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Javier CILLERUELO[†] & Jean-Marc DESHOUILLERS

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GAPS IN SUMSETS OF s PSEUDO s -TH POWERS

by Javier CILLERUELO[†] & Jean-Marc DESHOUILERS (*)

ABSTRACT. — We study the length of the gaps between consecutive members in the sumset sA when A is a pseudo s -th power sequence, with $s \geq 2$. We show that, almost surely, $\limsup(b_{n+1} - b_n)/\log b_n = s^s s!/\Gamma^s(1/s)$, where b_n are the elements of sA .

RÉSUMÉ. — On étudie la taille des différences entre les termes consécutifs de la suite sA où A est une suite de pseudo-puissances s -ièmes avec $s \geq 2$. On montre qu'on a presque sûrement $\limsup(b_{n+1} - b_n)/\log b_n = s^s s!/\Gamma^s(1/s)$, où les b_n sont les éléments de la suite sA .

1. Introduction

Erdős and Rényi [3] proposed in 1960 a probabilistic model for sequences A growing like the s -th powers: they build a probability space $(\mathcal{U}, \mathcal{T}, P)$ and a sequence of independent random variables $(\xi_n)_{n \in \mathbb{N}}$ with values in $\{0, 1\}$ and $P(\xi_n = 1) = \frac{1}{s}n^{-1+1/s}$; to any $u \in \mathcal{U}$, they associate the sequence of positive integers $A = A_u$ such that $n \in A_u$ if and only if $\xi_n(u) = 1$. In short, the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s}n^{-1+1/s}$. The counting function of these random sequences A satisfies almost surely the asymptotic relation $|A \cap [1, x]| \sim x^{1/s}$, whence the terminology *pseudo s -th powers*. Erdős and Rényi studied the random variable $r_s(A, n)$ which counts the number of representations of n in the form $n = a_1 + \dots + a_s$,

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$a_1 \leq \dots \leq a_s$, $a_i \in A$. For the simplest case $s = 2$ they proved that $r_2(A, n)$ converges to a Poisson distribution with parameter $\pi/8$, when $n \rightarrow \infty$. They also claimed the analogous result for $s > 2$ but their analysis did not take into account the dependence of some events. J. H. Goguel [4] proved indeed that for each integer d , the sequence of the integers n such that $r_s(A, n) = d$ has almost surely the density $\lambda_s^d e^{-\lambda_s} / d!$, where $\lambda_s = \Gamma^s(1/s) / (s^s s!)$. B. Landreau [5] gave a proof of this result based on correlation inequalities and also showed that the sequence of random variables $(r_s(A, n))_n$ converges in law towards the Poisson distribution with parameter λ_s .

In particular, both the sets of the integers belonging, or not belonging, to $sA = \{a_1 + \dots + a_s : a_i \in A\}$ have almost surely a positive density and it makes sense to study the length of the gaps in sA . The aim of the paper is to obtain a precise estimate for the maximal length of such gaps.

THEOREM 1.1. — *For any $s \geq 2$ the sequence $sA = (b_n)_n$, sum of s copies of a pseudo s -th power sequence A , satisfies almost surely*

$$(1.1) \quad \limsup_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{\log b_n} = \frac{s^s s!}{\Gamma^s(1/s)}.$$

We remark that this result is heuristically consistent with the fact that for a random sequence S with $P(n \in S) = 1 - e^{-\lambda}$, we have $\limsup (s_{m+1} - s_m) / \log s_m = 1/\lambda$, an exercise on Borel–Cantelli Lemma.

2. Notation, hint of the proof and general lemmas

2.1. Notation

We retain the notation of the introduction, for the probability space $(\mathcal{U}, \mathcal{T}, P)$ and the definition of the random sequences $A = A_u$, where the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s} n^{-1+1/s}$. We further use the following notation.

