_{X,x}$ at a point x of the rigid analytic variety associated to X. We say that a K-analytic space X is *equidimensional of dimension d* provided that every K-affinoid domain in X has dimension d. The analytification of a K-scheme X of dimension d (resp. of equidimension d) has dimension d (resp. equidimension d) by [12, Lemma A.1.2 (2)].

Let X be a good K-analytic space and let $x \in X$. The *local dimension* $\dim_x(X)$ is the infimum of $\dim(V)$ for $V \subset X$ an affinoid neighborhood of x. One has $\dim(X) = \max_{x\in X} \dim_x(X)$, and X is equidimensional if and only if $\dim_x(X) = \dim(X)$ for all $x \in X$.

Let $X = \mathscr{M}(\mathcal{A})$ be a K-affinoid space. The *irreducible components* of X are the reduced Zariski-closed subspaces of X defined by the minimal prime ideals of $\mathcal{A}$. Each irreducible component is equidimensional, and X is equidimensional if and only if its irreducible components have the same dimension by [4, Proposition 2.3.5]. If X is K-affinoid then $\dim_x(X)$ is the maximal dimension of an irreducible component containing x.

See [12] for a global theory of irreducible components. We need to extend the following result, found in ibid., to the case of non-strict K-affinoid domains.

Proposition 2.5. — *Let X be a finite-type K-scheme, let $Y \subset X$ be an irreducible component, and let $U \subset X^{\mathrm{an}}$ be a (possibly non-strict) K-affinoid domain. Then $Y^{\mathrm{an}} \cap U$ is a union of irreducible components of U.*

Proof. — Let $K' \supset K$ be a valued extension field of K which is non-trivially valued and such that $U_{K'}$ is strictly K'-affinoid. Let $X' = X_{K'}$, $Y' = Y_{K'}$, and $U' = U_{K'}$. Then the reduced space underlying $Y'^{\mathrm{an}} \cap U'$ is a union of irreducible components of U' by [12, Corollary 2.2.9, Theorem 2.3.1]. Let $U = \mathscr{M}(\mathcal{A})$ and let $\mathcal{A}' = \mathcal{A}\widehat{\otimes}_K K'$, so $U' = \mathscr{M}(\mathcal{A}')$. Then $\mathcal{A}'$ is a faithfully flat $\mathcal{A}$-algebra by [5, Lemma 2.1.2]. Suppose that $Y \cap U$ is defined by the ideal $\mathfrak{a} \subset \mathcal{A}$, so $Y' \cap U'$ is defined by $\mathfrak{a}' = \mathfrak{a}\mathcal{A}'$. The question is now purely one of commutative algebra: if $\mathcal{A} \to \mathcal{A}'$ is a faithfully flat homomorphism of Noetherian rings and $\sqrt{\mathfrak{a}'}$ is an intersection of minimal prime ideals $\wp_i'$ of $\mathcal{A}'$, then $\sqrt{\mathfrak{a}}$ is the intersection of the minimal prime ideals $\wp_i' \cap\ cA$ of $\mathcal{A}$. This follows from [20, Proposition 2.3.4]. □

2.6. Admissible formal schemes

Suppose now that the valuation on K is non-trivial. An *admissible K°-algebra* in the sense of [10] is a K°-algebra A which is topologically finitely generated and flat (i.e. torsionfree) over K°. We will generally use Roman letters to denote admissible K°-algebras. An *admissible K°-formal scheme* is a formal scheme $\mathfrak{X}$ which has a cover by formal affine opens of the form $\mathrm{Spf}(A)$ for A an admissible K°-algebra. The special fiber of $\mathfrak{X}$ is denoted $\mathfrak{X}_s = \mathfrak{X} \otimes_{K^\circ} \widetilde{K}$; this is a $\widetilde{K}$-scheme locally of finite type. If $\mathfrak{X}$ has a locally finite atlas, the *analytic generic fiber* $\mathfrak{X}_\eta$ of $\mathfrak{X}$ is the strictly K-analytic space defined locally by $\mathrm{Spf}(A)_\eta = \mathscr{M}(A \otimes_{K^\circ} K)$. Here we recall several facts about admissible K°-algebras and affine admissible K°-formal schemes.

Proposition 2.7. — *Let $f\colon A \to B$ be a homomorphism of admissible K°-algebras, let $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$, and let $\phi\colon \mathfrak{Y} \to \mathfrak{X}$ be the induced morphism.*

(1) *f is finite if and only if $f_K\colon A_K \to B_K$ is finite.*
(2) *Suppose that $f_K\colon A_K \to B_K$ is finite and dominant, i.e. that $\ker(f_K)$ is nilpotent. Then $\phi_s\colon \mathfrak{Y}_s \to \mathfrak{X}_s$ is finite and surjective.*
(3) *If $\mathfrak{X}_\eta$ has dimension d (resp. is equidimensional of dimension d), then $\mathfrak{X}_s$ has dimension d (resp. is equidimensional of dimension d).*
(4) *Suppose that $f_K\colon A_K \to B_K$ is finite and dominant, and that $\mathfrak{X}_\eta$ and $\mathfrak{Y}_\eta$ are equidimensional (necessarily of the same dimension). Then $\phi_s\colon \mathfrak{Y}_s \to \mathfrak{X}_s$ maps generic points to generic points.*

Proof. — These are all found in [3, §3], except for the "dimension d" part of (3), which uses a similar argument to the "equidimension d" part of [3, Proposition 3.23]. Note that the proofs in loc. cit. do not use the standing assumption there that K is algebraically closed. □

2.8. Power bounded elements

Assume that the valuation on K is non-trivial. Let $\mathcal{A}$ be a strictly K-affinoid algebra and let $X = \mathscr{M}(\mathcal{A})$. We let $\mathcal{A}^\circ \subset \mathcal{A}$ denote the subring of power-bounded elements and we let $\mathcal{A}^{\circ\circ} \subset \mathcal{A}^\circ$ denote the ideal of topologically nilpotent elements. The ring $\mathcal{A}^\circ$ is flat over K°, but it may not be topologically of finite type, see [3, Theorem 3.17]. If $|\cdot|_{\mathrm{sup}}$ is the supremum seminorm on $\mathcal{A}$, then $\mathcal{A}^\circ = \{f \in \mathcal{A} \mid |f|_{\mathrm{sup}} \leqslant 1\}$ and $\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} \mid |f|_{\mathrm{sup}} < 1\}$. We set $\widetilde{\mathcal{A}} = \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$.

2.9. The canonical reduction

With the notation in §2.9 the *canonical model* of $X = \mathscr{M}(\mathcal{A})$ is the affine formal K°-scheme $\mathfrak{X}^{\mathrm{can}} := \mathrm{Spf}(\mathcal{A}^\circ)$. The *canonical reduction* of X is $\widetilde{X} := \mathrm{Spec}(\widetilde{\mathcal{A}})$. This is a reduced affine $\widetilde{K}$-scheme of finite type, and the association $X \mapsto \widetilde{X}$ is functorial. Since the radical of the ideal $K^{\circ\circ}\mathcal{A}^\circ$ is equal to $\mathcal{A}^{\circ\circ}$, the canonical reduction $\widetilde{X}$ is the reduced scheme underlying the special fiber of the canonical model $\mathfrak{X}^{\mathrm{can}}_s$. If $\Gamma = v(K^\times)$ is divisible (e.g. if K is algebraically closed) then $\widetilde{X} = \mathfrak{X}^{\mathrm{can}}_s$.

We have the following analogue of Proposition 2.7 for the canonical reduction.

Proposition 2.10. — *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of strictly K-affinoid algebras, let $X = \mathscr{M}(\mathcal{A})$ and $Y = \mathscr{M}(\mathcal{B})$, and let $\phi\colon Y \to X$ be the induced morphism.*

1. *The following are equivalent: f is finite; $f^\circ\colon \mathcal{A}^\circ \to \mathcal{B}^\circ$ is integral; $\widetilde{f}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is finite.*
2. *Suppose that f is finite and dominant, i.e. that* $\ker(f)$ *is nilpotent. Then $\widetilde{\phi}\colon \widetilde{Y} \to \widetilde{X}$ is finite and surjective.*
3. *If X has dimension d (resp. is equidimensional of dimension d), then $\widetilde{X}$ has dimension d (resp. is equidimensional of dimension d).*
4. *Suppose that f is finite and dominant, and that X and Y are equidimensional (necessarily of the same dimension). Then $\widetilde{\phi}\colon \widetilde{Y} \to \widetilde{X}$ maps generic points to generic points.*

Proof. — Part (1) is [9, Theorem 6.3.5/1]. In the situation of (2), we know from (1) that $\widetilde{\phi}$ is finite. Let $\mathfrak{X}^{\mathrm{can}} = \mathrm{Spf}(\mathcal{A}^\circ)$ and $\mathfrak{Y}^{\mathrm{can}} = \mathrm{Spf}(\mathcal{B}^\circ)$. Then $\widetilde{\phi}\colon \widetilde{Y} \to \widetilde{X}$ is the morphism of reduced schemes underlying $\phi_s\colon \mathfrak{Y}^{\mathrm{can}}_s \to \mathfrak{X}^{\mathrm{can}}_s$, so ϕ_s is a closed map. Hence to show $\widetilde{\phi}$ is surjective, it suffices to show that $\ker(f^\circ_{\widetilde{K}}\colon \mathcal{A}^\circ_{\widetilde{K}} \to \mathcal{B}^\circ_{\widetilde{K}})$ is nilpotent. Let $a \in \mathcal{A}^\circ$ have zero image in $\mathcal{B}^\circ_{\widetilde{K}}$. Then $f(a) = \varpi b$ for some $b \in \mathcal{B}^\circ$ and $\varpi \in K^{\circ\circ}$. Let

$$b^n + f(c_{n-1})b^{n-1} + \cdots + f(c_1)b + f(c_0) = 0 \qquad (c_i \in \mathcal{A}^\circ)$$

be an equation of integral dependence for b over $\mathcal{A}^\circ$. Then

$$a^n + \varpi\big(c_{n-1}a^{n-1} + \cdots + \varpi^{n-2}c_1 a + \varpi^{n-1}c_0\big) \in \ker(f),$$

so the image of a^n in $\mathcal{A}^\circ_{\widetilde{K}}$ is nilpotent, as desired.

Now suppose that X has dimension d. By Noether normalization [9, Theorem 6.1.2/1], there exists a finite injection $K\langle x_1,\ldots,x_d\rangle \hookrightarrow \mathcal{A}$, which by (2) yields a finite, surjective morphism $\widetilde{X} \to \mathbf{A}^d_{\widetilde{K}}$. Hence $\widetilde{X}$ has dimension d. Suppose that X is equidimensional of dimension d. Choose a generic point $\widetilde{x} \in \widetilde{X}$, and choose $a \in \mathcal{A}^\circ$ such that $\widetilde{a}(\widetilde{x}) \neq 0$ and $\widetilde{a}$ vanishes on all other generic points of $\widetilde{X}$. Let $X' \subset X$ be the Laurent domain $\mathscr{M}(\mathcal{A}\langle a^{-1}\rangle)$. Then $\dim(X') = \dim(X)$ and $\widetilde{X}' = \operatorname{Spec}(\widetilde{\mathcal{A}}[\widetilde{a}^{-1}])$ by [9, Proposition 7.2.6/3]. By the above, $\widetilde{X}'$ has dimension d, and hence $\widetilde{X}$ has equidimension d.

Part (4) follows immediately from Parts (1)–(3). □

2.11. Relating the two reductions

We continue to assume the valuation on K is non-trivial. Let A be an admissible K°-algebra and let $\mathcal{A} = A_K$, a strictly K-affinoid algebra. Put $\mathfrak{X} = \operatorname{Spf}(A)$, $X = \mathscr{M}(\mathcal{A})$, and $\mathfrak{X}^{\mathrm{can}} = \operatorname{Spf}(\mathcal{A}^\circ)$, as above. Then $A \subset \mathcal{A}^\circ$ since by definition A is generated by power-bounded elements, so we obtain morphisms

$$\mathfrak{X}^{\mathrm{can}} \to \mathfrak{X} \quad \text{and} \quad \widetilde{X} \hookrightarrow \mathfrak{X}^{\mathrm{can}}_s \to \mathfrak{X}_s\,. \tag{2.11.1}$$

These morphisms are functorial in $\mathfrak{X}$. The next Proposition relates the two finite-type $\widetilde{K}$-schemes canonically associated with A.

Proposition 2.12. — *With the above notation, the natural inclusion $A \hookrightarrow \mathcal{A}^\circ$ is an integral homomorphism, and the morphism $\widetilde{X} \to \mathfrak{X}_s$ is finite and surjective.*

Proof. — Choose a surjection $K^\circ\langle x_1,\ldots,x_n\rangle \twoheadrightarrow A$. Tensoring with K, we get a surjection $K\langle x_1,\ldots,x_n\rangle \twoheadrightarrow \mathcal{A}$, so by [9, Theorem 6.3.5/1], the composition

$$K^\circ\langle x_1,\ldots,x_n\rangle \to A \to \mathcal{A}^\circ$$

is an integral homomorphism. Hence $A \to \mathcal{A}^\circ$ is integral. It follows that $A_{\widetilde{K}} \to \mathcal{A}^\circ_{\widetilde{K}} \twoheadrightarrow \widetilde{\mathcal{A}}$ is an integral homomorphism of finitely generated $\widetilde{K}$-algebras, so $A_{\widetilde{K}} \to \widetilde{\mathcal{A}}$ and $\widetilde{X} \to \mathfrak{X}_s$ are finite. In particular, $\widetilde{X} \to \mathfrak{X}_s$ is closed, and hence $\mathfrak{X}^{\mathrm{can}}_s \to \mathfrak{X}_s$ is closed, as $\widetilde{X}$ and $\mathfrak{X}^{\mathrm{can}}_s$ have the same underlying topological space. Thus to show surjectivity it is enough to prove that $\ker(A_{\widetilde{K}} \to \mathcal{A}^\circ_{\widetilde{K}})$ is nilpotent. This is done exactly as in the proof of Proposition 2.10 (2). □

2.13. The Shilov boundary and the reduction map

Here the valuation on K is allowed to be trivial. The *Shilov boundary* of a K-affinoid space $X = \mathscr{M}(\mathcal{A})$ is the unique minimal subset $B(X) \subset X$ on which every $f \in \mathcal{A}$ achieves its maximum. It exists and is finite and nonempty for any K-affinoid space by [4, Corollary 2.4.5]. The Shilov boundary is insensitive to nilpotent elements of $\mathcal{A}$.

We postpone the proof of the following Lemma until after Proposition 2.17.

Lemma 2.14. — *Let $X = \mathscr{M}(\mathcal{A})$ be a K-affinoid space, let $x \in B(X)$ be a Shilov boundary point, and let $|\cdot|_x \colon \mathcal{A} \to \mathbb{R}_{\geqslant 0}$ be the corresponding seminorm. Then $\ker|\cdot|_x$ is a minimal prime ideal of $\mathcal{A}$.*

Proposition 2.15. — *Let $X = \mathscr{M}(\mathcal{A})$ be a K-affinoid space and let $X = X_1 \cup \cdots \cup X_n$ be its decomposition into irreducible components. Then each Shilov boundary point of X is contained in a unique irreducible component X_i. Moreover, we have $B(X) = B(X_1) \sqcup \cdots \sqcup B(X_n)$, and if $x \in B(X_i)$ then $\dim_x(X) = \dim(X_i)$.*

Proof. — It follows from Lemma 2.14 that a Shilov boundary point $x \in B(X)$ is contained in a unique irreducible component X_i, namely, the one defined by the prime ideal $\ker|\cdot|_x \subset \mathcal{A}$. For $f \in \mathcal{A}$, by definition the restriction of f to X_i achieves its maximum value on $B(X_i)$, so it is clear that $B(X) \subset \bigcup_{i=1}^n B(X_i)$.

Let $x \in B(X_i)$ for some $i = 1, \dots, n$. Choose $f_i \in \mathcal{A}$ which vanishes identically on $\bigcup_{j \neq i} X_j$ but not on X_i. By Lemma 2.14, $|f_i(x)| \neq 0$. Choose also $g \in \mathcal{A}$ such that $|g|$ attains its maximum value on X_i only at x, i.e. such that $|g(x)| > |g(x')|$ for all $x' \in B_i \setminus \{x\}$. Using that $B(X)$ is finite, $g^n f_i \in \mathcal{A}$ achieves its maximum only on x for $n \gg 0$, so $x \in B$. Thus $B(X) = \bigcup_{i=1}^n B(X_i)$, and this union is disjoint because any $x \in B(X)$ lies on only one X_i. The final assertion is clear because $x \in B(X_i)$ admits an equidimensional K-affinoid neighborhood contained in X_i, namely, $\{|f_i| \geqslant \epsilon\}$ for f_i as above and ϵ small. □

Now we assume that the valuation on K is non-trivial. Let $X = \mathscr{M}(\mathcal{A})$ be a strictly K-affinoid space. By [4, §2.4], there is a canonical reduction map

$$\mathrm{red}\colon X \longrightarrow \widetilde{X}\,. \tag{2.15.1}$$

We have the following relationship between the reduction map and the Shilov boundary, proved in [4, Proposition 2.4.4].

Proposition 2.16. — *Let* $X = \mathscr{M}(\mathcal{A})$ *be a strictly* K*-affinoid space, and let* $\mathrm{red}\colon X \to \widetilde{X}$ *be the reduction map.*

(1) red *is surjective, anti-continuous,*[(1)] *and functorial in* X*.*
(2) *If* $\widetilde{x} \in \widetilde{X}$ *is a generic point, then* $\mathrm{red}^{-1}(\widetilde{x})$ *consists of a single point.*
(3) *The inverse image under* red *of the set of generic points of* $\widetilde{X}$ *is equal to the Shilov boundary of* X*.*

Now let $\mathfrak{X} = \mathrm{Spf}(A)$ be an affine admissible K°-formal scheme, and let $X = \mathfrak{X}_{\eta}$. Recall from (2.11.1) that we have a natural finite, surjective morphism $\widetilde{X} \to \mathfrak{X}_s$. We define the *reduction map*

$$\mathrm{red}\colon X \to \mathfrak{X}_s \tag{2.16.1}$$

to be the composition of $\mathrm{red}\colon X \to \widetilde{X}$ with the morphism $\widetilde{X} \to \mathfrak{X}_s$. This construction globalizes: if $\mathfrak{X}$ is any admissible K°-formal scheme with generic fiber $X = \mathfrak{X}_{\eta}$, then one obtains a reduction map $\mathrm{red}\colon X \to \mathfrak{X}_s$ by working on formal affines and gluing.

Proposition 2.17. — *Let* $\mathfrak{X}$ *be an admissible* K°*-formal scheme with generic fiber* $X = \mathfrak{X}_{\eta}$*. Then the reduction map* $\mathrm{red}\colon X \to \mathfrak{X}_s$ *is surjective, anti-continuous, and functorial in* $\mathfrak{X}$*.*

Proof. — We reduce immediately to the case of an affine formal scheme, where the result follows from Propositions 2.12 and 2.16. □

Proof of Lemma 2.14. — By passing to an irreducible component of X containing x, we may assume that $\mathcal{A}$ is an integral domain. First we suppose that the valuation on K is non-trivial and that X is strictly K-affinoid. Then $\mathrm{red}(x)$ is the generic point of the canonical reduction $\widetilde{X}$ by Proposition 2.16 (3). The canonical reduction is an equidimensional scheme of the same dimension as X by Proposition 2.10 (3), so x cannot be contained in a smaller-dimensional Zariski-closed subspace of X by functoriality of the reduction map.

Now we suppose that X is not strictly K-affinoid or that the valuation on K is trivial (or both). There exists a non-trivially-valued non-Archimedean extension field $K' \supset K$, namely, the field $K' = K_{r_1,\ldots,r_n}$ for suitable $r_1,\ldots,r_n \in \mathbb{R}_{>0}$ as in [4, §2.1], such that $X' = X\widehat{\otimes}_K K'$ is strictly K'-affinoid. Let $\mathcal{A}' = \mathcal{A}\widehat{\otimes}_K K'$, so $X' = \mathscr{M}(\mathcal{A}')$. Then $\mathcal{A}'$ is an integral domain by [4, Proposition 2.1.4 (iii)] and the proof of [4, Proposition 2.1.4 (ii)]. Let $\pi\colon X' \to X$ be the structural morphism. We have $\pi(B(X')) = B(X)$ by the proof of [4, Proposition 2.4.5], so there exists $x' \in B(X')$ with $\pi(x') = x$.

(1) The inverse image of an open set is closed.

The above argument in the strictly K-affinoid case shows that the seminorm $|\cdot|_{x'}\colon \mathcal{A}' \to \mathbb{R}_{\geqslant 0}$ is a norm, i.e., has trivial kernel. But $\mathcal{A} \to \mathcal{A}'$ is injective by [4, Proposition 2.1.2 (i)], so $|\cdot|_x$ is a norm. $\square$

2.18. Extension of scalars

In the sequel it will be important to understand the behavior of extension of the ground field with respect to the underlying topological space and the Shilov boundary. First we make a simple topological observation. Here the valuation on K is allowed to be trivial.

LEMMA 2.19. — *Let $K' \supset K$ be a valued extension of K and let X be a good K-analytic space. Then the structural morphism $\pi\colon X_{K'} \to X$ is a proper and closed map on underlying topological spaces.*

Proof. — Since X is good, every point $\xi \in X$ has a K-affinoid neighborhood U, which is compact. The inverse image $U' = \pi^{-1}(U) = U_{K'}$ is also affinoid, hence compact. Thus every point of X^{an} admits a compact neighborhood whose inverse image under π is compact. It is clear that any such map is both proper and closed. $\square$

The following Lemma is an analogue of [35, Proposition 3.1 (v)].

