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Jonathan ECKHARDT, Fritz GESZTESY, Helge HOLDEN,
Aleksey KOSTENKO & Gerald TESCHL

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REAL-VALUED ALGEBRO-GEOMETRIC SOLUTIONS OF THE TWO-COMPONENT CAMASSA–HOLM HIERARCHY

by Jonathan ECKHARDT, Fritz GESZTESY, Helge HOLDEN, Aleksey KOSTENKO & Gerald TESCHL (*)

Abstract. — We provide a construction of the two-component Camassa–Holm (CH-2) hierarchy employing a new zero-curvature formalism and identify and describe in detail the isospectral set associated to all real-valued, smooth, and bounded algebro-geometric solutions of the nth equation of the stationary CH-2 hierarchy as the real n-dimensional torus $\mathbb{T}^n$. We employ Dubrovin-type equations for auxiliary divisors and certain aspects of direct and inverse spectral theory for self-adjoint singular Hamiltonian systems. In particular, we employ Weyl–Titchmarsh theory for singular (canonical) Hamiltonian systems.

While we focus primarily on the case of stationary algebro-geometric CH-2 solutions, we note that the time-dependent case subordinates to the stationary one with respect to isospectral torus questions.

Résumé. — Nous présentons une construction de la hiérarchie de l'équation de Camassa–Holm à deux composantes (CH-2) en utilisant un nouveau formalisme de courbure nulle. Nous décrivons en détail et identifions l'ensemble isospectral associé à toutes les solutions algébro-géométriques à valeur réelle, réguliéres et bornées de la n-ème équation de l'équation stationnaire de la hiérarchie CH-2 au tore $\mathbb{T}^n$ de dimension n. Nous utilisons des équations de type Dubrovin pour les diviseurs auxiliaires et certains aspects de la théorie spectrale et d'inversion spectrale pour les systèmes Hamiltoniens singuliers auto-adjoints. En particulier, nous utilisons la théorie de Weyl–Titchmarsh pour les systèmes (canoniques) Hamiltoniens singuliers.

Bien que nous nous concentrons principalement sur le cas des solutions algébro-géométriques stationnaires pour CH-2, nous remarquons que le cas de la solution évolutive qui dépend du temps est subordonné au cas stationnaire en ce qui concernent les questions isospectrales liées au tore.

Keywords: Two-component Camassa–Holm hierarchy, real-valued algebro-geometric solutions, isospectral tori, self-adjoint Hamiltonian systems, Weyl–Titchmarsh theory.
Math. classification: 35Q51, 35Q53, 37K15, 37K10, 37K20.
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1. Introduction

The principal purpose of this paper is two-fold: first, we provide a construction of the two-component Camassa–Holm (CH-2) hierarchy based on a new zero-curvature pair, and second, identify and describe in detail the isospectral set associated to all real-valued, smooth, and bounded algebro-geometric solutions of the nth equation of the stationary CH-2 hierarchy as the real n-dimensional torus $\mathbb{T}^n$.

The first nonlinear partial differential equation of the two-component Camassa–Holm hierarchy, the two-component Camassa–Holm system [50], can be written in the form

$$\begin{aligned} 4u_t - u_{xxt} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x + w_x = 0,& \\ w_t + 4wu_x + 2w_x u = 0,& \quad (x,t) \in \mathbb{R}^2. \end{aligned} \tag{1.1}$$

When studying weak solutions of the Cauchy problem one writes the second equation in conservative form, that is, $\rho_t + 2(\rho u)_x = 0$ where $w = \rho^2$. For smooth solutions like those studied in the present paper, the two formulations are equivalent. This two-component system extends the Camassa–Holm equation, also known as the dispersive shallow water equation [5] (the special case $w \equiv 0$ of (1.1)) given by

$$4u_t - u_{xxt} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x = 0, \quad (x,t) \in \mathbb{R}^2 \tag{1.2}$$

(choosing a convenient scaling of x and t). The two-component CH-2 system (1.1) has generated much interest over the past decades. For instance, its relevance to shallow water theory is discussed in [10], [37], well-posedness and blow-up are studied in [10], [17], [19], [28], [41], various types of solutions (global, dissipative, conservative, etc.) are treated in [17], [24]–[27], [29], the inverse scattering transform is applied to the CH-2 system in [12], [34], N solitary waves are discussed in [10], [33], [34], [45], traveling waves are studied in [47], [49], the geometry of CH-2 is investigated in [16], [33], [34], [42], the periodic CH-2 system is discussed in [25], [36], [51]. For connections to other integrable systems see [3], [8], [18]. Various multicomponent extensions of the Camassa–Holm equation and its generalizations are discussed in, e.g., [6], [7], [19], [38], [40], [46], [48], [52]–[54]. Closest to the investigations in this paper is the derivation of the CH-2 hierarchy and its algebro-geometric solutions in [35].

In Section 2 we recall the basic polynomial recursion formalism that defines the CH-2 hierarchy using a new zero-curvature approach based on the 2×2 matrix pair (U, V_n), $n \in \mathbb{N}_0$ (with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), given by

$$U(z,x,t) = -z^{-1}\begin{pmatrix} \alpha(x,t) & -1 \\ \alpha(x,t)^2 + w(x,t) & -\alpha(x,t) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C}\backslash\{0\},\ (x,t) \in \mathbb{R}^2, \tag{1.3}$$

where

$$\alpha(x,t) = u_x(x,t) + 2u(x,t), \quad (x,t) \in \mathbb{R}^2, \tag{1.4}$$

and

$$V_n(z,x,t) = z^{-1}\begin{pmatrix} -G_{n+1}(z,x,t) & F_n(z,x,t) \\ H_n(z,x,t) & G_{n+1}(z,x,t) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\{0\},\ (x,t) \in \mathbb{R}^2, \tag{1.5}$$

assuming F_n, H_n, and G_{n+1} to be polynomials of degree n and $n+1$, respectively, with respect to (the spectral parameter) z and C^∞ in x, t (for simplicity). In addition, F_n and G_{n+1} are chosen to be monic with respect to $z \in \mathbb{C}$. The zero-curvature condition

$$U_t(z,x,t) - V_{n,x,t}(z,x,t) + [U(z,x,t), V_n(z,x,t)] = 0, \tag{1.6}$$

is then shown to generate the CH-2 hierarchy associated to the system (1.1). In fact, (1.1) corresponds to the first nonlinear system $n = 1$ (the case $n = 0$ represents a linear system). Actually, we derive the corresponding stationary (i.e., t-independent) hierarchy first as the latter will be most instrumental in determining the isospectral torus of all real-valued, smooth, and bounded algebro-geometric solutions of the CH-2 hierarchy. The stationary hierarchy is derived from the corresponding zero-curvature equation

$$-V_{n,x}(z,x) + [U(z,x), V_n(z,x)] = 0, \tag{1.7}$$

and it in turn naturally leads to the identity,

$$G_{n+1}(z,x)^2 + F_n(z,x)H_n(z,x) = R_{2n+2}(z), \tag{1.8}$$

where R_{2n+2} is an x-independent monic polynomial with respect to z of degree $2n+2$. The polynomial R_{2n+2} is fundamental as it defines the hyperelliptic curve $\mathcal{K}_n$ (cf. (3.4)) underlying the stationary CH-2 hierarchy.

Section 3 is devoted to the stationary CH-2 hierarchy and the associated algebro-geometric formalism. In particular, the underlying hyperelliptic curve $\mathcal{K}_n$ (defined in terms of the polynomial R_{2n+2}), an associated fundamental meromorphic function ϕ on $\mathcal{K}_n$, its divisor of zeros and poles, the Baker–Akhiezer vector Ψ, basic properties of ϕ and Ψ, Dubrovin-type equations for auxiliary Dirichlet divisors (in fact, zeros $\hat{\mu}_j \in \mathcal{K}_n$, $j = 1, \dots, n$, of ϕ), trace formulas for u and w in terms of the projections $\mu_j \in \mathbb{C}$, $j = 1, \dots, n$, and asymptotic properties of ϕ and Ψ are derived in detail. We conclude this section with a proof of the fact that solutions of the Dubrovin equations generate stationary (algebro-geometric) solutions of the stationary CH-2 hierarchy via the trace formulas (3.42), (3.43) for the pair (u, w).

Section 4 provides a brief summary of self-adjoint singular canonical systems as needed in the subsequent Section 5, and introduces (scalar-valued) half-line Weyl–Titchmarsh functions as well as their 2×2 matrix-valued generalizations for the entire real line.

Finally, Section 5 contains the principal result of this paper, the identification and description of the isospectral set of all real-valued, smooth, and bounded algebro-geometric solutions of the nth equation of the stationary CH-2 hierarchy as the real n-dimensional torus $\mathbb{T}^n$. We start this section by noticing that the basic stationary equation (3.29),

$$\Psi_x(-z,x) = U(-z,x)\Psi(-z,x), \quad \Psi = (\psi_1, \psi_2)^\top, \ (z,x) \in \mathbb{C} \times \mathbb{R}, \tag{1.9}$$

is equivalent to the following singular Hamiltonian (canonical) system

$$J\widetilde{\Psi}_x(\tilde{z},x) = [\tilde{z}A(x) + B(x)]\widetilde{\Psi}(\tilde{z},x), \ \ \widetilde{\Psi} = \big(\widetilde{\psi}_1, \widetilde{\psi}_2\big)^\top, \ \big(\tilde{z},x\big) \in \mathbb{C} \times \mathbb{R}, \tag{1.10}$$

where

$$\begin{gathered} J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\Psi}(\tilde{z},x) = \Psi(-z,x), \ \ \tilde{z} = -z^{-1}, \\ A(x) = \begin{pmatrix} \alpha(x)^2 + w(x) & -\alpha(x) \\ -\alpha(x) & 1 \end{pmatrix} > 0, \ \ B(x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = B(x)^*. \end{gathered} \tag{1.11}$$

We emphasize, in particular, that the new zero-curvature matrix $U(-z, \cdot)$ (cf. [12, App. A]) renders the Hamiltonian system (1.10) linear with respect to the spectral parameter $\tilde{z}$ and hence amenable to standard spectral theory (more precisely, Weyl–Titchmarsh theory and all its ramifications; see Section 4). In particular, with (u, w) subject to conditions (5.3) the Hamiltonian system (1.10) is in the limit point case at $x = \pm\infty$. Other known examples of zero-curvature matrices $U(-z, \cdot)$ (e.g., the one employed in [35]) lead to Hamiltonian systems quadratic in $\tilde{z}$ and hence their spectral theory cannot be handled by the methods indicated in Section 4. Upon characterizing certain classes of Nevanlinna–Herglotz functions defined in terms of polynomials and their square roots (cf. Lemma 5.2), we derive in detail the half-line Weyl–Titchmarsh functions corresponding to the Hamiltonian system (1.10) in connection with the stationary algebro-geometric solutions (u, w) discussed in Section 3. This then enables us to derive the corresponding 2×2 matrix Weyl–Titchmarsh functions and the associated 2×2 matrix spectral function in the Nevanlinna–Herglotz representation of the former on the entire real line (cf. Theorem 5.3), again in the context of stationary algebro-geometric solutions (u, w) of the s-CH-2 hierarchy. Here we just remark that these 2×2 matrix functions are both expressed in terms of the polynomials $F_n(z, \cdot)$, $G_{n+1}(z, \cdot)$, $H_n(z, \cdot)$, and $R_{2n+2}(z)$ (cf. (5.40)–(5.43)). The limit point (i.e., self-adjointness) property of the Hamiltonian system corresponding to real-valued, bounded stationary, algebro-geometric CH-2 solutions then restricts the motion of the zeros and poles of the fundamental function ϕ to real intervals (the closure of spectral gaps, cf. Theorem 5.4). Together with the Dubrovin initial value problem treated in Theorem 5.8, this finally leads to the determination of the isospectral set of all real-valued, smooth and bounded algebro-geometric solutions of the stationary CH-2 equation, $\text{s-CH-2}_n(u, w) = 0$, as the real n-dimensional torus $\mathbb{T}^n$ in Corollary 5.9.

We focus primarily on the case of stationary CH-2 hierarchy solutions as the time-dependent case subordinates to the stationary one with respect to isospectral torus questions, a fact that is briefly commented on at the end of Section 5.

As noted, the special case $w \equiv 0$ reduces the two-component Camassa–Holm hierarchy, CH-2, to the standard (i.e., one-component) Camassa–Holm hierarchy (CH-1). This special case was treated in detail in [1], [2], [20], [21, Ch. 5]. The corresponding isospectral torus of all real-valued, smooth, and bounded algebro-geometric solutions of the one-component CH-1 hierarchy has been derived in [22].

2. The CH-2 Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we review the basic construction of the two-component Camassa–Holm hierarchy (CH-2) using an appropriate zero-curvature approach. An alternative approach to the CH-2 hierarchy was first derived in [35]. Both approaches follow standard arguments first developed in [20] (cf. also [21, Ch. 5]).

Throughout this section we will suppose the following hypothesis.

Hypothesis 2.1. — *Suppose that* $u, w \colon \mathbb{R} \to \mathbb{C}$.
In the stationary case we assume that

$$u, w \in C^\infty(\mathbb{R}),\ u^{(m)}, w^{(m)} \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0. \tag{2.1}$$

In the time-dependent case (cf. (2.30)–(2.37)*) we suppose*

$$\begin{aligned} &u(\,\cdot\,,t), w(\,\cdot\,,t) \in C^\infty(\mathbb{R}),\ \frac{\partial^m u}{\partial x^m}(\,\cdot\,,t), \frac{\partial^m w}{\partial x^m}(\,\cdot\,,t) \in L^\infty(\mathbb{R}), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m\in\mathbb{N}_0,\ t\in\mathbb{R}, \\ &u(x,\,\cdot\,), u_x(x,\,\cdot\,), w(x,\,\cdot\,) \in C^1(\mathbb{R}),\ x\in\mathbb{R}. \end{aligned} \tag{2.2}$$

We start by formulating the basic polynomial setup. One defines $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_0 = 1, \qquad f_1 = -2u + c_1,$$

$$f_{\ell,x} = -2\mathcal{G}\big(2(4u-u_{xx})f_{\ell-1,x} + (4u_x - u_{xxx})f_{\ell-1} - 2wf_{\ell-2,x} - w_x f_{\ell-2}\big), \quad \ell\in\mathbb{N}\backslash\{1\}, \tag{2.3}$$

where c_1 is an integration constant and $\mathcal{G}$ is given by

$$\mathcal{G}\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (\mathcal{G}v)(x) = \frac{1}{4}\int_{\mathbb{R}} dy\, e^{-2|x-y|}v(y), \quad x\in\mathbb{R}, \tag{2.4}$$

for every $v \in L^\infty(\mathbb{R})$. One observes that $\mathcal{G}$ is the resolvent of minus the one-dimensional Laplacian when the spectral parameter is equal to -4, that is,

$$\mathcal{G} = \bigg(-\frac{d^2}{dx^2} + 4\bigg)^{-1}. \tag{2.5}$$

The coefficients f_ℓ, $\ell\in\mathbb{N}$, $\ell\geqslant 2$, are non-local with respect to u. At each level a new integration constant, denoted by c_ℓ, is introduced. Moreover, abbreviating

$$\alpha = u_x + 2u, \tag{2.6}$$

we introduce coefficients $\{g_\ell\}_{\ell\in\mathbb{N}_0}$ and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ by

$$g_\ell = f_\ell + \alpha f_{\ell-1} + \frac{1}{2}f_{\ell,x}, \quad h_\ell = -\big(\alpha^2 + w\big)f_\ell - g_{\ell+2,x}, \quad \ell\in\mathbb{N}_0, \tag{2.7}$$

with the convention $f_{-1} = 0$. Explicitly, one computes

$$\begin{gathered} f_0 = 1, \quad f_1 = -2u + c_1, \\ f_2 = 2u^2 + 2\mathcal{G}\big(u_x^2 + 8u^2 + w\big) + c_1(-2u) + c_2, \\ g_0 = 1, \quad g_1 = c_1, \\ g_2 = -2u^2 + 2\mathcal{G}\big(u_x^2 + u_xu_{xx} + 8uu_x + 8u^2 + w + 2^{-1}w_x\big) + c_2, \\ h_0 = -2\mathcal{G}\big(16uu_x + 2u_x^2 + 2u_xu_{xx} + 16u^2 + 2^{-1}w_{xx} + w_x\big) + 4u^2 - w, \\ \text{etc.} \end{gathered} \tag{2.8}$$

For later use we also note

$$h_{\ell,x} - 2h_\ell - 2\alpha h_{\ell-1} - 2\big(\alpha^2 + w\big)g_\ell = 0, \quad \ell\in\mathbb{N}_0, \tag{2.9}$$

again using the convention $h_{-1}=0$. This can be easily seen by first using (2.7) to eliminate g_ℓ, h_ℓ which eventually reduces (2.9) to (2.3).

