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CORRIGENDUM TO “MATHER DISCREPANCY AND
THE ARC SPACES”

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Abstract. — This paper gives a correction of a theorem in “Mather discrepancy and the arc spaces”.

Résumé. — Dans cette note nous corrigeons un théorème de « Discrépance de Mather et les espaces d’arcs ».

In this paper, we make a correction of the statement of Theorem 4.7 in [2], where (v) was misstated as:

“The tangent cone of \((X, x)\) has a reduced irreducible component.”

This statement should be corrected as:

“Let \(\overline{b} : \overline{Y} \to X\) be the composite of the blow up \(b : Y \to X\) at the point \(x \in X\) and the normalization \(\nu : \overline{Y} \to Y\). Then, the fiber scheme \(\overline{b}^{-1}(x) = E\) has a reduced irreducible component.”

The whole statement of the corrected theorem is as follows:

**Theorem 4.7. — For a singularity \((X, x)\) of dimension \(n\) the following are equivalent:**

(i) \(\text{mld}(x; X, \mathcal{O}_X) = n;\)

(ii) \(\lambda_m = 0\) for every \(m \in \mathbb{N};\)

(iii) \(\lambda^0_m = 0\) for every \(m \in \mathbb{N};\)

(iv) \(\lambda^1_0 = 0;\)

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(v) Let $\overline{b} : \overline{Y} \to X$ be the composite of the blow up $b : Y \to X$ at the point $x \in X$ and the normalization $\nu : \overline{Y} \to Y$. Then, the fiber scheme $\overline{b}^{-1}(x) = \overline{E}$ has a reduced irreducible component, where a reduced irreducible component means an irreducible component which is reduced at the generic point.

Here, we give a proof for the relevant parts under this alteration.

Proof. — The proof in [2] of equivalence among (i), (ii) and (iii) is not affected by the change in (v). The implication (iii) $\Rightarrow$ (iv) is obvious. The implication (iv) $\Rightarrow$ (v) is proved as follows:

Let $E \subset Y$ be the scheme theoretic fiber of $x$ by the blow up $b : Y \to X$ and let $g : \tilde{Y} \to Y$ be a log resolution of $(Y, E)$ and let $g$ be factored as $\tilde{Y} \xrightarrow{h} Y \xrightarrow{\nu} Y$. Then, by the same argument in the corresponding part of the proof in the paper [2], we obtain

$$\dim g(E'_{\text{reg}}) = n - 1,$$

where $E'$ is the scheme theoretic fiber of $x$ by the morphism $b \circ g : \tilde{Y} \to X$ and $E'_{\text{reg}}$ is the locus of non-singular points of $E'$. Therefore we obtain

$$\dim h(E'_{\text{reg}}) = n - 1.$$ 

As $h$ is isomorphic at the generic point of each irreducible component of $h(E'_{\text{reg}})$, this shows that $\overline{E} = \overline{b}^{-1}(x)$ is reduced at an irreducible component, which implies (v).

For the proof of (v) $\Rightarrow$ (iii), we show that we can reduce the discussion into the case that $E$ has a reduced component and $Y$ is non-singular at the generic point of the component. Then the discussion in the proof of the corresponding part in [2] would work.

Let $E'_0 \subset \overline{Y}$ be an irreducible component of $\overline{E}$ with the coefficient 1 in $\overline{E}$ and let $E_0 \subset Y$ be the irreducible components of $E$ corresponding to $E'_0$. Let $e$ and $\overline{e}$ be the generic points of $E_0$ and $\overline{E}_0$, respectively. The normalization $\nu : \overline{Y} \to Y$ induces a homomorphism

$$\widehat{\nu}^* : \widehat{\mathcal{O}}_{\overline{Y}, e} \to \widehat{\mathcal{O}}_{\overline{Y}, \overline{e}}.$$ 

of $k$-algebras. Let $\mathcal{O}_0$ be the image of $\widehat{\nu}^*$:

$$\widehat{\mathcal{O}}_{\overline{Y}, e} \twoheadrightarrow \mathcal{O}_0 \subset \widehat{\mathcal{O}}_{\overline{Y}, \overline{e}}.$$ 

Then $\text{Spec} \mathcal{O}_0$ is an analytic branch of $Y$ at $e$ that is dominated by $\text{Spec} \widehat{\mathcal{O}}_{\overline{Y}, \overline{e}}$. Here, as $\overline{Y}$ is non-singular at $\overline{e}$ we have

$$\widehat{\mathcal{O}}_{\overline{Y}, \overline{e}} = K[[s]],$$

where $K$ is a field.
for some extension field $K$ of $k$. For $f = \sum a_i s^i \in K[[s]]$, we denote the lowest degree $i$ with $a_i \neq 0$ by $\text{ord}_s f$ and call it the order of $f$ with respect to the variable $s$.

We will show that $O_0$ is a regular local ring. For that, we first prove that $O_0$ contains an element of order 1 with respect to $s$. Assume contrary, then every element of $O_0$ is either a unit or an element of order greater than 1. Let $\ell \in \hat{O}_{Y,e}$ be the defining equation of $E$ in $Y$ around $e$. We also denote by $\ell$ the images of $\ell$ in $O_0$ and in $\hat{O}_{Y,e}$ by abuse of notation. Then, in particular, $\text{ord}_s \ell \geq 2$. As $\ell$ is also the defining equation of $E$ in $\overline{Y}$ around $\overline{e}$ by the assumption on $\overline{E}$. Then, the above inequality shows that $E$ is not reduced at $\overline{e}$, which yields a contradiction.

Now we may assume there is an element $s' \in O_0$ with order 1 with respect to $s$. As $\hat{O}_{Y,e} = K[[s]] = K[[s']]$, we may assume that $s \in O_0$, by replacing $s$ by $s'$. By Cohen’s structure theorem, the residue field $K'$ of the complete local ring $O_0$ is contained in $O_0$ and therefore we obtain

$$K'[s] \subset O_0.$$  

Note that the base field $k$ is of characteristic 0. Then the extension $K' \hookrightarrow K$ of fields is separable, therefore it is étale. Now as $K'[s] \rightarrow K[[s]]$ is étale and $O_0 \rightarrow K[[s]]$ is flat, it follows that

$$K'[s] \rightarrow O_0$$

is étale by [1, IV, 17.7.7]. Therefore, $O_0$ is also regular and $\text{ord}_s(\ell) = 1$.

Now one branch of $Y$ at $e$ is non-singular and $E$ is reduced at the the generic point. We restrict the discussion onto this branch. So, we may assume that $Y$ is non-singular at $e$ and $E$ is reduced at $e$. Then, the proof of (v) $\Rightarrow$ (iii) in [2] completes the proof. \qed

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Weichen Gu kindly provides us the following example which shows a contradiction to the previous statement in the Theorem 4.7 in [2]. The author would like to thank him.

**Example.** — Let $X \subset \mathbb{A}^5$ be a hypersurface defined by

$$y^2 - x_1x_2x_3x_4 = 0.$$  

Then, the tangent cone has no reduced component, but $(X, 0)$ satisfies (iv). We should also note that $X$ satisfies the condition (v).
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