LEMMA 2.20. — *Suppose that the valuation on K is non-trivial. Let $X = \mathscr{M}(\mathcal{A})$ be a strictly K-affinoid space, let K' be a valued field extension of K, and let $X' = X_{K'}$. Then the canonical morphism*

$$\widetilde{X}' \to \widetilde{X} \otimes_{\widetilde{K}} \widetilde{K}'$$

is finite.

Proof. — There exists an admissible K°-algebra $A \subset \mathcal{A}^\circ$ such that $A_K = \mathcal{A}$ (let A be the image of $K\langle x_1, \dots, x_n\rangle^\circ$ under a surjection $K\langle x_1, \dots, x_n\rangle \twoheadrightarrow \mathcal{A}$). Let $A' = A \widehat{\otimes}_{K^\circ} K'^\circ$, let $\mathfrak{X} = \mathrm{Spf}(A)$, and let $\mathfrak{X}' = \mathrm{Spf}(A')$. Then A' is an admissible K'°-algebra and $\mathfrak{X}'_\eta = X'$. We have a commutative square of affine $\widetilde{K}'$-schemes

$$\begin{tikzcd} \widetilde{X}' \arrow[r] \arrow[d] & \widetilde{X} \otimes_{\widetilde{K}} \widetilde{K}' \arrow[d] \\ \mathfrak{X}'_s \arrow[r] & \mathfrak{X}_s \otimes_{\widetilde{K}} \widetilde{K}' \end{tikzcd}$$

where the left and right arrows are the morphisms coming from (2.11.1). They are finite by Proposition 2.12. Since the bottom arrow is an isomorphism, the top morphism is also finite. $\square$

Recall that the Shilov boundary of a K-affinoid space X is denoted $B(X)$.

Proposition 2.21. — *Let $K' \supset K$ be valued extension of K and let X be a (possibly non-strict) K-affinoid space, let $X' = X_{K'}$, and let $\pi\colon X' \to X$ be the structural map. Then $\pi(B(X')) = B(X)$.*

Proof. — It is clear a priori that $\pi(B(X')) \supset B(X)$ since there are more analytic functions on X' than on X. Therefore we are free to replace K' by a valued extension field.

Suppose that X is not strictly K-affinoid or that the valuation on K is trivial (or both). There exists a non-trivially-valued non-Archimedean extension field $K_{\mathbf{r}} := K_{r_1,\dots,r_n} \supset K$ for suitable $r_1,\dots,r_n \in \mathbb{R}_{>0}$ as in [4, §2.1], such that $X\widehat{\otimes}_K K_{\mathbf{r}}$ is strictly $K_{\mathbf{r}}$-affinoid and such that the image of $B(X\widehat{\otimes}_K K_{\mathbf{r}})$ under the structural morphism $X\widehat{\otimes}_K K_{\mathbf{r}} \to X$ is equal to $B(X)$. By [17, (0.3.2)], there exists a valued extension $K'' \supset K$ which is simultaneously a valued extension field of K' and $K_{\mathbf{r}}$. Replacing K by $K_{\mathbf{r}}$ and K' by K'', we may assume that X is strictly K-affinoid and that the valuation on K is non-trivial.

Let $X = X_1 \cup \cdots \cup X_n$ be the irreducible decomposition of X. Then $B(X) = \bigcup_{i=1}^n B(X_i)$ and $B(X') = \bigcup_{i=1}^n B(\pi^{-1}(X_i))$ by Proposition 2.15, since $\pi^{-1}(X_i)$ is a union of irreducible components of X'. Replacing X by X_i and X' by $\pi^{-1}(X_i) = (X_i)_{K'}$, we may assume X is irreducible.

Consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{red}"] \arrow[d, "\pi"'] & \widetilde{X}' \arrow[r] \arrow[d, "\widetilde{\pi}"'] & \widetilde{X} \otimes_{\widetilde{K}} \widetilde{K}' \arrow[dl] \\ X \arrow[r, "\mathrm{red}"] & \widetilde{X} & \end{tikzcd}$$

By [12, Lemma 2.1.5], X' is equidimensional of dimension $d = \dim(X)$, so the same is true of $\widetilde{X}'$ by Proposition 2.10 (3). Also $\widetilde{X} \otimes_{\widetilde{K}} \widetilde{K}'$ is equidimensional of dimension d, and $\widetilde{X} \otimes_{\widetilde{K}} \widetilde{K}' \to \widetilde{X}$ sends generic points to generic points. As the morphism $\widetilde{X}' \to \widetilde{X} \otimes_{\widetilde{K}} \widetilde{K}'$ is finite by Lemma 2.20, it takes generic points to generic points, so $\widetilde{\pi}\colon \widetilde{X}' \to \widetilde{X}$ takes generic points to generic points. It follows from Proposition 2.16 that $\pi\colon X' \to X$ takes Shilov boundary points to Shilov boundary points. $\square$

3. Toric varieties and tropicalizations

In this section we present the notions and notations that we will use for toric varieties and their tropicalizations. We generally follow [22, 33]. See Appendix A for a list of notations.

3.1. Toric varieties

Fix a finitely generated free abelian group $M \cong \mathbb{Z}^n$, and let $N = \mathrm{Hom}(M, \mathbb{Z})$. We write $\langle\,\cdot\,,\cdot\,\rangle\colon M \times N \to \mathbb{Z}$ for the evaluation pairing. For an additive subgroup $G \subset \mathbb{R}$ we set $M_G = M \otimes_{\mathbb{Z}} G$ and $N_G = N \otimes_{\mathbb{Z}} G = \mathrm{Hom}(M, G)$, and we extend $\langle\,\cdot\,,\cdot\,\rangle$ to a pairing $M_G \times N_G \to \mathbb{R}$.

For a ring R and a monoid S we write $R[S]$ for the monoid ring on S; for $u \in S$ we let $\chi^u \in R[S]$ denote the corresponding character. We set $T = \mathrm{Spec}(K[M]) \cong \mathbf{G}_{m,K}^n$, the split K-torus with character lattice M and cocharacter lattice N, where K is our fixed non-Archimedean field. For a rational cone $\sigma \subset N_{\mathbb{R}}$ we let $S_\sigma = \sigma^\vee \cap M$ and $Y_\sigma = \mathrm{Spec}(K[S_\sigma])$, an affine toric variety. Given a rational pointed fan Δ in $N_{\mathbb{R}}$ we let $Y_\Delta = \bigcup_{\sigma\in\Delta} Y_\sigma$ denote the corresponding K-toric variety with dense torus T. For $\sigma \in \Delta$ we write $M(\sigma) = \sigma^\perp \cap M$ and $N(\sigma) = N/\langle\sigma\rangle \cap N$, where $\langle\sigma\rangle \subset N_{\mathbb{R}}$ is the linear span of σ. For $G \subset \mathbb{R}$ as above we write $M_G(\sigma) = M(\sigma) \otimes_{\mathbb{Z}} G$, etc., and by abuse of notation we use $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pairing $M_G(\sigma) \times N_G(\sigma) \to \mathbb{R}$. The torus orbit [13, §3.2] corresponding to σ is $O(\sigma) \coloneqq \mathrm{Spec}(K[M(\sigma)])$, and $Y_\Delta = \bigsqcup_{\sigma\in\Delta} O(\sigma)$ as sets.

We denote by $\pi_\sigma\colon N_{\mathbb{R}} \to N_{\mathbb{R}}(\sigma) = N_{\mathbb{R}}/\langle\sigma\rangle$ the natural projection.

For $\tau \in \Delta$, the closure of $O(\tau)$ in Y_Δ is the toric variety Y_{Δ_τ} with dense torus $O(\tau)$, where Δ_τ is the fan $\{\pi_\tau(\sigma) \mid \sigma \in \Delta,\ \tau \prec \sigma\}$. We have

$$Y_{\Delta_\tau} = \bigsqcup_{\substack{\sigma\in\Delta\\ \tau\prec\sigma}} O(\sigma) = \bigsqcup_{\pi_\tau(\sigma)\in\Delta_\tau} O(\pi_\tau(\sigma)),$$

where $O(\sigma) = O(\pi_\tau(\sigma))$ for $\sigma \in \Delta$ such that $\tau \prec \sigma$.

3.2. Kajiwara–Payne compactifications

Much of this paper will be concerned with extended tropicalizations, which take place in a Kajiwara–Payne partial compactification of $N_{\mathbb{R}}$. We briefly introduce these partial compactifications here; see [31, 33] for details. Put $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and for a rational pointed cone $\sigma \subset N_{\mathbb{R}}$ we let $\overline{N}_{\mathbb{R}}^\sigma$ denote the space of monoid homomorphisms $\mathrm{Hom}(S_\sigma, \overline{\mathbb{R}})$. As usual we let $\langle\,\cdot\,,\cdot\,\rangle$ denote the evaluation pairing $S_\sigma \times \overline{N}_{\mathbb{R}}^\sigma \to \overline{\mathbb{R}}$. For a face $\tau \prec \sigma$, a point $\omega \in N_{\mathbb{R}}(\tau)$ gives rise to a point of $\overline{N}_{\mathbb{R}}^\sigma$ by the rule

$$u \mapsto \begin{cases} \langle u, \omega\rangle & \text{if } u \in \tau^\perp \\ \infty & \text{if not.} \end{cases}$$

This yields a decomposition $\overline{N}^{\sigma}_{\mathbb{R}} = \coprod_{\tau\prec\sigma} N_{\mathbb{R}}(\tau)$ as sets, but not as topological spaces. In particular, $N_{\mathbb{R}}$ is a subset of $\overline{N}^{\sigma}_{\mathbb{R}}$. For a rational pointed fan Δ in $N_{\mathbb{R}}$, the spaces $\overline{N}^{\sigma}_{\mathbb{R}}$ glue to give a partial compactification $\overline{N}^{\Delta}_{\mathbb{R}}$, which is to $N_{\mathbb{R}}$ as Y_Δ is to T. We have a decomposition $\overline{N}^{\Delta}_{\mathbb{R}} = \coprod_{\sigma\in\Delta} N_{\mathbb{R}}(\sigma)$. For an additive subgroup $G \subset \mathbb{R}$ we let $\overline{N}^{\sigma}_{G} = \coprod_{\tau\prec\sigma} N_G(\tau) \subset \overline{N}^{\sigma}_{\mathbb{R}}$ and $\overline{N}^{\Delta}_{G} = \coprod_{\sigma\in\Delta} N_G(\sigma) \subset \overline{N}^{\Delta}_{\mathbb{R}}$.

If we pass to the fan Δ_τ defined above, the associated partial compactification $\overline{N(\tau)}^{\Delta_\tau}_{\mathbb{R}}$ of $N_{\mathbb{R}}(\tau)$ is the disjoint union of all $N(\sigma)$ for $\tau \prec \sigma$.

3.3. Tropicalization

Recall that K is a non-Archimedean field whose valuation is allowed to be trivial. For a rational cone $\sigma \subset N_{\mathbb{R}}$ we define the *tropicalization map* $\operatorname{trop}\colon Y^{\mathrm{an}}_\sigma \to \overline{N}^{\sigma}_{\mathbb{R}}$ by

$$\langle u, \operatorname{trop}(\xi)\rangle = -\log|\chi^u(\xi)|,$$

where we interpret $-\log(0) = \infty$. This map is continuous, surjective, closed, and proper, in the sense that the inverse image of a compact subset is compact. Choosing a basis for M gives an isomorphism $K[M] \cong K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$; when $\sigma = \{0\}$ the tropicalization map $\operatorname{trop}\colon \mathbf{G}_m^n = T \to N_{\mathbb{R}} = \mathbb{R}^n$ is given by

$$\operatorname{trop}(\xi) = \big(-\log|x_1(\xi)|, \ldots, -\log|x_n(\xi)|\big).$$

The tropicalization map on Y^{an}_σ has a natural continuous section $s\colon \overline{N}^{\sigma}_{\mathbb{R}} \to Y^{\mathrm{an}}_\sigma$ given by $s(\omega) = |\cdot|_\omega$, where $\omega \in \operatorname{Hom}(S_\sigma, \overline{\mathbb{R}})$ and $|\cdot|_\omega\colon K[S_\sigma] \to \mathbb{R}$ is the multiplicative seminorm defined by

$$\Big|\sum_{u\in S_\sigma} a_u \chi^u\Big|_\omega = \max_u \big\{|a_u|\,\exp(-\langle u,\omega\rangle)\big\}, \tag{3.3.1}$$

where we put $\exp(-\infty) = 0$.

The image $s(\overline{N}^{\sigma}_{\mathbb{R}})$ is by definition the *skeleton* $S(Y^{\mathrm{an}}_\sigma) \subset Y^{\mathrm{an}}_\sigma$ of the affine toric variety Y^{an}_σ. The image of a continuous section of a continuous map between Hausdorff spaces is closed, so $S(Y^{\mathrm{an}}_\sigma)$ is closed in Y^{an}_σ.

If Δ is a rational pointed fan in $N_{\mathbb{R}}$, then the maps trop glue to give a tropicalization map $\operatorname{trop}\colon Y^{\mathrm{an}}_\Delta \to \overline{N}^{\Delta}_{\mathbb{R}}$. This map is again continuous and proper. The sections also glue to a continuous section $s\colon \overline{N}^{\Delta}_{\mathbb{R}} \to Y^{\mathrm{an}}_\Delta$, whose image is by definition the *skeleton* $S(Y^{\mathrm{an}}_\Delta)$ of the toric variety Y^{an}_Δ. Again the skeleton $S(Y^{\mathrm{an}}_\Delta)$ is closed in Y^{an}_Δ. The tropicalization and section are compatible with the decompositions $Y_\Delta = \coprod_{\sigma\in\Delta} O(\sigma)$ and $\overline{N}^{\Delta}_{\mathbb{R}} =$

$\coprod_{\sigma\in\Delta} N_{\mathbb{R}}(\sigma)$, in that trop (resp. s) restricts to the tropicalization map $O(\sigma)^{\mathrm{an}} \to N_{\mathbb{R}}(\sigma)$ (resp. the section $N_{\mathbb{R}}(\sigma) \to O(\sigma)^{\mathrm{an}}$) defined on the torus $O(\sigma) = \mathrm{Spec}(K[M(\sigma)])$ with cocharacter lattice $N(\sigma)$.

For a closed subscheme $X \subset Y_\Delta$, the *tropicalization* of X is

$$\mathrm{Trop}(X) := \mathrm{trop}(X^{\mathrm{an}}) \subset \overline{N}_{\mathbb{R}}^{\Delta}.$$

For any $\sigma \in \Delta$ we have $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma) = \mathrm{Trop}(X \cap O(\sigma))$, so that $\mathrm{Trop}(X)$ may be defined on each torus orbit separately. The tropicalization is insensitive to extension of scalars: if $K' \supset K$ is a valued field extension then $\mathrm{Trop}(X_{K'}) = \mathrm{Trop}(X)$ as subsets of $\overline{N}_{\mathbb{R}}^{\Delta}$.

3.4. Fibers of tropicalization

For $\omega \in N_{\mathbb{R}}$ the subset $U_\omega := \mathrm{trop}^{-1}(\omega) \subset T^{\mathrm{an}}$ is a K-affinoid domain, which is strictly K-affinoid when $\omega \in N_{\sqrt{\Gamma}}$. See [21, Proposition 4.1] and its proof. The ring of analytic functions on U_ω is

$$K\langle U_\omega\rangle = \Big\{ \sum_{u\in M} a_u \chi^u \mid a_u \in K,\, v(a_u) + \langle u,\omega\rangle \to \infty \Big\},$$

where the sums are infinite and the limit is taken on the complement of finite subsets of M. The supremum semi-norm $|\cdot|_{\sup}$ on $K\langle U_\omega\rangle$ is multiplicative, and is given by the formula

$$-\log \Big| \sum_{u\in M} a_u \chi^u \Big|_{\sup} = \min\big\{ v(a_u) + \langle u,\omega\rangle \mid a_u \neq 0 \big\}.$$

Compare (3.3.1). Suppose now that the valuation on K is non-trivial and that $\omega \in N_{\sqrt{\Gamma}}$, so that U_ω is strictly affinoid. Then the ring of power-bounded elements in $K\langle U_\omega\rangle$ is

$$K\langle U_\omega\rangle^\circ = \Big\{ \sum_{u\in M} a_u \chi^u \in K\langle U_\omega\rangle \mid v(a_u) + \langle u,\omega\rangle \geqslant 0 \text{ for all } u \in M \Big\}.$$

If the value group Γ is discrete, then $K\langle U_\omega\rangle^\circ$ is an admissible K°-algebra (i.e., it is topologically of finite presentation) for all $\omega \in N_{\sqrt{\Gamma}}$ by [22, Proposition 6.7]. If Γ is not discrete, then $K\langle U_\omega\rangle^\circ$ is an admissible K°-algebra if and only if $\omega \in N_\Gamma$. This follows from [22, Proposition 6.9] by noting that $K\langle U_\omega\rangle^\circ$ is the completion of what is denoted $K[M]^{\{\omega\}}$ in ibid.

If $K\langle U_\omega\rangle^\circ$ is an admissible K°-algebra, then we set $\mathfrak{U}_\omega = \mathrm{Spf}(K\langle U_\omega\rangle^\circ)$, which is in our terminology the canonical model of U_ω.

Let Δ be a pointed rational fan in $N_{\mathbb{R}}$ and assume that $\omega \in \overline{N}_{\mathbb{R}}^{\Delta}$ is contained in $N_{\mathbb{R}}(\sigma)$ for $\sigma \in \Delta$. Then we define U_ω, $K\langle U_\omega\rangle$, and $\mathfrak{U}_\omega$ as above, with the torus orbit $O(\sigma)$ replacing the torus T.

3.5. Initial degeneration

Let Δ be a pointed rational fan in $N_{\mathbb{R}}$ and let $X \subset Y_\Delta$ be a closed subscheme. For $\omega \in \overline{N}_{\mathbb{R}}^{\Delta}$ we set $X_\omega = U_\omega \cap X^{\mathrm{an}}$. This is an affinoid domain in $(X \cap O(\sigma))^{\mathrm{an}}$, where $\omega \in N_{\mathbb{R}}(\sigma)$; moreover, X_ω is strictly K-affinoid when $\omega \in \overline{N}_{\sqrt{\Gamma}}^{\Delta}$, and is a Zariski-closed subspace of U_ω in any case. Suppose now that the valuation on K is non-trivial and that $\omega \in \overline{N}_{\Gamma}^{\Delta}$. Let $\mathfrak{a}_\omega \subset K\langle U_\omega\rangle$ be the ideal defining X_ω. The *tropical formal model* of X_ω is the admissible formal scheme $\mathfrak{X}_\omega$ defined as the closed formal subscheme of $\mathfrak{U}_\omega$ given by the ideal $\mathfrak{a}_\omega \cap K\langle U_\omega\rangle^\circ$. The *initial degeneration* of X at ω is by definition the special fiber of $\mathfrak{X}_\omega$:

$$\mathrm{in}_\omega(X) := (\mathfrak{X}_\omega)_s\,.$$

This closed subscheme of $(\mathfrak{U}_\omega)_s$ is defined by the ω-initial forms of the elements of $\mathfrak{a}_\omega$. See [3, §4.13] and [22, §5] for details.

Now let $\omega \in \overline{N}_{\mathbb{R}}^{\Delta}$ be any point and let $K' \supset K$ be an algebraically closed valued field extension whose value group $\Gamma' = v(K'^\times)$ is non-trivial and large enough that $\omega \in \overline{N}_{\Gamma'}^{\Delta}$. Let $X' = X_{K'}$. Let $Z \subset \mathrm{in}_\omega(X')$ be an irreducible component with generic point ζ and let m_Z be the multiplicity of Z, i.e. the length of the local ring $\mathscr{O}_{\mathrm{in}_\omega(X'),\zeta}$. The *tropical multiplicity* of X at ω is

$$m_{\mathrm{Trop}}(\omega) = m_{\mathrm{Trop}}(X,\omega) := \sum_Z m_Z\,,$$

where the sum is taken over all irreducible components Z of $\mathrm{in}_\omega(X')$. This quantity is independent of the choice of K': see [22, §13].

Remark 3.6. — Assuming the valuation on K is non-trivial, for $\omega \in \overline{N}_{\Gamma}^{\Delta}$ we let

$$\mathfrak{X}_\omega^{\mathrm{can}} = \mathrm{Spf}\left((K\langle U_\omega\rangle/\mathfrak{a}_\omega)^\circ\right)$$

be the canonical model of X_ω. Then (2.11.1) gives an integral morphism $\mathfrak{X}_\omega^{\mathrm{can}} \to \mathfrak{X}_\omega$ and a finite, surjective morphism

$$\widetilde{X}_\omega \to (\mathfrak{X}_\omega)_s = \mathrm{in}_\omega(X)$$

by Proposition 2.12. Suppose now that K is algebraically closed and that X is reduced, so that $\mathfrak{X}_\omega^{\mathrm{can}}$ is an admissible affine formal scheme with special fiber $\widetilde{X}_\omega$ (see [3, Theorem 3.17] and §2.9). The *projection formula* in this case [3, (3.34.2)] says that for every irreducible component Z of $\mathrm{in}_\omega(X)$, we have

$$m_Z = \sum_{Z' \twoheadrightarrow Z} [Z' : Z], \tag{3.6.1}$$

where the sum is taken over all irreducible components Z' of $\widetilde{X}_\omega$ surjecting onto Z, and $[Z' : Z]$ is the degree of the finite dominant morphism of integral schemes $Z' \to Z$.