Given Hypothesis 2.1, one introduces the 2×2 matrix U by

$$U(z,x)=-z^{-1}\begin{pmatrix}\alpha(x) & -1\\ \alpha(x)^2+w(x) & -\alpha(x)\end{pmatrix}+\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad z\in\mathbb{C}\backslash\{0\},\ x\in\mathbb{R}, \tag{2.10}$$

and for each $n\in\mathbb{N}_0$ the following 2×2 matrix V_n by

$$V_n(z,x)=z^{-1}\begin{pmatrix}-G_{n+1}(z,x) & F_n(z,x)\\ H_n(z,x) & G_{n+1}(z,x)\end{pmatrix},\quad n\in\mathbb{N}_0,\quad z\in\mathbb{C}\backslash\{0\},\ x\in\mathbb{R}, \tag{2.11}$$

assuming F_n, H_n, and G_{n+1} to be polynomials of degree n and $n+1$, respectively, with respect to z and C^∞ in x. In addition, we will choose F_n and G_{n+1} to be monic in z. Postulating the zero-curvature condition

$$-V_{n,x}(z,x)+[U(z,x),V_n(z,x)]=0, \tag{2.12}$$

one finds

$$-zF_{n,x}(z,x)-2[\alpha(x)+z]F_n(z,x)+2G_{n+1}(z,x)=0, \tag{2.13}$$

$$-zG_{n+1,x}(z,x)-\big[\alpha(x)^2+w(x)\big]F_n(z,x)-H_n(z,x)=0, \tag{2.14}$$

$$-zH_{n,x}(z,x)+2[\alpha(x)+z]H_n(z,x)+2\big[\alpha(x)^2+w(x)\big]G_{n+1}(z,x)=0. \tag{2.15}$$

In addition, employing (2.13) and (2.14), one infers that (2.15) is equivalent to

$$H_{n,x}(z,x)+2[\alpha(x)+z]G_{n+1,x}(z,x)-\big[\alpha(x)^2+w(x)\big]F_{n,x}(z,x)=0. \tag{2.16}$$

From (2.13)–(2.15) one infers that

$$\frac{d}{dx}\det(V_n(z,x))=-z^{-2}\frac{d}{dx}\big[G_{n+1}(z,x)^2+F_n(z,x)H_n(z,x)\big]=0, \tag{2.17}$$

and hence

$$G_{n+1}(z,x)^2+F_n(z,x)H_n(z,x)=R_{2n+2}(z), \tag{2.18}$$

where R_{2n+2} is an x-independent monic polynomial with respect to z of degree $2n+2$ and hence of the form

$$R_{2n+2}(z)=\prod_{m=0}^{2n+1}(z-E_m),\quad \{E_m\}_{m=0}^{2n+1}\subset\mathbb{C}. \tag{2.19}$$

Using equations (2.13)–(2.15) one can also derive individual differential equations for F_n and H_n. Focusing on F_n only, one obtains

$$\begin{aligned} F_{n,xxx}(z,x) - 4F_{n,x} - 4\big[z^{-1}(4u(x) - u_{xx}(x)) - z^{-2}w\big]F_{n,x}(z,x) \\ - 2z^{-1}\big[(4u_x(x) - u_{xxx}(x)) - z^{-1}w_x\big]F_n(z,x) = 0, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} -(z^2/2)F_{n,xx}(z,x)F_n(z,x) + (z^2/4)F_{n,x}(z,x)^2 \\ + \big[z^2 + z(4u(x) - u_{xx}(x)) - w\big]F_n(z,x)^2 = R_{2n+2}(z). \end{aligned} \tag{2.21}$$

Next, we connect the recursion relations (2.3), (2.7) with the polynomials F_n, H_n, and G_{n+1} by making the ansatz

$$\begin{aligned} F_n(z,x) &= \sum_{\ell=0}^{n} f_{n-\ell}(x)z^\ell, \\ G_{n+1}(z,x) &= \sum_{\ell=0}^{n+1} g_{n+1-\ell}(x)z^\ell - f_{n+1} - \frac{1}{2}f_{n+1,x}, \\ H_n(z,x) &= \sum_{\ell=0}^{n} h_{n-\ell}(x)z^\ell + g_{n+2,x}. \end{aligned} \tag{2.22}$$

Inserting the ansatz (2.22) into (2.13) and comparing coefficients shows that this equation holds due to (2.7). Similarly, inserting (2.22) into (2.14) shows that the latter equation holds due to (2.7) and $g_0' = g_1' = 0$ if and only if the term linear in z vanishes,

$$f_{n+1,x} + \frac{1}{2}f_{n+1,xx} = 0. \tag{2.23}$$

Finally, inserting (2.22) into (2.15) all coefficients of z^ℓ for $\ell \geqslant 2$ cancel due to (2.9). For the constant (i.e., z^0) term one gets, using (2.7),

$$2\alpha\left(g_{n+2,x} + h_n\right) + 2\left(\alpha^2 + w\right)\left(g_{n+1} - f_{n+1} - \frac{1}{2}f_{n+1,x}\right) = 0. \tag{2.24}$$

Similarly, for the z^1-term one gets using (2.9), (2.7), and (2.3) (in this order),

$$\begin{aligned} h_{n,x} + g_{n+2,xx} - 2\alpha h_{n-1} - 2\left(g_{n+2,x} + h_n\right) - 2\left(\alpha^2 + w\right)g_n \\ = -2g_{n+2,x} + g_{n+2,xx} \\ = -w_x f_n - 2wf_{n,x} + 2\alpha\left(f_{n+1} + \frac{1}{2}f_{n+1,x}\right)_x. \end{aligned} \tag{2.25}$$

Hence (2.15) holds if and only if the final right-hand side of (2.25) vanishes.

In summary, the zero-curvature condition (2.12) will hold if and only if

$$\left(f_{n+1,x} + \frac{1}{2} f_{n+1,xx}\right) = 0 \text{ and } w_x f_n + 2w f_{n,x} = 0. \tag{2.26}$$

For reasons to become clear in connection with the time-dependent formulation, we will replace the first equation in (2.26) by the equivalent one

$$\left(\frac{d}{dx} + 2\right)^{-1} \left(f_{n+1} + \frac{1}{2} f_{n+1,x}\right)_x = \frac{1}{2} f_{n+1,x} = 0. \tag{2.27}$$

Thus, the zero-curvature condition (2.12) is equivalent to

$$\text{s-CH-2}_n(u,w) = \begin{pmatrix} \frac{1}{2} f_{n+1,x} \\ -w_x f_n - 2w f_{n,x} \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{2.28}$$

Varying $n \in \mathbb{N}_0$ in (2.28) then defines the stationary CH-2 hierarchy.

We record the first two equations explicitly,

$$\begin{aligned} &\text{s-CH-2}_0(u,w) = \begin{pmatrix} -u_x \\ -w_x \end{pmatrix} = 0, \\ &\text{s-CH-2}_1(u,w) = \begin{pmatrix} \mathcal{G}(2u_x u_{xx} + 16 u u_x + w_x) + 2u u_x - c_1 u_x \\ 2w_x u + 4w u_x + c_1(-w_x) \end{pmatrix} = 0, \\ &\text{etc.} \end{aligned} \tag{2.29}$$

By definition, the set of solutions of (2.28), with n ranging in $\mathbb{N}_0$, represents the class of algebro-geometric CH-2 solutions. If (u,w) satisfies one of the stationary CH-2 equations in (2.28) for a particular value of n, then it satisfies infinitely many such equations of order higher than n for certain choices of integration constants c_ℓ (see [21, Remark 1.5] for the corresponding argument for the KdV equation).

Next, we turn to the time-dependent CH-2 hierarchy. Introducing a deformation parameter $t_n \in \mathbb{R}$ into u and w (i.e., replacing $(u(x), w(x))$ by $(u(x,t_n), w(x,t_n)))$, the definitions (2.10), (2.11), and (2.22) of U, V_n, and F_n, G_{n+1}, and H_n, respectively, still apply. The corresponding zero-curvature relation reads

$$U_{t_n}(z,x,t_n) - V_{n,x}(z,x,t_n) + [U(z,x,t_n), V_n(z,x,t_n)] = 0, \quad n \in \mathbb{N}_0, \tag{2.30}$$

which results in the following set of time-dependent equations

$$
\begin{aligned}
&(2.31) \quad zF_{n,x}(z,x,t_n) = -2[\alpha(x,t_n)+z]F_n(z,x,t_n) + 2G_{n+1}(z,x,t_n),\\
&(2.32) \quad z\alpha_{t_n}(x,t_n) = zG_{n+1,x}(z,x,t_n)\\
&\qquad\qquad + \big[\alpha(x,t_n)^2 + w(x,t_n)\big]F_n(z,x,t_n) + H_n(z,x,t_n),\\
&(2.33) \quad z[2\alpha(x,t_n)\alpha_{t_n}(x,t_n) + w_{t_n}(x,t_n)] = -zH_{n,x}(z,x,t_n)\\
&\qquad\qquad + 2\big[\alpha(x,t_n)+z\big]H_n(z,x,t_n)\\
&\qquad\qquad + 2\big[\alpha(x,t_n)^2 + w(x,t_n)\big]G_{n+1}(z,x,t_n) = 0.
\end{aligned}
$$

Now one proceeds as in the stationary case to conclude that these equations hold if and only if

$$
\alpha_{t_n} + f_{n+1,x} + \frac{1}{2} f_{n+1,xx} = 0 \tag{2.34}
$$

and

$$
2\alpha\alpha_{t_n} + w_{t_n} - w_x f_n - 2w f_{n,x} + 2\alpha\left(f_{n+1} + \frac{1}{2}f_{n+1,x}\right)_x = 0. \tag{2.35}
$$

Hence one arrives at the corresponding time-dependent hierarchy

$$
\text{CH-2}_n(u,w) = \begin{pmatrix} u_{t_n} + \frac{1}{2}f_{n+1,x} \\ w_{t_n} - w_x f_n - 2w f_{n,x} \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{2.36}
$$

Varying $n \in \mathbb{N}_0$ in (2.36) then defines the time-dependent CH-2 hierarchy. We record the first few equations explicitly,

$$
\begin{aligned}
&\text{CH-2}_0(u,w) = \begin{pmatrix} u_{t_0} - u_x \\ w_{t_0} - w_x \end{pmatrix} = 0,\\
&\text{CH-2}_1(u,w) = \begin{pmatrix} u_{t_1} + \mathcal{G}(2u_x u_{xx} + 16uu_x + w_x) + 2uu_x - c_1 u_x \\ w_{t_1} + 2w_x u + 4wu_x + c_1(-w_x) \end{pmatrix} = 0,\\
&\text{etc.}
\end{aligned} \tag{2.37}
$$

Up to an inessential scaling of the (x,t_1) variables, $\text{CH-2}_1(u) = 0$ with $c_1 = 0$ represents *the two-component Camassa–Holm* equation as discussed, for instance in [33], [35]. In this respect we remark that the first component is more frequently written in the literature as

$$
\begin{aligned}
&(2.38) \quad \mathcal{G}^{-1}\left(u_{t_n} + \frac{1}{2}f_{n+1,x}\right)\\
&\qquad = 4u_{t_n} - u_{xxt_n} + (u_{xxx} - 4u_x)f_n + 2(u_{xx} - 4u)f_{n,x}\\
&\qquad\qquad + w_x f_{n-1} + 2w f_{n-1,x}, \quad n \in \mathbb{N}.
\end{aligned}
$$

3. The Stationary Algebro-Geometric CH-2 Formalism

This section is devoted to a quick review of the stationary CH-2 hierarchies and the corresponding algebro-geometric formalism. This topic has first been discussed in [35] using a different zero-curvature pair (U, V_n). These approaches are standard and follow the lines developed in [20] (see also [21, Ch. 5]).

We start with the stationary hierarchy and hence impose the following assumptions:

Hypothesis 3.1. — *Suppose that $u, w\colon \mathbb{R} \to \mathbb{C}$ satisfy*

$$u, w \in C^\infty(\mathbb{R}),\ u^{(m)}, w^{(m)} \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0, \tag{3.1}$$

and let all associated quantities (2.3), (2.7), (2.22) *be defined as in the previous section. Moreover, suppose (cf.* (2.18), (2.19)*)*

$$\{E_m\}_{m=0}^{2n+1} \subset \mathbb{C}\backslash\{0\}. \tag{3.2}$$

Recalling (2.19),

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \tag{3.3}$$

we introduce the (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of arithmetic genus n defined by

$$\mathcal{K}_n\colon \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0. \tag{3.4}$$

We compactify $\mathcal{K}_n$ by adding two points at infinity, $P_{\infty_+}, P_{\infty_-}$, with $P_{\infty_+} \neq P_{\infty_-}$, still denoting its projective closure by $\mathcal{K}_n$. Hence $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of arithmetic genus n. Points P on $\mathcal{K}_n\backslash\{P_{\infty_\pm}\}$ are denoted by $P = (z, y)$, where $y(\,\cdot\,)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$ (the case $n = 0$ being trivial).

In the following the roots of the polynomials F_n and H_n will play a special role and hence we introduce on $\mathbb{C} \times \mathbb{R}$

$$F_n(z, x) = \prod_{j=1}^{n} [z - \mu_j(x)], \quad H_n(z, x) = h_0(x) \prod_{j=1}^{n} [z - \nu_j(x)], \tag{3.5}$$

temporarily assuming

$$h_0(x) \neq 0, \quad x \in \mathbb{R}. \tag{3.6}$$

Moreover, we introduce

$$\hat{\mu}_j(x) = (\mu_j(x), -G_{n+1}(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n, \ x \in \mathbb{R}, \tag{3.7}$$

$$\hat{\nu}_j(x) = (\nu_j(x), G_{n+1}(\nu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{3.8}$$

The branch of $y(\,\cdot\,)$ near $P_{\infty_\pm}$ is fixed according to

$$\frac{y(P)}{z(P)^{n+1}} \underset{\substack{|z(P)|\to\infty \\ P\to P_{\infty_\pm}}}{=} \mp\big[1 + c_1(\underline{E})z(P)^{-1} + O\big(z(P)^{-2}\big)\big]. \tag{3.9}$$

Due to assumption (3.1), u is smooth and bounded, and hence $F_n(z,\,\cdot\,)$ and $H_n(z,\,\cdot\,)$ share the same property. Thus, one concludes

$$\mu_j, \nu_k \in C(\mathbb{R}), \ j,k = 1, \dots, n, \tag{3.10}$$

taking multiplicities (and appropriate reordering) of the zeros of F_n and H_n into account.

Equation (2.21) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in the stationary CH-2 equations (2.28) in terms of the zeros E_m, $m = 0, \dots, 2n+1$, of the associated polynomial R_{2n+2} in (2.19), as follows: Choosing $P = (z, y) \in \Pi_{n,+}$ (cf. (5.16), (5.17)) and inserting

$$\frac{F_n(z,x)}{y(P)} = -\sum_{\ell=0}^{\infty} \hat{f}_\ell(x) z^{-\ell-1} \tag{3.11}$$

into (2.21), one obtains the nonlinear recursion

$$\begin{aligned} &\hat{f}_0 = 1, \quad \hat{f}_1 = -2u, \\ &\hat{f}_\ell = -\mathcal{G}\bigg(\sum_{m=1}^{\ell-1}\Big[\frac{1}{2}\hat{f}_{m,x}\hat{f}_{\ell-m,x} + \hat{f}_m\big(2\hat{f}_{\ell-m} - \hat{f}_{\ell-m,xx}\big)\Big] \\ &\qquad + 2\sum_{m=0}^{\ell-1}\hat{f}_m\Big[(4u - u_{xx})\hat{f}_{\ell-m-1} - w\hat{f}_{\ell-m-2}\Big]\bigg), \quad \ell \in \mathbb{N}\backslash\{1\}. \end{aligned} \tag{3.12}$$

Furthermore, inserting (3.11) into (2.20) one sees that $\hat{f}_\ell$ also satisfies (2.3), and by homogeneity considerations one infers

$$f_\ell = \sum_{m=0}^{\ell} c_{\ell-m}\hat{f}_m. \tag{3.13}$$

Using again (3.11) and (2.22) one finally obtains

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n, \tag{3.14}$$

where $c_k(\underline{E})$, $k \in \mathbb{N}_0$, denote the asymptotic expansion coefficients of $y(P)^{-1} = -\sum_{\ell=0}^{\infty} c_\ell(\underline{E}) z^{-n-\ell-1}$. Explicitly (cf. [20, App. D]),

$$c_0(\underline{E}) = 1, \quad c_1(\underline{E}) = -\frac{1}{2}\sum_{m=0}^{2n+1} E_m,$$

$$c_k(\underline{E}) = -\sum_{\substack{j_1,\dots,j_{2n+1}=0 \\ j_1+\cdots+j_{2n+1}=k}}^{k} \frac{(2j_1)!\cdots(2j_{2n+1})!}{2^{2k}(j_1!)^2\cdots(j_{2n+1}!)^2(2j_1-1)\cdots(2j_{2n+1}-1)} \times E_1^{j_1}\cdots E_{2n+1}^{j_{2n+1}}, \quad k \in \mathbb{N}. \tag{3.15}$$

Next, we introduce the fundamental meromorphic function $\phi(\,\cdot\,,x)$ on $\mathcal{K}_n$ by

$$\phi(P,x) = \frac{y - G_{n+1}(z,x)}{F_n(z,x)} = \frac{H_n(z,x)}{y + G_{n+1}(z,x)}, \quad P=(z,y)\in\mathcal{K}_n,\ x\in\mathbb{R}. \tag{3.16}$$

Assuming (3.6), the divisor $(\phi(\,\cdot\,,x))$ of $\phi(\,\cdot\,,x)$ is given by

$$(\phi(\,\cdot\,,x)) = \mathcal{D}_{P_{\infty_-}\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}(x)}, \tag{3.17}$$

taking into account (3.9). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1,\dots,\hat{\nu}_n\} \in \sigma^n\mathcal{K}_n, \tag{3.18}$$

where $\sigma^m\mathcal{K}_n$, $m\in\mathbb{N}$, denotes the mth symmetric product of $\mathcal{K}_n$. Moreover, we used the following convenient notation for a positive divisor $\mathcal{D}_{\underline{Q}}$ of degree n on $\mathcal{K}_n$,

(3.19)

$$\mathcal{D}_{\underline{Q}}\colon \mathcal{K}_n \to \mathbb{N}_0,\ P \mapsto \mathcal{D}_{\underline{Q}}(P) = \begin{cases} m & \text{if } P \text{ occurs } m \text{ times in } \{Q_1,\dots,Q_n\}, \\ 0 & \text{if } P \notin \{Q_1,\dots,Q_n\}, \end{cases} \quad \underline{Q} = \{Q_1,\dots,Q_n\}\in\sigma^n\mathcal{K}_n,$$

and used the following notation for divisors of degree $n+1$ on $\mathcal{K}_n$,

$$\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad Q_0\in\mathcal{K}_n, \tag{3.20}$$

where for any $Q\in\mathcal{K}_n$,

$$\mathcal{D}_Q\colon \mathcal{K}_n\to\mathbb{N}_0, \quad P\mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P=Q, \\ 0 & \text{for } P\in\mathcal{K}_n\backslash\{Q\}. \end{cases} \tag{3.21}$$

If h_0 is permitted to vanish at a point $x_1\in\mathbb{R}$, then for $x=x_1$, the polynomial $H_n(\,\cdot\,,x_1)$ is at most of degree $n-1$ (cf. (2.22)). Since this can be viewed as a limiting case of (3.17), we will henceforth not particularly distinguish the case $h_0 \neq 0$ from the more general situation where h_0 is permitted to vanish.