- (1) We write ω to denote a set of distinct positive integers and we denote by E_ω and E_ω^c the events

$$E_\omega = \{\omega \subset A\} \quad \text{and} \quad E_\omega^c = \{\omega \not\subset A\}$$

respectively. We write $\omega \sim \omega'$ to mean that $\omega \cap \omega' \neq \emptyset$ but $\omega \neq \omega'$; we remark that $\omega \sim \omega'$ if and only if the events E_ω and $E_{\omega'}$ are distinct and dependent.

If $\omega = \{x_1, \dots, x_r\}$ we write

$$\sigma(\omega) = \{a_1x_1 + \dots + a_rx_r : a_1 + \dots + a_r = s, a_i \geq 1\}$$

for the set of all integers which can be written as a sum of s integers using all the integers x_1, \dots, x_r . For an integer z we let

$$\Omega_z = \{\omega : z \in \sigma(\omega)\}.$$

- (2) Given $\alpha > 0$, we denote by I_i the interval $[i, i + \alpha \log i]$ and we denote by $F_i^{(\alpha)}$, or simply F_i when the context is clear, the event

$$F_i = F_i^{(\alpha)} = \{sA \cap I_i = \emptyset\}.$$

We denote by Ω_{I_i} the family of sets

$$\Omega_{I_i} = \{\omega : \sigma(\omega) \cap I_i \neq \emptyset\}.$$

- (3) We let $\lambda_s = \frac{\Gamma^s(1/s)}{s!s^s}$.

- (4) We use Vinogradov's notation \ll , where $f \ll g$ is equivalent to Landau's notation $f = O(g)$.

2.2. Hints for the proof to Theorem 1.1

We wish to prove that for $\alpha > \lambda_s^{-1}$, the event $F_i^{(\alpha)}$, defined in Section 2.1, occurs — almost surely — for only finitely many i 's and that for $\alpha < \lambda_s^{-1}$ it occurs — almost surely — for infinitely many i 's. There is a flavour of Borel–Cantelli and indeed a key point in the proof is Lemma 3.5 which asserts relation (3.2), namely

$$(2.1) \quad P(F_i^{(\alpha)}) = i^{-\alpha\lambda_s + o(1)}.$$

Let us first see how we can obtain that relation. Here α is fixed and we do not mention it anylonger. By the definition, the event F_i occurs if and only if for any family of s non necessarily distinct integers which sum up to an integer in I_i , at least one of them is not in A ; with our notation, this leads to

$$F_i = \bigcap_{\omega \in \Omega_{I_i}} E_\omega^c.$$

If the ω 's which are involved had pairwise empty intersections, the events E_ω^c would be independent and we would have

$$P(F_i) = \prod_{\omega \in \Omega_{I_i}} P(E_\omega^c).$$

Although this is not the case, the structure of the events E_ω , which are finite intersections of events taken from an independent family, permits us to use Harris' inequality (or FKG inequality, cf. Theorem 2.2 below) to get the lower bound

$$P(F_i) \geq \prod_{\omega \in \Omega_{I_i}} P(E_\omega^c).$$

It also permits us, thanks to Janson's Correlation Inequality (cf. Theorem 2.2 below), to get the upper bound

$$P(F_i) \leq \prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) \times \exp \left(2 \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \right),$$

where the notation $\omega \sim \omega'$ is defined in Section 2.1. It is then a matter of computation, based on Lemma 2.3, to get the central inequality (2.1).

When $\alpha > \lambda_s^{-1}$ the series $\sum_i P(F_i^{(\alpha)})$ converges and the Borel–Cantelli lemma immediately implies that for such α the events $F_i^{(\alpha)}$ almost surely occur for only finitely many i 's.