4. The tropical skeleton

Let X be a closed subscheme of a K-toric scheme Y_Δ. In this section we define a canonical locally closed subset $S_{\mathrm{Trop}}(X) \subset X^{\mathrm{an}}$ which we call the *tropical skeleton*. We prove that $S_{\mathrm{Trop}}(X)$ is closed in every torus orbit $O(\sigma)^{\mathrm{an}}$. For a polyhedron in $\mathrm{Trop}(X)$ and $\xi \in S_{\mathrm{Trop}}(X)$, we introduce the notions of d-maximality at ξ and relevance for ξ which will be crucial for studying limit points of $S_{\mathrm{Trop}}(X)$ in X^{an} in the next section.

4.1. Definition and basic properties

For the rest of this section, we fix a pointed rational fan Δ in $N_{\mathbb{R}}$. Let $X \subset Y_\Delta$ be a closed subscheme. Recall that trop maps Y_Δ^{an} to $\overline{N}_{\mathbb{R}}^{\Delta}$ and that $X_\omega \coloneqq \mathrm{trop}^{-1}(\omega) \cap X^{\mathrm{an}}$ for $\omega \in \overline{N}_{\mathbb{R}}^{\Delta}$.

Definition 4.2. — *The* tropical skeleton *of a closed subscheme $X \subset Y_\Delta$ is the set*

$$S_{\mathrm{Trop}}(X) \coloneqq \left\{\xi \in X^{\mathrm{an}} \mid \xi \textit{ is a Shilov boundary point of } X_{\mathrm{trop}(\xi)}\right\}.$$

In other words, the tropical skeleton is the set of all Shilov boundary points of fibers of tropicalization on X^{an}. It is clear that for $\sigma \in \Delta$ we have $S_{\mathrm{Trop}}(X \cap O(\sigma)) = S_{\mathrm{Trop}}(X) \cap O(\sigma)^{\mathrm{an}}$, where the left side of the equation is defined with respect to the closed subscheme $X \cap O(\sigma)$ of the torus $O(\sigma)$. In other words, the tropical skeleton is defined independently on each torus orbit. It is also clear that trop maps $S_{\mathrm{Trop}}(X)$ surjectively onto $\mathrm{Trop}(X) = \mathrm{trop}(X^{\mathrm{an}})$.

Lemma 4.3. — *Let $X \subset Y_\Delta$ be a closed subscheme and let X_{red} be the underlying reduced subscheme. Then $S_{\mathrm{Trop}}(X) = S_{\mathrm{Trop}}(X_{\mathrm{red}})$.*

Proof. — The Shilov boundary of an affinoid space $\mathscr{M}(\mathcal{A})$ is obviously equal to the Shilov boundary of $\mathscr{M}(\mathcal{A}_{\mathrm{red}})$. □

Hence when discussing the tropical skeleton, we may always assume X is reduced. The remainder of this subsection is devoted to showing that $S_{\mathrm{Trop}}(X) \cap O(\sigma)^{\mathrm{an}}$ is closed in $X^{\mathrm{an}} \cap O(\sigma)^{\mathrm{an}}$ for each $\sigma \in \Delta$. This is clear when $X = Y_\Delta^{\mathrm{an}}$, as in this case $S_{\mathrm{Trop}}(X)$ coincides with the usual skeleton $S(Y_\Delta^{\mathrm{an}})$, which is closed in Y_Δ^{an}, as we saw in §3.3.

LEMMA 4.4. — *Let $X \subset Y_\Delta$ be a closed subscheme, let $K' \supset K$ be a valued extension field, and let $\pi\colon X^{\mathrm{an}}_{K'} \to X^{\mathrm{an}}$ be the structural map. Then $\pi(S_{\mathrm{Trop}}(X_{K'})) = S_{\mathrm{Trop}}(X)$.*

Proof. — This is an immediate consequence of Proposition 2.21. □

Remark 4.5. — In the situation of Lemma 4.4, it is certainly not true in general that $\pi^{-1}(S_{\mathrm{Trop}}(X)) = S_{\mathrm{Trop}}(X_{K'})$. Indeed, take $K = \mathbb{C}_p$ and $X = T = \mathbf{G}_m$. Then for $\omega \in \mathbb{R} \setminus \mathbb{Q}$ the non-strict affinoid domain $X_\omega = \mathscr{M}(K\langle \omega^{-1}T, \omega T^{-1}\rangle)$ is a single point (see [4, §2.1]), and if $K' \supset \mathbb{C}_p$ is an extension containing ω in its value group, then X'_ω is a "modulus-zero" closed annulus, where $X' = X_{K'}$. We conclude that $\pi^{-1}(S_{\mathrm{Trop}}(X)) \cap X'_\omega = X'_\omega$ while $S_{\mathrm{Trop}}(X') \cap X'_\omega$ contains only one point.

PROPOSITION 4.6. — *Let $X \subset T$ be an equidimensional closed subscheme. Then the tropical skeleton $S_{\mathrm{Trop}}(X)$ is closed in X^{an}.*

Proof. — This is a minor variation of the argument used in [23, Theorem 10.6], so we only provide a sketch. By Lemmas 4.4 and 2.19, we may assume that K is algebraically closed and that $\Gamma = \mathbb{R}$, i.e. that $v\colon K^\times \to \mathbb{R}$ is surjective. By Lemma 4.3 we may also assume X is reduced. Let $d = \dim(X)$. A generic homomorphism $\psi\colon T \to \mathbf{G}_m^d$ has the property that the induced homomorphism $f\colon N_{\mathbb{R}} \to \mathbb{R}^d$ is finite-to-one on $\mathrm{Trop}(X)$. Using this and properness of the tropicalization maps, one deduces easily that the analytification of the composition $\phi\colon X \hookrightarrow T \to \mathbf{G}_m^d$ is proper as a map of topological spaces. Since it also boundaryless [4, Theorem 3.4.1], it is a proper morphism of analytic spaces. We conclude that φ is proper and hence finite as it is an affine morphism. Let $\omega \in N_{\mathbb{R}}$, let $\omega' = f(\omega) \in \mathbb{R}^d$, and let $U'_{\omega'} = \mathrm{trop}^{-1}(\omega') \subset \mathbf{G}_m^{d,\mathrm{an}}$. Then $\psi^{\mathrm{an}}\colon X_\omega \to U'_{\omega'}$ is finite, so the map on canonical reductions $\widetilde{X}_\omega \to \widetilde{U}'_{\omega'}$ is finite as well by [9, Theorem 6.3.4/2]. Since X is equidimensional of dimension d, the same is true of X_ω, and hence of $\widetilde{X}_\omega$ by Proposition 2.10, so generic points of $\widetilde{X}_\omega$ map to the generic point of $\widetilde{U}'_{\omega'}$. By functoriality of the reduction map, this implies that the ϕ-inverse image of the Shilov boundary point of $U'_{\omega'}$ is the Shilov boundary of X_ω. Therefore $\phi^{-1}(S_{\mathrm{Trop}}(\mathbf{G}_m^d)) = S_{\mathrm{Trop}}(X)$. Since the skeleton $S(\mathbf{G}_m^{d,\mathrm{an}}) = S_{\mathrm{Trop}}(\mathbf{G}_m^d)$ is closed in $\mathbf{G}_m^{d,\mathrm{an}}$ as we have seen in §3.3, this proves that $S_{\mathrm{Trop}}(X)$ is closed. □

In Corollary 7.7 we will prove a more general statement about closed subschemes in toric varietes.

Proposition 4.7. — *Let $X \subset T$ be a closed subscheme and let $X = X_1 \cup \cdots \cup X_n$ be its decomposition into irreducible components. Then*

$$S_{\mathrm{Trop}}(X) = S_{\mathrm{Trop}}(X_1) \sqcup \cdots \sqcup S_{\mathrm{Trop}}(X_n)$$

as topological spaces, i.e., the $S_{\mathrm{Trop}}(X_i)$ are disjoint open and closed subsets of $S_{\mathrm{Trop}}(X)$.

Proof. — For each i, $(X_i)_\omega$ is a union of irreducible components of X_ω by Proposition 2.5. We claim that for $i \neq j$, $(X_i)_\omega$ and $(X_j)_\omega$ do not share any irreducible components. Let $X = \mathrm{Spec}(A)$ and $X_\omega = \mathscr{M}(\mathcal{A}_\omega)$. Then $A \to \mathcal{A}_\omega$ is flat, so $\mathrm{Spec}(\mathcal{A}_\omega) \to X$ is flat. Thus the image of a generic point of $\mathrm{Spec}(\mathcal{A}_\omega)$ is a generic point of X, so we have shown that any irreducible component of $(X_i)_\omega$ is dense in X_i. This proves our claim.

By Proposition 2.15, the Shilov boundary of X_ω is the disjoint union of the Shilov boundaries of $(X_1)_\omega, \ldots, (X_n)_\omega$. Hence $S_{\mathrm{Trop}}(X)$ is the disjoint union of the $S_{\mathrm{Trop}}(X_i)$. Each $S_{\mathrm{Trop}}(X_i)$ is closed in X_i^{an} (and hence in X^{an}) by Proposition 4.6. Hence for all i the finite union $\bigsqcup_{j \neq i} S_{\mathrm{Trop}}(X_j)$ is closed, so the complement of $S_{\mathrm{Trop}}(X_i)$ in $S_{\mathrm{Trop}}(X)$ is also closed. This implies our claim. $\square$

Corollary 4.8. — *Let $X \subset Y_\Delta$ be a closed subscheme. Then $S_{\mathrm{Trop}}(X) \cap O(\sigma)^{\mathrm{an}}$ is closed in $X^{\mathrm{an}} \cap O(\sigma)^{\mathrm{an}}$ for every $\sigma \in \Delta$. Therefore, $S_{\mathrm{Trop}}(X)$ is locally closed in X^{an}.*

Proof. — Since $S_{\mathrm{Trop}}(X)$ is defined independently on each torus orbit, we may assume $O(\sigma) = T$. Now apply Propositions 4.6 and 4.7. $\square$

In general, $S_{\mathrm{Trop}}(X)$ is not closed in X^{an} when the ambient toric variety Y_Δ is not a torus, even when X itself is irreducible, as the following example shows.

Example 4.9. — For simplicity, we let K be an algebraically closed non-Archimedean field with value group $\mathbb{R}$. Let $Y_\Delta = Y_\sigma$ be the affine toric variety $\mathbf{A}^3$, so $\sigma = \mathbb{R}^3_+$. Let x_1, x_2, x_3 be coordinates on $\mathbf{A}^3$ and let $X \subset \mathbf{A}^3$ be the closed subscheme defined by the equation $(x_1 - 1)x_2 + x_3 = 0$. This is an irreducible hypersurface.

The partial compactification $\overline{N}^\Delta_{\mathbb{R}}$ in this case is $\overline{\mathbb{R}}^3 = (\mathbb{R} \cup \{\infty\})^3$, and the tropicalization map $\mathrm{trop}\colon \mathbf{A}^{3,\mathrm{an}} \to \overline{\mathbb{R}}^3$ takes ξ to $-(\log|\xi(x_1)|, \log|\xi(x_2)|, \log|\xi(x_3)|)$. Let $\omega_1, \omega_2, \omega_3$ be coordinates on $\mathbb{R}^3$. Then $\mathrm{Trop}(X) \cap \mathbb{R}^3$ is the union of the three cones

$$P_1 = \{\omega \in \mathbb{R}^3 \mid \omega_1 \geqslant 0,\ \omega_3 = \omega_2\}$$
$$P_2 = \{\omega \in \mathbb{R}^3 \mid \omega_1 = 0,\ \omega_3 \geqslant \omega_2\}$$
$$P_3 = \{\omega \in \mathbb{R}^3 \mid \omega_1 \leqslant 0,\ \omega_3 = \omega_1 + \omega_2\}.$$

Let $\sigma_1 = \{\omega \in \mathbb{R}_+^3 \mid \omega_2 = \omega_3 = 0\}$, so $O(\sigma_1) = \{x_1 = 0, x_2x_3 \neq 0\}$ and we identify $N_\mathbb{R}(\sigma_1)$ with $\{\infty\} \times \mathbb{R}^2$. Then $X \cap O(\sigma_1)$ is defined by the equation $x_3 = x_2$ in $\{0\} \times \mathbf{G}_m^2$, so

$$\mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma_1) = \{\infty\} \times \{\omega_3 = \omega_2\}.$$

Let $\sigma_2 = \{\omega \in \mathbb{R}_+^3 \mid \omega_1 = \omega_3 = 0\}$, so $O(\sigma_2) = \{x_2 = 0, x_1x_3 \neq 0\}$ and we identify $N_\mathbb{R}(\sigma_2)$ with $\mathbb{R} \times \{\infty\} \times \mathbb{R}$. Then $X \cap O(\sigma_2) = \emptyset$ and therefore

$$\mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma_2) = \emptyset.$$

Let $\sigma_3 = \{\omega \in \mathbb{R}_+^3 \mid \omega_1 = \omega_2 = 0\}$, so $O(\sigma_3) = \{x_3 = 0, x_1x_2 \neq 0\}$ and we identify $N_\mathbb{R}(\sigma_3)$ with $\mathbb{R}^2 \times \{\infty\}$. Then $X \cap O(\sigma_3)$ is defined by the equation $x_1 = 1$ in $\mathbf{G}_m^2 \times \{0\}$, so

$$\mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma_3) = \{0\} \times \mathbb{R} \times \{\infty\}.$$

Let $\sigma_{12} = \{\omega \in \mathbb{R}_+^3 \mid \omega_3 = 0\}$, let $\sigma_{13} = \{\omega \in \mathbb{R}_+^3 \mid \omega_2 = 0\}$, and let $\sigma_{23} = \{\omega \in \mathbb{R}_+^3 \mid \omega_1 = 0\}$. Then $X \cap O(\sigma_{12}) = \emptyset$ and $X \cap O(\sigma_{13}) = \emptyset$, so

$$\mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma_{12}) = \emptyset \quad \text{and} \quad \mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma_{13}) = \emptyset.$$

However, X contains $O(\sigma_{23}) = \{x_1 \neq 0, x_2 = x_3 = 0\}$, so

$$\mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma_{23}) = \mathbb{R} \times \{\infty\} \times \{\infty\}.$$

Finally, $O(\sigma) = \{(0,0,0)\}$ is contained in X, so

$$\mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma) = \{(\infty,\infty,\infty)\}.$$

One checks that the initial form of the defining equation $(x_1 - 1)x_2 + x_3$ at every $\omega \in \overline{\mathbb{R}}^3$ is irreducible, and hence that each X_ω contains a unique Shilov boundary point $s_X(\omega)$ by [23, Lemma 10.3] or Proposition 8.3 below. In the following, $\mathbf{B}(r)$ denotes the closed disk with center 0 and radius r in $\mathbf{A}^{3,\mathrm{an}}$. For $\omega = (0, \omega_2, \infty) \in \mathrm{Trop}(X) \cap N_\mathbb{R}(\sigma_3)$, the tropical fiber X_ω is the modulus-zero annulus $\{(1, \xi, 0) \mid v(\xi) = \omega_2\}$, so the Shilov point $s_X(\omega)$ is the Gauss point of the disk $\{1\} \times \mathbf{B}(\exp(-\omega_2)) \times \{0\}$. Taking $\omega_2 \to \infty$, the Shilov points $s_X(\omega)$ converge to $(1,0,0)$. However, for $\omega = (0,\infty,\infty) \in N_\mathbb{R}(\sigma_{23})$ the tropical fiber X_ω is all of $\{(\xi,0,0) \mid v(\xi) = 0\}$, so the Shilov point $s_X(\omega)$ is the Gauss point of the disk $\mathbf{B}(1) \times \{0\} \times \{0\}$. Hence $s_X(\omega) \neq (1,0,0)$, so $S_{\mathrm{Trop}}(X)$ does not contain the limit point $(1,0,0)$.

With a bit more work, one can show that if $\omega(r) = (0, r, r) \in P_2$ for $r \geqslant 0$ then $s_X(\omega(r)) \to s_X(0,\infty,\infty)$ as $r \to \infty$, but that if $\omega'(r) = (0, r, 2r)$ then $s_X(\omega'(r)) \to (1,0,0)$.

5. Relevant polyhedra and d-maximality

In this section we introduce two technical notions, those of relevant and d-maximal polyhedra, which will be used to prove our main results in §6.

5.1. The local dimension

In Example 4.9, the "problem" with the "incorrect" limit point $(1,0,0) \in O(\sigma_{23})$ of $S_{\mathrm{Trop}}(X)$ is that there exists a maximal polyhedron P_2 in the tropicalization of X intersected with the dense torus orbit, such that the closure of P_2 intersects the tropicalization of $X \cap O(\sigma_{23})$ in a non-maximal polyhedron. Note that in our example, X has codimension one in $\mathbf{A}^3$, but $X \cap O(\sigma_{23})$ has codimension zero in $O(\sigma_{23})$ The main theorem of the next section, Theorem 6.1, says that, in a precise sense, this dimensional incompatibility of polyhedra across torus orbits is the only possible reason for a limit point of $S_{\mathrm{Trop}}(X)$ not to be contained in $S_{\mathrm{Trop}}(X)$.

In what follows, we fix a rational pointed fan Δ in $N_{\mathbb{R}}$ and a closed subscheme $X \subset Y_\Delta$. For $\omega \in \mathrm{Trop}(X)$ we let $\mathrm{LC}_\omega(\mathrm{Trop}(X))$ denote the local cone of ω in $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$, where $N_{\mathbb{R}}(\sigma)$ is the orbit containing ω. See [22, §A.6].

Definition 5.2. — *The* local dimension of $\mathrm{Trop}(X)$ at $\omega \in \mathrm{Trop}(X)$ *is defined as*

$$d(\omega) := \dim(\mathrm{LC}_\omega(\mathrm{Trop}(X)))\,.$$

Note that $d(\omega)$ only depends on $X \cap O(\sigma)$ and $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$. If $X \cap O(\sigma)$ is equidimensional of dimension d then $d(\omega) = d$ for all $\omega \in \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ by the Bieri–Groves theorem.

Lemma 5.3. — *Let $\sigma \in \Delta$ and let $\omega \in \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$. Let $K' \supset K$ be a valued field extension whose value group $\Gamma' = v(K'^\times)$ is non-trivial and large enough that $\omega \in N_{\Gamma'}(\sigma)$, and let $X' = X_{K'}$. The following numbers coincide:*

$$d(\omega) = \dim(\mathrm{in}_\omega(X')) = \dim(\widetilde{X}'_\omega) = \dim(X'_\omega) = \dim(X_\omega)\,.$$

Proof. — The final equality holds by definition of $\dim(X_\omega)$. As all numbers in question depend only on the torus orbit whose tropicalization is $N_{\mathbb{R}}(\sigma)$, we may assume $Y_\Delta = T$, and as $\mathrm{Trop}(X) = \mathrm{Trop}(X')$, we may assume $K = K'$ and $X = X'$. By [22, Proposition 10.15], the local cone of ω in $\mathrm{Trop}(X)$ is equal to the tropicalization of $\mathrm{in}_\omega(X)$, considered as a scheme over the trivially valued field $\widetilde{K}'$, so the first equality follows from the Bieri–Groves theorem. The other two equalities are a result of Proposition 2.10 (3) and Remark 3.6. □

Definition 5.4. — *For $\sigma \in \Delta$, we call a polyhedron $P \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$* d-maximal at $\omega \in N_{\mathbb{R}}(\sigma)$ *provided that $\omega \in P$ and $d(\omega) = \dim(P)$.*

Let $\omega \in \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ for $\sigma \in \Delta$. Choose a polyhedral complex structure Σ on $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$. Then there is a $P \in \Sigma$ which is d-maximal at ω, and any such P is a maximal element of Σ with respect to inclusion.

We will be concerned with limit points of $S_{\mathrm{Trop}}(X)$ contained in the boundary $Y_{\Delta}^{\mathrm{an}} \setminus T^{\mathrm{an}}$. First we recall what the limit points are in $\overline{N}_{\mathbb{R}}^{\Delta}$ of a polyhedron in $N_{\mathbb{R}}$. The following lemma is [29, Lemma 3.9].

Lemma 5.5. — *Let $P \subset N_{\mathbb{R}}$ be a polyhedron and let $\overline{P}$ be its closure in $\overline{N}_{\mathbb{R}}^{\Delta}$. Let $\sigma \in \Delta$, and recall that $\pi_\sigma \colon N_{\mathbb{R}} \to N_{\mathbb{R}}(\sigma)$ denotes the projection map.*

(1) *We have $\overline{P} \cap N_{\mathbb{R}}(\sigma) \neq \emptyset$ if and only if the recession cone $\rho(P)$ of P intersects the relative interior of σ.*
(2) *If $\overline{P} \cap N_{\mathbb{R}}(\sigma) \neq \emptyset$, then $\overline{P} \cap N_{\mathbb{R}}(\sigma) = \pi_\sigma(P)$.*

Proposition 5.6. — *Let X be a closed subscheme of the toric variety Y_Δ. Suppose that $X \cap T$ is dense in X and write $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$ as union of a finite set Σ of polyhedra in $N_{\mathbb{R}}$. Then $\mathrm{Trop}(X)$ is the closure of $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$ in $\overline{N}_{\mathbb{R}}^{\Delta}$ and for any $\sigma \in \Delta$ we have*

$$\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma) = \bigcup_{\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset} \pi_\sigma(P),$$

where P ranges over all polyhedra in Σ with $\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset$.