Given the meromorphics function $\phi(\,\cdot\,,x)$, one defines the associated Baker–Akhiezer vector $\Psi(\,\cdot\,,x,x_0)$ on $\mathcal{K}_n\backslash\{P_{\infty_+},P_{\infty_-}\}$ by

$$\Psi(P,x,x_0) = \begin{pmatrix} \psi_1(P,x,x_0) \\ \psi_2(P,x,x_0) \end{pmatrix}, \quad P \in \mathcal{K}_n\backslash\{P_{\infty_+},P_{\infty_-}\},\ (x,x_0) \in \mathbb{R}^2, \tag{3.22}$$

where

$$\begin{aligned} \psi_1(P,x,x_0) = \exp\bigg(& - z^{-1}\int_{x_0}^x dx'\,\phi(P,x') - (x-x_0) \\ & - z^{-1}\int_{x_0}^x dx'\,\alpha(x')\bigg), \end{aligned} \tag{3.23}$$

$$\psi_2(P,x,x_0) = -\psi_1(P,x,x_0)\phi(P,x). \tag{3.24}$$

The basic properties of ϕ and Ψ then read as follows.

Lemma 3.2. — *Assume Hypothesis 3.1 and that the nth stationary* CH-2 *equation* (2.28) *holds on some open interval $\Omega \subseteq \mathbb{R}$. Moreover, suppose that $P = (z,y) \in \mathcal{K}_n\backslash\{P_{\infty_+},P_{\infty_-}\}$, $(x,x_0) \in \Omega^2$. Then ϕ satisfies the Riccati-type equation*

$$\begin{aligned} \phi_x(P,x) - z^{-1}\phi(P,x)^2 - 2z^{-1}(\alpha(x)+z)\phi(P,x) & \\ - 2z^{-1}[\alpha(x)^2 + w(x)] = 0, & \end{aligned} \tag{3.25}$$

as well as

$$\phi(P,x)\phi(P^*,x) = -\frac{H_n(z,x)}{F_n(z,x)}, \tag{3.26}$$

$$\phi(P,x) + \phi(P^*,x) = -2\frac{G_{n+1}(z,x)}{F_n(z,x)}, \tag{3.27}$$

$$\phi(P,x) - \phi(P^*,x) = \frac{2y}{F_n(z,x)}, \tag{3.28}$$

while Ψ fulfills

$$\Psi_x(P,x,x_0) = U(z,x)\Psi(P,x,x_0), \tag{3.29}$$

$$- y\Psi(P,x,x_0) = zV_n(z,x)\Psi(P,x,x_0), \tag{3.30}$$

$$\psi_1(P,x,x_0) = \left(\frac{F_n(z,x)}{F_n(z,x_0)}\right)^{1/2} \exp\bigg(- (y/z)\int_{x_0}^x dx' F_n(z,x')^{-1}\bigg), \tag{3.31}$$

$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0) = \frac{F_n(z,x)}{F_n(z,x_0)}, \tag{3.32}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0) = -\frac{H_n(z,x)}{F_n(z,x_0)}, \tag{3.33}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0)+\psi_1(P^*,x,x_0)\psi_2(P,x,x_0)=2\frac{G_{n+1}(z,x)}{F_n(z,x_0)}, \tag{3.34}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0)-\psi_1(P^*,x,x_0)\psi_2(P,x,x_0)=\frac{2y}{F_n(z,x_0)}. \tag{3.35}$$

In addition, as long as the zeros of $F_n(\,\cdot\,,x)$ are all simple for $x\in\Omega$, $\Psi(\,\cdot\,,x,x_0)$, $x,x_0\in\Omega$, is meromorphic on $\mathcal{K}_n$.

Proof. — The proof of Lemma 3.2 is standard and follows that of [20, Lem. 3.1] line by line (cf. also [21, Lem. 5.2]). In particular, (3.26)–(3.28) are clear from the definition (3.16) of ϕ and from the fact that $y(P^*)=-y(P)$, similarly, (3.29)–(3.35) are immediate from (3.23), (3.24), and (3.26)–(3.28). The Riccati-type equation (3.25) follows from combining the first equality in (3.16) with (2.13), (2.14) and (2.18). Meromorphy of $\Psi(\,\cdot\,,x,x_0)$, on $\mathcal{K}_n$ as long as the zeros of $F_n(\,\cdot\,,x)$ are all simple follows from

$$-\frac{1}{z}\phi(P,x') \underset{P\to\hat{\mu}_j(x')}{=} \frac{\partial}{\partial x'}\ln(F_n(z,x'))+O(1) \text{ as } z\to\mu_j(x'), \tag{3.36}$$

(cf. (2.13), (3.7), and (3.16)) and (3.23). □

Next, we recall the Dubrovin-type equations for μ_j. In the remainder of this section we will frequently assume that $\mathcal{K}_n$ has a nonsingular affine part, that is, we suppose that

$$E_m\in\mathbb{C}\backslash\{0\},\ E_m\neq E_{m'} \text{ for } m\neq m',\ m,m'=0,\dots,2n+1. \tag{3.37}$$

Lemma 3.3. — *Assume Hypothesis 3.1 and that the nth stationary* CH-2 *equation* (2.28) *holds subject to the constraint* (3.37) *on an open interval $\widetilde{\Omega}_\mu\subseteq\mathbb{R}$. Moreover, suppose that the zeros μ_j, $j=1,\dots,n$, of $F_n(\,\cdot\,)$ remain distinct and nonzero on $\widetilde{\Omega}_\mu$. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, defined by* (3.7), *satisfies the following first-order system of differential equations*

$$\mu_{j,x}(x)=2\frac{y(\hat{\mu}_j(x))}{\mu_j(x)}\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}[\mu_j(x)-\mu_\ell(x)]^{-1},\quad j=1,\dots,n,\ x\in\widetilde{\Omega}_\mu. \tag{3.38}$$

Next, assume the affine part of $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n}\subset\mathcal{K}_n \tag{3.39}$$

for some $x_0\in\mathbb{R}$, where $\mu_j(x_0)\neq 0$, $j=1,\dots,n$, are assumed to be distinct. Then there exists an open interval $\Omega_\mu\subseteq\mathbb{R}$, with $x_0\in\Omega_\mu$, such that the initial value problem (3.38), (3.39) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n}\subset\mathcal{K}_n$ satisfying*

$$\hat{\mu}_j\in C^\infty(\Omega_\mu,\mathcal{K}_n),\quad j=1,\dots,n, \tag{3.40}$$

and μ_j, $j = 1, \dots, n$, *remain distinct and nonzero on* Ω_μ.

Proof. — Since $y(\hat{\mu}_j) = -G_{n+1}(\mu_j) = -(\mu_j/2)F_{n,x}(\mu_j)$ by (2.13) and (3.7), one computes

$$F_{n,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (\mu_j - \mu_\ell) = -(2/\mu_j) y(\hat{\mu}_j), \quad j = 1, \dots, n, \tag{3.41}$$

from which the rest follows by standard arguments (cf. [20, Lem. 3.2], [21, Lem. 5.3]). □

Combining the polynomial approach in Section 2 with (3.5) yields trace formulas for the CH-2 invariants. For simplicity we just record two simple cases.

Lemma 3.4. — *Assume Hypothesis 3.1 and that the* n*th stationary* CH-2 *equation* (2.28) *holds on some set* Ω_μ *as in Lemma 3.3, and let* $x \in \Omega_\mu$. *Then*

$$u(x) = \frac{1}{2} \sum_{j=1}^{n} \mu_j(x) - \frac{1}{4} \sum_{m=0}^{2n+1} E_m, \tag{3.42}$$

$$w(x) = -\Bigg(\prod_{m=0}^{2n+1} E_m \Bigg) \Bigg(\prod_{j=1}^{n} \mu_j(x)^{-2} \Bigg). \tag{3.43}$$

Proof. — For the proof of Lemma 3.4 one can follow [20, Lem. 3.3] (equivalently, [21, Lem. 5.4]) line by line. Indeed,

$$f_1 = -2u + c_1, \quad f_1 = -\sum_{j=1}^{n} \mu_j \tag{3.44}$$

(cf. (2.8) and (3.5)), and

$$c_1 = -2^{-1} \sum_{m=0}^{2n+1} E_m \tag{3.45}$$

(cf. (3.15)), prove (3.42). Combining

$$\begin{gathered} f_n = (-1)^n \prod_{j=1}^{n} \mu_j, \quad g_{n+1} - f_{n+1} - \frac{1}{2} f_{n,x} = \alpha f_n, \\ h_n + g_{n+2,x} = -\big(\alpha^2 + w\big) f_n \end{gathered} \tag{3.46}$$

(cf. (2.7) and (3.5)), with

$$\begin{aligned}(3.47)\quad \big[g_{n+1} - f_{n+1} - 2^{-1} f_{n+1,x}\big]^2 + f_n[h_n + g_{n+2,x}]\\ = \alpha^2 f_n^2 - \big(\alpha^2 + w\big) f_n^2 = \prod_{m=0}^{2n+1} E_m\end{aligned}$$

(cf. (2.18) and (2.22)), prove (3.43). By Lemma 3.3 one concludes that $\mu_j(x) \neq 0$ for all $j = 1, \dots, n$, $x \in \Omega_\mu$. □

One notes that both, u and w, are uniquely determined by μ_j, $j = 1, \dots, n$. Moreover, $w \to 0$ if some $E_m \to 0$, hence we excluded the latter situation.

Remark 3.5. — The trace (actually, product) formula for w in (3.43) is somewhat familiar from the CH-1 context where $w \equiv 0$. Indeed, combining relations (2.28), (2.29), and (3.7) in [20] yields

$$(3.48)\qquad 4u - u_{xx} = -\left(\prod_{m=0}^{2n+1} E_m\right)\left(\prod_{j=1}^{n} \mu_j(x)^{-2}\right),$$

an identity derived earlier in the periodic context in [11].

Next we turn to asymptotic properties of ϕ and ψ_j, $j = 1, 2$.

Lemma 3.6. — *Assume Hypothesis 3.1 and assume that the nth stationary* CH-2 *equation* (2.28) *holds on some open interval $\Omega \subseteq \mathbb{R}$. In addition, let $P = (z, y) \in \mathcal{K}_n \backslash \{P_{\infty_+}\}$, $x \in \Omega$. Then*

$$(3.49)\quad \phi(P,x) \underset{\zeta\to 0}{=} \begin{cases} -2\zeta^{-1} + [-4u(x) + c_1] + O(\zeta), & P \to P_{\infty_+}, \\ O(\zeta), & P \to P_{\infty_-}, \end{cases} \zeta = z^{-1},$$

and

$$(3.50)\quad \psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp(\pm(x-x_0))(1+O(\zeta)), \quad P \to P_{\infty_\pm}, \quad \zeta = z^{-1},$$

$$(3.51)\quad \psi_2(P,x,x_0) \underset{\zeta\to 0}{=} \exp(\pm(x-x_0)) \begin{cases} -2\zeta^{-1} + O(1), & P \to P_{\infty_+}, \\ O(\zeta), & P \to P_{\infty_-}, \end{cases} \zeta = z^{-1}.$$

Proof. — This is an immediate consequence of (3.9), (3.16),(3.17), (3.23), and (3.24). □

Since the representations of ϕ and u in terms of the Riemann theta function associated with $\mathcal{K}_n$ (assuming the affine part of $\mathcal{K}_n$ to be nonsingular) are not explicitly needed in this paper (yet can be derived as in [20] and [21, Ch. 5]), we omit the corresponding details. We note that reference [35] derives these representations adapted to their framework.

Finally, we note that solvability of the Dubrovin equations (3.38) on $\Omega_\mu \subseteq \mathbb{R}$ in fact yields the nth stationary CH-2 equation (2.28) on Ω_μ.

Theorem 3.7. — *Fix $n \in \mathbb{N}$ and assume (3.37). Suppose also that $\{\hat\mu_j\}_{j=1,\dots,n}$ satisfies the stationary Dubrovin equations (3.38) on an open interval $\Omega_\mu \subseteq \mathbb{R}$ such that μ_j, $j = 1,\dots,n$, remain distinct and nonzero on Ω_μ. Then $u, w \in C^\infty(\Omega_\mu)$ defined by*

$$\begin{aligned} u(x) &= \frac{1}{2}\sum_{j=1}^{n} \mu_j(x) - \frac{1}{4}\sum_{m=0}^{2n+1} E_m, \\ w(x) &= -\left(\prod_{m=0}^{2n+1} E_m\right)\left(\prod_{j=1}^{n} \mu_j(x)^{-2}\right), \end{aligned} \tag{3.52}$$

satisfy the nth stationary CH-2 *equation* (2.28), *that is,*

$$\text{s-CH-2}_n(u,w) = 0 \text{ on } \Omega_\mu. \tag{3.53}$$

Proof. — Given the solutions $\hat\mu_j = (\mu_j, y(\hat\mu_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1,\dots,n$ of (3.38) we introduce

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j), \tag{3.54}$$

$$G_{n+1}(z) = (\alpha + z)F_n(z) + (z/2)F_{n,x}(z) \tag{3.55}$$

on $\mathbb{C} \times \Omega_\mu$. The Dubrovin equations imply

$$y(\hat\mu_j) = \frac{1}{2}\mu_j \mu_{j,x} \prod_{\substack{\ell=1 \\ \ell\neq j}}^{n} (\mu_j - \mu_\ell) = -\frac{1}{2}\mu_j F_{n,x}(\mu_j) = -G_{n+1}(\mu_j). \tag{3.56}$$

Thus,

$$R_{2n+2}(\mu_j) - G_{n+1}(\mu_j)^2 = 0, \quad j = 1,\dots,n, \tag{3.57}$$

and one can write

$$R_{2n+2}(z) - G_{n+1}(z)^2 = F_n(z)H(z), \tag{3.58}$$

for some polynomial H with respect to z. Investigating the leading asymptotics of H as $|z| \to \infty$ reveals that the degree of H equals at most n and we thus write $H = H_n$ from now on. Indeed, one computes (for $n \in \mathbb{N}$,

$n \geqslant 2$, and analogously for $n = 0, 1$),

$$
\begin{aligned}
(3.59)\quad G_{n+1}(z) &= z^{n+1} + \Bigg[-\frac{1}{2}\sum_{j=1}^{n}\mu_{j,x} - \sum_{j=1}^{n}\mu_j + \alpha\Bigg]z^n \\
&+ \Bigg[\frac{1}{2}\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{n}[\mu_{j_1}\mu_{j_2,x} + \mu_{j_1,x}\mu_{j_2} + 2\mu_{j_1}\mu_{j_2}] - \alpha\sum_{j=1}^{n}\mu_j\Bigg]z^{n-1} + O\big(z^{n-2}\big) \\
&= z^{n+1} - \frac{1}{2}\Bigg(\sum_{m=0}^{2n+1}E_m\Bigg)z^n \\
&+ \Bigg[\frac{1}{2}\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{n}[\mu_{j_1}\mu_{j_2,x} + \mu_{j_1,x}\mu_{j_2} + 2\mu_{j_1}\mu_{j_2}] - \alpha\sum_{j=1}^{n}\mu_j\Bigg]z^{n-1} + O\big(z^{n-2}\big),
\end{aligned}
$$

where we used $\alpha = u_x + 2u$ and the trace formula for u (and hence for u_x) in (3.52). Insertion of (3.59) into (3.58) confirms that H has degree at most n as a polynomial in z. Next, we introduce the polynomial P in z via

$$
(3.60)\qquad P(z) = -zG_{n+1,x}(z) - \big(\alpha^2 + w\big)F_n(z) - H_n(z)
$$

on $\mathbb{C}\times\Omega_\mu$. Applying once more (3.59) shows that P also has at most degree n in z and hence we write $P = P_n$ in the following. One then computes,

$$
\begin{aligned}
(3.61)\quad & G_{n+1}(z)P_n(z) \\
&= -(z/2)\partial_x\big[G_{n+1}(z)^2\big] - \big(\alpha^2 + w\big)F_n(z)G_{n+1}(z) \\
&\quad - G_{n+1}(z)H_n(z) \\
&= (z/2)[F_{n,x}(z)H_n(z) + F_n(z)H_{n,x}(z)] - \big(\alpha^2 + w\big)F_n(z)G_{n+1}(z) \\
&\quad - G_{n+1}(z)H_n(z) \\
&= (z/2)H_{n,x}(z)F_n(z) - \big(\alpha^2 + w\big)G_{n+1}(z)F_n(z) \\
&\quad + H_n(z)[(z/2)F_{n,x}(z) - G_{n+1}(z)] \\
&= \big[(z/2)H_{n,x}(z) - \big(\alpha^2 + w\big)G_{n+1}(z) - (\alpha + z)H_n\big]F_n(z).
\end{aligned}
$$