When $\alpha < \lambda_s^{-1}$ the series $\sum_i P(F_i)$ diverges, but this is not enough to conclude directly since the events F_i 's are not independent. However, P. Erdős and A. Rényi proved that a weak dependence among the F_i 's is sufficient for obtaining an “inverse Borel–Cantelli” result (cf. Theorem 2.1 below). It is thus important to have a small upper bound for $P(F_i \cap F_j) - P(F_i)P(F_j)$ in average. With our notation, we have

$$F_i \cap F_j = \bigcap_{\omega \in I_i \cup I_j} E_\omega^c,$$

and here again Janson's inequality will help us to obtain a suitable bound.

2.3. Probabilistic results

We use the following generalization of the Borel–Cantelli Lemma, proved indeed by P. Erdős and A. Rényi in 1959 [2].

THEOREM 2.1 (Borel–Cantelli Lemma). — *Let $(F_i)_{i \in \mathbb{N}}$ be a sequence of events and let $Z_n = \sum_{i \leq n} P(F_i)$.*

If the sequence $(Z_n)_n$ is bounded, then, with probability 1, only finitely many of the events F_i occur.

If the sequence $(Z_n)_n$ tends to infinity and

$$\lim_{n \rightarrow \infty} \frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,$$

then, with probability 1, infinitely many of the events F_i occur.

The next result, which combines Harris' inequality and Janson's Correlation Inequality, can be found in [1].

THEOREM 2.2. — *Let $(E_\omega)_{\omega \in \Omega}$ be a finite collection of events which are intersections of elementary independent events and assume that $P(E_\omega) \leq 1/2$ for any $\omega \in \Omega$. Then*

$$\prod_{\omega \in \Omega} P(E_\omega^c) \leq P\left(\bigcap_{\omega \in \Omega} E_\omega^c\right) \leq \prod_{\omega \in \Omega} P(E_\omega^c) \times \exp\left(2 \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'})\right),$$

where $\omega \sim \omega'$ means that the events E_ω and $E_{\omega'}$ are dependent events.

2.4. A technical lemma

LEMMA 2.3. — *Given $1 \leq t \leq s - 1$ and positive integers a_1, \dots, a_t we have, as z tends to infinity:*

$$(1) \quad \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} \ll z^{-1+t/s}.$$

$$(2) \quad \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t < z}} (x_1 \cdots x_t)^{-1+1/s} (z - (a_1 x_1 + \dots + a_t x_t))^{-2t/s} \ll z^{-t/s} \log z.$$

$$(3) \quad \sum_{\substack{1 \leq x_1 < \dots < x_s \\ x_1 + \dots + x_s = z}} (x_1 \cdots x_s)^{-1+1/s} \sim s^s \lambda_s.$$

Proof. — (1) We have

$$\begin{aligned} & \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} \\ &= (a_1 \cdots a_t)^{1-1/s} \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = z}} (a_1 x_1 \cdots a_t x_t)^{-1+1/s} \\ &\leq (a_1 \cdots a_t)^{1-1/s} \sum_{\substack{1 \leq y_1, \dots, y_t \\ y_1 + \dots + y_t = z}} (y_1 \cdots y_t)^{-1+1/s}. \end{aligned}$$

If $y_1 + \dots + y_t = z$ then at least one of them, say y_t , is greater than z/t and is determined by y_1, \dots, y_{t-1} . Thus,

$$\begin{aligned} \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} &\ll z^{-1+1/s} \sum_{1 \leq y_1, \dots, y_{t-1} < z} (y_1 \cdots y_{t-1})^{-1+1/s} \\ &\ll z^{-1+1/s} \left(\sum_{1 \leq y < z} y^{-1+1/s} \right)^{t-1} \\ &\ll z^{-1+1/s} (z^{1/s})^{t-1} \\ &\ll z^{-1+t/s}. \end{aligned}$$