Proof. — By [28, Lemma 3.1.1], we have that $\mathrm{Trop}(X)$ is the closure of $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$ in $\overline{N}_{\mathbb{R}}^{\Delta}$, and by Lemma 5.5, for $P \in \Sigma$ such that $\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset$, the closure $\overline{P}$ of P in $\overline{N}_{\mathbb{R}}^{\Delta}$ satisfies $\overline{P} \cap N_{\mathbb{R}}(\sigma) = \pi_\sigma(P)$. Hence we get the last claim. □

Example 5.7. — We continue with Example 4.9. It is clear that P_i is d-maximal at all $\omega \in P_i$ for $i = 1, 2, 3$. Consider now the orbit $O(\sigma_1) = \{x_1 = 0, x_2 x_3 \neq 0\}$. By Lemma 5.5 we have $\overline{P}_2 \cap N_{\mathbb{R}}(\sigma_1) = \overline{P}_3 \cap N_{\mathbb{R}}(\sigma_1) = \emptyset$, and

$$\overline{P}_1 \cap N_{\mathbb{R}}(\sigma_1) = \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma_1) = \{\infty\} \times \{\omega_3 = \omega_2\}.$$

Therefore $\overline{P}_1 \cap N_{\mathbb{R}}(\sigma_1)$ is d-maximal at all of its points. Similarly, $\overline{P}_1 \cap N_{\mathbb{R}}(\sigma_3) = \overline{P}_3 \cap N_{\mathbb{R}}(\sigma_3) = \emptyset$ and

$$\overline{P}_2 \cap N_{\mathbb{R}}(\sigma_3) = \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma_3) = \{0\} \times \mathbb{R} \times \{\infty\},$$

which is d-maximal at all of its points. Clearly

$$\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma) = N_{\mathbb{R}}(\sigma) = \{(\infty, \infty, \infty)\}$$

is d-maximal at its only point. We have $X \cap O(\sigma_2) = X \cap O(\sigma_{12}) = X \cap O(\sigma_{13}) = \emptyset$.

Now consider $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma_{23}) = \mathbb{R} \times \{\infty\} \times \{\infty\}$, which we identify with $\mathbb{R}$. We have

$$\begin{aligned}\pi_{\sigma_{23}}(P_1) &= \overline{P}_1 \cap N_{\mathbb{R}}(\sigma_{23}) = [0, \infty)\\ \pi_{\sigma_{23}}(P_2) &= \overline{P}_2 \cap N_{\mathbb{R}}(\sigma_{23}) = \{0\}\\ \pi_{\sigma_{23}}(P_3) &= \overline{P}_3 \cap N_{\mathbb{R}}(\sigma_{23}) = (-\infty, 0].\end{aligned}$$

Let $Q_i = \pi_{\sigma_{23}}(P_i) = \overline{P}_i \cap N_{\mathbb{R}}(\sigma_{23})$ for $i = 1, 2, 3$. Then Q_1 and Q_3 are d-maximal at all of their points, but Q_2 is not d-maximal at its point. We will see in Theorem 6.1 that this is related to the fact in Example 4.9 than $s_X(\omega'(r))$ does not approach a point of $S_{\mathrm{Trop}}(X)$ as $r \to \infty$ (note that $\omega'(r) \in \mathrm{relint}(P_2)$ for $r > 0$).

5.8. Relevant polyhedra for Shilov boundary points

Let us consider a point $\xi \in S_{\mathrm{Trop}}(X)$ with tropicalization $\omega \in \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$. We will see that not every polyhedron $P \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ is "relevant" for ξ for the purposes of checking limit points.

First, we assume that $\omega \in N_{\Gamma}(\sigma)$ and that the valuation is non-trivial. Then X_ω is a strictly affinoid domain of dimension $d(\omega)$ (see Lemma 5.3) and we have a finite surjective morphism $\iota : \widetilde{X}_\omega \to \mathrm{in}_\omega(X)$ of $d(\omega)$-dimensional affine schemes of finite type over $\widetilde{K}$ (see 3.6). It is a fact of tropical geometry [22, Proposition 10.15] that the local cone of the tropical variety at ω decomposes as

$$\mathrm{LC}_\omega(\mathrm{Trop}(X)) = \mathrm{Trop}(\mathrm{in}_\omega(X)) = \bigcup_Z \mathrm{Trop}(Z), \tag{5.8.1}$$

where Z ranges over the irreducible components of $\mathrm{in}_\omega(X)$ and where the tropical varieties $\mathrm{Trop}(\mathrm{in}_\omega(X))$ and $\mathrm{Trop}(Z)$ are taken with respect to the trivial valuation on the residue field.

Definition 5.9.

(1) *Assume that the valuation on K is non-trivial and that $\omega \in N_\Gamma(\sigma)$. Let ξ be a Shilov boundary point of X_ω. Then the reduction* $\mathrm{red}(\xi)$ *is a generic point of the canonical reduction $\widetilde{X}_\omega$ and hence* $\iota(\mathrm{red}(\xi))$ *is the generic point of an irreducible component Z of* $\mathrm{in}_\omega(X)$. *Put*

$\operatorname{trop}(\xi) = \omega \in N_\Gamma$. We say that a polyhedron $P \subset \operatorname{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ is relevant for ξ *if $\omega \in P$ and $P \subset \operatorname{Trop}(Z) + \omega$. Equivalently, P has a non-empty local cone at ω contained in* $\operatorname{Trop}(Z)$ *(see Lemma 5.11).*

(2) *For general $\omega \in N_{\mathbb{R}}(\sigma)$, we choose a valued extension field $K' \supset K$ whose value group Γ' is non-trivial and large enough that $\omega \in N_{\Gamma'}(\sigma)$. Let $X' = X_{K'}$ and let $\pi : X' \to X$ be the canonical morphism. Then it follows from Proposition 2.21 that $\pi(B(X'_\omega)) = B(X_\omega)$. We conclude that there is $\xi' \in B(X'_\omega)$ with $\pi(\xi') = \xi$. We say that the polyhedron $P \subset \operatorname{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ is* relevant for ξ *if there is such a ξ' with P relevant for ξ' in the sense of* (1).

Lemma 5.10. — *The above definition of ξ-relevance for a polyhedron $P \subset \operatorname{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ does not depend on the choice of K'.*

Proof. — Suppose that P is relevant for ξ with respect to $\xi' \in B(X'_\omega)$ as in Definition 5.9 (2). We choose another valued extension field $K'' \supset K$ with value group Γ'' non-trivial and with $\omega \in N_{\Gamma''}(\sigma)$. Setting $X'' := X_{K''}$, we have to show that there is a $\xi'' \in B(X''_\omega)$ over ξ such that P is relevant for ξ'' in the sense of Definition 5.9 (1). By [17, (0.3.2)], there is a complete valued extension field K''' of K' and K'' simultaneously; we let $X''' = X_{K'''}$. It follows from Proposition 2.21 that the Shilov boundary of X'''_ω maps onto $B(X'_\omega)$ and also onto $B(X''_\omega)$. We choose a preimage $\xi''' \in B(X'''_\omega)$ of ξ' and let Z''' be the irreducible component of $\operatorname{in}_\omega(X''')$ with generic point $\iota(\operatorname{red}(\xi'''))$. It follows from functoriality of the reduction and the initial degenerations that the canonical map

$$\operatorname{in}_\omega(X''') = \operatorname{in}_\omega(X') \otimes_{\widetilde{K}'} \widetilde{K}''' \to \operatorname{in}_\omega(X')$$

maps Z''' onto Z', the closure of $\iota(\xi')$. As $Z''' \to Z'$ is surjective, we have $\operatorname{Trop}(Z''') = \operatorname{Trop}(Z')$. By assumption, P is contained in $\operatorname{Trop}(Z')$. We conclude that P is relevant for ξ'''. Let $\xi'' \in B(X''_\omega)$ be the image of ξ''' under the analytification of the canonical map $X''' \to X''$. A similar argument as above shows that Z''' surjects onto an irreducible component Z'' of $\operatorname{in}_\omega(X'')$ and that $\operatorname{red}(\xi'')$ maps to the generic point of Z''. Since we have $\operatorname{Trop}(Z'') = \operatorname{Trop}(Z''')$, we conclude that P is relevant for ξ''. Since ξ'' is lying over ξ, this proves the claim. $\square$

Lemma 5.11. — *Let U be a closed d-dimensional subscheme of the torus T and let Σ be a polyhedral complex with support equal to* $\operatorname{Trop}(U)$. *If Y is a d-dimensional irreducible component of U, then there is a subcomplex of Σ with support equal to* $\operatorname{Trop}(Y)$.

Proof. — By the Bieri–Groves theorem, $\operatorname{Trop}(Y)$ is a finite union of d-dimensional polyhedra and hence $\operatorname{Trop}(Y)$ is covered by the d-dimensional polyhedra in Σ. It is enough to show that any d-dimensional $P \in \Sigma$ with $\mathrm{relint}(P) \cap \mathrm{Trop}(Y) \neq \emptyset$ is necessarily contained in $\mathrm{Trop}(Y)$. If not, then $P \cap \mathrm{Trop}(Y)$ has a boundary point ω with respect to P. We may assume that ω is in $\mathrm{relint}(P)$ and also in the relative interior of a $(d-1)$-dimensional face τ of a d-dimensional polyhedron $Q \subset P \cap \mathrm{Trop}(Y)$. Let H be a supporting hyperplane of the face τ of Q. By construction, there is a neighbourhood Ω of ω in $N_{\mathbb{R}}$ such that $\Omega \cap P \cap \mathrm{Trop}(Y) = \Omega \cap Q$ is contained in a half space H^+ bounded by H.

Bieri–Groves [8, Theorem D] proved that $\mathrm{Trop}(Y)$ is totally concave in ω which means that there is a polyhedron $R \subset \mathrm{Trop}(Y)$ through ω and intersecting the complement of H^+. This can also be deduced from the balancing condition. Since $\Omega \cap P \cap \mathrm{Trop}(Y) \subset H^+$, we conclude that $R \cap P \subset H^+$. Using that R expands from ω into the complement of H^+ and that $\mathrm{Trop}(Y) \subset \mathrm{Trop}(U)$, we deduce that there is a face $S \in \Sigma$ through ω with $S \neq P$. Since Σ is a polyhedral complex, this contradicts $\omega \in \mathrm{relint}(P)$. This proves $P \subset \mathrm{Trop}(Y)$. $\square$

Proposition 5.12. — *Let $\xi \in S_{\mathrm{Trop}}(X)$ with tropicalization $\omega \in N_{\mathbb{R}}(\sigma)$. Let $K' \supset K$ be a valued extension field with non-trivial value group Γ' and with $\omega \in N_{\Gamma'}$.*

1. *A polyhedron $P \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ is relevant for ξ with respect to the closed subscheme X of the toric variety Y_Δ if and only if P is relevant for ξ with respect to the affine closed subscheme $X \cap O(\sigma)$ of the orbit $O(\sigma)$.*
2. *We set $X' := X_{K'}$. The local cone at ω of the union of all ξ-relevant polyhedra is the tropical variety of an irreducible component Z of $\mathrm{in}_\omega(X')$ with respect to the trivial valuation and with $\dim(Z) = \dim_\xi(X)$.*
3. *Let Σ be a polyhedral complex in $N_{\mathbb{R}}(\sigma)$ with support equal to $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$. If $\dim_\xi(X^{\mathrm{an}}) = d(\omega)$, then in (2) it is enough to consider ξ-relevant $d(\omega)$-dimensional polyhedra in Σ. Moreover, the above local cone and Z are both of dimension $d(\omega)$. In particular, there is a polyhedron in Σ which is d-maximal at ω and which is relevant for ξ.*
4. *If $m_{\mathrm{Trop}}(\omega) = 1$, then every polyhedron in $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ containing ω is relevant for ξ.*

Proof. — Property (1) is obvious from the definition. By Lemma 4.4, there is $\xi' \in S_{\mathrm{Trop}}(X')$ over ξ for the given valued extension field $K' \supset K$. Then the canonical reduction $\mathrm{red}(\xi') \in \widetilde{X}'_\omega$ maps to the generic point of an irreducible component Z of $\mathrm{in}_\omega(X')$. By Proposition 4.7, ξ is contained in a unique irreducible component and hence X' is equidimensional in a neighbourhood of ξ' of dimension $\dim_\xi(X)$. By Proposition 2.10 and Proposition 2.12, we deduce that $\dim(Z) = \dim_\xi(X)$. It follows from (5.8.1) that

$$\mathrm{Trop}(Z) = \bigcup_{P \subset \mathrm{Trop}(Z)+\omega} \mathrm{LC}_\omega(P), \tag{5.12.1}$$

where P ranges over all polyhedra contained in $\mathrm{Trop}(X)$ with $\omega \in P \subset \mathrm{Trop}(Z) + \omega$. This proves (2).

If $\dim_\xi(X^{\mathrm{an}}) = d(\omega)$, then (2) shows that Z is $d(\omega)$-dimensional and the same is true for $\mathrm{in}_\omega(X')$ by Lemma 5.3. We apply Lemma 5.11 to the fan $\{\mathrm{LC}_\omega(P) \mid P \in \Sigma\}$ which has support $\mathrm{Trop}(\mathrm{in}_\omega(X'))$ by (5.8.1). This shows that $\mathrm{Trop}(Z)$ is the union of the $d(\omega)$-dimensional $(\mathrm{LC}_\omega(P))_{P\in\Sigma}$ contained in $\mathrm{Trop}(Z)$, which proves (3).

Let P be a polyhedron in $\mathrm{Trop}(X)$ containing ω. Then $\mathrm{LC}_\omega(P)$ is non-empty and contained in $\mathrm{Trop}(\mathrm{in}_\omega(X)) = \mathrm{LC}_\omega(\mathrm{Trop}(X))$ by (5.8.1). If $m_{\mathrm{Trop}}(\omega) = 1$, then $Z \coloneqq \mathrm{in}_\omega(X)$ is irreducible, and (4) follows. □

In the following important lemma, X is any closed subscheme of the multiplicative torus T over K with character lattice N. Recall that $s(\omega)$ denotes the point of $S(T^{\mathrm{an}}) = S_{\mathrm{Trop}}(T)$ lying above $\omega \in N_\mathbb{R}$, that $X_\omega = \mathrm{trop}^{-1}(\omega) \cap X^{\mathrm{an}}$, and that $B(X_\omega)$ denotes the Shilov boundary of the affinoid space X_ω.

Lemma 5.13. — *Choose a polyhedral complex structure Σ on $\mathrm{Trop}(X)$. Let $\omega \in \mathrm{Trop}(X)$ and let $d \coloneqq d(\omega)$. We consider a homomorphism $\psi\colon T \to \mathbf{G}_m^d$, let $f\colon N_\mathbb{R} \to \mathbb{R}^d$ be the induced linear map, let $\phi = \psi|_X$, and let $\omega' = f(\omega)$. Then $\phi^{-1}(s(\omega')) \cap X_\omega$ is equal to the set of Shilov boundary points $\xi \in B(X_\omega)$ for which there is a $P \in \Sigma$ satisfying the following three conditions:*

(1) *P is d-maximal at ω;*
(2) *P is relevant for ξ;*
(3) *f is injective on P.*

For such a ξ, we have always $\dim_\xi(X_\omega) = d$.

Proof. — Assume to begin that the valuation on K is non-trivial and that $\omega \in N_\Gamma$. Let $U'_{\omega'} = \mathrm{trop}^{-1}(\omega') \subset \mathbf{G}_m^{d,\mathrm{an}}$. We have $\phi(X_\omega) \subset U'_{\omega'}$ due to the commutativity of the square

$$\begin{array}{ccc} X^{\mathrm{an}} & \xrightarrow{\phi} & \mathbf{G}_m^{d,\mathrm{an}} \\ {\scriptstyle \mathrm{trop}}\downarrow & & \downarrow{\scriptstyle \mathrm{trop}} \\ \mathrm{Trop}(X) & \xrightarrow{f} & \mathbb{R}^d \end{array}$$

Consider the morphism $\phi\colon X_\omega \to U'_{\omega'}$ and the induced morphism on canonical reductions $\widetilde{\phi}\colon \widetilde{X}_\omega \to \widetilde{U}'_{\omega'} \cong \mathbf{G}^d_{m,\widetilde{K}}$. If $\xi \in X_\omega$ maps to $s(\omega')$ then by functoriality of the reduction map, we have that $\mathrm{red}(\xi) \in \widetilde{X}_\omega$ maps to the generic point of $\mathbf{G}^d_{m,\widetilde{K}}$. Since $d(\omega) = \dim(\widetilde{X}_\omega) = d$ by Lemma 5.3, it follows that $\mathrm{red}(\xi)$ is a generic point of $\widetilde{X}_\omega$, so ξ is a Shilov boundary point of X_ω by Proposition 2.16. This proves that $\phi^{-1}(s(\omega')) \cap X_\omega$ is contained in the Shilov boundary $B(X_\omega)$. Since $\mathrm{red}(\xi)$ is the generic point of an irreducible component Z' of $\widetilde{X}_\omega$ of dimension d, the irreducible component of X_ω containing ξ must also have dimension d. Hence $\dim_\xi(X_\omega) = d$.

Next, we show that ξ satisfies properties (1)–(3). Note that $\widetilde{\phi}\colon \widetilde{X}_\omega \to \widetilde{U}'_{\omega'} \cong \mathbf{G}^d_{m,\widetilde{K}}$ has a canonical factorization

$$\widetilde{X}_\omega \longrightarrow \mathrm{in}_\omega(X) \xrightarrow{\mathrm{in}_\omega(\phi)} \mathrm{in}_{\omega'}(\mathbf{G}^d_{m,K}) \cong \mathbf{G}^d_{m,\widetilde{K}}. \tag{5.13.1}$$

We have seen in Remark 3.6 that the first map is surjective and finite. We conclude that Z' maps onto a d-dimensional irreducible component Z of $\mathrm{in}_\omega(X)$. Since $\mathrm{red}(\xi)$ maps to the generic point of $\mathbf{G}^d_{m,\widetilde{K}}$, the restriction of $\mathrm{in}_\omega(\phi)$ to Z is dominant. By functoriality of the tropicalization, we get a commutative square

$$\begin{tikzcd} Z^{\mathrm{an}} \arrow[r, "\mathrm{in}_\omega(\phi)"] \arrow[d, "\mathrm{trop}"'] & \mathbf{G}^{d,\mathrm{an}}_{m,\widetilde{K}} \arrow[d, "\mathrm{trop}"] \\ \mathrm{Trop}(Z) \arrow[r, "f"'] & \mathbb{R}^d \end{tikzcd} \tag{5.13.2}$$

using that the tropicalization of $\mathrm{in}_\omega(\phi)$ agrees with the restriction of f. The Bieri–Groves theorem shows that $\mathrm{Trop}(Z)$ is a d-dimensional polyhedral fan. Since the tropicalization maps are surjective and since $\mathrm{in}_\omega(\phi)(Z)$ is dense in $\mathbf{G}^d_{m,\widetilde{K}}$, commutativity of the diagram (5.13.2) shows that $f(\mathrm{Trop}(Z)) = \mathbb{R}^d$. By Proposition 5.12 (3), there is a d-dimensional polyhedron $P \subset \mathrm{Trop}(Z)$ in Σ containing ω such that $f(P)$ is d-dimensional. In other words, P is d-maximal at ω and relevant for ξ. Moreover, $f|_P$ is injective. This proves (1)–(3).

Conversely, we assume that $\xi \in B(X_\omega)$ has a polyhedron $P \in \Sigma$ satisfying (1)–(3). We must show that $\widetilde{\phi}(\mathrm{red}(\xi))$ is the generic point of $\mathbf{G}^d_{m,\widetilde{K}}$. The first map in (5.13.1) is finite and surjective, hence it maps Z' onto an irreducible component Z of $\mathrm{in}_\omega(X)$. Since P is relevant for ξ by (2), we conclude that $P \subset \mathrm{Trop}(Z) + \omega$. Property (1) says that P is d-maximal at ω which means that $\dim(P) = d(\omega) = d$. It follows from Lemma 5.3 and the Bieri–Groves theorem that $\dim(Z) = d$. Using again the commutative diagram (5.13.2), we deduce from Property (3) that the tropicalization of $\mathrm{in}_\omega(\phi)(Z)$ contains a d-dimensional polyhedron. This is only possible if the generic point of Z maps to the generic point of $\mathbf{G}^d_{m,\widetilde{K}}$. Using the factorization (5.13.1) of $\widetilde{\varphi}$ and that the first map takes $\mathrm{red}(\xi)$ to the generic point of Z, we deduce that $\widetilde{\varphi}$ maps $\mathrm{red}(\xi)$ to the generic point of $\mathbf{G}^d_{m,\widetilde{K}}$. This proves $\xi \in \phi^{-1}(s(\omega')) \cap X_\omega$ and hence we have shown the Lemma in case of a non-trivial valuation on K with $\omega \in N_\Gamma$.

To deal with the general case, we choose a valued extension field $K' \supset K$ whose value group $\Gamma' = v(K'^\times)$ is non-trivial and large enough that $\omega \in N_{\Gamma'}$, and let $X' = X_{K'}$. Let $\xi \in \phi^{-1}(s(\omega')) \cap X_\omega$. We have a commutative square

$$\begin{tikzcd}
X'_\omega \arrow[r, "\phi_{K'}"] \arrow[d, "\pi"'] & \mathbf{G}^{d,\mathrm{an}}_{m,K'} \arrow[d, "\pi"] \\
X_\omega \arrow[r, "\phi"'] & \mathbf{G}^{d,\mathrm{an}}_{m,K}
\end{tikzcd}$$

where the vertical arrows are the structural morphisms. By [4, Thm. 1.2.1] and [17, (0.3.2)], the spectrum $F = \mathscr{M}\big(\mathscr{H}(\xi)\widehat{\otimes}_{\mathscr{H}(s(\omega'))}\mathscr{H}(s'(\omega'))\big)$ of the Banach ring $\mathscr{H}(\xi)\widehat{\otimes}_{\mathscr{H}(s(\omega'))}\mathscr{H}(s'(\omega'))$ is nonempty, where $s' \colon \mathbb{R}^d \to \mathbf{G}^{d,\mathrm{an}}_{m,K'}$ is the section of tropicalization. If ξ' is in the image of the natural morphism $F \to X'_\omega$, then $\phi_{K'}(\xi') = s'(\omega')$ and $\pi(\xi') = \xi$. By the case handled above, $\xi' \in B(X'_\omega)$ and there is $P \in \Sigma$ satisfying (1)–(3). Then $\xi \in B(X_\omega)$ by Proposition 2.21 and P is also relevant for ξ by definition. Moreover, one always has $\dim_{\xi'}(X'_\omega) \leqslant \dim_\xi(X_\omega)$, so $\dim_{\xi'}(X'_\omega) = d(\omega) = \dim(X_\omega)$ implies $\dim_\xi(X_\omega) = d(\omega)$. This proves one inclusion of the Lemma in the general case and the last claim.