Temporarily restricting $x \in \widetilde{\Omega}_\mu$, where

$$
\begin{aligned}
(3.62)\quad \widetilde{\Omega}_\mu &= \{x \in \Omega_\mu \mid \mu_j(x)F_{n,x}(\mu_j(x),x)/2 = -y(\hat{\mu}_j(x)) = G_{n+1}(\mu_j(x),x) \neq 0, \\
&\qquad\qquad j = 1,\dots,n\} \\
&= \{x \in \Omega_\mu \mid \mu_j(x) \notin \{E_0,\dots,E_{2n+1}\},\ j = 1,\dots,n\},
\end{aligned}
$$

one infers that

$$P_n(z,x) = \gamma(x) F_n(z,x) \tag{3.63}$$

for some continuous function γ on $\widetilde{\Omega}_\mu$. Taking $z = 0$ in (3.58) then yields

$$\bigg(\prod_{m=0}^{2n+1} E_m\bigg) - \bigg[\alpha \prod_{j=1}^{n} \mu_j\bigg]^2 = (-1)^n \bigg(\prod_{j=1}^{n} \mu_j\bigg) H_n(0) \tag{3.64}$$

and employing the trace (resp., product) formula for w in (3.52), (3.64) is equivalent to

$$\alpha(x)^2 + w(x) = -H_n(0,x)/F_n(0,x), \quad x \in \widetilde{\Omega}_\mu. \tag{3.65}$$

Next, choosing $z = 0$ in (3.61) implies (with $G_n(0) = \alpha F_n(0)$ by (3.55))

$$\begin{aligned} 2G_{n+1}(0,x)P_n(0,x) &= 2\alpha(x)\gamma(x)F_n(0,x)^2 \\ &= \big[-2\big(\alpha(x)^2 + w(x)\big)G_{n+1}(0,x) - 2\alpha(x)H_n(0,x)\big]F_n(0,x) \\ &= \{2[H_n(0,x)/F_n(0,x)]\alpha(x)F_n(0,x) - 2\alpha(x)H_n(0,x)\}F_n(0,x) \\ &= 0, \quad x \in \widetilde{\Omega}_\mu. \end{aligned} \tag{3.66}$$

Thus,

$$\gamma(x) = 0 \text{ for } x \in \widetilde{\Omega}_\mu \text{ such that } \alpha(x) \neq 0. \tag{3.67}$$

Since $\alpha = (u_x + 2u) \in C^\infty(\Omega_\mu)$ by hypothesis, and $u(x) \neq e^{-2x}$, one concludes that $\gamma(x) = 0$, $x \in \widetilde{\Omega}_\mu$. At this point one can follow the final part of [20, Thm. 3.11] (or [21, Ch. 5]) to conclude that

$$\gamma(x) = 0 \text{ and hence } P_n(z,x) = 0 \text{ for } x \in \Omega_\mu. \tag{3.68}$$

Thus,

$$H_n(z) = -zG_{n+1}(z) - \big(\alpha^2 + w\big)F_n(z) \tag{3.69}$$

on $\mathbb{C} \times \Omega_\mu$. Finally, differentiating

$$R_{2n+2}(z) - G_{n+1}(z,x)^2 = F_n(z,x)H_n(z,x) \tag{3.70}$$

with respect to $x \in \Omega_\mu$ and employing (3.55) and (3.69) yields

$$\begin{aligned} F_n H_{n,x} &= -2G_{n+1}G_{n+1,x} - F_{n,x}H_n \\ &= F_n\big[-2(\alpha + z)G_{n+1,x} + \big(\alpha^2 + w\big)F_{n,x}\big] \end{aligned} \tag{3.71}$$

and hence also

$$H_{n,x} = -2(\alpha + z)G_{n+1,x} + \big(\alpha^2 + w\big)F_{n,x} \tag{3.72}$$

on $\mathbb{C} \times \Omega_\mu$. Thus, the zero-curvature equations (2.13)–(2.15) have been established on Ω_μ and one can now follow the discussion in Section 2 to arrive at (3.53). □

4. Basic Facts on Self-Adjoint Hamiltonian Systems

We now turn to the Weyl–Titchmarsh theory for singular Hamiltonian (canonical) systems and briefly recall the basic material needed in the following section. This material is standard and can be found, for instance, in [9], [30], [31], [32], [39], [43], and the references therein.

Hypothesis 4.1.

(i) *Define the* 2×2 *matrix* $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *and suppose* $a_{j,k}, b_{j,k} \in L^1_{\mathrm{loc}}(\mathbb{R})$, $j,k = 1,2$ *and* $A(x) = \big(a_{j,k}(x)\big)_{j,k=1,2} \geqslant 0$, $B(x) = \big(b_{j,k}(x)\big)_{j,k=1,2} = B(x)^*$ *for a.e.* $x \in \mathbb{R}$. *We consider the Hamiltonian system*

$$J\Psi'(z,x) = [zA(x) + B(x)]\Psi(z,x), \quad z \in \mathbb{C} \tag{4.1}$$

for a.e. $x \in \mathbb{R}$, *where* z *plays the role of the spectral parameter, and where*

$$\Psi(z,x) = (\psi_1(z,x)\ \psi_2(z,x))^\top, \quad \psi_j(z,\,\cdot\,) \in AC_{\mathrm{loc}}(\mathbb{R}),\ j = 1,2. \tag{4.2}$$

Here $AC_{\mathrm{loc}}(\mathbb{R})$ *denotes the set of locally absolutely continuous functions on* $\mathbb{R}$ *and the* M^* *and* $M^\top$ *denote the adjoint and transpose of a matrix* M, *respectively.*

(ii) *For all nontrivial solutions* Ψ *of* (4.1) *we assume the positive definiteness hypothesis (cf.* [4, Sect. 9.1]*)*

$$\int_c^d dx\, \Psi(z,x)^* A(x) \Psi(z,x) > 0\,, \tag{4.3}$$

on every interval $(c,d) \subset \mathbb{R}$, $c < d$.

Next, we introduce the vector space ($-\infty \leqslant a < b \leqslant \infty$)

$$L^2_A((a,b)) = \left\{\Phi : (a,b) \to \mathbb{C}^2 \text{ measurable} \,\middle|\, \int_a^b dx\, \Phi(x)^* A(x) \Phi(x) < \infty\right\}. \tag{4.4}$$

Fix a point $x_0 \in \mathbb{R}$. Then the Hamiltonian system (4.1) is said to be in the *limit point case* at ∞ (resp., $-\infty$) if for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$, precisely one solution of (4.1) lies in $L^2_A((x_0,\infty))$ (resp., $L^2_A((-\infty,x_0))$). (By the analog of Weyl's alternative, if (4.1) is not in the limit point case at $\pm\infty$, all solutions of (4.1) lie in $L^2_A((x_0,\pm\infty))$ for all $z \in \mathbb{C}$. In the latter case the Hamiltonian system (4.1) is said to be in the *limit circle case* at $\pm\infty$.)

To simplify matters for the remainder of this section, we will always suppose the limit point case at $\pm\infty$ from now on.

Hypothesis 4.2. — *Assume Hypothesis 4.1 and suppose that the Hamiltonian system* (4.1) *is in the limit point case at* $\pm\infty$.

An elementary example of a Hamiltonian system satisfying Hypothesis 4.2 is given by the case where all entries of A and B are essentially bounded on $\mathbb{R}$ (cf. Section 5).

When considering the Hamiltonian system (4.1) on the half-line $[x_0, \infty)$ (resp., $(-\infty, x_0]$), a self-adjoint (separated) boundary condition at the point x_0 is of the type

$$\beta\Psi(x_0) = 0, \tag{4.5}$$

where $\beta = (\beta_1\ \beta_2) \in \mathbb{C}^{1\times 2}$ satisfies

$$\beta\beta^* = 1, \quad \beta J\beta^* = 0 \quad (\text{equivalently, } |\beta_1|^2 + |\beta_2|^2 = 1,\ \mathrm{Im}(\beta_2\overline{\beta_1}) = 0). \tag{4.6}$$

In particular, the boundary condition (4.5) (with β satisfying (4.6)) is equivalent to $\beta_1\psi_1(x_0) + \beta_2\psi_2(x_0) = 0$ with $\beta_1/\beta_2 \in \mathbb{R}$ if $\beta_2 \neq 0$ and $\beta_2/\beta_1 \in \mathbb{R}$ if $\beta_1 \neq 0$. The special case $\beta_0 = (1\ 0)$ will be of particular relevance in Section 5. Due to our limit point assumption at $\pm\infty$ in Hypothesis 4.2, no additional boundary condition at $\pm\infty$ needs to be introduced when considering (4.1) on the half-lines $[x_0, \infty)$ and $(-\infty, x_0]$. The resulting full-line and half-line Hamiltonian systems are said to be self-adjoint on $\mathbb{R}$, $[x_0, \infty)$, and $(-\infty, x_0]$, respectively (assuming of course a boundary condition of the type (4.5) in the two half-line cases).

Next we digress a bit and briefly turn to Nevanlinna–Herglotz functions and their representations in terms of measures, the focal point of Weyl–Titchmarsh theory (and hence spectral theory) of self-adjoint Hamiltonian systems.

Definition 4.3. — *Any analytic map* $m\colon \mathbb{C}_+ \to \mathbb{C}$ *is called a* Nevanlinna–Herglotz *function if* $\mathrm{Im}(m(z)) \geqslant 0$ *for all* $z \in \mathbb{C}_+$ *(here* $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$*). Similarly, any analytic map* $M\colon \mathbb{C}_+ \to \mathbb{C}^{k\times k}$, $k \in \mathbb{N}$, *is called a* $k\times k$ *matrix-valued Nevanlinna–Herglotz function if* $\mathrm{Im}(M(z)) \geqslant 0$ *for all* $z \in \mathbb{C}_+$.

Nevanlinna–Herglotz functions are characterized by a representation of the form

$$m(z)=a+bz+\int_{-\infty}^{\infty} d\omega(\lambda)\left((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\right), \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{4.7}$$

$$a\in\mathbb{R},\ b\geqslant 0, \quad \int_{-\infty}^{\infty} d\omega(\lambda)\,(1+\lambda^2)^{-1}<\infty, \tag{4.8}$$

$$\omega((\lambda_1,\lambda_2])=\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\nu\,\mathrm{Im}\left(m(\nu+i\epsilon)\right), \tag{4.9}$$

in the following sense: Every Nevanlinna–Herglotz function admits a representation of the type (4.7), (4.8) and conversely, any function of the type (4.7), (4.8) is a Nevanlinna–Herglotz function. Moreover, local singularities and zeros of m are necessarily located on the real axis and at most of first order in the sense that

$$\omega(\{\lambda\})=\lim_{\varepsilon\downarrow 0}(\omega(\lambda+\varepsilon)-\omega(\lambda-\varepsilon))=-\lim_{\varepsilon\downarrow 0} i\varepsilon\, m(\lambda+i\varepsilon)\geqslant 0, \quad \lambda\in\mathbb{R}, \tag{4.10}$$

$$\lim_{\varepsilon\downarrow 0} i\varepsilon\, m(\lambda+i\varepsilon)^{-1}\geqslant 0, \quad \lambda\in\mathbb{R}. \tag{4.11}$$

In particular, isolated poles of m are simple and located on the real axis, the corresponding residues being negative. Analogous results hold for matrix-valued Nevanlinna–Herglotz functions (see, e.g., [23] and the literature cited therein).

For subsequent purpose in Section 5 we also note that $-z^{-1}$ is a Nevanlinna–Herglotz function and that compositions of Nevanlinna–Herglotz functions remain Nevanlinna–Herglotz functions (as long as this composition is well-defined). In addition, diagonal elements of a matrix-valued Nevanlinna–Herglotz function are Nevanlinna–Herglotz functions.

Returning to Hamiltonian systems on half-lines $[x_0,\pm\infty)$ satisfying Hypotheses 4.1 and 4.2, we now denote by $\Psi_\pm(z,x,x_0)$ the unique solution of (4.1) satisfying $\Psi_\pm(z,\,\cdot\,,x_0)\in L^2_A([x_0,\pm\infty))$, $z\in\mathbb{C}\backslash\mathbb{R}$, normalized by $\psi_{1,\pm}(z,x_0,x_0)=1$. Then the half-line Weyl–Titchmarsh function $m_\pm(z,x)$, associated with the Hamiltonian system (4.1) on $[x,\pm\infty)$ and the fixed boundary condition $\beta_0=(1\ 0)$ at the point $x\in\mathbb{R}$, is defined by

$$m_\pm(z,x)=\psi_{2,\pm}(z,x,x_0)/\psi_{1,\pm}(z,x,x_0), \quad z\in\mathbb{C}\backslash\mathbb{R},\ \pm x\geqslant\pm x_0. \tag{4.12}$$

The actual normalization of $\Psi_\pm(z,x,x_0)$ was chosen for convenience only and is clearly immaterial in the definition of $m_\pm(z,x)$ in (4.12). For later use in Section 5 we also recall that

$$\begin{aligned} \Psi_{\pm}(z,x,x_0) &= \begin{pmatrix} \psi_{1,\pm}(z,x,x_0) \\ \psi_{2,\pm}(z,x,x_0) \end{pmatrix} \\ &= \begin{pmatrix} \vartheta_1(z,x,x_0) & \varphi_1(z,x,x_0) \\ \vartheta_2(z,x,x_0) & \varphi_2(z,x,x_0) \end{pmatrix} \begin{pmatrix} 1 \\ m_{\pm}(z,x_0) \end{pmatrix}, \end{aligned} \tag{4.13}$$

with $\vartheta_j(z,x,x_0)$, and $\varphi_j(z,x,x_0)$, $j=1,2$, defined such that

$$\Upsilon(z,x,x_0) = \begin{pmatrix} \vartheta_1(z,x,x_0) & \varphi_1(z,x,x_0) \\ \vartheta_2(z,x,x_0) & \varphi_2(z,x,x_0) \end{pmatrix} \tag{4.14}$$

represents a normalized fundamental system of solutions of (4.1) at some $x_0 \in \mathbb{R}$, satisfying

$$\Upsilon(z,x_0,x_0) = I_2. \tag{4.15}$$

One recalls that for fixed $x, x_0 \in \mathbb{R}$,

$$\vartheta_j(z,x,x_0) \text{ and } \varphi_j(z,x,x_0),\ j=1,2, \text{ are entire in } z \in \mathbb{C}. \tag{4.16}$$

In addition, one verifies that $m_{\pm}(z,x)$ satisfies the following Riccati-type differential equation,

$$\begin{aligned} m'(z,x) &+ [b_{2,2}(x) + a_{2,2}(x)z]m(z,x)^2 \\ &+ [b_{1,2}(x) + b_{2,1}(x) + (a_{1,2}(x) + a_{2,1}(x))z]m(z,x) \\ &+ b_{1,1}(x) + a_{1,1}(x)z = 0. \end{aligned} \tag{4.17}$$

Finally, the 2×2 Weyl–Titchmarsh matrix $M(z,x)$ associated with the Hamiltonian system (4.1) on $\mathbb{R}$ is then defined in terms of the half-line Weyl–Titchmarsh functions $m_{\pm}(z,x)$ by

$$M(z,x) = \big(M_{j,j'}(z,x)\big)_{j,j'=1,2}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{4.18}$$

$$\begin{aligned} M_{1,1}(z,x) &= [m_-(z,x) - m_+(z,x)]^{-1}, \\ M_{1,2}(z,x) &= M_{2,1}(z,x) \\ &= 2^{-1}[m_-(z,x) - m_+(z,x)]^{-1}[m_-(z,x) + m_+(z,x)], \\ M_{2,2}(z,x) &= [m_-(z,x) - m_+(z,x)]^{-1} m_-(z,x) m_+(z,x). \end{aligned} \tag{4.19}$$

One verifies that $M(z,x)$ is a 2×2 matrix-valued Nevanlinna–Herglotz function. We emphasize that for any fixed $x_0 \in \mathbb{R}$, $M(z,x_0)$ contains all the spectral information of the self-adjoint Hamiltonian system (4.1) on $\mathbb{R}$ (assuming Hypotheses 4.1 and 4.2).

5. Real-Valued Algebro-Geometric CH-2 Solutions and the Associated Isospectral Torus

In our final and principal section we study real-valued algebro-geometric solutions of the CH-2 hierarchy associated with curves $\mathcal{K}_n$ whose affine part is nonsingular and prove that the isospectral manifold of smooth bounded solutions of the nth stationary CH-2 equation can be characterized as a real n-dimensional torus $\mathbb{T}^n$. We focus on the stationary case as this is the primary concern in this context and briefly comment on the time-dependent case at the end of this section.

To study the direct spectral problem we first introduce the following assumptions.