(2) We have

$$\begin{aligned} &\sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t < z}} (x_1 \cdots x_t)^{-1+1/s} (z - (a_1 x_1 + \dots + a_t x_t))^{-2t/s} \\ &= \sum_{1 \leq m < z} (z - m)^{-2t/s} \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = m}} (x_1 \cdots x_t)^{-1+1/s} \\ \text{(by (1)) } &\ll \sum_{1 \leq m < z} (z - m)^{-2t/s} m^{-1+t/s} \\ &\ll \sum_{1 \leq m \leq z/2} (z - m)^{-2t/s} m^{-1+t/s} + \sum_{z/2 < m < z} (z - m)^{-2t/s} m^{-1+t/s} \\ &\ll z^{-2t/s} z^{t/s} + z^{-1+t/s} \sum_{z/2 < m < z} (z - m)^{-2t/s} \\ &\ll z^{-t/s} + z^{-1+t/s} \left(z^{1-2t/s} \log z \right) \\ &\ll z^{-t/s} \log z. \end{aligned}$$

Remark 2.4. — Except in the case when $s = 2$ and $t = 1$, the upper bound in (2) may be replaced by $z^{-t/s}$.

(3) It follows from Lemma 3 of [5]. □

3. Proof of Theorem 1.1

3.1. Combinatorial lemmas

LEMMA 3.1. — We have

$$\sum_{\omega \in \Omega_z} P(E_\omega) \sim \lambda_s$$

as $z \rightarrow \infty$.

Proof. — We have

$$(3.1) \quad \sum_{\omega \in \Omega_z} P(E_\omega) = \sum_{\substack{\omega \in \Omega_z \\ |\omega|=s}} P(E_\omega) + \sum_{\substack{\omega \in \Omega_z \\ |\omega| \leq s-1}} P(E_\omega).$$

The main contribution comes from the first sum.

$$\sum_{\substack{\omega \in \Omega_z \\ |\omega|=s}} P(E_\omega) = \frac{1}{s^s} \sum_{\substack{1 \leq x_1 < \dots < x_s \\ x_1 + \dots + x_s = z}} (x_1 \dots x_s)^{-1+1/s} \sim \lambda_s$$

as $z \rightarrow \infty$, by Lemma 2.3(3). For the second sum we have

$$\begin{aligned} \sum_{\substack{\omega \in \Omega_z \\ |\omega| \leq s-1}} P(E_\omega) &\leq \sum_{1 \leq r \leq s-1} \sum_{\substack{a_1, \dots, a_r \\ a_1 + \dots + a_r = s}} \sum_{\substack{1 \leq x_1, \dots, x_r \\ a_1 x_1 + \dots + a_r x_r = z}} (x_1 \dots x_r)^{-1+1/s} \\ (\text{Lem. 2.3(1)}) &\ll \sum_{1 \leq r \leq s-1} z^{\frac{r}{s}-1} \ll z^{-1/s}. \quad \square \end{aligned}$$

LEMMA 3.2. — For any $z \leq z'$ we have

$$\sum_{\substack{\omega \sim \omega' \\ \omega \in \Omega_z, \omega' \in \Omega_{z'}}} P(E_\omega \cap E_{\omega'}) \ll z^{-1/s} \log z.$$

Proof. — If $\omega \in \Omega_z$ then there exist some $r \leq s$ and some positive integers a_1, \dots, a_r with $a_1 + \dots + a_r = s$ such that $a_1 x_1 + \dots + a_r x_r = z$. Thus, any pair of sets $\omega \sim \omega'$ with $\omega \in \Omega_z, \omega' \in \Omega_{z'}, z \leq z'$ is of the form

$$\begin{aligned} \omega &= \{x_1, \dots, x_t, u_{t+1}, \dots, u_r\} \\ \omega' &= \{x_1, \dots, x_t, v_{t+1}, \dots, v_{r'}\} \end{aligned}$$

with $1 \leq t \leq r \leq r' \leq s$ and positive integers a_1, \dots, a_r and $b_1, \dots, b_{r'}$ with

$$\begin{aligned} a_1 x_1 + \dots + a_t x_t + a_{t+1} u_{t+1} + \dots + a_r u_r &= z \\ b_1 x_1 + \dots + b_t x_t + b_{t+1} v_{t+1} + \dots + b_{r'} v_{r'} &= z'. \end{aligned}$$

Of course if $r = t$ then $\omega = \{x_1, \dots, x_r\}$ and $r' \geq t + 1$. Otherwise $\omega = \omega'$. Similarly, when $r' = t$, we have $r \geq t + 1$.