Conversely, let $\xi \in B(X_\omega)$ with $P \in \Sigma$ satisfying (1)–(3). Then there exists $\xi' \in B(X'_\omega)$ mapping to ξ by Lemma 5.10 such that P is also relevant for ξ'. By the special case considered above, we have $\phi_{K'}(\xi') = s'(\omega')$, so $\phi(\xi) = \pi(s'(\omega')) = s(\omega')$. This proves $\xi \in \phi^{-1}(s(\omega')) \cap X_\omega$. $\square$

6. Limit points of the tropical skeleton

We have seen in Example 4.9 that the tropical skeleton $S_{\mathrm{Trop}}(X)$ is not necessarily closed in the toric variety Y_Δ^{an}. In this section, we give conditions under which a limit of a sequence of points of $S_{\mathrm{Trop}}(X)$ is contained in $S_{\mathrm{Trop}}(X)$. Our goal is to study the accumulation points of $S_{\mathrm{Trop}}(X)$ for the closed subscheme X of the toric variety Y_Δ. By [32, Théorème 5.3], every limit point of $S_{\mathrm{Trop}}(X)$ is the limit of a sequence of points.[(2)] We use the notation from §3.1.

THEOREM 6.1. — *Let Δ be a rational pointed fan in $N_{\mathbb{R}}$ and let $X \subset Y_\Delta$ be a closed subscheme. By T we denote the dense torus in Y_Δ. We assume that $X \cap T$ is equidimensional of dimension d. Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence in $S_{\mathrm{Trop}}(X)$ which is contained in $(X\cap T)^{\mathrm{an}}$ such that ξ_i converges to a point $\xi \in (X \cap O(\sigma))^{\mathrm{an}}$ for some torus orbit $O(\sigma) \subset Y_\Delta$. Put $\omega_i = \mathrm{trop}(\xi_i) \in N_{\mathbb{R}}$ and $\omega = \mathrm{trop}(\xi) \in N_{\mathbb{R}}(\sigma)$.*

If there exists a polyhedron $P \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}$ of dimension d containing all ω_i and relevant for all ξ_i such that $\pi_\sigma(P)$ is d-maximal at ω, we have $\xi \in S_{\mathrm{Trop}}(X)$ and $\dim_\xi(X_\omega) = d(\omega)$.

Remark 6.2. — The hypotheses in Theorem 6.1 are sufficient but not necessary. For example, let the notation be as in Examples 4.9 and 5.7, and for $r \geqslant 1$ let $\omega''(r) = (0, r, r + r^{-1}) \in \mathrm{relint}(P_2)$. One can show that $s_X(\omega''(r)) \to s_X(0,\infty,\infty)$ as $r \to \infty$ even though $\omega''(r)$ is contained in $P_2 \setminus (P_1 \cup P_3)$ and $\pi_{\sigma_{23}}(P_2)$ is not d-maximal at $(0,\infty,\infty)$.

At a basic level, the proof of Theorem 6.1 uses a similar idea to [23, Theorem 10.6] (and Proposition 4.6), in that we compare $S_{\mathrm{Trop}}(X)$ with the inverse image of the skeleton of a smaller-dimensional toric variety. However, one must be much more careful in constructing the smaller toric variety.

Proof of Theorem 6.1. — If $O(\sigma) = T$, our claim follows from Corollary 4.8. Hence we may assume that $\sigma \neq 0$.

Since $\omega = \lim \omega_i \in N_{\mathbb{R}}(\sigma)$, we have $\overline{P} \cap N_{\mathbb{R}}(\sigma) \neq \emptyset$, where $\overline{P}$ is the closure of P in $\overline{N}_{\mathbb{R}}^{\Delta}$. Hence by Lemma 5.5, the recession cone of P intersects the relative interior of σ, and $\overline{P} \cap N_{\mathbb{R}}(\sigma) = \pi_\sigma(P)$. This set contains ω. Let H be a rational supporting hyperplane of the face $\{0\} \prec \sigma$: that is, H is rational and $H \cap \sigma = \{0\}$, so σ is contained in one of the half-spaces bounded by H. Let $\langle\sigma\rangle \subset N_{\mathbb{R}}$ be the linear span of σ, and let $\langle P\rangle_0 \subset N_{\mathbb{R}}$ be the linear (as opposed to the affine) span of P, i.e., $\langle P\rangle_0$ is the linear span of $P - \eta$ for any $\eta \in P$. The subspaces $\langle\sigma\rangle$ and $\langle P\rangle_0$ are rational. Note that $H + (\langle\sigma\rangle \cap \langle P\rangle_0) = N_{\mathbb{R}}$ since the recession cone of P intersects $\mathrm{relint}(\sigma)$.

(2) This is mainly for convenience, as one could work with nets.

We claim that there exists a rational subspace $L \subset N_{\mathbb{R}}$ such that

(a) $$L \subset H$$

(b) $$N_{\mathbb{R}} = L \oplus \langle P \rangle_0$$

(c) $$N_{\mathbb{R}}(\sigma) = \pi_\sigma(L) \oplus \pi_\sigma(\langle P \rangle_0).$$

This is pure linear algebra. Let $n = \dim(N_{\mathbb{R}})$, recall $d = \dim(P)$ and let $d' = d(\omega)$. By hypothesis, $d' = \dim(\pi_\sigma(P))$, and hence $\dim(\langle\sigma\rangle \cap \langle P \rangle_0) = d - d'$. Since $\langle\sigma\rangle \not\subset H$, we have that H surjects onto $N_{\mathbb{R}}(\sigma)$. Choose a rational subspace $V \subset H$ mapping isomorphically onto $N_{\mathbb{R}}(\sigma)$, so $H = (\langle\sigma\rangle \cap H) \oplus V$ and $N_{\mathbb{R}} = \langle\sigma\rangle \oplus V$. As $H + (\langle\sigma\rangle \cap \langle P \rangle_0) = N_{\mathbb{R}}$, we have $(\langle\sigma\rangle \cap H) + (\langle\sigma\rangle \cap \langle P \rangle_0) = \langle\sigma\rangle$, so $\dim(\langle\sigma\rangle \cap H \cap \langle P \rangle_0) = d - d' - 1$, and therefore,

$$\begin{aligned}\dim(\langle\sigma\rangle \cap H) - \dim(\langle\sigma\rangle \cap H \cap \langle P \rangle_0) &= (\dim(\langle\sigma\rangle) - 1) - (d - d' - 1)\\ &= \dim(\langle\sigma\rangle) - (d - d')\\ &= \dim(\langle\sigma\rangle) - \dim(\langle\sigma\rangle \cap \langle P \rangle_0).\end{aligned}$$

We conclude that

$$\begin{aligned}\operatorname{codim}(\langle\sigma\rangle \cap H \cap \langle P \rangle_0,\, \langle\sigma\rangle \cap H) &= \operatorname{codim}(\langle\sigma\rangle \cap \langle P \rangle_0,\, \langle\sigma\rangle)\\ &= \dim(\langle\sigma\rangle) - (d - d'),\end{aligned}$$

and

$$\operatorname{codim}(\pi_\sigma(\langle P \rangle_0),\, N_{\mathbb{R}}(\sigma)) = n - \dim(\langle\sigma\rangle) - d'.$$

It is possible to choose (generic) rational subspaces $L_1 \subset \langle\sigma\rangle \cap H$ of dimension $\dim(\langle\sigma\rangle) - (d - d')$ and $L_2 \subset V$ of dimension $n - \dim(\langle\sigma\rangle) - d'$ such that $L_1 \cap (\langle\sigma\rangle \cap \langle P \rangle_0) = \{0\}$ and $\pi_\sigma(L_2) \cap \pi_\sigma(\langle P \rangle_0) = \{0\}$. The subspace $L = L_1 \oplus L_2$ satisfies our requirements (a)–(c).

To prove the theorem, we may replace Y_Δ by Y_σ and X by $X \cap Y_\sigma$ without loss of generality. Let $N' = N/(N \cap L)$ and let $f\colon N \to N'$ be the quotient homomorphism. Let $M' = \operatorname{Hom}(N', \mathbb{Z})$ and let $T' = \operatorname{Spec}(K[M']) \cong \mathbf{G}_m^d$ be the torus with cocharacter lattice N'. The map f induces a homomorphism $\psi\colon T \to T'$. Let $\sigma' = f(\sigma)$. This is a pointed cone in $N'_{\mathbb{R}}$ since the supporting hyperplane H contains L. Thus $\psi\colon T \to T'$ extends to a toric morphism of affine toric varieties $\psi\colon Y_\sigma \to Y'_{\sigma'}$, $f\colon N_{\mathbb{R}} \to N'_{\mathbb{R}}$ extends to a continuous map $f\colon \overline{N}^\sigma_{\mathbb{R}} \to \overline{N}'^{\sigma'}_{\mathbb{R}}$, and the following squares commute:

$$\begin{tikzcd} T \arrow[r, "\mathrm{trop}"] \arrow[d, "\psi"'] & N_{\mathbb{R}} \arrow[d, "f"] \\ T' \arrow[r, "\mathrm{trop}"'] & N'_{\mathbb{R}} \end{tikzcd} \qquad \begin{tikzcd} Y_\sigma \arrow[r, "\mathrm{trop}"] \arrow[d, "\psi"'] & \overline{N}^\sigma_{\mathbb{R}} \arrow[d, "f"] \\ Y'_{\sigma'} \arrow[r, "\mathrm{trop}"'] & \overline{N}'^{\sigma'}_{\mathbb{R}} \end{tikzcd}$$

Note that $N'_{\mathbb{R}}(\sigma') = N_{\mathbb{R}}/(\langle\sigma\rangle + L) = N_{\mathbb{R}}(\sigma)/\pi_\sigma(L)$. Condition (b) for L implies f is injective on P with $\dim(P) = \dim(N'_{\mathbb{R}})$, and condition (c) implies $f\colon N_{\mathbb{R}}(\sigma) \to N'_{\mathbb{R}}(\sigma')$ is injective on $\pi_\sigma(P)$ with $d' = \dim(\pi_\sigma(P)) = \dim(N'_{\mathbb{R}}(\sigma'))$.

Let $\phi = \psi|_X$, and define $S = \phi^{-1}\big(S(Y'^{\mathrm{an}}_{\sigma'})\big)$, the inverse image of the skeleton of the affine toric variety $Y'_{\sigma'}$. This is a closed subset of X^{an} because $S(Y'^{\mathrm{an}}_{\sigma'})$ is closed in $Y'^{\mathrm{an}}_{\sigma'}$, as we saw in §3.3. Lemma 5.13 implies that $\xi_i \in S \cap X_{\omega_i}$ for all i. Hence the limit point ξ lies in $S \cap X_\omega$. By construction, $d' = d(\omega) = \dim(N'_{\mathbb{R}}(\sigma'))$, which is equal to the dimension of the torus orbit $O'(\sigma') \subset Y'_{\sigma'}$.

Again by Lemma 5.13, this time applied to $T = O(\sigma)$, $X = X \cap O(\sigma)$, and the map $\psi\colon O(\sigma) \to O'(\sigma')$ on torus orbits, we see that $S \cap X_\omega$ is contained in the Shilov boundary of X_ω, and therefore that $\xi \in S_{\mathrm{Trop}}(X)$. Moreover, the local dimension of X_ω at any point of $S \cap X_\omega$ is $d(\omega)$ by Lemma 5.13. □

If X intersects all torus orbits equidimensionally, we can deduce the following result.

Theorem 6.3. — *Let X be a closed subscheme of the toric variety Y_Δ such that $X \cap O(\sigma)$ is equidimensional of dimension d_σ for any $\sigma \in \Delta$. We suppose that for all faces $\tau \prec \sigma$ of Δ there exists a finite polyhedral complex structure Σ on $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\tau)$ with the following property: For every d_τ-dimensional polyhedron P in Σ such that its recession cone intersects the relative interior of $\pi_\tau(\sigma)$ in $N_{\mathbb{R}}(\tau)$, put $\overline{\sigma} = \pi_\tau(\sigma)$ and assume that the canonical projection $\pi_{\overline{\sigma}}(P)$ has dimension d_σ in $N_{\mathbb{R}}(\sigma)$. Then $S_{\mathrm{Trop}}(X)$ is closed.*

Proof. — Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence in $S_{\mathrm{Trop}}(X)$ converging to some point $\xi \in X^{\mathrm{an}}$. We have to show that $\xi \in S_{\mathrm{Trop}}(X)$. We may always pass to a subsequence and so we may assume that the sequence $\{\xi_i\}$ is contained in a single torus orbit $O(\tau)^{\mathrm{an}}$. Let $\sigma \in \Delta$ be the cone such that $\xi \in O(\sigma)^{\mathrm{an}}$. Then $\tau \prec \sigma$. Let $\omega_i = \mathrm{trop}(\xi_i) \in N_{\mathbb{R}}(\tau)$ and let $\omega = \mathrm{trop}(\xi) \in N_{\mathbb{R}}(\sigma)$, so $\omega = \lim \omega_i$ by continuity of trop.

Since $X \cap O(\tau)$ is equidimensional of dimension d_τ, we have $\dim_{\xi_i}(X) = d_\tau = d(\omega_i)$ for every i. By Proposition 5.12 (3), there is a polyhedron $P_i \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\tau)$ in Σ which is d-maximal at ω_i and which is relevant for ξ_i. After passing to a subsequence, we may assume that all $P_i = P$ for a single d_τ-dimensional polyhedron P in Σ. Since ω lies in the closure of P, the recession cone of P meets the relative interior of $\pi_\tau(\sigma)$ by Lemma 5.5. Therefore the projection of P to $N_{\mathbb{R}}(\sigma)$ is d-maximal by assumption. Hence our claim follows from Theorem 6.1. □

7. Proper intersection with orbits

In this section we discuss common dimensionality conditions under which the hypotheses of Theorem 6.1 are automatically satisfied. We consider a closed subscheme X of a toric variety Y_Δ with dense torus T. We assume throughout that $X \cap T$ is equidimensional of dimension d and that $X \cap T$ is dense in X.

DEFINITION 7.1. — *Let $\sigma \in \Delta$. We say that X* intersects $O(\sigma)$ properly *provided that* $\dim(X \cap O(\sigma)) = \dim(X) - \dim(\sigma)$.

Note that if X intersects $O(\sigma)$ properly then $X \cap O(\sigma) \neq \emptyset$ when $\dim(\sigma) \leqslant \dim(X)$, and $X \cap O(\sigma) = \emptyset$ when $\dim(\sigma) > \dim(X)$.

LEMMA 7.2. — *If $\dim(\sigma) \leqslant \dim(X)$ and X intersects $O(\sigma)$ properly, then $X \cap O(\sigma)$ is equidimensional and*

$$\operatorname{codim}(X \cap O(\sigma),\, O(\sigma)) = \operatorname{codim}(X \cap T,\, T)\,.$$

Therefore, $\operatorname{Trop}(X \cap O(\sigma)) = \operatorname{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ *has pure dimension* $\dim(X) - \dim(\sigma)$.

Proof. — This follows from [22, Proposition 14.7] and the fact that the dimension of σ is $\operatorname{codim}(O(\sigma), T)$. The last statement is a consequence of the Bieri–Groves theorem. □

Remark 7.3. — For a polyhedron $P \subset N_{\mathbb{R}}$, we let $\rho(P) \subset N_{\mathbb{R}}$ denote the recession cone of P. Let U be a closed subscheme of the torus T over K. Then the Bieri–Groves theorem shows that we may write $\operatorname{Trop}(U)$ as a finite union of integral Γ-affine polyhedra P (see [23, 2.2] for the definition of integral Γ-affine polyhedra for a subgroup Γ of $\mathbb{R}$). If U is of pure dimension d, then we can choose all P d-dimensional. Let Σ be the collection of these polyhedra and let $\operatorname{Trop}_0(U)$ be the tropical variety of X with respect to the trivial valuation. Then we recall from [22, Corollary 11.13] the non-trivial fact that

$$\operatorname{Trop}_0(U) = \bigcup_{P \in \Sigma} \rho(P)\,. \tag{7.3.1}$$

The next proposition shows that the condition that X intersects $O(\sigma)$ properly can be checked on tropicalizations.

Proposition 7.4. — *Choose a finite collection Σ of integral $\mathbb{R}$-affine d-dimensional polyhedra whose union is* $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$*. Fix $\sigma \in \Delta$ with* $\dim(\sigma) \leqslant d$*. Then the following are equivalent:*

(1) *X intersects $O(\sigma)$ properly.*
(2) *There exists $P \in \Sigma$ such that $\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset$, and for all such P,*

$$\dim(\rho(P) \cap \sigma) = \dim(\sigma).$$

(3) *There exists $P \in \Sigma$ such that $\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset$, and for all such P,*

$$\dim(\pi_\sigma(P)) = d - \dim(\sigma).$$

Proof. — The equivalence of (1) and (2) is [22, Corollary 14.4, Remark 14.5], using (7.3.1) and noting that $\dim(\rho(P) \cap \sigma) = \dim(\sigma)$ if and only if $\dim(\rho(P) \cap \mathrm{relint}(\sigma)) = \dim(\sigma)$. For $P \in \Sigma$ such that $\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset$, condition (3) is equivalent to having $\langle\sigma\rangle \subset \langle P\rangle_0$, where $\langle\sigma\rangle$ is the span of σ and $\langle P\rangle_0$ is the linear span of P, as in the proof of Theorem 6.1. As $\rho(P) \subset \langle P\rangle_0$, it is clear that (2) implies (3). For (3) implies (1), we have

$$\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma) = \bigcup_{\rho(P)\cap\mathrm{relint}(\sigma)\neq\emptyset} \pi_\sigma(P)$$

by Proposition 5.6, so

$$\dim(X \cap O(\sigma)) = \dim(\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)) = d - \dim(\sigma)$$

proving (1). □

The following Corollary is a special case of Theorem 6.1.

Corollary 7.5. — *Let $\sigma \in \Delta$ be a cone such that X intersects $O(\sigma)$ properly. If $(\xi_i)_{i\in\mathbb{N}}$ is a sequence in $S_{\mathrm{Trop}}(X) \cap T^{\mathrm{an}}$ converging to a point $\xi \in O(\sigma)^{\mathrm{an}}$, then $\xi \in S_{\mathrm{Trop}}(X)$.*

Proof. — Since X^{an} is closed in Y_Δ^{an}, we have $\xi \in X^{\mathrm{an}} \cap O(\sigma)^{\mathrm{an}}$. In particular, $X \cap O(\sigma) \neq \emptyset$, so $\dim(\sigma) \leqslant d$. Let $\omega_i = \mathrm{trop}(\xi_i) \in N_{\mathbb{R}}$ and let $\omega = \mathrm{trop}(\xi) \in N_{\mathbb{R}}(\sigma)$. Choose a finite collection Σ of integral $\mathbb{R}$-affine d-dimensional polyhedra whose union is $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$. By Proposition 5.12, for every ξ_i there exists a polyhedron in Σ, which has dimension d and is relevant for ξ_i. After passing to a subsequence, we may assume that the same polyhedron P works for all ξ_i. By Lemma 5.5, this implies that $\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset$, so by Proposition 7.4, the dimension of $\pi_\sigma(P)$ is

$$\dim(\pi_\sigma(P)) = d - \dim(\sigma) = \dim(X \cap O(\sigma)) = \dim(\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)).$$

It follows that $\pi_\sigma(P)$ is d-maximal at all of its points, so we can apply Theorem 6.1. □

The case when X intersects *all* torus orbits properly is even more special. As above we assume that $X \cap T$ is equidimensional of dimension d and that $X \cap T$ is dense in X.