Hypothesis 5.1. — *Suppose*

$$E_0 < E_1 < \cdots < E_{2n} < E_{2n+1}, \quad 0 \in (E_{2m_0}, E_{2m_0+1}) \tag{5.1}$$

for some $m_0 \in \{0, \dots, n\}$, and let u, w be a real-valued solution of the nth stationary CH-2 *equation* (2.28),

$$\text{s-CH-2}_n(u, w) = 0 \tag{5.2}$$

satisfying

$$u, w \in C^\infty(\mathbb{R}),\ w > 0,\ u^{(m)}, w^{(m)} \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0. \tag{5.3}$$

We start by noticing that the basic stationary equation (3.29),

$$\Psi_x(-z, x) = U(-z, x)\Psi(-z, x), \quad \Psi = (\psi_1, \psi_2)^\top,\ (z, x) \in \mathbb{C} \times \mathbb{R}, \tag{5.4}$$

is equivalent to the following Hamiltonian (canonical) system

$$J\widetilde{\Psi}_x(\tilde{z}, x) = [\tilde{z}A(x) + B(x)]\widetilde{\Psi}(\tilde{z}, x),\ \ \widetilde{\Psi} = \big(\widetilde{\psi}_1, \widetilde{\psi}_2\big)^\top,\ \big(\tilde{z}, x\big) \in \mathbb{C} \times \mathbb{R}, \tag{5.5}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\Psi}(\tilde{z}, x) = \Psi(-z, x), \quad \tilde{z} = -z^{-1}, \tag{5.6}$$

$$A(x) = \begin{pmatrix} \alpha(x)^2 + w(x) & -\alpha(x) \\ -\alpha(x) & 1 \end{pmatrix} > 0, \quad B(x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = B(x)^*. \tag{5.7}$$

In particular, due to assumptions (5.1)–(5.3), the Hamiltonian system (5.5) satisfies Hypotheses 4.1 and 4.2. Explicitly, the Hamiltonian system (5.5) boils down to

$$\widetilde{\psi}_{1,x}(\tilde{z}, x) = -\widetilde{\psi}_1(\tilde{z}, x) - \tilde{z}\alpha(x)\widetilde{\psi}_1(\tilde{z}, x) + \tilde{z}\widetilde{\psi}_2(\tilde{z}, x), \tag{5.8}$$

$$\widetilde{\psi}_{2,x}(\tilde{z}, x) = \widetilde{\psi}_2(\tilde{z}, x) + \tilde{z}\alpha(x)\widetilde{\psi}_2(\tilde{z}, x) - \tilde{z}\big(\alpha(x)^2 + w(x)\big)\widetilde{\psi}_1(\tilde{z}, x), \tag{5.9}$$

$$\big(z = -\tilde{z}^{-1}, x\big) \in \mathbb{C} \times \mathbb{R},$$

and upon eliminating $\widetilde{\psi}_2$ results in a particular case of a quadratic weighted Sturm–Liouville pencil (cf. [8], [10], [12]–[15], [34]) of the type

$$\begin{aligned} -\widetilde{\psi}_{1,xx}(\widetilde{z},x)+\widetilde{\psi}_1(\tilde{z},x) = \tilde{z}^2 w(x)\widetilde{\psi}_1(\tilde{z},x) - \tilde{z}(4u(x)-u_{xx}(x))\widetilde{\psi}_1(\tilde{z},x),& \\ \big(z=-\tilde{z}^{-1},x\big)\in\mathbb{C}\times\mathbb{R}.& \end{aligned} \tag{5.10}$$

Introducing

$$\Sigma_n = \bigcup_{\ell=0}^{n} [E_{2\ell}, E_{2\ell+1}], \tag{5.11}$$

we define

$$\begin{aligned} R_{2n+2}(\lambda)^{1/2} &= |R_{2n+2}(\lambda)^{1/2}| \\ &\times \begin{cases} -1 & \text{for } \lambda\in(E_{2n+1},\infty), \\ (-1)^{n+j} & \text{for } \lambda\in(E_{2j-1},E_{2j}),\ j=1,\dots,n, \\ (-1)^n & \text{for } \lambda\in(-\infty,E_0), \\ i(-1)^{n+j+1} & \text{for } \lambda\in(E_{2j},E_{2j+1}),\ j=0,\dots,n, \end{cases} \quad \lambda\in\mathbb{R}, \end{aligned} \tag{5.12}$$

and

$$R_{2n+2}(\lambda)^{1/2} = \lim_{\varepsilon\downarrow 0} R_{2n+2}(\lambda+i\varepsilon)^{1/2}, \quad \lambda\in\Sigma_n, \tag{5.13}$$

and analytically continue $R_{2n+2}(\,\cdot\,)^{1/2}$ to $\mathbb{C}\backslash\Sigma_n$. We also note the property

$$\overline{R_{2n+2}(\overline{z})^{1/2}} = R_{2n+2}(z)^{1/2}. \tag{5.14}$$

For notational convenience we will occasionally call (E_{2j-1},E_{2j}) for $j = 1,\dots,n$, spectral gaps and E_{2j-1},E_{2j} the corresponding spectral gap end-points.

Next, we introduce the cut plane

$$\Pi_n = \mathbb{C}\backslash\Sigma_n, \tag{5.15}$$

and the upper, respectively, lower sheets $\Pi_{n,\pm}$ of $\mathcal{K}_n$ by

$$\Pi_{n,\pm} = \big\{(z,\pm R_{2n+2}(z)^{1/2})\in\mathcal{K}_n \mid z\in\Pi_n\big\}, \tag{5.16}$$

with the associated charts

$$\zeta_\pm\colon \Pi_{n,\pm}\to\Pi_n, \quad P=(z,\pm R_{2n+2}(z)^{1/2})\mapsto z. \tag{5.17}$$

The two branches $\Psi_\pm(z,x,x_0)$ of the Baker–Akhiezer vector $\Psi(P,x,x_0)$ in (3.22) are then given by

$$\Psi_\pm(z,x,x_0) = \Psi(P,x,x_0), \quad P=(z,y)\in\Pi_{n,\pm}, \quad \Psi_\pm = (\psi_{1,\pm},\psi_{2,\pm})^\top, \tag{5.18}$$

and one infers from (3.50) (note that the error term is uniform in x in the case where $\mu_j(x)$ remains within its respective gaps) that

$$\psi_{1,\pm}(z,\,\cdot\,,x_0) \in L^2((x_0,\mp\infty)) \text{ for } |z| \text{ sufficiently large.} \tag{5.19}$$

Thus, introducing

$$\widetilde{\Psi}_\pm(\tilde{z},x,x_0) = \Psi_\mp(-z,x,x_0), \quad \widetilde{\Psi}_\pm = \big(\widetilde{\psi}_{1,\pm},\widetilde{\psi}_{2,\pm}\big)^\top, \quad \tilde{z} = -z^{-1}, \tag{5.20}$$

and the two branches $\phi_\pm(z,x)$ of $\phi(P,x)$ on $\Pi_{n,\pm}$ by

$$\phi_\pm(z,x) = \phi(P,x), \quad P=(z,y)\in\Pi_{n,\pm}, \tag{5.21}$$

one infers from (4.12) and (5.19) that the Weyl–Titchmarsh functions $\widetilde{m}_\pm(\tilde{z},x)$ associated with the self-adjoint Hamiltonian system (5.5) on the half-lines $[x,\pm\infty)$ and the Dirichlet boundary condition indexed by $\beta_0 = (1\ 0)$ at the point $x\in\mathbb{R}$, are given by

$$\begin{aligned}\widetilde{m}_\pm(\tilde{z},x) &= \widetilde{\psi}_{2,\pm}(\widetilde{z},x,x_0)/\widetilde{\psi}_{1,\pm}(\tilde{z},x,x_0)\\ &= \psi_{2,\mp}(-z,x,x_0)/\psi_{1,\mp}(-z,x,x_0)\\ &= -\phi_\mp(-z,x), \quad z\in\mathbb{C}\backslash\Sigma_n.\end{aligned} \tag{5.22}$$

More precisely, (5.19) yields (5.22) only for $|z|$ sufficiently large. However, since by general principles $\widetilde{m}_\pm(\,\cdot\,,x)$ are analytic in $\mathbb{C}\backslash\mathbb{R}$, and by (3.16), $\phi_\pm(\,\cdot\,,x)$ are analytic in $\mathbb{C}\backslash\Sigma_n$, one infers (5.22). An application of (4.13) and (4.16) then shows that (5.19) extends to all $z\in\mathbb{C}\backslash\Sigma_n$, that is,

$$\psi_{1,\pm}(z,\,\cdot\,,x_0) \in L^2((x_0,\mp\infty)), \quad z\in\mathbb{C}\backslash\Sigma_n. \tag{5.23}$$

Next, we mention a useful fact concerning a special class of Nevanlinna–Herglotz functions closely related to the problem at hand. The result must be well-known to experts, but since we could not quickly locate a proof in the literature, we provide the simple contour integration argument below.

Lemma 5.2. — *Assume that P_N is a monic polynomial of degree N. Then $P_N/R_{2n+2}^{1/2}$, respectively, $-P_N/R_{2n+2}^{1/2}$ is a Nevanlinna–Herglotz function if and only if one of the following three cases applies:*

(i) *$N=n$ and*

$$P_n(z) = \prod_{j=1}^n (z-a_j), \quad a_j\in[E_{2j-1},E_{2j}],\ j=1,\dots,n. \tag{5.24}$$

If (5.24) is satisfied, then $P_n/R_{2n+2}^{1/2}$ admits the Nevanlinna–Herglotz representation

$$\frac{P_n(z)}{R_{2n+2}(z)^{1/2}} = \frac{1}{\pi}\int_{\Sigma_n} \frac{|P_n(\lambda)|\,d\lambda}{|R_{2n+2}(\lambda)^{1/2}|}\frac{1}{\lambda-z}, \quad z\in\mathbb{C}\backslash\Sigma_n. \tag{5.25}$$

(ii) $N = n+1$ *and*

$$P_{n+1}(z) = \prod_{\ell=0}^{n}(z - b_\ell), \qquad b_0 \in (-\infty, E_0],\ b_j \in [E_{2j-1}, E_{2j}],\ j = 1, \dots, n. \tag{5.26}$$

If (5.26) *is satisfied, then* $P_{n+1}/R_{2n+2}^{1/2}$ *admits the Nevanlinna–Herglotz representation*

$$\frac{P_{n+1}(z)}{R_{2n+2}(z)^{1/2}} = \mathrm{Re}\bigg(\frac{P_{n+1}(i)}{R_{2n+2}(i)^{1/2}}\bigg) + \frac{1}{\pi}\int_{\Sigma_n} \frac{|P_{n+1}(\lambda)|\, d\lambda}{|R_{2n+2}(\lambda)^{1/2}|}\bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\bigg), \ z \in \mathbb{C}\backslash\Sigma_n. \tag{5.27}$$

(iii) $N = n+1$ *and*

$$P_{n+1}(z) = \prod_{\ell=0}^{n}(z - d_\ell), \qquad d_0 \in [E_{2n+1}, \infty),\ d_j \in [E_{2j-1}, E_{2j}],\ j = 1, \dots, n. \tag{5.28}$$

If (5.28) *is satisfied, then* $-P_{n+1}/R_{2n+2}^{1/2}$ *admits the Nevanlinna–Herglotz representation*

$$\frac{-P_{n+1}(z)}{R_{2n+2}(z)^{1/2}} = -\,\mathrm{Re}\bigg(\frac{P_{n+1}(i)}{R_{2n+2}(i)^{1/2}}\bigg) + \frac{1}{\pi}\int_{\Sigma_n} \frac{\big|P_{n+1}(\lambda)\big|\, d\lambda}{|R_{2n+2}(\lambda)^{1/2}|}\bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\bigg), \ z \in \mathbb{C}\backslash\Sigma_n. \tag{5.29}$$

Proof. — Except for Case (iii), this has been proven in [22, Lem. 5.1]. For convenience of the reader we repeat the argument here. Since Nevanlinna–Herglotz functions are $O(z)$ as $|z| \to \infty$ and cannot vanish faster than $O(z^{-1})$ as $|z| \to \infty$, we can confine ourselves to the range $N \in \{n, n+1, n+2\}$. We start with the case $N = n$ and employ the following contour integration approach. Consider a closed oriented contour $\Gamma_{R,\varepsilon}$ which consists of the clockwise oriented semicircle $C_\varepsilon = \{z \in \mathbb{C} \,|\, z = E_0 - \varepsilon \exp(-i\theta),\ -\pi/2 \leqslant \theta \leqslant \pi/2\}$ centered at E_0, the straight line $L_+ = \{z \in \mathbb{C}_+ \,|\, z = E_0 + x + i\varepsilon,\ 0 \leqslant x \leqslant R\}$ (oriented from left to right), the following part of the counterclockwise oriented circle of radius $(R^2+\varepsilon^2)^{1/2}$ centered at E_0, $C_R = \{z \in \mathbb{C} \,|\, z = E_0 + (R^2+\varepsilon^2)^{1/2} \exp(i\theta),\ \arctan(\varepsilon/R) \leqslant \theta \leqslant 2\pi - \arctan(\varepsilon/R)\}$, and the straight line $L_- = \{z \in \mathbb{C}_- \,|\, z = E_0 + x - i\varepsilon,\ 0 \leqslant x \leqslant R\}$ (oriented from right to left). Then, for $\varepsilon > 0$ small enough and

$R > 0$ sufficiently large, one infers

$$\begin{aligned}\frac{P_n(z)}{R_{2n+2}(z)^{1/2}} &= \frac{1}{2\pi i}\oint_{\Gamma_{R,\varepsilon}} \frac{1}{\zeta - z}\frac{P_n(\zeta)}{R_{2n+2}(\zeta)^{1/2}}d\zeta \\ &\underset{\varepsilon\downarrow 0, R\uparrow\infty}{=} \frac{1}{\pi}\int_{\Sigma_n}\frac{1}{\lambda - z}\frac{P_n(\lambda)d\lambda}{iR_{2n+2}(\lambda)^{1/2}}.\end{aligned} \tag{5.30}$$

Here we used (5.12) to compute the contributions of the contour integral along $[E_0, R]$ in the limit $\varepsilon \downarrow 0$ and note that the integral over C_R tends to zero as $R \uparrow \infty$ since

$$\frac{P_n(\zeta)}{R_{2n+2}(\zeta)^{1/2}} \underset{\zeta\to\infty}{=} O\big(|\zeta|^{-1}\big). \tag{5.31}$$

Next, utilizing the fact that P_n is monic and using (5.12) again, one infers that $P_n(\lambda)d\lambda/[iR_{2n+2}(\lambda)^{1/2}]$ represents a positive measure supported on Σ_n if and only if P_n has precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $j = 1, \dots, n$. In other words,

$$\frac{P_n(\lambda)}{iR_{2n+2}(\lambda)^{1/2}} = \frac{|P_n(\lambda)|}{|R_{2n+2}(\lambda)^{1/2}|} \geqslant 0 \ \text{ on } \Sigma_n \tag{5.32}$$

if and only if P_n has precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $j = 1, \dots, n$. The Nevanlinna–Herglotz representation (4.7), (4.8) then finishes the proof of (5.25).

In the case where $N = n+1$, the proofs of (5.26) and (5.28) follow along similar lines taking into account the additional residues at $\pm i$ inside $\Gamma_{R,\varepsilon}$ which are responsible for the real parts on the right-hand sides of (5.27) and (5.29).

Finally, in the case $N = n+2$, assume that $P_{n+2}/R_{2n+2}^{1/2}$ is a Nevanlinna–Herglotz function. Then for some $a \in \mathbb{R}$, $b \geqslant 0$, and some finite (positive) measure ω supported on $[E_0, E_{2n+2}]$,

$$\frac{P_{n+2}(z)}{R_{2n+2}(z)^{1/2}} = a + bz + \int_{E_0}^{E_{2n+2}} d\omega(\lambda)\,(\lambda - z)^{-1}, \quad z \in \mathbb{C}\backslash\Sigma_n, \tag{5.33}$$

since

$$\lim_{\varepsilon\downarrow 0}\operatorname{Im}(P_{n+2}(\lambda)R_{2n+2}(\lambda+i\varepsilon)^{-1/2}) = 0 \quad \text{for } \lambda > E_{2n+2} \text{ and } \lambda < E_0. \tag{5.34}$$

In particular, (5.33) implies

$$P_{n+2}(z)R_{2n+2}(z)^{-1/2} \underset{|z|\to\infty}{=} bz + O(1), \quad b \geqslant 0. \tag{5.35}$$

However, by (5.12), one immediately infers

$$P_{n+2}(\lambda)R_{2n+2}(\lambda)^{-1/2} \underset{\lambda\uparrow\infty}{=} -\lambda + O(1). \tag{5.36}$$

This contradiction dispenses with the case $N = n+2$. □

Now we are in position to state the following result concerning the half-line and full-line Weyl–Titchmarsh functions associated with the self-adjoint Hamiltonian system (5.5). We denote by $\widetilde{m}_\pm(\tilde z, x)$ the Weyl–Titchmarsh m-functions corresponding to (5.5) associated with the half-lines $(x, \pm\infty)$ and the Dirichlet boundary condition indexed by $\beta_0 = (1\ 0)$ at the point $x \in \mathbb{R}$, and by $\widetilde{M}(\tilde z, x)$ the 2×2 Weyl–Titchmarsh matrix corresponding to (5.5) on $\mathbb{R}$ (cf. (4.12), (4.18), and (4.19)). Moreover, Σ_n^o denotes the open interior of Σ_n and the real part of a matrix M is defined as usual by $\operatorname{Re}(M) = (M + M^*)/2$.