Given positive integers $z, z', t, r, r', a_1, \dots, a_r, b_1, \dots, b_{r'}$ we estimate the sum

$$\sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'})$$

where the symbol \sum^* means that the sum is extended to the pairs $\omega \sim \omega'$ satisfying the above conditions. We distinguish several cases according to the values of r and r' .

Case $r \geq t + 1$ and $r' \geq t + 1$. — We have

$$\begin{aligned} \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) &\leq \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t < z \\ b_1 x_1 + \dots + b_t x_t < z'}} (x_1 \cdots x_t)^{-1+1/s} \\ &\quad \times \sum_{\substack{1 \leq u_{t+1}, \dots, u_r, \\ a_{t+1} u_{t+1} + \dots + a_r u_r \\ = z - (a_1 x_1 + \dots + a_t x_t)}} (u_{t+1} \cdots u_r)^{-1+1/s} \\ &\quad \times \sum_{\substack{1 \leq v_{t+1}, \dots, v_{r'}, \\ b_{t+1} v_{t+1} + \dots + b_{r'} v_{r'} \\ = z' - (b_1 x_1 + \dots + b_t x_t)}} (v_{t+1} \cdots v_{r'})^{-1+1/s}. \end{aligned}$$

By Lemma 2.3(1) we have

$$\begin{aligned} \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) &\ll \sum_{\substack{x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t < z \\ b_1 x_1 + \dots + b_t x_t < z'}} (x_1 \cdots x_t)^{-1+1/s} \\ &\quad \times \left(z - (a_1 x_1 + \dots + a_t x_t) \right)^{\frac{r-t}{s} - 1} \\ &\quad \times \left(z' - (b_1 x_1 + \dots + b_t x_t) \right)^{\frac{r'-t}{s} - 1}. \end{aligned}$$

Using the inequality $AB \leq A^2 + B^2$, we get

$$\begin{aligned} \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) &\leq \sum_{\substack{x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t < z}} (x_1 \cdots x_t)^{-1+1/s} \left(z - (a_1 x_1 + \dots + a_t x_t) \right)^{2(r-t-s)/s} \\ &\quad + \sum_{\substack{x_1, \dots, x_t \\ b_1 x_1 + \dots + b_t x_t < z'}} (x_1 \cdots x_t)^{-1+1/s} \left(z' - (b_1 x_1 + \dots + b_t x_t) \right)^{2(r'-t-s)/s} \end{aligned}$$

(Lem. 2.3(2))

$$\ll z^{-t/s} \log z \ll z^{-1/s} \log z.$$

Case $r = t$ and $r' \geq t + 1$. — In this case we have

$$\begin{aligned} & \sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) \\ & \leq \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = z \\ b_1 x_1 + \dots + b_t x_t < z'}} (x_1 \cdots x_t)^{-1+1/s} \\ & \quad \times \sum_{\substack{1 \leq v_{t+1}, \dots, v_{r'} \\ b_{t+1} v_{t+1} + \dots + b_{r'} v_{r'} \\ = z' - (b_1 x_1 + \dots + b_t x_t)}} (v_{t+1} \cdots v_{r'})^{-1+1/s} \\ (\text{Lem. 2.3(1)}) & \leq \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = z \\ b_1 x_1 + \dots + b_t x_t < z'}} (x_1 \cdots x_t)^{-1+1/s} \left(z' - (b_1 x_1 + \dots + b_t x_t) \right)^{\frac{r'-t}{s}-1} \\ & \leq \sum_{\substack{1 \leq x_1, \dots, x_t \\ a_1 x_1 + \dots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} \\ & \ll z^{\frac{t}{s}-1} \ll z^{-1/s}. \end{aligned}$$