LEMMA 7.6. — *Suppose that for all $\sigma \in \Delta$, either $X \cap O(\sigma) = \emptyset$ or X intersects $O(\sigma)$ properly. Fix $\tau \in \Delta$, and let $X_\tau \subset Y_{\Delta_\tau}$ be the closure of $X \cap O(\tau)$. Then for all $\overline{\sigma} = \pi_\tau(\sigma) \in \Delta_\tau$, either $X_\tau \cap O(\overline{\sigma}) = \emptyset$ or X_τ intersects $O(\overline{\sigma})$ properly.*

Proof. — If $X \cap O(\tau) = \emptyset$ then the assertion is trivial. Otherwise, $X \cap O(\tau)$ is equidimensional of dimension $d - \dim(\tau)$ by Lemma 7.2. Choose a finite collection Σ of integral $\mathbb{R}$-affine d-dimensional polyhedra whose union is $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$, so

$$\mathrm{Trop}(X \cap O(\tau)) = \bigcup_{\rho(P)\cap \mathrm{relint}(\tau)\neq\emptyset} \pi_\tau(P)$$

as seen in Proposition 5.6. By Proposition 7.4, we have $\dim(\pi_\tau(P)) = d - \dim(\tau)$ for all such P. Fix $\sigma \in \Delta$ with $\tau \prec \sigma$, let $\overline{\sigma} = \pi_\tau(\sigma) \in \Delta_\tau$, and suppose that $X_\tau \cap O(\overline{\sigma}) \neq \emptyset$. This implies $X \cap O(\sigma) \neq \emptyset$, so $\dim(\sigma) \leqslant d$. If $P \in \Sigma$ with $\rho(P) \cap \mathrm{relint}(\tau) \neq \emptyset$ has $\rho(\pi_\tau(P)) \cap \mathrm{relint}(\overline{\sigma}) \neq \emptyset$ then the closure $\overline{\pi_\tau(P)}$ of $\pi_\tau(P)$ in $\overline{N(\tau)}_{\mathbb{R}}^{\Delta_\tau}$ satisfies

$$\overline{\pi_\tau(P)} \cap N(\tau)_{\mathbb{R}}(\overline{\sigma}) = \overline{\pi_\tau(P)} \cap N_{\mathbb{R}}(\sigma) = \pi_{\overline{\sigma}}(\pi_\tau(P)) = \pi_\sigma(P)$$

by Lemma 5.5. In particular, the closure $\overline{P}$ of P in $\overline{N}_{\mathbb{R}}^{\Delta}$ intersects $N_{\mathbb{R}}(\sigma)$, so Lemma 5.5 again shows $\rho(P) \cap \mathrm{relint}(\sigma) \neq \emptyset$. Hence $\dim(\pi_\sigma(P)) = d - \dim(\sigma)$ by Proposition 7.4, since X intersects $O(\sigma)$ properly, so

$$\dim(\pi_{\overline{\sigma}}(\pi_\tau(P))) = \dim(\pi_\sigma(P)) = d - \dim(\sigma) = (d - \dim(\tau)) - \dim(\overline{\sigma}).$$

This is true for all P with $\rho(\pi_\tau(P)) \cap \mathrm{relint}(\overline{\sigma}) \neq \emptyset$, so again by Proposition 7.4, X_τ intersects $O(\overline{\sigma})$ properly. □

COROLLARY 7.7. — *If, for all $\sigma \in \Delta$, either $X \cap O(\sigma) = \emptyset$ or X intersects $O(\sigma)$ properly, then $S_{\mathrm{Trop}}(X)$ is closed in X^{an}.*

Proof. — Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence in $S_{\mathrm{Trop}}(X)$ converging to some $\xi \in X^{\mathrm{an}}$. We wish to show $\xi \in S_{\mathrm{Trop}}(X)$. Passing to a subsequence, we may assume that the sequence is contained in a single torus orbit $O(\tau)$. By Lemma 7.6, the closure X_τ of $X \cap O(\tau)$ in the toric variety Y_{Δ_τ} satisfies the same hypotheses as X. Hence for dimension reasons, if $\sigma \in \Delta$, $\tau \prec \sigma$, and $\overline{\sigma} = \pi_\tau(\sigma)$, then $X_\tau \cap O(\overline{\sigma})$ is a union of irreducible components of

$X \cap O(\sigma)$. By Proposition 4.7 as applied to $X \cap O(\sigma)$, then, we may replace X by X_τ to assume $\{\xi_i\} \subset T^{\mathrm{an}}$. Now we apply Corollary 7.5 to conclude $\xi \in S_{\mathrm{Trop}}(X)$. $\square$

Remark 7.8. — In the situation of Corollary 7.7, suppose that $Y_\Delta = Y_\sigma$ is an affine toric variety. In this case it is possible using the techniques of Theorem 6.1 to produce a morphism $\psi\colon Y_\sigma \to Y'_{\sigma'}$ of affine toric varieties such that the composite morphism $\phi\colon X \hookrightarrow Y_\sigma \to Y'_{\sigma'}$ is finite and surjective over every torus orbit $O(\tau') \subset Y'_{\sigma'}$ such that $\phi^{-1}(O(\tau')) \neq \emptyset$. From this it follows exactly as in Proposition 4.6 that $\phi^{-1}(S(Y'_{\sigma'})) = S_{\mathrm{Trop}}(X)$. This gives another proof of Corollary 7.7, and also shows that $S_{\mathrm{Trop}}(X)$ is a kind of generalization of a *c*-skeleton in the sense of [16, 18]. Compare [23, Remark 10.7].

In particular, one should be able to use the results of Ducros to prove that each torus orbit in the tropical skeleton has a natural $\mathbb{Q}$-affine structure. Forthcoming work of Ducros–Thuillier may allow for stronger assertions.

Remark 7.9. — The hypotheses of Corollary 7.7 are commonly satisfied in the context of tropical compactifications. Let $\mathrm{Trop}_0(X)$ denote the tropicalization of X, considered as a subscheme of Y_Δ over the field K endowed with the trivial valuation. If Σ is a finite collection of integral $\mathbb{R}$-affine d-dimensional polyhedra whose union is $\mathrm{Trop}(X \cap T)$, then $\mathrm{Trop}_0(X \cap T)$ is the union of the recession cones of the polyhedra in Σ as we have seen in Remark 7.3.

Suppose that the support of the fan Δ is equal to $\mathrm{Trop}_0(X \cap T)$. This happens for instance if Δ is a *tropical fan* for $X \cap T$ as defined by Tevelev [36] for integral $X \cap T$ and generalized in [22, §12] to arbitrary closed subschemes of T. Then X is proper over K by [36, Proposition 2.3], and X intersects each torus orbit $O(\sigma)$ properly by [22, Theorem 14.9]. In this case, $S_{\mathrm{Trop}}(X)$ is closed by Corollary 7.7, so it is even compact.

8. Section of tropicalization

In this section we prove that there is a section of the tropicalization map on the locus of tropical multiplicity one, and we use the results of §4 to examine when this section is continuous.

8.1. Existence of the section

Let Δ be a rational pointed fan in $N_{\mathbb{R}}$ and let $X \subset Y_\Delta$ be a closed subscheme. Suppose that $\omega \in \mathrm{Trop}(X)$ has $m_{\mathrm{Trop}}(\omega) = 1$. We will show that in this case, there is a distinguished Shilov boundary point of $X_\omega = \operatorname{trop}^{-1}(\omega) \cap X^{\mathrm{an}}$, which will be the image of the section evaluated at ω. However, as the following example shows, X_ω may still have multiple Shilov boundary points.

Example 8.2. — Suppose that the valuation on K is non-trivial. Let X be the closed subscheme of $T = \operatorname{Spec}(K[x_1^{\pm 1}, x_2^{\pm 1}])$ given by the ideal $\mathfrak{a} = (x_1-1, x_2-1) \cap (x_1-1-\varpi)$ for $\varpi \in K^\times$ with $|\varpi| < 1$. Then X is the disjoint union of the line $\{x_1 = 1+\varpi\}$ with the point $(1,1)$. The initial degeneration at $\omega = 0$ is defined by the ideal $\operatorname{in}_w(\mathfrak{a}) = \big((x_1-1)^2, (x_1-1)(x_2-1)\big)$ over $\widetilde{K}$. This is a generically reduced line with an associated point at $(1,1)$. It has tropical multiplicity 1, but the canonical reduction is the disjoint union of a point and a line, so that X_ω has *two* Shilov boundary points. Note however that one of these points is contained in an irreducible component of dimension one, and the other in a component of dimension zero.

Recall from Definition 5.2 that for $\omega \in \operatorname{Trop}(X)$, the local dimension of $\operatorname{Trop}(X)$ at ω is denoted $d(\omega) = \dim\big(\operatorname{LC}_\omega(\operatorname{Trop}(X))\big)$.

Proposition 8.3. — *Let $X \subset Y_\Delta$ be a closed subscheme and let $\omega \in \operatorname{Trop}(X)$ be a point with $m_{\mathrm{Trop}}(\omega) = 1$. Let $\sigma \in \Delta$ be the cone such that $\omega \in N_{\mathbb{R}}(\sigma)$. Then there is a unique irreducible component C of $X \cap O(\sigma)$ of dimension $d(\omega)$ such that $\omega \in \operatorname{Trop}(C)$, and there is a unique Shilov boundary point of C_ω. Moreover, $m_{\mathrm{Trop}}(C, \omega) = 1$.*

Proof. — We immediately reduce to the case $Y_\Delta = O(\sigma) = T$. First suppose that K is algebraically closed, non-trivially valued and $\omega \in N_\Gamma$. By hypothesis, $\operatorname{in}_\omega(X)$ is irreducible and generically reduced, and its dimension is $d \coloneqq d(\omega) = \dim(X_\omega) = \dim(\widetilde{X}_\omega)$ by Lemma 5.3. Replacing X by its underlying reduced subscheme X_{red} has the effect of replacing $\operatorname{in}_\omega(X)$ by the closed subscheme defined by a nilpotent ideal sheaf, so the same is true of $\operatorname{in}_\omega(X_{\mathrm{red}})$. The conclusions of the Proposition (except the last one) are insensitive to nilpotents, so we may assume without loss of generality that X is reduced (coming back to the last claim at the end).

The projection formula of §3.6 gives

$$1 = \sum_{Z \twoheadrightarrow \operatorname{in}_\omega(X)} [Z : \operatorname{in}_\omega(X)],$$

where the sum is taken over all irreducible components Z of $\widetilde{X}_\omega$ dominating $\operatorname{in}_\omega(X)$. Since $\widetilde{X}_\omega \to (\mathfrak{X}_\omega)_s$ is finite in any case by Proposition 2.12, this proves that there is a unique irreducible component Z of $\widetilde{X}_\omega$ of dimension

d. Let $\xi \in X_\omega$ be the Shilov boundary point reducing to the generic point of Z.

Let D be an irreducible component of X_ω of dimension $d = \dim(X_\omega)$. The inclusion $D \hookrightarrow X_\omega$ gives a finite morphism of canonical reductions $\widetilde{D} \to \widetilde{X}_\omega$. As $\widetilde{D}$ is equidimensional of dimension d by Proposition 2.10(3), every generic point of $\widetilde{D}$ maps to the generic point of Z, so every Shilov boundary point of D maps to ξ by Proposition 2.16. Since $D \to X_\omega$ is injective, its Shilov boundary is $\{\xi\}$. By Proposition 2.15, D is the unique irreducible component of X_ω containing ξ, thus is the unique irreducible component of X_ω of dimension d.

Let C be an irreducible component of X containing the Shilov point ξ in its analytification. Then C_ω is a union of irreducible components of X_ω of the same dimension as C by Proposition 2.5. Since $\xi \in C_\omega$ we have $D \subset C_\omega$, so $\dim(C) = \dim(D) = d$, and therefore $C_\omega = D$. Finally, if C' is another irreducible component of X of dimension d such that $C'_\omega \neq \emptyset$, then $D \subset C'_\omega$, which is impossible since $\dim(C \cap C') < d$. Thus C is the unique irreducible component of X of dimension d such that $\omega \in \operatorname{Trop}(C)$, and ξ is the unique Shilov boundary point of $C_\omega = D$.

Now we allow K and $\omega \in N_{\mathbb{R}}$ to be arbitrary. Let $K' \supset K$ be an algebraically closed valued extension field whose value group $\Gamma' = v(K^\times)$ is non-trivial and large enough that $\omega \in N_{\Gamma'}$, let $X' = X_{K'}$, and let $\pi\colon X' \to X$ be the structural morphism. By the above, there is a unique irreducible component C' of X' of dimension $d = d(\omega)$ such that $\omega \in \operatorname{Trop}(C')$, and there is a unique Shilov boundary point of C'_ω. Then $C = \pi(C')$ is the unique irreducible component of X of dimension d containing ω in its tropicalization, and $C' = (\pi^{-1}(C))_{\mathrm{red}} = C_{K',\mathrm{red}}$. Hence $C'_\omega = ((C_\omega)_{K'})_{\mathrm{red}}$, so $B(C_\omega) = \pi(B(C'_\omega))$ by Proposition 2.21, so $B(C_\omega)$ has only one point.

We come now to the last claim no longer assuming that X is reduced. By definition, $m_{\mathrm{Trop}}(C,\omega) = 1$ provided that $\mathrm{in}_\omega(C_{K'})$ is irreducible and generically reduced. The map on initial degenerations $\mathrm{in}_\omega(C_{K'}) \to \mathrm{in}_\omega(X')$ is a closed immersion, as both are closed subschemes of $\mathrm{in}_\omega(T_{K'}) \cong \mathbf{G}^n_{m,\widetilde{K}'}$. Since $\mathrm{in}_\omega(X')$ is irreducible and generically reduced, and $\dim(\mathrm{in}_\omega(C_{K'})) = \dim(\mathrm{in}_\omega(X'))$, this shows that $\mathrm{in}_\omega(C_{K'})$ is also irreducible and generically reduced. $\square$

Definition 8.4. — *Let $X \subset Y_\Delta$ be a closed subscheme. Write*

$$\operatorname{Trop}(X)_{m_{\mathrm{Trop}}=1} \coloneqq \{\omega \in \operatorname{Trop}(X) \mid m_{\mathrm{Trop}}(\omega) = 1\}$$

for the tropical multiplicity-1 locus in $\operatorname{Trop}(X)$. *If $\omega \in \operatorname{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}(\sigma)$ for $\sigma \in \Delta$, we let $C(\omega)$ be the unique irreducible component of*

$X \cap O(\sigma)$ of dimension $d(\omega)$ with $\omega \in \mathrm{Trop}(C(\omega))$, and we define

$$\begin{aligned} s_X(\omega) &= \text{the unique Shilov boundary point of } C(\omega)_\omega \\ &= \text{the unique Shilov boundary point } \xi \text{ of } X_\omega \\ &\qquad\qquad\qquad \text{such that } \dim_\xi(X_\omega) = d(\omega). \end{aligned}$$

We regard s_X as a map $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \to S_{\mathrm{Trop}}(X) \subset X^{\mathrm{an}}$.

It follows immediately from the above definition that the image of s_X is contained in $S_{\mathrm{Trop}}(X)$. By construction, $\mathrm{trop} \circ s_X$ is the identity, so s_X is a section of trop defined on $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$. If $X \cap O(\sigma)$ is equidimensional of dimension d then $d(\omega) = d$ for all $\omega \in \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ by the Bieri–Groves theorem, so $s_X(\omega)$ is the unique Shilov boundary point of X_ω in this case: $S_{\mathrm{Trop}}(X) \cap \mathrm{trop}^{-1}(\omega) = \{s_X(\omega)\}$. Therefore our s_X coincides with the section considered in [23, §10] for $X \subset T$ irreducible. It also coincides with the section $s\colon \overline{N}_{\mathbb{R}}^{\Delta} \to Y_{\Delta}^{\mathrm{an}}$ introduced in §3.3 when $X = Y_\Delta$.

Remark 8.5 (Behavior with respect to extension of scalars). — Let K' be a valued extension field of K, let $X' = X_{K'}$, and let $\pi\colon X'^{\mathrm{an}} \to X^{\mathrm{an}}$ be the structural map. Then $\mathrm{Trop}(X')_{m_{\mathrm{Trop}}=1} = \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ by the definition of m_{Trop}. It is clear from the proof of Proposition 8.3 that $\pi \circ s_{X'} = s_X$.

The uniqueness of $C(\omega)$ for $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ gives us the following decomposition of $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$. Supposing for simplicity that X is a closed subscheme of T, for an irreducible component C of X let

$$Z(C) = \big\{\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \mid C(\omega) = C\big\}. \tag{8.5.1}$$

In other words, $Z(C)$ is the set of all multiplicity-1 points ω such that C is the unique irreducible component of X of dimension $d(\omega)$ with $\omega \in \mathrm{Trop}(C)$. Hence $s_X(\omega)$ is the Shilov boundary point of C_ω. Then by definition, $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ is the disjoint union $\coprod_C Z(C)$, and $s_X = s_C$ on $Z(C)$ (which makes sense by the final assertion of Proposition 8.3). This observation, along with the following Lemma, will allow us to reduce topological questions about $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ to the case when X is irreducible.

Lemma 8.6. — *Let $X \subset T$ be a closed subscheme, let C be an irreducible component of X, and define $Z(C)$ as in* (8.5.1)*. Then $Z(C)$ is open and closed in $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$.*

Proof. — Since $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} = \coprod_C Z(C)$, it is enough to prove that $Z(C)$ is closed. Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence in $Z(C)$ converging to a point $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$. By passing to a subsequence, we may assume that all ω_i are contained in a single polyhedron $P \subset \mathrm{Trop}(C)$ of dimension

$d = \dim(C)$. Then $\omega \in P$ as well. We claim that $d(\omega) = d$. If not, then there exists a polyhedron $P' \subset \mathrm{Trop}(X)$ of dimension $d' > d$ also containing ω. We note that $\mathrm{LC}_\omega(P)$ is not included in $\mathrm{LC}_\omega(P')$ as we have $d(\omega_i) = d$ for all i. But then $\mathrm{LC}_\omega(\mathrm{Trop}(X)) = \mathrm{Trop}(\mathrm{in}_\omega(X))$ is not equidimensional, so $\mathrm{in}_\omega(X)$ is not irreducible, which contradicts $m_{\mathrm{Trop}}(\omega) = 1$. This proves the claim. We have $\omega \in P \subset \mathrm{Trop}(C)$, so C is the unique irreducible component of X of dimension $d = d(\omega)$ containing ω in its tropicalization, and therefore $\omega \in Z(C)$. □

8.7. Continuity on torus orbits

The following analogue of Corollary 4.8 is a generalization of [23, Theorem 10.6] to reducible X and a general non-Archimedean ground field K.

Proposition 8.8. — *For $\sigma \in \Delta$, the section of tropicalization s_X is continuous on the multiplicity-1 locus $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}(\sigma)$. Moreover, if $Z \subset \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}(\sigma)$ is contained in the closure of its interior in $\mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$, then s_X is the unique continuous partial section of $\mathrm{trop}\colon X^{\mathrm{an}} \to \mathrm{Trop}(X)$ defined on Z.*

Proof. — The statement of the Proposition is intrinsic to the torus orbit $O(\sigma)$, so we immediately reduce to the case $Y_\Delta = O(\sigma) = T$. Since $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ is the disjoint union (as a topological space) of the subspaces $Z(C)$ by Lemma 8.6, it is enough to prove continuity and uniqueness on $Z(C)$ for a fixed irreducible component C of X. Since $s_X = s_C$ on $Z(C)$, we may replace X by C to assume X irreducible.

Let $Z = \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$. It suffices to show that $s_X(Z)$ is closed in $\mathrm{trop}^{-1}(Z) \cap X^{\mathrm{an}}$ (endowed with its relative topology), since $\mathrm{trop}\colon \mathrm{trop}^{-1}(Z) \cap X^{\mathrm{an}} \to Z$ is a proper map to a first-countable topological space, thus is a closed map by [30]. In this case, $s_X(\omega)$ is the unique Shilov boundary point of X_ω for all $\omega \in Z$. Thus $s_X(Z) = \mathrm{trop}^{-1}(Z) \cap S_{\mathrm{Trop}}(X)$, which is closed in $\mathrm{trop}^{-1}(Z) \cap X^{\mathrm{an}}$ by Proposition 4.6. This settles the continuity assertion.

Now let $Z \subset \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ be a subset which is contained in the closure of its interior in $\mathrm{Trop}(X)$, still assuming (as we may) that X is irreducible. The proof of uniqueness of the section s_X on Z goes through exactly as in [23, Theorem 10.6], which only uses that in the situation of Proposition 4.6, we have

$$\phi^{-1}(S_{\mathrm{Trop}}(\mathbf{G}_m^d)) = S_{\mathrm{Trop}}(X),$$

and that $S_{\mathrm{Trop}}(X) \cap \mathrm{trop}^{-1}(\omega) = \{s_X(\omega)\}$ for $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$. □

Remark 8.9. — In the proof of Proposition 8.8, after reducing to the case of irreducible X, we could have applied [23, Theorem 10.6] after an extension of scalars to prove continuity. However, it is instructive to see why continuity follows from the more general results of §4.

8.10. A sequential continuity criterion

The section s_X need not be continuous on all of $\mathrm{Trop}(X)$ when X is a closed subscheme of a toric variety Y_Δ.

Example 8.11. — In Example 4.9, $s_X(0, r, 2r)$ does not tend to $s_X(0, \infty, \infty)$ as $r \to \infty$.

It is easy to see that $\overline{N}_{\mathbb{R}}^{\Delta}$ is a metric space. Hence s_X is continuous if and only if it is sequentially continuous. Given a sequence $(\omega_i)_{i\in\mathbb{N}}$ in $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$ converging to a point $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1}$, if ω and all ω_i are contained in the same torus orbit then $s_X(\omega_i) \to s_X(\omega)$ by Proposition 8.8. If $(\omega_i)_{i\in\mathbb{N}}$ is contained in several different torus orbits, one checks sequential continuity on each torus orbit separately. Hence verifying sequential continuity amounts to showing that if $(\omega_i)_{i\in\mathbb{N}}$ is a sequence in $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}$ converging to $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}(\sigma)$, then $s_X(\omega_i)$ converges to $s_X(\omega)$. Decomposing by local dimension, one breaks $(\omega_i)_{i\in\mathbb{N}}$ into several subsequences, each one of which is contained in a single polyhedron in $\mathrm{Trop}(X)$ which is d-maximal (Definition 5.4) at each point of the subsequence.

The main ingredient in the proof of the following result is Theorem 6.1.