Theorem 5.3. — *Assume Hypothesis 5.1 and let* $(z, x) \in (\mathbb{C}\backslash\Sigma_n) \times \mathbb{R}$, $\tilde z = -z^{-1}$. *Then*

$$\widetilde{m}_\pm(\tilde z, x) = \frac{\pm R_{2n+2}(-z)^{1/2} + G_{n+1}(-z,x)}{F_n(-z,x)} \tag{5.37}$$

$$\begin{aligned} &= \pm z - z + c_1 + \sum_{j=1}^n \mu_j(x) \pm \operatorname{Re}\left(\frac{R_{2n+2}(-i)^{1/2}}{iF_n(-i,x)}\right) \\ &\quad - \sum_{j=1}^n \frac{G_{n+1}(-\mu_j(x),x)\big[1 \mp \widetilde{\varepsilon}_j(x)\big]}{F_{n,z}(-\mu_j(x),x)} \frac{1}{z+\mu_j(x)} \\ &\quad \pm \frac{1}{\pi}\int_{\Sigma_n} \frac{|R_{2n+2}(-\lambda)^{1/2}|\, d\lambda}{|F_n(-\lambda,x)|}\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \end{aligned} \tag{5.38}$$

where $\widetilde{\varepsilon}_j(x) \in \{1, -1\}$, $j = 1, \dots, n$, *is chosen such that*

$$\frac{G_{n+1}(-\mu_j(x),x)\widetilde{\varepsilon}_j(x)}{F_{n,z}(-\mu_j(x),x)} \geqslant 0, \quad j = 1, \dots, n. \tag{5.39}$$

Moreover,

$$\widetilde{M}(\tilde z, x) = \frac{-1}{2R_{2n+2}(-z)^{1/2}}\begin{pmatrix} F_n(-z,x) & G_{n+1}(-z,x) \\ G_{n+1}(-z,x) & -H_n(-z,x)\end{pmatrix} \tag{5.40}$$

$$= \operatorname{Re}\big(\widetilde{M}(-i,x)\big) + \int_{\Sigma_n} d\Omega(\lambda,x)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \tag{5.41}$$

where

$$\Omega(\lambda,x) = \frac{-1}{2\pi i R_{2n+2}(-\lambda)^{1/2}}\begin{pmatrix} F_n(-\lambda,x) & G_{n+1}(-\lambda,x) \\ G_{n+1}(-\lambda,x) & -H_n(-\lambda,x)\end{pmatrix}, \quad \lambda \in \Sigma_n^o. \tag{5.42}$$

The essential spectrum of the Hamiltonian systems (5.5) *on* $[x, \pm\infty)$ *(with any self-adjoint boundary condition at* x*) as well as the essential spectrum* *of the Hamiltonian system* (5.5) *on* $\mathbb{R}$ *is purely absolutely continuous and given by*

$$\big(-\infty, -E_{2m_0+1}^{-1}\big] \cup \bigcup_{\substack{\ell=0\\ \ell\neq m_0}}^{n-1} \big[-E_{2l}^{-1}, -E_{2\ell+1}^{-1}\big] \cup \big[-E_{2m_0}^{-1}, \infty\big). \tag{5.43}$$

The spectral multiplicities are simple in the half-line cases and of uniform multiplicity two in the full-line case.

Proof. — Equation (5.37) follows from (3.16), (5.12), and (5.22). Equation (5.40) is then a consequence of (3.26)–(3.28), (4.18), (4.19), (5.22), and (5.37). Different self-adjoint boundary conditions at the point x lead to different half-line Hamiltonian systems whose Weyl–Titchmarsh functions are related by a linear fractional transformation (cf., e.g., [9]), which leads to the invariance of the essential spectrum with respect to the boundary condition at x. In order to prove the Nevanlinna–Herglotz representation (5.38) one can follow the corresponding computation for Schrödinger operators with algebro-geometric potentials in [44, Sect. 8.1]. For this purpose one first notes that by (5.27) also $-R_{2n+2}(z)^{1/2}/F_n(z,x)$ is a Nevanlinna–Herglotz function. A contour integration as in the proof of Lemma 5.2 then proves

$$\begin{aligned} \frac{R_{2n+2}(z)^{1/2}}{F_n(z,x)} &= z+\mathrm{Re}\bigg(\frac{R_{2n+2}(i)^{1/2}}{F_n(i,x)}\bigg)-\sum_{j=1}^{n}\frac{|R_{2n+2}(\mu_j(x))^{1/2}|}{|F_{n,z}(\mu_j(x),x)|}\,\frac{1}{z-\mu_j(x)} \\ &\quad+\frac{1}{\pi}\int_{\Sigma_n}\frac{|R_{2n+2}(\lambda)^{1/2}|\,d\lambda}{|F_n(\lambda,x)|}\bigg(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\bigg) \end{aligned} \tag{5.44}$$

$$\begin{aligned} &= z+\mathrm{Re}\bigg(\frac{R_{2n+2}(i)^{1/2}}{F_n(i,x)}\bigg)-\sum_{j=1}^{n}\frac{G_{n+1}(\mu_j(x),x)\varepsilon_j(x)}{F_{n,z}(\mu_j(x),x)}\,\frac{1}{z-\mu_j(x)} \\ &\quad+\frac{1}{\pi}\int_{\Sigma_n}\frac{|R_{2n+2}(\lambda)^{1/2}|\,d\lambda}{|F_n(\lambda,x)|}\bigg(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\bigg). \end{aligned} \tag{5.45}$$

The only difference compared to the corresponding argument in the proof of Lemma 5.2 concerns additional (approximate) semicircles of radius ε centered at each $\mu_j(x)$, $j=1,\dots,n$, in the upper and lower complex half-planes. Whenever, $\mu_j(x)\in(E_{2j-1},E_{2j})$, the limit $\varepsilon\downarrow 0$ picks up a residue contribution displayed in the sum over j in (5.44). This contribution vanishes, however, if $\mu_j(x)\in\{E_{2j-1},E_{2j}\}$. In this case $F_{n,z}(\mu_j(x),x)\neq 0$ by (4.10) and $R_{2n+2}(\mu_j(x))=0$ by (2.19). Equation (5.45) then follows from (3.7) and the sign of $\varepsilon_j(x)$ must be chosen according to

$$\frac{G_{n+1}(\mu_j(x),x)\varepsilon_j(x)}{F_{n,z}(\mu_j(x),x)} \geqslant 0, \quad j=1,\dots,n, \tag{5.46}$$

in order to guarantee nonpositive residues in (5.45) (cf. (4.10)).

In order to analyze the term G_{n+1}/F_n in $\widetilde{m}_\pm$ we turn to Lagrange-type interpolation formulas. If Q_{n-1} is a polynomial of degree $n-1$, then

$$Q_{n-1}(z) = F_n(z)\sum_{j=1}^n \frac{Q_{n-1}(\mu_j)}{F_{n,z}(\mu_j)}\frac{1}{z-\mu_j}, \quad z\in\mathbb{C}. \tag{5.47}$$

Since F_n and G_{n+1} are monic polynomials of degree n and $n+1$, respectively, and $g_1 = c_1$ (cf. (2.8)), we can apply (5.47) to $Q_{n-1} = G_{n+1} - z^{n+1} - c_1F_n$ and hence obtain via (5.47),

$$\frac{G_{n+1}(z,x)}{F_n(z,x)} = \frac{z^{n+1}}{F_n(z,x)} + c_1 + \sum_{j=1}^n \frac{G_{n+1}(\mu_j(x),x)-\mu_j(x)^{n+1}}{F_{n,z}(\mu_j(x),x)}\frac{1}{z-\mu_j(x)}. \tag{5.48}$$

Next we recall some more Lagrange-type interpolation formulas: if $\{\mu_j\}_1^n \subset \mathbb{C}$, $\mu_j \neq \mu_k$ for $j\neq k$, $j,k=1,\dots,n$, and $F_n(z)=\prod_{j=1}^n(z-\mu_j)$, then

$$F_{n,z}(z) = \sum_{j=1}^n \prod_{\substack{\ell=1\\ \ell\neq j}}^n (z-\mu_\ell), \quad F_{n,z}(\mu_j) = \prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell), \tag{5.49}$$

$$\sum_{j=1}^n \frac{\mu_j^{k-1}}{F_{n,z}(\mu_j)} = \delta_{k,n},\ k=1,\dots,n, \quad \sum_{j=1}^n \frac{\mu_j^n}{F_{n,z}(\mu_j)} = \sum_{j=1}^n \mu_j \tag{5.50}$$

(see, e.g., [20], [21, App. E]). Then

$$\begin{aligned}
&\frac{z^{n+1}}{F_n(z)} - \sum_{j=1}^n \frac{\mu_j^{n+1}}{F_{n,z}(\mu_j)}\frac{1}{z-\mu_j}\\
&\quad = \sum_{j=1}^n \frac{\big[z^{n+1}-\mu_j^{n+1}\big]}{F_{n,z}(\mu_j)}\frac{1}{z-\mu_j}\\
&\quad = \sum_{j=1}^n \frac{\big[z^n + z^{n-1}\mu_j + z^{n-2}\mu_j^2 + \cdots + z^2\mu_j^{n-2} + z\mu_j^{n-1} + \mu_j^n\big]}{F_{n,z}(\mu_j)}\\
&\quad = z + \sum_{j=1}^n \mu_j,
\end{aligned} \tag{5.51}$$

applying $a^{n+1}-b^{n+1}=(a-b)\big[a^n+a^{n-1}b+\dots+ab^{n-1}+b^n\big]$ and (5.50). Thus,

$$\frac{G_{n+1}(z,x)}{F_n(z,x)}=z+\sum_{j=1}^{n}\mu_j(x)+c_1+\sum_{j=1}^{n}\frac{G_{n+1}(\mu_j(x),x)}{F_{n,z}(\mu_j(x),x)}\frac{1}{z-\mu_j(x)}. \tag{5.52}$$

Equivalently, employing (3.7), (3.15) for c_1, the trace formula (3.42) for u, and (5.49),

$$G_{n+1}(z)-(z+2u)F_n(z)=-\sum_{j=1}^{n}\frac{y(\hat{\mu}_j)}{\prod_{\substack{\ell=1\\ \ell\neq j}}(\mu_j-\mu_\ell)}\prod_{\substack{k=1\\ k\neq j}}^{n}(z-\mu_k). \tag{5.53}$$

Alternatively, (5.53) can be proved directly as follows: By the asymptotic behavior (cf. (3.59) for a refinement),

$$G_{n+1}(z)-(z+2u)F_n(z)\underset{|z|\to\infty}{=}O\big(|z|^{n-1}\big), \tag{5.54}$$

both sides of (5.53) are polynomials in z of degree $n-1$ which coincide (with the value $-y(\hat{\mu}_j)$ applying (3.7) again) at the n points μ_j, $j=1,\dots,n$.

Employing (5.52) then yields

$$\begin{aligned}\frac{\mp R_{2n+2}(z)^{1/2}+G_{n+1}(z,x)}{F_n(z,x)}&=\frac{\mp R_{2n+2}(z)^{1/2}}{F_n(z,x)}+z+\sum_{j=1}^{n}\mu_j(x)+c_1\\&\quad+\sum_{j=1}^{n}\frac{G_{n+1}(\mu_j(x),x)\{\varepsilon_j(x)+[1-\varepsilon_j(x)]\}}{F_{n,z}(\mu_j(x),x)}\frac{1}{z-\mu_j(x)},\end{aligned} \tag{5.55}$$

and hence (5.38) follows by inserting (5.45) into (5.55) and changing z into $-z$. Equations (5.41) and (5.42) are clear from the matrix analog of (4.9).

The statement (5.43) for the essential half-line spectra then follows from the fact that the measure in the Nevanlinna–Herglotz representation (5.38) of $\widetilde{m}_\pm$ (as a function of z) is supported on the set Σ_n in (5.11), with a strictly positive density on the open interior Σ_n^o of Σ_n. The transformation $z\to -z^{-1}$ then yields (5.43) and since half-line spectra with a regular endpoint x have always simple spectra this completes the proof of our half-line spectral assertions. The full-line case follows in exactly the same manner since the corresponding 2×2 matrix-valued measure Ω in the Nevanlinna–Herglotz representation (5.41) of $\widetilde{M}$ (as a function of z) also has support Σ_n and rank equal to two on Σ_n^o. $\square$

Returning to direct spectral theory, we note that the two spectral problems (5.5) on $\mathbb{R}$ associated with the vanishing of the first and second component of $\widetilde{\Psi}$ at x, respectively, are clearly self-adjoint since they correspond to the choices $\beta = (1\ 0)$ and $\beta = (0\ 1)$ in (4.5). Hence a comparison with (3.5), (3.32), and (3.33) necessarily yields $\{\mu_j(x)\}_{j=1,\dots,n}, \{\nu_j(x)\}_{j=1,\dots,n} \subset \mathbb{R}$. Thus we will assume the convenient eigenvalue orderings

$$\mu_j(x) < \mu_{j+1}(x), \quad \nu_j(x) < \nu_{j+1}(x) \quad \text{for } j = 1, \dots, n-1,\ x \in \mathbb{R}. \tag{5.56}$$

Combining Lemma 5.2 with the Nevanlinna–Herglotz property of the 2×2 Weyl–Titchmarsh matrix $\widetilde{M}(\,\cdot\,, x)$ then yields the following refinement of Lemma 3.3.

Theorem 5.4. — *Assume Hypothesis 5.1. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, with the projections $\mu_j(x)$, $j = 1, \dots, n$, the zeros of $F_n(\,\cdot\,, x)$ in* (3.5)*, satisfies the first-order system of differential equations* (3.38) *on $\Omega_\mu = \mathbb{R}$ and*

$$\hat{\mu}_j \in C^\infty(\mathbb{R}, \mathcal{K}_n), \quad j = 1, \dots, n. \tag{5.57}$$

Moreover,

$$\mu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n,\ x \in \mathbb{R}. \tag{5.58}$$

In particular, $\hat{\mu}_j(x)$ changes sheets whenever it hits E_{2j-1} or E_{2j} and its projection $\mu_j(x)$ remains trapped in $[E_{2j-1}, E_{2j}]$ for all $j = 1, \dots, n$ and $x \in \mathbb{R}$. The analogous statements apply to $\hat{\nu}_j(x)$ and one infers

$$\nu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n,\ x \in \mathbb{R}. \tag{5.59}$$

Proof. — Since $\widetilde{M}(\,\cdot\,, x)$ is a 2×2 Nevanlinna–Herglotz matrix, its diagonal elements are Nevanlinna–Herglotz functions. Thus,

$$\begin{aligned} \widetilde{M}_{1,1}(\tilde{z}, x) &= \frac{-F_n(-z, x)}{2R_{2n+2}(-z)^{1/2}}, \\ \widetilde{M}_{2,2}(\tilde{z}, x) &= \frac{H_n(-z, x)}{2R_{2n+2}(-z)^{1/2}}, \quad \tilde{z} = -z^{-1}, \end{aligned} \tag{5.60}$$

are Nevanlinna–Herglotz functions (the left-hand sides with respect to $\tilde{z}$, the right-hand sides with respect to z) and the interlacing properties (5.58), (5.59) then follow from (5.24) and (5.28). □

Remark 5.5. — The Nevanlinna–Herglotz property of $\widetilde{M}_{2,2}(\,\cdot\,,x)$ necessitates the inequality $h_0(x)<0$, $x\in\mathbb{R}$, which appears to be difficult to verify directly. In particular, the explicit expression (cf. (3.58) with $H=H_n$, and (3.59))

$$\begin{aligned}(5.61)\quad h_0 = \sum_{j=0}^{2n+1}\sum_{k=j+1}^{2n+1} E_jE_k - \left(\frac{1}{2}\sum_{j=0}^{2n+1} E_j\right)^2 \\ - \Bigg[\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{n} [\mu_{j_1}\mu_{j_2,x}+\mu_{j_1,x}\mu_{j_2}+2\mu_{j_1}\mu_{j_2}] - 2\alpha\sum_{j=1}^{n}\mu_j\Bigg]\end{aligned}$$

does not necessarily shed any light on this issue.

Remark 5.6. — The zeros $\mu_j(x)\in(E_{2j-1},E_{2j})$, $j=1,\dots,n$ of $F_n(\,\cdot\,,x)$ which are related to eigenvalues of the Hamiltonian system (5.5) on $\mathbb{R}$ associated with the boundary condition $\widetilde{\psi}_1(x)=0$, in fact, are related to left and right half-line eigenvalues of the corresponding Hamiltonian system restricted to the half-lines $(-\infty,x]$ and $[x,\infty)$, respectively. Indeed, by (5.20) and (5.23), depending on whether $\hat{\mu}_j(x)\in\Pi_{n,+}$ or $\hat{\mu}_j(x)\in\Pi_{n,-}$, $\mu_j(x)$ is related to a left or right half-line eigenvalue associated with the Dirichlet boundary condition $\widetilde{\psi}_1(x)=0$. A careful investigation of the sign of the right-hand sides of the Dubrovin equations (3.37) (combining (5.1), (5.12), and (5.16)), then proves that the $\mu_j(x)$ related to right (resp., left) half-line eigenvalues of the Hamiltonian system (5.5) associated with the boundary condition $\widetilde{\psi}_1(x)=0$, are strictly monotone increasing (resp., decreasing) with respect to x, as long as the μ_j stay away from the right (resp., left) endpoints of the corresponding spectral gaps (E_{2j-1},E_{2j}). Here we purposely avoided the limiting case where some of the $\mu_k(x)$ hit the boundary of the spectral gaps, $\mu_k(x)\in\{E_{2k-1},E_{2k}\}$, since the half-line eigenvalue interpretation is lost as there is no $L^2((x,\pm\infty))^2$ eigenfunction $\widetilde{\Psi}(x)$ satisfying $\widetilde{\psi}_1(x)=0$ in this case. In fact, whenever an eigenvalue $\mu_k(x)$ hits a spectral gap endpoint, the associated point $\hat{\mu}_j(x)$ on $\mathcal{K}_n$ crosses over from one sheet to the other (equivalently, the corresponding left half-line eigenvalue becomes a right half-line eigenvalue and vice versa) and accordingly, strictly increasing half-line eigenvalues become strictly decreasing half-line eigenvalues and vice versa. In particular, using the appropriate local coordinate $(z-E_{2k})^{1/2}$ (resp., $(z-E_{2k-1})^{1/2}$) near E_{2k} (resp., E_{2k-1}), one verifies that $\mu_k(x)$ does not pause at the endpoints E_{2k} and E_{2k-1}.