Case $r' = t$ and $r \geq t + 1$. — This case is similar to the previous one. \square

LEMMA 3.3. — Let $\alpha > 0$ and let I_i be the interval $[i, i + \alpha \log i]$. For any $i \leq j$ we have

$$\sum_{\substack{\omega \sim \omega' \\ \omega \in \Omega_{I_i}, \omega' \in \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s} (\log i)^2 (\log j).$$

Proof. — We have

$$\begin{aligned} & \sum_{\substack{\omega \sim \omega' \\ \omega \in \Omega_{I_i}, \omega' \in \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) \leq \sum_{z \in I_i, z' \in I_j} \sum_{\substack{\omega \sim \omega' \\ \omega \in \Omega_z, \omega' \in \Omega_{z'}}} P(E_\omega \cap E_{\omega'}) \\ (\text{Lem. 3.2}) & \ll \sum_{z \in I_i, z' \in I_j} z^{-1/s} \log z \\ & \ll (\log i)^2 (\log j) i^{-1/s}. \quad \square \end{aligned}$$

LEMMA 3.4. — We have

$$\prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) = i^{-\alpha \lambda_s + o(1)}.$$

Proof. — We observe that

$$\prod_{z \in I_i} \prod_{\omega \in \Omega_z} P(E_\omega^c) \leq \prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) \leq \prod_{\substack{\omega \in \Omega_{I_i} \\ |\omega|=s}} P(E_\omega^c) = \prod_{z \in I_i} \prod_{\substack{\omega \in \Omega_z \\ |\omega|=s}} P(E_\omega^c).$$

Writing $P(E_\omega^c) = 1 - P(E_\omega)$ and taking logarithms we have

$$\begin{aligned} \log \left(\prod_{z \in I_i} \prod_{\omega \in \Omega_z} P(E_\omega^c) \right) &= \sum_{z \in I_i} \sum_{\omega \in \Omega_z} \log(1 - P(E_\omega)) \sim - \sum_{z \in I_i} \sum_{\omega \in \Omega_z} P(E_\omega) \\ &\quad (\text{Lem. 3.1}) \sim - \sum_{z \in I_i} \lambda_s \sim -\alpha \lambda_s \log i. \end{aligned}$$

On the other hand,

$$\begin{aligned} \log \left(\prod_{z \in I_i} \prod_{\substack{\omega \in \Omega_z \\ |\omega|=s}} P(E_\omega^c) \right) &= \sum_{z \in I_i} \sum_{\substack{\omega \in \Omega_z \\ |\omega|=s}} \log(1 - P(E_\omega)) \sim - \sum_{z \in I_i} \sum_{\substack{\omega \in \Omega_z \\ |\omega|=s}} P(E_\omega) \\ &= - \sum_{z \in I_i} \sum_{\substack{x_1 < \dots < x_s \\ x_1 + \dots + x_s = z}} \frac{1}{s^s} (x_1 \cdots x_s)^{-1+1/s} \\ &\quad (\text{Lem. 2.3(3)}) \sim -\lambda_s \alpha \log i. \end{aligned} \quad \square$$

LEMMA 3.5. — We have

$$(3.2) \quad P(F_i) = i^{-\alpha \lambda_s + o(1)}.$$

Proof. — As we noticed it in Section 2.2, we have

$$F_i = \bigcap_{\omega \in \Omega_{I_i}} E_\omega^c.$$

Since $P(E_\omega) \leq 1/2$ for any ω , Theorem 2.2 applies and we have

$$\prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) \leq P(F_i) \leq \prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) \times \exp \left(2 \sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i}}} P(E_\omega \cap E_{\omega'}) \right).$$

After Lemma 3.4 we only need to prove

$$\sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i}}} P(E_\omega \cap E_{\omega'}) = o(1).$$