Theorem 8.12. — *Let Δ be a pointed rational fan in $N_{\mathbb{R}}$ and let $X \subset Y_\Delta$ be a closed subscheme. Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}$ converging to a point $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}(\sigma)$ for $\sigma \in \Delta$. Suppose that there exists a polyhedron $P \subset \mathrm{Trop}(X) \cap N_{\mathbb{R}}$ which is d-maximal at each ω_i. If $\pi_\sigma(P)$ is d-maximal at ω, then $s_X(\omega_i) \to s_X(\omega)$.*

Proof. — First we reduce to the case where $X \cap T$ is irreducible and dense in X. In fact, equidimensionality would be enough to apply Theorem 6.1 later.

Let $d = \dim(P)$, so $d(\omega_i) = d$ for all i. By Proposition 8.3, there exists a unique irreducible component $C_i = C(\omega_i)$ of $X \cap T$ of dimension d with $\omega_i \in \mathrm{Trop}(C_i)$. As all other irreducible components of X meeting X_{ω_i} have smaller dimension, we must have $\mathrm{LC}_{\omega_i}(P) \subset \mathrm{LC}_{\omega_i}(\mathrm{Trop}(C_i))$. Let C be a d-dimensional irreducible component of X occurring as C_i for infinitely many i. Replacing P by $\mathrm{Trop}(C) \cap P$, we may asssume that $P \subset \mathrm{Trop}(C)$. Note that d-maximality of $\pi_\sigma(P)$ at ω is preserved. The limit point ω lies in the closure $\overline{P}$ of P intersected with $N_{\mathbb{R}}(\sigma)$, and hence in $\pi_\sigma(P)$ by Lemma 5.5, which is d-maximal at ω. Letting $\overline{C} \subset X$ be the closure of C in Y_Δ, we have $\overline{P} \subset \mathrm{Trop}(\overline{C})$, so Lemma 5.3 yields

$$\dim(X_\omega) = d(\omega) = \dim(\mathrm{LC}_\omega(\mathrm{Trop}(\overline{C}))) = \dim(\overline{C}_\omega).$$

Therefore using Proposition 8.3, the unique irreducible component of $X \cap O(\sigma)$ containing $s_X(\omega)$ is an irreducible component of $\overline{C} \cap O(\sigma)$. This is true for all irreducible components C occurring as C_i for infinitely many i, so we may assume that $X \cap T = C$ is irreducible and dense in $X = \overline{C}$.

Let $W = \{\omega_i \mid i \in \mathbb{N}\}$ and hence $\overline{W} = \{\omega_i \mid i \in \mathbb{N}\} \cup \{\omega\}$. Proving the Theorem amounts to showing that s_X is continuous on the subspace $\overline{W}$ of $\mathrm{Trop}(X)$. As $\overline{W}$ is a first-countable Hausdorff space, as in the proof of Proposition 8.8 it suffices to show that $s_X(\overline{W})$ is closed in $\mathrm{trop}^{-1}(\overline{W}) \cap X^{\mathrm{an}}$. By [32, Théorème 5.3], X^{an} is a Fréchet–Urysohn space, so for every subspace, closure is the same as sequential closure. Since $s_X(W)$ is discrete, we only have to prove that $s_X(\omega)$ is the unique (sequential) limit point of $s_X(W)$ not contained in $s_X(W)$. Any such is contained in X_ω, so let $\xi \in X_\omega$ be a limit point of $s_X(W)$. By Theorem 6.1, $\xi \in S_{\mathrm{Trop}}(X)$ and $\dim_\xi(X_\omega) = d(\omega)$. The only point with these properties is $s_X(\omega)$. □

We can apply the preceeding result in the following situation.

Corollary 8.13. — *Let Δ be a pointed rational fan in $N_{\mathbb{R}}$ and let $X \subset Y_\Delta$ be a closed subscheme. Suppose that $X \cap T$ is equidimensional and let $\sigma \in \Delta$ be a cone such that X intersects $O(\sigma)$ properly. Then every sequence $(\omega_i)_{i\in\mathbb{N}}$ in $\mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}$ converging to a point $\omega \in \mathrm{Trop}(X)_{m_{\mathrm{Trop}}=1} \cap N_{\mathbb{R}}(\sigma)$ has the property that $s_X(\omega_i)$ converges to $s_X(\omega)$.*

Proof. — We choose a finite collection Σ of integral $\mathbb{R}$-affine d-dimensional polyhedra whose union is $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$. For every P in Σ such that $\rho(P)$ meets the relative interior of σ, Proposition 7.4 implies that $\pi_\sigma(P)$ has dimension $\dim(X) - \dim(\sigma) = d(\omega)$. Hence we can apply Theorem 8.12 to every P containing infinitely many ω_i. □

Theorem 8.14. — *Let Δ be a rational pointed fan in $N_{\mathbb{R}}$ and let $X \subset Y_\Delta$ be a closed subscheme. Suppose that $X \cap T$ is dense in T, and that for all $\sigma \in \Delta$ the subscheme $X \cap O(\sigma)$ of $O(\sigma)$ is either empty or equidimensional of dimension d_σ, where we put $d_0 = d$. Assume that $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$ can* *be covered by finitely many d-dimensional polyhedra P with the following property: If the recession cone of P meets the relative interior of σ, the projection $\pi_\sigma(P)$ has dimension d_σ. Then $s_X\colon \operatorname{Trop}(X)_{m_{\mathrm{Trop}}=1} \to X^{\mathrm{an}}$ is continuous.*

Proof. — Let $(\omega_i)_{i\in\mathbb{N}}$ be any sequence in $\operatorname{Trop}(X)_{m_{\mathrm{Trop}}=1}$ converging to $\omega \in \overline{N}^{\Delta}_{\mathbb{R}}$. We have to show that $s_X(\omega_i)$ converges to $s_X(\omega)$. By passing to subsequences we may assume that all ω_i are contained in $N_{\mathbb{R}}(\tau)$ for some face $\tau \in \Delta$. Then ω lies in $N_{\mathbb{R}}(\sigma)$ for a face σ with $\tau \prec \sigma$. According to Proposition 5.6, $\operatorname{Trop}(X) \cap N_{\mathbb{R}}(\tau)$ is covered by the polyhedra $\pi_\tau(P)$, where P runs over all polyhedra in our finite covering of $\operatorname{Trop}(X) \cap N_{\mathbb{R}}$ with $\rho(P) \cap \operatorname{relint}(\tau) \neq \emptyset$. Assume that infinitely many ω_i are contained in the same $\pi_\tau(P)$. By hypothesis, $\pi_\tau(P)$ has dimension d_τ. Now we apply Theorem 8.12 to the subscheme $X \cap Y_{\Delta_\tau}$ of Y_{Δ_τ}: We consider the face $\overline{\sigma} = \pi_\tau(\sigma)$ of Δ_τ. The projection $\pi_{\overline{\sigma}}(\pi_\tau(P)) = \pi_\sigma(P)$ has dimension d_σ, which implies that s_X applied to our subsequence converges to $s_X(\omega)$. This proves our claim. □

Theorem 8.15. — *Let X be a closed subscheme of Y_Δ such that $X \cap T$ is equidimensional and dense in X. Assume additionally that for all $\sigma \in \Delta$, either $X \cap O(\sigma) = \emptyset$ or X intersects $O(\sigma)$ properly. Then*

$$s_X\colon \operatorname{Trop}(X)_{m_{\mathrm{Trop}}=1} \to X^{\mathrm{an}}$$

is continuous.

Proof. — This follows from Theorem 8.14 using Lemma 7.2 and Proposition 7.4. □

Example 8.16. — Let us now briefly discuss the example of the Grassmannian of planes $X = \operatorname{Gr}(2,n)$ in n-space with its Plücker embedding $\varphi : \operatorname{Gr}(2,n) \hookrightarrow \mathbf{P}^{\binom{n}{2}-1}$. The toric variety in this example is given by projective space with projective coordinates p_{kl} indexed by pairs $k < l$ in $\{1,\ldots,n\}$. In this case, continuity for the section map on tropical Grassmannians was shown directly in [14]. See also [37] for an expository account of this construction. Note that in [14], we use log instead of $-\log$ for tropicalization.

Put $\mathcal{T}\operatorname{Gr}(2,n) = \operatorname{Trop}(\operatorname{Gr}(2,n))$. Let $N_{\mathbb{R}}$ be the cocharacter space of the dense torus in $\mathbf{P}^{\binom{n}{2}-1}$. Let J be a proper subset of the set of all projective coordinates $\{p_{kl}\}$, and let E_J be the subvariety of projective space, where precisely the coordinates in J vanish. These are the torus orbits in projective space. Hence the locally closed subvariety $\operatorname{Gr}_J(2,n)$ from [14] is the intersection of $\operatorname{Gr}(2,n)$ with a torus orbit. By [14, Lemma 5.2], $\operatorname{Gr}_J(2,n)$

is irreducible. In particular, the intersection of $\mathrm{Gr}(2,n)$ with every torus orbit is equidimensional.

Now let C_T be a maximal cone in $\mathcal{T}\mathrm{Gr}(2,n)\cap N_{\mathbb{R}}$. It has dimension $2(n-2)$. By [34], it corresponds to all phylogenetic trees on the trivalent combinatorial tree T. With the help of the results in [14, §4 and 5], one can show that the projection of C_T to the cocharacter space of the torus orbit E_J is d-maximal. Moreover, it is shown in [14, Corollary 6.5] that the tropical Grassmannian has tropical multiplicity one everywhere. Therefore we can apply Theorem 8.14 to deduce the existence of a continous section to the tropicalization map.

This provides a conceptual explanation for the existence of a section for the tropical Grassmannian. Note however that we need the combinatorial arguments developed in [14] in order to show that the tropical Grassmannian satisfies the prerequisites of Theorem 8.14.

9. Schön subvarieties

In this section, we will prove that when X is a so-called schön subvariety of a torus T, the tropical skeleton $S_{\mathrm{Trop}}(X)$ can be identified with the parameterizing complex of Helm–Katz [24]. It also coincides with the skeleton of a strictly semistable pair in the sense of [23]. Since the latter is a deformation retract of X^{an}, this answers a question of Helm–Katz.

In this section, K_0 is a discretely valued field with value group Γ_0. We write $\overline{K}_0$ for an algebraic closure, and K for the completion of K_0. We assume that the value group Γ of $\overline{K}_0$ and K is equal to $\mathbb{Q}$, so that $\Gamma_0 = v(K_0^{\times}) = r\mathbb{Z}$ for some $r \in \mathbb{Q}^{\times}$.

9.1. Tropical compactifications

Here we recall several standard facts about tropical compactifications. For more details, see [22, §7, §12].

Let $\mathbb{R}_+ = [0,\infty) \subset \mathbb{R}$. Let Σ be a pointed rational fan in $N_{\mathbb{R}} \times \mathbb{R}_+$. For $r \in \mathbb{R}_+$ we let $\Sigma_r = \{\sigma \cap N_{\mathbb{R}} \times \{r\} \mid \sigma \in \Sigma\}$; this is a polyhedral complex in $N_{\mathbb{R}}$, which is a fan when $r = 0$. The K°-toric scheme associated to Σ will be denoted $\mathscr{Y}_{\Sigma}$. This is a finitely presented, flat, normal, separated K°-scheme. The generic fiber of $\mathscr{Y}_{\Sigma}$ is the toric variety Y_{Σ_0}. The torus $T = \mathrm{Spec}(K[M])$ is dense in $\mathscr{Y}_{\Sigma}$, and the integral torus $\mathscr{T} := \mathrm{Spec}(K^{\circ}[M])$ acts on $\mathscr{Y}_{\Sigma}$. The torus orbits on the generic (resp. special) fiber correspond to cones (resp. polyhedra) in Σ_0 (resp. Σ_1); for $\sigma \in \Sigma_0$ (resp. $P \in \Sigma_1$) we let $O(\sigma) \subset Y_{\Sigma_0}$ (resp. $O(P) \subset (\mathscr{Y}_\Sigma)_s$) denote the corresponding orbit. For $\sigma \in \Sigma_0$ and $P \in \Sigma_1$, we have

$$\dim(\sigma) + \dim(O(\sigma)) = \dim(N_{\mathbb{R}}) \quad \text{and} \quad \dim(P) + \dim(O(P)) = \dim(N_{\mathbb{R}}). \tag{9.1.1}$$

By a *variety* we mean a geometrically integral, separated, finite-type scheme over a field. In the following, X is always a closed subvariety of the torus T.

Definition 9.2. — *Let Σ be a pointed rational tropical fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let $\mathscr{X}$ be the closure of X in $\mathscr{Y}_\Sigma$. We say that $\mathscr{X}$ is a* tropical compactification *of X provided that $\mathscr{X}$ is proper over K° and the multiplication map*

$$\mu\colon \mathscr{T} \times_{K^\circ} \mathscr{X} \longrightarrow \mathscr{Y}_\Sigma$$

is faithfully flat. In this case we call Σ a tropical fan *for X.*

We refer the reader to [22, §12] for proofs of the following facts in this context. We write $|\Sigma_r|$ for the support of Σ_r in $N_{\mathbb{R}}$.

Theorem 9.3. — *Let Σ be a tropical fan for X.*

1. *If Σ' is a rational fan which subdivides Σ, then Σ' is a tropical fan for X.*
2. *The support $|\Sigma_1|$ is equal to* $\mathrm{Trop}(X)$.
3. *If $O \subset \mathscr{Y}_\Sigma$ is any torus orbit, then $\mathscr{X} \cap O$ is a non-empty pure dimensional scheme and* $\operatorname{codim}(\mathscr{X} \cap O, O) = \operatorname{codim}(X, T)$.

It follows from Theorem 9.3 (3) that $\mathscr{X} \setminus X$ is a closed subscheme of pure codimension one, which we regard as a reduced Weil divisor on $\mathscr{X}$.

Definition 9.4. — *Let $\mathscr{X} \subset \mathscr{Y}_\Sigma$ be a tropical compactification of a variety $X \subset T$. The* boundary divisor *of $\mathscr{X}$ is the Weil divisor $\mathscr{X} \setminus X$. The* horizontal part *of the boundary divisor is the closure of $\mathscr{X}_K \setminus X$ in $\mathscr{X}$, and the* vertical part *is the special fiber $\mathscr{X}_s$.*

In case of a tropical compactification as above, replacing K_0 be a finite extension, we may always assume that all vertices of Σ_1 are in N_{Γ_0}. Then the toric scheme $\mathscr{Y}_\Sigma$ is defined over K_0° (see [22, Proposition 7.11]). If $X \subset T$ is defined over $\overline{K}_0$ (that is, X is the extension of scalars of a subvariety of $\mathrm{Spec}(\overline{K}_0[M])$), then after passing to a larger finite extension, we may assume that X is defined over K_0, so $\mathscr{X}$ is defined over K_0°. This means that the tropical compactification $\mathscr{X} \subset \mathscr{Y}_\Sigma$ is obtained by base change from the corresponding toric compactification over K_0°. This will be used to quote results proved over discrete valuation rings.

We have the following relationship between torus orbits and initial degenerations. Let $P \in \Sigma_1$, let $\mathscr{T}_P \subset \mathscr{T}$ be the subtorus that acts trivially on $O(P) \subset \mathscr{Y}_\Sigma$, and let $\omega \in \operatorname{relint}(P) \cap N_\Gamma$. Then by [24, §3], there is a natural map $\operatorname{in}_\omega(X) \to \mathscr{X} \cap O(P)$, and an isomorphism

$$\operatorname{in}_\omega(X) \xrightarrow{\sim} \mathscr{T}_P \times (\mathscr{X} \cap O(P)). \tag{9.4.1}$$

In particular, $\operatorname{in}_\omega(X)$ is smooth if and only if $\mathscr{X} \cap O(P)$ is smooth.

9.5. Schön subvarieties

The following class of subvarieties of tori are sometimes called "tropically smooth".

Definition 9.6. — *Let $X \subset T$ be a subvariety. We say that X (or more precisely, the embedding $X \hookrightarrow T$) is* schön *provided that there exists a tropical compactification $\mathscr{X} \subset \mathscr{Y}_\Sigma$ of X such that the multiplication map $\mu\colon \mathscr{T} \times_{K^\circ} \mathscr{X} \to \mathscr{Y}_\Sigma$ is smooth.*

Note that if X is schön then it is smooth. The following result is due to Luxton and Qu [27, Proposition 7.6].

Lemma 9.7. — *Let $X \subset T$ be a schön subvariety defined over $\overline{K}_0$ and let $\mathscr{X} \subset \mathscr{Y}_\Sigma$ be any tropical compactification. Then $\mu\colon \mathscr{T} \times_{K^\circ} \mathscr{X} \to \mathscr{Y}_\Sigma$ is smooth.*

Proposition 9.8 (Helm–Katz). — *Let $X \subset T$ be a subvariety defined over $\overline{K}_0$. The following are equivalent:*

(1) *X is schön.*
(2) *$\operatorname{in}_\omega(X)$ is smooth for all $\omega \in \operatorname{Trop}(X)$.*
(3) *For any tropical compactification $\mathscr{X} \subset \mathscr{Y}_\Sigma$ and any polyhedron $P \in \Sigma_1$, the intersection $\mathscr{X} \cap O(P)$ is smooth.*

Proof. — See [24, Proposition 3.9]. □

De Jong defined in [25, §6] the notion of a strictly semistable pair $(\mathscr{X}, H)$ over a complete discrete valuation ring R. Roughly, this consists of a pair $(\mathscr{X}, H)$, where $\mathscr{X}$ is an irreducible, proper, flat, separated R-scheme, H is an effective "horizontal" Cartier divisor on $\mathscr{X}$, and $H + \mathscr{X}_s$ is a divisor with strict normal crossings. Note that de Jong denotes such a strictly semistable pair by $(\mathscr{X}, H + \mathscr{X}_s)$, but we will omit the vertical part in the notation.

Proposition 9.9 (Helm–Katz). — *Let $X \subset T$ be a schön subvariety defined over $\overline{K}_0$. Then there exists a tropical compactification $\mathscr{X} \subset \mathscr{Y}_\Sigma$ such that, letting H be the horizontal part of the boundary divisor of $\mathscr{X} \subset \mathscr{Y}_\Sigma$ (Definition 9.4), then H is Cartier and the pair $(\mathscr{X}, H)$ is the base change a strictly semistable pair defined over the valuation ring of a finite subextension of K/K_0.*

Proof. — Proposition 3.10 in [24] says that $\mathscr{X}$ is strictly semistable over K° in the sense of de Jong [25, §2]. For the stronger fact that $(\mathscr{X}, H)$ is a strictly semistable pair, one has to refer back to the proof of [24, Proposition 2.3]. □

Remark 9.10. — The desingularization process [24, Proposition 2.3] used in the proof of Proposition 9.9 is the primary reason for the assumption that $X \subset T$ is defined over $\overline{K}_0$. It involves subdividing $\mathrm{Trop}(X)$ into unimodular simplicial polyhedra, which is not possible in general when the value group is too large, e.g. if $\Gamma = \mathbb{R}$. On the other hand, the proofs of Lemma 9.7 and Proposition 9.8 can be extended to any valued field.

9.11. Skeletons and the parameterizing complex

Following Helm–Katz, the pair consisting of a schön subvariety $X \subset T$ and a tropical compactification $\mathscr{X} \subset \mathscr{Y}_\Sigma$ satisfying the conclusions of Proposition 9.9 is called a *normal crossings pair*. We associate to a normal crossings pair $(X, \mathscr{Y}_\Sigma)$ a piecewise linear set $\mathrm{HK}(X, \mathscr{Y}_\Sigma)$ (denoted $\Gamma_{(X,\mathbf{P})}$ in [24, §4]), defined as follows. For $P \in \Sigma_1$ we let $\mathscr{X}_P$ be the closure of $\mathscr{X} \cap O(P)$. The k-cells of $\mathrm{HK}(X, \mathscr{Y}_\Sigma)$ are pairs (P, C), where P is a k-dimensional polyhedron in Σ_1 and C is an irreducible component (equivalently, a connected component) of $\mathscr{X}_P$. The cells on the boundary of (P, C) are the cells of the form (P', C'), where P' is a facet of P and C' is the irreducible component of $\mathscr{X}_{P'}$ containing C; there is only one such component as $\mathscr{X}_{P'}$ is smooth. The piecewise linear set $\mathrm{HK}(X, \mathscr{Y}_\Sigma)$ is called the *Helm–Katz parameterizing complex*; it maps naturally to Σ_1 by sending (P, C) to P. See [24, §4] for details. Note that although Helm–Katz work over a discretely valued field, their construction only depends on the (geometric) special fiber of $\mathscr{X}$, so one may as well pass to the completion of the algebraic closure first. The complex $\mathrm{HK}(X, \mathscr{Y}_\Sigma)$ inherits an integral Γ-affine structure [23, §2] from $\mathrm{Trop}(X)$.

Let $(\mathscr{X}, H)$ be the strictly semistable pair associated to the normal crossings pair $(X, \mathscr{Y}_\Sigma)$ in Proposition 9.9. It follows from [23, 3.2] that $(\mathscr{X}, H)$

is a strictly semistable pair in the sense of [23, Definition 3.1]. Such a pair admits a canonical skeleton $S(\mathscr{X},H) \subset X^{\mathrm{an}}$ as constructed in [23, §4]. By [23, Proposition 4.10] there is a bijective, order-reversing correspondence between strata S in $\mathscr{X}_s$ and polyhedra Δ_S in the skeleton $S(\mathscr{X},H)$. These strata are precisely the connected components of the intersections with torus orbits $\mathscr{X} \cap O(P)$ for $P \in \Sigma_1$: indeed, the stratum closures of $\mathscr{X}_s$ are the intersections with stratum closures in $\mathscr{Y}_\Sigma$ because they are smooth of the correct dimension by Theorem 9.3 (3) and Proposition 9.8 (3). Hence the polyhedra Δ_S correspond to cells in $\mathrm{HK}(X,\mathscr{Y}_\Sigma)$. This correspondence respects the facet relation, so that $\mathrm{HK}(X,\mathscr{Y}_\Sigma)$ and $S(\mathscr{X},H)$ are identified as piecewise linear sets. It is not obvious that this identification respects the respective integral $\mathbb{Q}$-affine structures (see Lemma 9.15 below).