Next, we turn to the inverse spectral problem and determine the isospectral manifold of real-valued, smooth, and bounded CH-2 solutions.

Our basic assumptions then will be the following:

Hypothesis 5.7. — *Suppose*

$$E_0 < E_1 < \cdots < E_{2n} < E_{2n+1}, \quad 0 \in (E_{2m_0}, E_{2m_0+1}) \tag{5.62}$$

for some $m_0 \in \{0, \dots, n\}$*. In addition, fix* $x_0 \in \mathbb{R}$*, and assume that the initial data*

$$\{\hat{\mu}_j(x_0) = (\mu_j(x_0), -G_{n+1}(\mu_j(x_0), x_0))\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{5.63}$$

for the Dubrovin equations (3.38) *are constrained by*

$$\mu_j(x_0) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n. \tag{5.64}$$

Theorem 5.8. — *Assume Hypothesis 5.7. Then the Dubrovin initial value problem* (3.38), (5.63), (5.64) *has a unique solution* $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ *satisfying*

$$\hat{\mu}_j \in C^\infty(\mathbb{R}, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{5.65}$$

and the projections μ_j *remain trapped in the intervals* $[E_{2j-1}, E_{2j}]$*,* $j = 1, \dots, n$*, for all* $x \in \mathbb{R}$*,*

$$\mu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{5.66}$$

Moreover, u, w *defined by the formulas* (3.42), (3.43),

$$\begin{aligned} u(x) &= \frac{1}{2} \sum_{j=1}^{n} \mu_j(x) - \frac{1}{4} \sum_{m=0}^{2n+1} E_m, \\ w(x) &= -\left(\prod_{m=0}^{2n+1} E_m \right) \left(\prod_{j=1}^{n} \mu_j(x)^{-2} \right), \quad x \in \mathbb{R}, \end{aligned} \tag{5.67}$$

satisfy Hypothesis 5.1, that is, (u, w) *is a real-valued solution of the* n*th stationary* CH-2 *equation* (2.28)*,*

$$\text{s-CH-2}_n(u, w) = 0, \tag{5.68}$$

with integration constants c_ℓ *in* (5.68) *given by* $c_\ell = c_\ell(\underline{E})$*,* $\ell = 1, \dots, n$*, according to* (3.14), (3.15)*, satisfying*

$$u, w \in C^\infty(\mathbb{R}), \ w > 0, \ u^{(m)}, w^{(m)} \in L^\infty(\mathbb{R}), \ m \in \mathbb{N}_0. \tag{5.69}$$

Proof. — Given initial data constrained by $\mu_j(x_0) \in (E_{2j-1}, E_{2j})$, $j = 1, \dots, n$, one concludes from the Dubrovin equations (3.38) and the sign properties of $R_{2n+2}^{1/2}$ on the intervals $[E_{2k-1}, E_{2k}]$, $k = 1, \dots, n$, described in (5.12), that the solution $\mu_j(x)$ remains in the interval $[E_{2j-1}, E_{2j}]$ as long as $\hat{\mu}_j(x)$ stays away from the branch points $(E_{2j-1},0),(E_{2j},0)$. In case $\hat{\mu}_j$ hits such a branch point, one can use the local chart around $(E_m,0)$, with local coordinate $\zeta=\sigma(z-E_m)^{1/2}$, $\sigma\in\{1,-1\}$, $m\in\{2j-1,2j\}$, to verify (5.65) and (5.66). Relations (5.67), (5.69) are then evident from (5.65), (5.66), and

$$|\partial_x^k\mu_j(x)|\leqslant C_k,\quad k\in\mathbb{N}_0,\ j=1,\dots,n,\ x\in\mathbb{R}. \tag{5.70}$$

In the course of the proof of Theorem 3.7 one constructs the polynomial formalism (F_n, G_{n+1}, H_n, R_{2n+2}, etc.) and then obtains identity (3.43) as an elementary consequence. Finally, (5.68) also follows from Theorem 3.7 (with $\Omega_\mu=\mathbb{R}$). □

Corollary 5.9. — *Fix $\{E_m\}_{m=0,\dots,2n+1}\subset\mathbb{R}$ and assume the ordering*

$$E_0<E_1<\dots<E_{2n}<E_{2n+1},\quad 0\in(E_{2m_0},E_{2m_0+1}) \tag{5.71}$$

for some $m_0\in\{0,\dots,2n+1\}$. Then the isospectral manifold of smooth, real-valued solutions $u,w\in C^\infty(\mathbb{R})$, $w>0$, of s-CH-$2_n(u,w)=0$ *is given by the real n-dimensional torus $\mathbb{T}^n$. (These smooth solutions necessarily satisfy $u^{(m)},w^{(m)}\in L^\infty(\mathbb{R})$, $m\in\mathbb{N}_0$.)*

Proof. — The discussion in Remark 5.6 and Theorem 5.8, shows that the motion of each $\hat{\mu}_j(x)$ on $\mathcal{K}_n$ proceeds topologically on a circle and is uniquely determined by the initial data $\hat{\mu}_k(x_0)$, $k=1,\dots,n$. More precisely, the initial data

$$\begin{aligned}\hat{\mu}_j(x_0)=(\mu_j(x_0),y(\hat{\mu}_j(x_0)))=(\mu_j(x_0),-G_{n+1}(\mu_j(x_0),x_0)),\\ \mu_j(x_0)\in[E_{2j-1},E_{2j}],\quad j=1,\dots,n,\end{aligned} \tag{5.72}$$

are topologically equivalent to data of the type

$$(\mu_j(x_0),\sigma_j(x_0))\in[E_{2j-1},E_{2j}]\times\{+,-\},\quad j=1,\dots,n, \tag{5.73}$$

the sign of $\sigma_j(x_0)$ depending on $\hat{\mu}_j(x_0)\in\Pi_{n,\pm}$. If, on the other hand, some of the $\mu_k(x_0)\in\{E_{2k-1},E_{2k}\}$, then the determination of the sheet $\Pi_{n,\pm}$ and hence the sign $\sigma_k(x_0)$ in (5.73) becomes superfluous and is eliminated from (5.73). Indeed, since by (2.18),

$$G_{n+1}(\mu_j(x_0),x_0)^2=R_{2n+2}(\mu_j(x_0)), \tag{5.74}$$

$G_{n+1}(\mu_j(x_0),x_0)$ is determined up to a sign unless $\mu_j(x_0)$ hits a spectral gap endpoint E_{2j-1},E_{2j} in which case

$$G_{n+1}(\mu_j(x_0),x_0)=R_{2n+2}(\mu_j(x_0))=0 \tag{5.75}$$

and the sign ambiguity disappears. The n data in (5.73) (properly interpreted if $\mu_j(x_0)\in\{E_{2j-1},E_{2j}\}$) can be identified with circles. Since the latter are independent of each other, the isospectral manifold of real-valued, smooth, and bounded solutions of s-CH-$2_n(u) = 0$ is given by the real n-dimensional torus $\mathbb{T}^n$. □

In summary, one observes that the reality problem for smooth bounded solutions of the CH-2 hierarchy, assuming the ordering (5.62), parallels that of the KdV hierarchy with the basic self-adjoint Lax operator (the one-dimensional Schrödinger operator) replaced by the self-adjoint Hamiltonian system (5.5).

Remark 5.10. — Since the focus of this paper centered around the two-component Camassa–Holm hierarchy CH-2, we assumed $w > 0$ throughout Section 5. The limit $w \to 0$, although straightforward in connection with the material in Section 2, requires some care in Sections 3 and 5. Indeed, formula (5.67) for w indicates the singular nature of such a limit: one infers from the z^1-term in (2.18),

$$f_n\big[2\alpha g_n + h_{n-1} - (\alpha^2 + w) f_{n-1}\big] = -\sum_{m=0}^{2n+1} \prod_{\substack{m'=0\\ m'\neq m}}^{2n+1} E_{m'}, \tag{5.76}$$

and recalling $f_n = (-1)^n \prod_{j=1}^n \mu_j$, one concludes

$$\big[2\alpha g_n + h_{n-1} - (\alpha^2 + w) f_{n-1}\big] = (-1)^{n+1} \Bigg(\sum_{m=0}^{2n+1} \prod_{\substack{m'=0\\ m'\neq m}}^{2n+1} E_{m'} \Bigg) \prod_{j=1}^{n} \mu_j^{-1}. \tag{5.77}$$

The case $w = 0$ has been analyzed in detail in [20], [21, Ch. 5] and under the assumption $[4u - u_{xx}] > 0$, it is known that necessarily

$$E_0 < E_1 < \cdots < E_{2n_0} < E_{2n_0+1} = 0, \quad \mu_{j_0}(x) \in [E_{2j_0-1}, E_{2j_0}], \quad j_0 = 1, \dots, n_0, \tag{5.78}$$

with n_0 the (topological) genus of the underlying curve $\mathcal{K}_{n_0}$. (If $[4u - u_{xx}] < 0$ all E_m are reflected with respect to $E = 0$.) A comparison with the case when $w > 0$ and $[4u - u_{xx}] > 0$,

$$E_0 < E_1 < \cdots < E_{2n} < E_{2n+1}, \quad 0 \in (E_{2m_0}, E_{2m_0+1}), \quad \mu_j(x) \in [E_{2j-1}, E_{2j}],\ j = 1, \dots, n \tag{5.79}$$

for some $m_0 \in \{0, \dots, n\}$, shows that in order to guarantee a smooth limit $w \to 0$ in (5.77), the following situation must occur,

$$E_{2m_0+2k-1}, \mu_{m_0+k}, E_{2m_0+2k}, E_{2n+1} \downarrow 0 \text{ as } w \to 0, \quad k = 1, \dots, n - m_0. \tag{5.80}$$

The limit $w \to 0$ thus yields a singular curve $\mathcal{K}_n$ associated with

$$y^2 = R_{2n+2}(z) = \left(\prod_{m=0}^{2m_0+1}(z - E_m)\right) z^{2(n-m_0)}. \tag{5.81}$$

Desingularizing this curve yields $y^2 = \prod_{m=0}^{2m_0+1}(z - E_m)$ and hence corresponds to $m_0 = n_0$ in connection with (5.78).

Finally, we briefly turn to the time-dependent case.

HYPOTHESIS 5.11. — *Suppose that $u, w\colon \mathbb{R}^2 \to \mathbb{C}$ satisfy*

$$\begin{aligned} &u(\,\cdot\,,t),\, w(\,\cdot\,,t) \in C^\infty(\mathbb{R}), \quad \frac{\partial^m u}{\partial x^m}(\,\cdot\,,t),\, \frac{\partial^m w}{\partial x^m}(\,\cdot\,,t) \in L^\infty(\mathbb{R}), \\ &m \in \mathbb{N}_0, \;\; t \in \mathbb{R}, \;\; u(x,\,\cdot\,),\, u_x(x,\,\cdot\,),\, w(x,\,\cdot\,) \in C^1(\mathbb{R}), \;\; x \in \mathbb{R}. \end{aligned} \tag{5.82}$$

The basic problem in the analysis of algebro-geometric solutions of the CH-2 hierarchy consists in solving the time-dependent rth CH-2 flow with initial data a stationary solution of the nth equation in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, consider a solution $\left(u^{(0)}, w^{(0)}\right)$ of the nth stationary CH-2 equation, that is, s-CH-$2_n\left(u^{(0)}, w^{(0)}\right) = 0$ associated with $\mathcal{K}_n$ and a given set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution u, w of the rth CH-2 flow CH-$2_r(u,w) = 0$ with $u(t_{0,r}) = u^{(0)}$, $w(t_{0,r}) = w^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the integration constants in the definitions of the stationary and the time-dependent CH-2 equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde{V}_r$, $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_s$, $\tilde{g}_s$, $\tilde{h}_s$, $\tilde{c}_s$, etc., in order to distinguish them from V_n, F_n, G_{n+1}, H_n, f_ℓ, g_ℓ, h_ℓ, c_ℓ, etc., in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth CH-2 flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution u of

$$\widetilde{\text{CH-2}}_r(u,w) = \begin{pmatrix} u_{t_r} + \frac{1}{2}\tilde{f}_{r+1,x} \\ w_{t_r} - w_x\tilde{f}_r - 2w\tilde{f}_{r,x} \end{pmatrix} = 0, \tag{5.83}$$

$$u(x,t_{0,r}) = u^{(0)}(x), \quad w(x,t_{0,r}) = w^{(0)}(x), \quad x \in \mathbb{R},$$

$$\text{s-CH-}2_n\left(u^{(0)}, w^{(0)}\right) = \begin{pmatrix} \frac{1}{2} f^{(0)}_{n+1,x} \\ -w^{(0)}_x f^{(0)}_n - 2w^{(0)} f^{(0)}_{n,x} \end{pmatrix} = 0, \tag{5.84}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where (u,w) satisfy (5.82).

We pause for a moment to reflect on the pair of equations (5.83), (5.84): As it turns out (cf. [20], [21, Sect. 5.4] in the special case $w = 0$), it represents a dynamical system on the set of algebro-geometric solutions isospectral to the initial value $\big(u^{(0)}, w^{(0)}\big)$. By isospectrality we here allude to the fact that for any fixed $t_r \in \mathbb{R}$, the solution $(u(\,\cdot\,, t_r), w(\,\cdot\,, t_r))$ of (5.83), (5.84) is a stationary solution of (5.84),

$$\begin{aligned}(5.85)\quad \text{s-CH-2}_n(u(\,\cdot\,,t_r), w(\,\cdot\,,t_r)) &= \begin{pmatrix} \frac{1}{2} f^{(0)}_{n+1,x}(\,\cdot\,,t_r) \\ -w^{(0)}_x(\,\cdot\,,t_r) f^{(0)}_n(\,\cdot\,,t_r) - 2w^{(0)}(\,\cdot\,,t_r) f^{(0)}_{n,x}(\,\cdot\,,t_r) \end{pmatrix} \\ &= 0,\end{aligned}$$

associated with the fixed underlying algebraic curve $\mathcal{K}_n$. Put differently, the solution $(u(\,\cdot\,, t_r), w(\,\cdot\,, t_r))$ is an isospectral deformation of $\big(u^{(0)}, w^{(0)}\big)$ with t_r the corresponding deformation parameter. In addition, $(u(\,\cdot\,, t_r), w(\,\cdot\,, t_r))$ traces out a curve in the set of algebro-geometric solutions isospectral to $\big(u^{(0)}, w^{(0)}\big)$.