But it is a consequence of Lemma 3.3 with $j = i$.

$$\sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i}}} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s + o(1)}. \quad \square$$

LEMMA 3.6. — *If $i < j$ and $I_i \cap I_j = \emptyset$ then*

$$\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) \leq P(F_i)P(F_j)(1 + O(j^{-1/s} \log j)).$$

Proof. — It is clear that

$$\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) = \left(\prod_{\omega \in \Omega_{I_i}} P(E_\omega^c) \right) \left(\prod_{\omega \in \Omega_{I_j}} P(E_\omega^c) \right) \left(\prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega^c) \right)^{-1}.$$

Harris' inequality, applied to the first two products, gives

$$\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) \leq P(F_i)P(F_j) \left(\prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega^c) \right)^{-1}.$$

The logarithm of the last factor is

$$- \sum_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} \log(1 - P(E_\omega)) \sim \sum_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega)$$

Since $I_i \cap I_j = \emptyset$, if $\omega \in \Omega_{I_i} \cap \Omega_{I_j}$ then $|\omega| \leq s - 1$. Thus,

$$\begin{aligned} \sum_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega) &\leq \sum_{\substack{\omega \in \Omega_{I_j} \\ |\omega| \leq s-1}} P(E_\omega) \\ &\leq \sum_{z \in I_j} \sum_{1 \leq r \leq s-1} \sum_{a_1 + \dots + a_r = s} \sum_{\substack{1 \leq x_1 < \dots < x_r \\ a_1 x_1 + \dots + a_r x_r = z}} (x_1 \cdots x_r)^{-1+1/s} \end{aligned}$$

(Lem. 2.3(1)) $\ll j^{-1/s} \log j$.

Thus,

$$\left(\prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E_\omega^c) \right)^{-1} \leq 1 + O(j^{-1/s} \log j)$$

which ends the proof of the Lemma. □

3.2. End of the proof

After those Lemmas we are ready to finish the proof of Theorem 1.1.

If $\alpha > 1/\lambda_s$ then

$$\sum_i P(F_i) = \sum_i i^{-\alpha\lambda_s + o(1)} < \infty$$

and Theorem 2.1 implies that with probability 1 only finitely many events F_i occur. This proves that

$$\limsup_{k \rightarrow \infty} \frac{b_{k+1} - b_k}{\log b_k} \leq 1/\lambda_s.$$

If $\alpha < 1/\lambda_s$ then

$$Z_n = \sum_{i \leq n} P(F_i) = \sum_{i \leq n} i^{-\alpha\lambda_s + o(1)} = n^{1-\alpha\lambda_s + o(1)} \rightarrow \infty.$$

If in addition

$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,$$

Theorem 2.1 implies that with probability 1 infinitely many events F_i occur and

$$\limsup_{k \rightarrow \infty} \frac{b_{k+1} - b_k}{\log b_k} \geq 1/\lambda_s.$$

Note that $P(F_i \cap F_j) \geq P(F_i)P(F_j)$ for all $1 \leq i < j \leq n$, so the limit (3.3) is not negative.

We next prove (3.3). We observe that

$$F_i \cap F_j = \bigcap_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} E_\omega^c,$$

so we can use Janson inequality to get

$$P(F_i \cap F_j) \leq \prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) \times \exp \left(2 \sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i} \cup \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) \right).$$

Observe that

$$\begin{aligned} \sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i} \cup \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) &\leq \sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i}}} P(E_\omega \cap E_{\omega'}) + \sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) \\ &\quad + \sum_{\substack{\omega \sim \omega' \\ \omega \in \Omega_{I_i}, \omega' \in \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}). \end{aligned}$$