Theorem 9.12. — *Let $X \subset T$ be a schön subvariety defined over $\overline{K}_0$, and let $(X,\mathscr{Y}_\Sigma)$ be a normal crossings pair. Let $(\mathscr{X},H)$ be the associated strictly semistable pair (Proposition 9.9), with skeleton $S(\mathscr{X},H)$ as above. Then*

(1) *$S(\mathscr{X},H) = S_{\mathrm{Trop}}(X)$ as subsets of X^{an}, and*
(2) *there is a canonical isomorphism $S(\mathscr{X},H) \xrightarrow{\sim} \mathrm{HK}(X,\mathscr{Y}_\Sigma)$ of abstract integral $\mathbb{Q}$-affine piecewise linear sets, making the following triangle commute:*

$$\begin{tikzcd} S(\mathscr{X},H) \arrow[rr, "\sim"] \arrow[dr, "\mathrm{trop}"'] & & \mathrm{HK}(X,\mathscr{Y}_\Sigma) \arrow[dl] \\ & \mathrm{Trop}(X) & \end{tikzcd}$$

One consequence of Theorem 9.12 is that for schön X, one can construct the (Berkovich) skeleton $S(\mathscr{X},H)$ using only tropical data, namely, $\mathrm{Trop}(X)$ and the initial degenerations of X. This is a kind of "faithful tropicalization" result. As a consequence, any invariant of X that can be recovered from $S(\mathscr{X},H)$ can be calculated tropically. For example, the skeleton $S(\mathscr{X},H)$ is a strong deformation retraction of X^{an} by [23, §4.9], so we have the following Corollary.

Corollary 9.13. — *With the notation in Theorem 9.12, there is a canonical homotopy equivalence $X^{\mathrm{an}} \to \mathrm{HK}(X,\mathscr{Y}_\Sigma)$.*

Remark 9.14. — Suppose now that $X \subset T$ is a schön subvariety and that K_0 is a *local* field (i.e., $\widetilde{K}_0$ is finite). Assume that X arises as the base change of a subvariety $X_{\overline{K}_0}$ of $\mathrm{Spec}(\overline{K}_0[M])$. The main theorem of the paper of Helm–Katz [24, Theorem 6.1], which relates the weight-zero étale cohomology of $X_{\overline{K}_0}$ with the singular cohomology of the parameterizing complex of X, is now seen in the light of Theorem 9.12 to be a consequence[(3)] of a general result of Berkovich [7], relating the weight-zero étale cohomology of $X_{\overline{K}_0}$ with the singular cohomology of X^{an}. This answers a question posed in the introduction of [24].

Before beginning the proof of Theorem 9.12, we need the following Lemma, which roughly says that $S(\mathscr{X}, H) \to \mathrm{Trop}(X)$ is an "unramified covering" of integral $\mathbb{Q}$-affine piecewise linear sets.

Lemma 9.15. — *With the notation in Theorem 9.12, let $P \in \Sigma_1$, let S be a connected component of $\mathscr{X} \cap O(P)$, and let $\Delta_S \subset S(\mathscr{X}, H)$ be the corresponding cell. Then* trop *maps Δ_S bijectively onto P, and this map is unimodular in the sense of* [23, §2].

Proof. — First suppose that $X = T$ and $\mathscr{X} = \mathscr{Y}_\Sigma$. In this case, it is straightforward to check that the skeleton $S(\mathscr{X}, H)$ is canonically identified with Σ_1 (considered as a polyhedral subdivision of $N_\mathbb{R} = \mathrm{Trop}(T)$), and that the retraction to the skeleton $\tau\colon X^{\mathrm{an}} \to S(\mathscr{X}, H)$ is identified with $\mathrm{trop}\colon T^{\mathrm{an}} \to N_\mathbb{R}$. In particular, by [23, Proposition 4.10] or [22, Proposition 8.8], all points of $\mathrm{trop}^{-1}(\mathrm{relint}\, P)$ reduce to the stratum S under the reduction map $\mathrm{red}\colon X^{\mathrm{an}} \to \mathscr{X}_s$, and all points of X^{an} reducing to S are contained in $\mathrm{trop}^{-1}(\mathrm{relint}\, P)$.

No longer assuming $\mathscr{X} = \mathscr{Y}_\Sigma$, but restricting the previous sentence to $\mathscr{X} \subset \mathscr{Y}_\Sigma$, we see that a point $x \in X^{\mathrm{an}}$ reduces to $\mathscr{X} \cap O(P)$ if and only if $\mathrm{trop}(x) \in \mathrm{relint}\, P$. Applying [23, Proposition 4.10] to $(\mathscr{X}, H)$, this shows that $\mathrm{trop}^{-1}(\mathrm{relint}\, P) \cap S(\mathscr{X}, H)$ is the disjoint union of the interiors $\mathrm{relint}\, \Delta_{S'}$ for S' a component of $\mathscr{X} \cap O(P)$. In particular, trop maps $\mathrm{relint}\, \Delta_S$ (but not the boundary of Δ_S) into $\mathrm{relint}\, P$. Taking closures, we see that trop maps Δ_S into P. The map $\mathrm{trop}\colon \Delta_S \to P$ is integral $\mathbb{Q}$-affine by [23, Proposition 8.2]. This is enough to ensure trop is injective on Δ_S. The dimensions of S and P are complementary by Theorem 9.3 (3) and (9.1.1), so $\dim(\Delta_S) = \dim(P)$. Hence $\mathrm{trop}(\mathrm{relint}\, \Delta_S)$ is open in $\mathrm{relint}\, P$, and it is also closed since the boundary of Δ_S does not map into $\mathrm{relint}\, P$. It follows that trop maps $\mathrm{relint}\, \Delta_S$ (resp. Δ_S) bijectively onto $\mathrm{relint}\, P$ (resp. P).

Now we treat unimodularity. Since $\mathrm{Trop}(X)$ is pure dimensional, the same is true of $S(\mathscr{X}, H)$. Hence it suffices to consider P of maximal dimension. Let $S_1, \ldots, S_r$ be the connected components of $\mathscr{X} \cap O(P)$. Choose

(3) At least over local fields; Helm–Katz work over slightly more general K_0.

$\omega \in (\operatorname{relint} P) \cap N_\Gamma$. By the skeletal Sturmfels–Tevelev multiplicity formula [23, Theorem 8.4], we have

$$m_{\mathrm{Trop}}(\omega) = \sum_{i=1}^{r} [N_P : N_{\Delta_{S_i}}] \geqslant r\,, \tag{9.15.1}$$

where the terms $[N_P : N_{\Delta_{S_i}}]$ are the lattice indices introduced in [23, §2]. By (9.4.1), we have $\mathrm{in}_\omega(X) \cong \mathscr{T}_P \times \bigsqcup_{i=1}^r S_i$, so it is clear that $m_{\mathrm{Trop}}(\omega) = r$. It follows that $[N_P : N_{\Delta_{S_i}}] = 1$ for all i, which is to say that trop is unimodular on Δ_{S_i}. □

Proof of Theorem 9.12. — We have already argued that the cells and facet relations of $S(\mathscr{X}, H)$ are in natural bijection with those of $\mathrm{HK}(X, \mathscr{Y}_\Sigma)$. The integral $\mathbb{Q}$-affine structures of the cells of $\mathrm{HK}(X, \mathscr{Y}_\Sigma)$ are inherited from those of Σ_1, and Lemma 9.15 proves that the same is true for the cells of $S(\mathscr{X}, H)$. Moreover, the cell of $S(\mathscr{X}, H)$ corresponding to a cell (P, C) of $\mathrm{HK}(X, \mathscr{Y}_\Sigma)$ maps to P under trop by Lemma 9.15, so we have proved (2).

To prove (1), we will adapt the argument of [23, Proposition 10.8]. Let K' be an algebraically closed, complete valued field extension of K whose value group is all of $\mathbb{R}$, let $X' := X \otimes_K K'$ and let $\pi\colon X'^{\mathrm{an}} \to X^{\mathrm{an}}$ be the structural morphism. By Lemma 4.4, we have $\pi(S_{\mathrm{Trop}}(X')) = S_{\mathrm{Trop}}(X)$. Let $(\mathscr{X}', H')$ be the strictly semistable pair over K'° in the sense of [23, Definition 3.1] given by base change $(\mathscr{X}, H)$ to K'°. It is easy to see from the construction [23, §4] that π induces an isomorphism $S(\mathscr{X}', H') \cong S(\mathscr{X}, H)$ identifies faces and strata. Therefore to prove (1), it is enough to show that $S_{\mathrm{Trop}}(X') = S(\mathscr{X}', H')$.

Let $\omega \in \mathrm{Trop}(X)$. By hypothesis $\mathrm{in}_\omega(X') = \mathrm{in}_\omega(X) \otimes_{\widetilde{K}} \widetilde{K}'$ is smooth, so its irreducible components are connected components. As $\mathrm{in}_\omega(X')$ is the special fiber of the tropical formal model $\mathfrak{X}'_\omega$, it follows from [3, Proposition 3.18] that the canonical model of X'_ω coincides with $\mathfrak{X}'_\omega$, so the canonical reduction $\widetilde{X}'_\omega$ is equal to $\mathrm{in}_\omega(X')$. Let $C \subset \mathrm{in}_\omega(X')$ be a connected component. Then $X'_C := \mathrm{red}^{-1}(C)$ is a connected component of X'_ω by anti-continuity of red, and it has a unique Shilov boundary point ξ_C by Proposition 2.16. We claim that $X'_C \cap S(\mathscr{X}', H') = \{\xi_C\}$; this suffices to prove (1), as clearly $X'_C \cap S_{\mathrm{Trop}}(X') = \{\xi_C\}$, and X'^{an} is covered (set-theoretically) by the affinoid domains X'_C.

We introduce a partial ordering $\leqslant$ on X'^{an} by declaring that $x \leqslant y$ if $|f(x)| \leqslant |f(y)|$ for all $f \in K'[M]$, where M is the character lattice of $T' = T \otimes_K K'$. The tropicalization $\mathrm{trop}\colon X'^{\mathrm{an}} \to N_{\mathbb{R}}$ factors through the retraction to the skeleton $\tau\colon (X' \setminus H')^{\mathrm{an}} \to S(\mathscr{X}', H')$ by [23, Proposition 8.2], so $\tau(\xi_C) \in X'_\omega$. By [6, Theorem 5.2 (ii)] (as applied to a suitable affinoid neighborhood of ξ_C; see the proof of [23, Proposition 10.8]), we have $\xi_C \leqslant \tau(\xi_C)$. As ξ_C is by definition maximal with respect to $\leqslant$, this implies $\xi_C = \tau(\xi_C)$, so $\xi_C \in S(\mathscr{X}', H')$.

Let $P \in \Sigma_1$ be the polyhedron containing ω in its relative interior, and let $C_1, \ldots, C_r$ be the connected components of $\mathrm{in}_\omega(X')$. We have shown the inclusion $\{\xi_{C_1}, \ldots, \xi_{C_r}\} \subset S(\mathscr{X}', H') \cap \mathrm{trop}^{-1}(\omega)$. By (9.4.1), $C_1, \ldots, C_r$ correspond to the open strata $S_1, \ldots, S_r$ of $\mathscr{X}'$ lying on $\mathscr{X}' \cap O(P)$, and by Lemma 9.15, each cell $\Delta_{S_i} \subset S(\mathscr{X}', H')$ maps bijectively onto P under trop, with no other such cells mapping into $\mathrm{relint}(P)$. It follows that $S(\mathscr{X}', H') \cap \mathrm{trop}^{-1}(\omega)$ contains exactly r points, so $\{\xi_{C_1}, \ldots, \xi_{C_r}\} = S(\mathscr{X}', H') \cap \mathrm{trop}^{-1}(\omega)$. This completes the proof. □

Remark 9.16. — With the notation in Theorem 9.12, suppose in addition that all initial degenerations $\mathrm{in}_\omega(X)$ are irreducible, or equivalently, that $m_{\mathrm{Trop}}(\omega) = 1$ for all $\omega \in \mathrm{Trop}(X)$. Then Theorem 9.12 and its proof imply that $\mathrm{trop}\colon S_{\mathrm{Trop}}(X) = S(\mathscr{X}, H) \to \mathrm{Trop}(X)$ is an isomorphism of integral Γ-affine piecewise linear sets, and that the canonical section $\mathrm{Trop}(X) \to S_{\mathrm{Trop}}(X)$ of §8 is the inverse isomorphism. Compare [23, Proposition 10.8].

Appendix A. Summary of notations

A.1. Analytic spaces and formal schemes

$X_{K'}$	The extension of scalars of an object X/K to K'/K.	2.1, p. 1911
K	A non-Archimedean field.	2.2, p. 1911
v	$: K \to \mathbb{R} \cup \{\infty\}$, a complete non-Archimedean valuation.	2.2, p. 1911
$\lvert\cdot\rvert$	$= \exp(-v(\,\cdot\,))$, an associated absolute value.	2.2, p. 1911
K°	The valuation ring in K.	2.2, p. 1911
$K^{\circ\circ}$	The maximal ideal in K°.	2.2, p. 1911
$\widetilde{K}$	$= K^\circ/K^{\circ\circ}$, the residue field of K.	2.2, p. 1911
Γ	$= \Gamma_K = v(K^\times) \subset \mathbb{R}$, the value group of K.	2.2, p. 1911
$\sqrt{\Gamma}$	The saturation of Γ in $\mathbb{R}$.	2.2, p. 1911
$K\langle r^{-1}x\rangle$	The generalized Tate algebra; also for n variables.	2.2, p. 1911
$\mathscr{H}(x)$	The completed residue field at a point x of a K-analytic space.	2.3, p. 1911
$\mathscr{M}(\mathcal{A})$	The Berkovich spectrum of a K-affinoid algebra $\mathcal{A}$.	2.3, p. 1911
X^{an}	The analytification of a locally finite-type K-scheme X.	2.3, p. 1911

$\mathfrak{X}_s$	$= \mathfrak{X} \otimes_{K^\circ} \widetilde{K}$, the special fiber of a formal K°-scheme $\mathfrak{X}$.	2.6, p. 1913
$\mathfrak{X}_\eta$	The analytic generic fiber of an admissible formal K°-scheme $\mathfrak{X}$.	2.6, p. 1913
$\mathcal{A}^\circ$	Ring of power-bounded elements in a strictly K-affinoid algebra $\mathcal{A}$.	2.8, p. 1913
$\mathcal{A}^{\circ\circ}$	The ideal of topologically nilpotent elements in $\mathcal{A}^\circ$.	2.8, p. 1913
$\widetilde{\mathcal{A}}$	$= \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$, a $\widetilde{K}$-algebra of finite type.	2.8, p. 1914
$\lvert\cdot\rvert_{\sup}$	The supremum (or spectral) semi-norm on a K-affinoid algebra $\mathcal{A}$.	2.8, p. 1914
$\mathfrak{X}^{\mathrm{can}}$	$= \mathrm{Spf}(\mathcal{A}^\circ)$, the canonical model of $X = \mathscr{M}(\mathcal{A})$.	2.9, p. 1914
$\widetilde{X}$	$= \mathrm{Spec}(\widetilde{\mathcal{A}})$, the canonical reduction of $X = \mathscr{M}(\mathcal{A})$.	2.9, p. 1914
$B(X)$	The Shilov boundary of a K-affinoid space X.	2.13, p. 1916
red	$: X \to \widetilde{X}$, the reduction map to the canonical reduction.	(2.15.1)
red	$: \mathfrak{X}_\eta \to \mathfrak{X}_s$, reduction map of an admissible formal K°-scheme $\mathfrak{X}$.	(2.16.1)

A.2. Toric varieties and tropicalizations

M	$\cong \mathbb{Z}^n$, a finitely generated free abelian group.	3.1, p. 1920
N	$= \mathrm{Hom}(M, \mathbb{Z})$, its dual.	3.1, p. 1920
M_G	$= M \otimes_{\mathbb{Z}} G \subset M_{\mathbb{R}}$ for $G \subset \mathbb{R}$ an additive subgroup.	3.1, p. 1920
N_G	Likewise.	3.1, p. 1920
$\langle\cdot,\cdot\rangle$	$: M_G \times N_G \to \mathbb{R}$, the evaluation pairing.	3.1, p. 1920
$R[S]$	The monoid ring over a ring R of a monoid S.	3.1, p. 1920
χ^u	$\in R[S]$, the character corresponding to $u \in S$.	3.1, p. 1920
T	$= \mathrm{Spec}(K[M]) \cong \mathbf{G}_{m,K}^n$, a split K-torus.	3.1, p. 1920
S_σ	$= \sigma^\vee \cap M$, the monoid associated to a cone $\sigma \subset N_{\mathbb{R}}$.	3.1, p. 1920
Y_σ	$= \mathrm{Spec}(K[S_\sigma])$, the affine toric variety from a rational cone $\sigma \subset N_{\mathbb{R}}$.	3.1, p. 1920
Y_Δ	The toric variety associated to a rational pointed fan Δ in $N_{\mathbb{R}}$.	3.1, p. 1920
$M_G(\sigma)$	$= (\sigma^\perp \cap M) \otimes_{\mathbb{Z}} G$ for $\sigma \subset N_{\mathbb{R}}$ a rational cone and $G \subset \mathbb{R}$.	3.1, p. 1920
$N_G(\sigma)$	$= (N/\langle\sigma\rangle \cap N) \otimes_{\mathbb{Z}} G$.	3.1, p. 1920
π_σ	$: N_{\mathbb{R}} \to N_{\mathbb{R}}(\sigma)$, the projection.	3.1, p. 1920
$O(\sigma)$	$= \mathrm{Spec}(K[M(\sigma)])$, the torus orbit in Y_Δ coming from $\sigma \in \Delta$.	3.1, p. 1920
$\overline{\mathbb{R}}$	$= \mathbb{R} \cup \{\infty\}$, an additive monoid.	3.2, p. 1920
$\overline{N}_G^\sigma$	$= \coprod_{\tau \prec \sigma} N_G(\tau)$.	3.2, p. 1920
$\overline{N}_G^\Delta$	$= \coprod_{\sigma \in \Delta} N_G(\sigma)$, the G-points in a partial compactification of $N_{\mathbb{R}}$.	3.2, p. 1921
trop	$: Y_\Delta^{\mathrm{an}} \to \overline{N}_{\mathbb{R}}^\Delta$ or $Y_\sigma^{\mathrm{an}} \to \overline{N}_{\mathbb{R}}^\sigma$, the tropicalization map.	3.3, p. 1921

s	$: \overline{N}_{\mathbb{R}}^{\Delta} \to Y_{\Delta}^{\text{an}}$ or $\overline{N}_{\mathbb{R}}^{\sigma} \to Y_{\sigma}^{\text{an}}$, the section of tropicalization.	3.3, p. 1921
$\lvert\cdot\rvert_{\omega}$	$= s(\omega)$ for $\omega \in \overline{N}_{\mathbb{R}}^{\sigma}$.	3.3, p. 1921
$S(Y_{\sigma}^{\text{an}})$	$= s(\overline{N}_{\mathbb{R}}^{\sigma})$, the skeleton of Y_{σ}^{an}.	3.3, p. 1921
$S(Y_{\Delta}^{\text{an}})$	$= s(\overline{N}_{\mathbb{R}}^{\Delta})$, the skeleton of Y_{Δ}^{an}.	3.3, p. 1921
$\text{Trop}(X)$	$\subset \overline{N}_{\mathbb{R}}^{\Delta}$ (resp. $\overline{N}_{\mathbb{R}}^{\sigma}$), the tropicalization of $X \subset Y_{\Delta}$ (resp. Y_{σ}).	3.3, p. 1922
U_{ω}	$= \text{trop}^{-1}(\omega) \subset Y_{\Delta}^{\text{an}}$ for $\omega \in \overline{N}_{\mathbb{R}}^{\Delta}$.	3.4, p. 1922
$\mathfrak{U}_{\omega}$	$= \text{Spf}(K\langle U_{\omega}\rangle^{\circ})$, the canonical model of U_{ω} for $\omega \in \overline{N}_{\Gamma}^{\Delta}$.	3.4, p. 1922
X_{ω}	$= U_{\omega} \cap X^{\text{an}}$ for $X \subset Y_{\Delta}$ a closed subscheme.	3.5, p. 1923
$\mathfrak{X}_{\omega}$	The tropical formal model of X_{ω}, a closed formal subscheme of $\mathfrak{U}_{\omega}$.	3.5, p. 1923
$\text{in}_{\omega}(X)$	$= (\mathfrak{X}_{\omega})_s$, the initial degeneration of X at ω.	3.5, p. 1923
m_Z	The multiplicity of an irreducible component Z of $\text{in}_{\omega}(X)$.	3.5, p. 1923
$m_{\text{Trop}}(\omega)$	The tropical multiplicity of X at ω.	3.5, p. 1923
$\text{LC}_{\omega}(\Pi)$	The local cone at ω of a polyhedral complex Π in $N_{\mathbb{R}}$.	5.1, p. 1928
$\rho(P)$	The recession cone of a polyhedron P.	(1), p. 1929
s_X	$: \text{Trop}(X)_{m_{\text{Trop}}=1} \to S_{\text{Trop}}(X)$, the section of tropicalization.	8.4, p. 1944

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Manuscrit reçu le 15 septembre 2015,
révisé le 11 janvier 2017,
accepté le 24 janvier 2017.

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