Applying Lemma 3.3 to the three sums we have

$$\sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i} \cup \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s} (\log i)^3 + j^{-1/s} (\log j)^3 + i^{-1/s} (\log i)^2 (\log j),$$

and so

$$(3.4) \quad \exp \left(2 \sum_{\substack{\omega \sim \omega' \\ \omega, \omega' \in \Omega_{I_i} \cup \Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) \right) \leq 1 + O \left(i^{-1/s} (\log i)^2 (\log j) \right).$$

Thus,

$$(3.5) \quad P(F_i \cap F_j) \leq \prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) \times (1 + O(i^{-1/s} (\log i)^2 (\log j))).$$

Since $\alpha < \lambda_s$, the number $\beta = (1 - \alpha \lambda_s)/2$ is positive. Now we split the sum in (3.3) into three sums:

$$\begin{aligned} \Delta_{1n} &= \sum_{\substack{1 \leq i < j \leq n \\ n^\beta < i < j - \alpha \log j}} P(F_i \cap F_j) - P(F_i)P(F_j) \\ \Delta_{2n} &= \sum_{\substack{1 \leq i < j \leq n \\ i \leq n^\beta}} P(F_i \cap F_j) - P(F_i)P(F_j) \\ \Delta_{3n} &= \sum_{\substack{1 \leq i < j \leq n \\ j - \log j \leq i \leq j}} P(F_i \cap F_j) - P(F_i)P(F_j) \end{aligned}$$

- (1) Estimate of Δ_{1n} . Since in this case we have $I_i \cap I_j = \emptyset$, we can apply Lemma 3.6 to (3.5) to get

$$\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E_\omega^c) \leq P(F_i)P(F_j)(1 + O(j^{-1/s} \log j)).$$

This inequality and (3.5) gives

$$P(F_i \cap F_j) \leq P(F_i)P(F_j) \times (1 + O(i^{-1/s} (\log i)^2 (\log j))),$$

so

$$\begin{aligned} P(F_i \cap F_j) - P(F_i)P(F_j) &\ll P(F_i)P(F_j) i^{-1/s} (\log i)^2 (\log j) \\ &\ll n^{-\beta/s + o(1)} P(F_i)P(F_j). \end{aligned}$$

Thus

$$(3.6) \quad \Delta_{1n} \ll n^{-\beta/s + o(1)} \sum_{i, j \leq n} P(F_i)P(F_j) \ll n^{-\beta/s + o(1)} Z_n^2.$$

(2) Estimate of Δ_{2n} . In this case we use the crude estimate

$$(3.7) \quad P(F_i \cap F_j) - P(F_i)P(F_j) \leq P(F_j).$$

We have

$$(3.8) \quad \Delta_{2n} \leq \sum_{j \leq n} \sum_{i \leq j^\beta} P(F_j) \leq \sum_{j \leq n} j^\beta P(F_j) \leq n^\beta Z_n \leq n^{-\beta+o(1)} Z_n^2,$$

since $Z_n = n^{1-\alpha\lambda_s+o(1)} = n^{2\beta+o(1)}$.

(3) Estimate of Δ_{3n} . Again we use (3.7) and we have

$$(3.9) \quad \Delta_{3n} \leq \sum_{j \leq n} \sum_{j-\alpha \log j \leq i \leq j} P(F_j) \leq \alpha \log n \sum_{j \leq n} P(F_j) \leq n^{-2\beta+o(1)} Z_n^2.$$

Finally, using the estimates in (3.6), (3.8) and (3.9) we have

$$\frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} \ll n^{-\beta/s+o(1)} + n^{-\beta+o(1)} + n^{-2\beta+o(1)} \rightarrow 0.$$

This ends the proof of (3.3) and hence that of Theorem 1.1. \square

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Javier CILLERUELO[†]
Instituto de Ciencias Matemáticas
(CSIC-UAM-UC3M-UCM) and
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049, Madrid (Spain)

Jean-Marc DESHOUILLEERS
Bordeaux INP, CNRS
Institut Mathématique de Bordeaux, UMR 5251
33405 Talence (France)
jean-marc.deshouillers@math.u-bordeaux.fr

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