Thus, relying on this isospectral property of the CH-2 flows, we will go a step further and assume (5.84) not only at $t_r = t_{0,r}$ but for all $t_r \in \mathbb{R}$. Hence, we start with

$$(5.86)\qquad U_{t_r}(z,x,t_r) - \widetilde{V}_{r,x}(z,x,t_r) + [U(z,x,t_r), \widetilde{V}_r(z,x,t_r)] = 0,$$

$$(5.87)\qquad -V_{n,x}(z,x,t_r) + [U(z,x,t_r), V_n(z,x,t_r)] = 0, \quad (z,x,t_r) \in \mathbb{C}\times\mathbb{R}^2,$$

where (cf. (2.22))

$$\begin{aligned} U(z,x,t_r) &= -z^{-1}\begin{pmatrix} \alpha(x,t_r) & -1 \\ \alpha(x,t_r)^2 + w(x,t_r) & -\alpha(x,t_r)\end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}, \\ (5.88)\quad \widetilde{V}_r(z,x,t_r) &= z^{-1}\begin{pmatrix} -\widetilde{G}_{r+1}(z,x,t_r) & \widetilde{F}_r(z,x,t_r) \\ \widetilde{H}_r(z,x,t_r) & \widetilde{G}_{r+1}(z,x,t_r)\end{pmatrix}, \\ V_n(z,x,t_r) &= z^{-1}\begin{pmatrix} -G_{n+1}(z,x,t_r) & F_n(z,x,t_r) \\ H_n(z,x,t_r) & G_{n+1}(z,x,t_r)\end{pmatrix},\end{aligned}$$

and

$$(5.89)\qquad F_n(z,x,t_r) = \sum_{\ell=0}^{n} f_{n-\ell}(x,t_r) z^\ell = \prod_{j=1}^{n} (z - \mu_j(x,t_r)),$$

$$(5.90)\quad G_{n+1}(z,x,t_r) = \sum_{\ell=0}^{n+1} g_{n+1-\ell}(x,t_r) z^\ell - f_{n+1}(x,t_r) - \frac{1}{2} f_{n+1,x}(x,t_r),$$

$$\begin{aligned} H_n(z,x,t_r) &= \sum_{\ell=0}^{n} h_{n-\ell}(x,t_r) z^\ell + g_{n+2,x}(x,t_r) \\ &= h_0(x,t_r) \prod_{j=1}^{n} (z-\nu_j(x,t_r)), \end{aligned} \tag{5.91}$$

$$\widetilde{F}_r(z,x,t_r) = \sum_{s=0}^{r} \tilde{f}_{r-s}(x,t_r) z^s, \tag{5.92}$$

$$\widetilde{G}_{r+1}(z,x,t_r) = \sum_{s=0}^{r+1} \tilde{g}_{r+1-s}(x,t_r) z^s - \tilde{f}_{r+1}(x,t_r) - \frac{1}{2}\tilde{f}_{r+1,x}(x,t_r), \tag{5.93}$$

$$\widetilde{H}_r(z,x,t_r) = \sum_{s=0}^{r} \tilde{h}_{r-s}(x,t_r) z^s + \tilde{g}_{r+2,x}(x,t_r), \tag{5.94}$$

for fixed $n, r \in \mathbb{N}_0$. Here $f_\ell(x,t_r)$, $\tilde{f}_s(x,t_r)$, $g_\ell(x,t_r)$, $\tilde{g}_s(x,t_r)$, $h_\ell(x,t_r)$, and $\tilde{h}_s(x,t_r)$ for $\ell = 0,\dots,n+1$, $s = 0,\dots,r+1$, are defined as in (2.3) and (2.7) with $u(x)$ replaced by $u(x,t_r)$, etc., and with appropriate integration constants. Explicitly, (5.86), (5.87) are equivalent to

$$z\widetilde{F}_{r,x}(z,x,t_r) = -2[\alpha(x,t_r)+z]\widetilde{F}_r(z,x,t_r) + 2\widetilde{G}_{r+1}(z,x,t_r), \tag{5.95}$$

$$\begin{aligned} z\widetilde{G}_{r+1,x}(z,x,t_r) &= z\alpha_{t_r}(x,t_r) \\ &\quad - \big[\alpha(x,t_r)^2 + w(x,t_r)\big]\widetilde{F}_r(z,x,t_r) - \widetilde{H}_r(z,x,t_r), \end{aligned} \tag{5.96}$$

$$\begin{aligned} z\widetilde{H}_{r,x}(z,x,t_r) &= -z[2\alpha(x,t_r)\alpha_{t_r}(x,t_r) + w_{t_r}(x,t_r)] \\ &\quad + 2[\alpha(x,t_r)+z]\widetilde{H}_r(z,x,t_r) \\ &\quad + 2\big[\alpha(x,t_r)^2 + w(x,t_r)\big]\widetilde{G}_{r+1}(z,x,t_r) = 0, \end{aligned} \tag{5.97}$$

and

$$zF_{n,x}(z,x,t_r) = -2[\alpha(x,t_r)+z]F_n(z,x,t_r) + 2G_{n+1}(z,x,t_r), \tag{5.98}$$

$$zG_{n+1,x}(z,x,t_r) = -\big[\alpha(x,t_r)^2 + w(x,t_r)\big]F_n(z,x,t_r) - H_n(z,x,t_r), \tag{5.99}$$

$$\begin{aligned} zH_{n,x}(z,x,t_r) &= 2[\alpha(x,t_r)+z]H_n(z,x,t_r) \\ &\quad + 2\big[\alpha(x,t_r)^2 + w(x,t_r)\big]G_{n+1}(z,x,t_r). \end{aligned} \tag{5.100}$$

One observes that equations (2.3)–(2.21) apply to F_n, G_{n+1}, H_n, f_ℓ, g_ℓ, and h_ℓ and (2.3)–(2.8), (2.22), with n replaced by r and c_ℓ replaced by $\tilde{c}_\ell$, apply to $\widetilde{F}_r$, $\widetilde{G}_{r+1}$, $\widetilde{H}_r$, $\tilde{f}_\ell$, $\tilde{g}_\ell$, and $\tilde{h}_\ell$. In particular, the fundamental identity (2.18),

$$G_{n+1}(z,x,t_r)^2 + F_n(z,x,t_r)H_n(z,x,t_r) = R_{2n+2}(z), \quad t_r \in \mathbb{R}, \tag{5.101}$$

holds as in the stationary context and the hyperelliptic curve $\mathcal{K}_n$ is still given by

$$\mathcal{K}_n \colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0, \quad R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0}^{2n+1} \subset \mathbb{C}. \tag{5.102}$$

Moreover, (5.86) and (5.87) also yield

$$-V_{n,t_r}(z,x,t_r) + \big[\widetilde{V}_r(z,x,t_r), V_n(z,x,t_r)\big] = 0, \quad (z,x,t_r) \in \mathbb{C} \times \mathbb{R}^2. \tag{5.103}$$

The independence of (5.101) of $t_r \in \mathbb{R}$ can be interpreted as follows: The rth CH-2 flow represents an isospectral deformation of the curve $\mathcal{K}_n$ in (5.102), in particular, the branch points of $\mathcal{K}_n$ remain invariant under these flows,

$$\partial_{t_r} E_m = 0, \quad m = 0, \dots, 2n+1. \tag{5.104}$$

Without going into further details we note that the time-dependent analog of the Dubrovin-type equations (3.38) now reads,

$$\mu_{j,x}(x,t_r) = 2\mu_j(x,t_r)^{-1} y(\hat{\mu}_j(x,t_r)) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} [\mu_j(x,t_r) - \mu_\ell(x,t_r)]^{-1}, \tag{5.105}$$

$$\begin{aligned} \mu_{j,t_r}(x,t_r) &= 2\tilde{F}_r(\mu_j(x,t_r),x,t_r) \\ &\quad \times \mu_j(x,t_r)^{-1} y(\hat{\mu}_j(x,t_r)) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} [\mu_j(x,t_r) - \mu_\ell(x,t_r)]^{-1}, \\ &\qquad j = 1, \dots, n, \ (x,t_r) \in \widetilde{\Omega}_\mu, \end{aligned} \tag{5.106}$$

with an appropriate open and connected set $\widetilde{\Omega}_\mu \subseteq \mathbb{R}^2$. In particular, higher-order CH-2_r flows drive each $\hat{\mu}_j(x,t_r)$ around the same circles as in the stationary case.

Together with the comments following (5.84), this shows that isospectral torus questions are conveniently reduced to the study of the stationary hierarchy of CH-2 flows since time-dependent solutions just trace out a curve in the isospectral torus defined by the stationary hierarchy. This is of course in complete agreement with other completely integrable 1 + 1-dimensional hierarchies such as the KdV, Toda, and AKNS hierarchies.

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BIBLIOGRAPHY

[1] M. S. Alber, R. Camassa, Y. N. Fedorov, D. D. Holm & J. E. Marsden, "The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDE's of shallow water and Dym type", *Comm. Math. Phys.* **221** (2001), no. 1, p. 197-227.

[2] M. S. Alber & Y. N. Fedorov, "Algebraic geometrical solutions for certain evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians", *Inverse Problems* **17** (2001), no. 4, p. 1017-1042, Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000).

[3] H. Aratyn, J. F. Gomes & A. H. Zimerman, "On a negative flow of the AKNS hierarchy and its relation to a two-component Camassa–Holm equation", *SIGMA Symmetry Integrability Geom. Methods Appl.* **2** (2006), p. Paper 070, 12.

[4] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Mathematics in Science and Engineering, vol. 8, Academic Press, New York-London, 1964, xiv+570 pages.

[5] R. Camassa & D. D. Holm, "An integrable shallow water equation with peaked solitons", *Phys. Rev. Lett.* **71** (1993), no. 11, p. 1661-1664.

[6] C. Chen & S. Wen, "Wave breaking phenomena and global solutions for a generalized periodic two-component Camassa–Holm system", *Discrete Contin. Dyn. Syst.* **32** (2012), no. 10, p. 3459-3484.

[7] D. Chen, Y. Li & W. Yan, "On well-posedness of two-component Camassa–Holm system in the critical Besov space", *Nonlinear Anal.* **120** (2015), p. 285-298.

[8] M. Chen, S.-Q. Liu & Y. Zhang, "A two-component generalization of the Camassa–Holm equation and its solutions", *Lett. Math. Phys.* **75** (2006), no. 1, p. 1-15.

[9] S. Clark & F. Gesztesy, "Weyl-Titchmarsh M-function asymptotics, local uniqueness results, trace formulas, and Borg-type theorems for Dirac operators", *Trans. Amer. Math. Soc.* **354** (2002), no. 9, p. 3475-3534 (electronic).

[10] A. Constantin & R. I. Ivanov, "On an integrable two-component Camassa–Holm shallow water system", *Phys. Lett. A* **372** (2008), no. 48, p. 7129-7132.

[11] A. Constantin & H. P. McKean, "A shallow water equation on the circle", *Comm. Pure Appl. Math.* **52** (1999), no. 8, p. 949-982.

[12] J. Eckhardt, "The inverse spectral transform for the conservative Camassa–Holm flow with decaying initial data", `https://arxiv.org/abs/1510.04916`, to appear in *Arch. Rat. Mech. Anal.*

[13] J. Eckhardt & A. Kostenko, "An isospectral problem for global conservative multi-peakon solutions of the Camassa–Holm equation", *Commun. Math. Phys.* **329** (2014), no. 3, p. 893-918.

[14] ———, "Quadratic operator pencils associated with the conservative Camassa–Holm flow", `https://arxiv.org/abs/1406.3703`, to appear in *Bull. Soc. Math. Fr.*, 2014.

[15] ———, "The inverse spectral problem for indefinite strings", *Invent. Math.* **204** (2016), no. 3, p. 939-977.

[16] J. Escher, M. Kohlmann & J. Lenells, "The geometry of the two-component Camassa–Holm and Degasperis-Procesi equations", *J. Geom. Phys.* **61** (2011), no. 2, p. 436-452.

[17] J. Escher, O. Lechtenfeld & Z. Yin, "Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation", *Discrete Contin. Dyn. Syst.* **19** (2007), no. 3, p. 493-513.

[18] G. Falqui, "On a Camassa–Holm type equation with two dependent variables", *J. Phys. A, Math. Theor.* **39** (2006), no. 2, p. 327-342.

[19] Y. Fu & C. Qu, "Well posedness and blow-up solution for a new coupled Camassa–Holm equations with peakons", *J. Math. Phys.* **50** (2009), no. 1, p. 012906, 25.

[20] F. Gesztesy & H. Holden, "Algebro-geometric solutions of the Camassa–Holm hierarchy", *Rev. Mat. Iberoam.* **19** (2003), no. 1, p. 73-142.

[21] ———, *Soliton Equations and Their Algebro-Geometric Solutions. Vol. I*, Cambridge Studies in Advanced Mathematics, vol. 79, Cambridge University Press, Cambridge, 2003, $(1+1)$-dimensional continuous models, xii+505 pages.

[22] ———, "Real-valued algebro-geometric solutions of the Camassa–Holm hierarchy", *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **366** (2008), no. 1867, p. 1025-1054.

[23] F. Gesztesy & E. Tsekanovskii, "On matrix-valued Herglotz functions", *Math. Nachr.* **218** (2000), p. 61-138.

[24] K. Grunert, H. Holden & X. Raynaud, "Global solutions for the two-component Camassa–Holm system", *Comm. Partial Differential Equations* **37** (2012), no. 12, p. 2245-2271.

[25] ———, "Periodic conservative solutions for the two-component Camassa–Holm system", in *Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy's 60th birthday*, Proc. Sympos. Pure Math., vol. 87, Amer. Math. Soc., Providence, RI, 2013, p. 165-182.

[26] ———, "Global dissipative solutions of the two-component Camassa–Holm system for initial data with nonvanishing asymptotics", *Nonlinear Anal., Real World Appl.* **17** (2014), p. 203-244.

[27] ———, "A continuous interpolation between conservative and dissipative solutions for the two-component Camassa–Holm system", *Forum Math. Sigma* **3** (2015), Art. ID e1, 73 p.

[28] C. Guan & Z. Yin, "Global existence and blow-up phenomena for an integrable two-component Camassa–Holm shallow water system", *J. Differ. Equations* **248** (2010), no. 8, p. 2003-2014.

[29] ———, "Global weak solutions for a two-component Camassa–Holm shallow water system", *J. Funct. Anal.* **260** (2011), no. 4, p. 1132-1154.

[30] D. B. Hinton & J. K. Shaw, "On Titchmarsh–Weyl $M(\lambda)$-functions for linear Hamiltonian systems", *J. Differ. Equations* **40** (1981), no. 3, p. 316-342.

[31] ———, "Hamiltonian systems of limit point or limit circle type with both endpoints singular", *J. Differ. Equations* **50** (1983), no. 3, p. 444-464.

[32] ———, "On boundary value problems for Hamiltonian systems with two singular points", *SIAM J. Math. Anal.* **15** (1984), no. 2, p. 272-286.

[33] D. D. Holm & R. I. Ivanov, "Multi-component generalizations of the CH equation: geometrical aspects, peakons and numerical examples", *J. Phys. A, Math. Theor.* **43** (2010), no. 49, p. 492001, 20.

[34] ———, "Two-component CH system: inverse scattering, peakons and geometry", *Inverse Problems* **27** (2011), no. 4, p. 045013, 19.

[35] Y. Hou & E. Fan, "Algebro-geometric solutions for the two-component Camassa–Holm hierarchy", `https://arxiv.org/abs/1406.6153`.

[36] Q. Hu & Z. Yin, "Well-posedness and blow-up phenomena for a periodic two-component Camassa–Holm equation", *Proc. Roy. Soc. Edinburgh Sect. A* **141** (2011), no. 1, p. 93-107.

[37] D. Ionescu-Kruse, "Variational derivation of two-component Camassa–Holm shallow water system", *Appl. Anal.* **92** (2013), no. 6, p. 1241-1253.

[38] R. I. Ivanov, "Extended Camassa–Holm hierarchy and conserved quantities", *Z. Nat.forsch., A: Phys. Sci.* **61** (2006), no. 3-4, p. 133-138.

[39] V. I. Kogan & F. S. Rofe-Beketov, "On square-integrable solutions of symmetric systems of differential equations of arbitrary order", *Proc. R. Soc. Edinb., Sect. A* **74** (1974/75), p. 5-40 (1976).

[40] M. Kohlmann, "On a two-component π-Camassa–Holm system", *J. Geom. Phys.* **62** (2012), no. 4, p. 832-838.

[41] ———, "The two-component Camassa–Holm system in weighted L_p spaces", *ZAMM, Z. Angew. Math. Mech.* **94** (2014), no. 3, p. 264-272.

[42] ———, "Curvature computations for a two-component Camassa–Holm equation with vorticity", `https://arxiv.org/abs/1509.00694`, to appear in *Appl. Anal.*, 2015.

[43] M. Lesch & M. Malamud, "On the deficiency indices and self-adjointness of symmetric Hamiltonian systems", *J. Differ. Equations* **189** (2003), no. 2, p. 556-615.

[44] B. M. Levitan, *Inverse Sturm–Liouville Problems*, VSP, Zeist, 1987, Translated from the Russian by O. Efimov, x+240 pages.

[45] X. Liu, "Stability of the train of N solitary waves for the two-component Camassa–Holm shallow water system", *J. Differ. Equations* **260** (2016), no. 12, p. 8403-8427.

[46] C. Ma, A. Alsaedi, T. Hayat & Y. Zhou, "On blow-up of solutions to the two-component π-Camassa–Holm system", *J. Math. Anal. Appl.* **426** (2015), no. 2, p. 1026-1039.

[47] K. Mohajer, "A note on traveling wave solutions to the two component Camassa–Holm equation", *J. Nonlinear Math. Phys.* **16** (2009), no. 2, p. 117-125.

[48] B. Moon, "Global solutions to a special case of the generalized weakly dissipative periodic two-component Camassa–Holm system", *Nonlinear Anal.* **117** (2015), p. 38-46.

[49] O. G. Mustafa, "On smooth traveling waves of an integrable two-component Camassa–Holm shallow water system", *Wave Motion* **46** (2009), no. 6, p. 397-402.

[50] P. J. Olver & P. Rosenau, "Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support", *Phys. Rev. E* **53** (1996), no. 2, p. 1900-1906.

[51] R. C. Thompson, "The periodic Cauchy problem for the 2-component Camassa–Holm system", *Differ. Integral Equ.* **26** (2013), no. 1-2, p. 155-182.

[52] Z. Wen, "Several new types of bounded wave solutions for the generalized two-component Camassa–Holm equation", *Nonlinear Dynam.* **77** (2014), no. 3, p. 849-857.

[53] B. Xia, Z. Qiao & R. Zhou, "A synthetical two-component model with peakon solutions", *Stud. Appl. Math.* **135** (2015), no. 3, p. 248-276.

[54] K. Yan, Z. Qiao & Z. Yin, "Qualitative analysis for a new integrable two-component Camassa–Holm system with peakon and weak kink solutions", *Comm. Math. Phys.* **336** (2015), no. 2, p. 581-617.

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Jonathan ECKHARDT
Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1
1090 Wien (Austria)
jonathan.eckhardt@univie.ac.at

Fritz GESZTESY
Department of Mathematics
University of Missouri
Columbia, MO 65211 (USA)
Current address:
Department of Mathematics
Baylor University
One Bear Place #97328
Waco, TX 76798-7328 (USA)
Fritz_Gesztesy@baylor.edu

Helge HOLDEN
Department of Mathematical Sciences
Norwegian University of Science and Technology
NO–7491 Trondheim (Norway)
holden@math.ntnu.no

Aleksey KOSTENKO
Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1
1090 Wien (Austria)
Oleksiy.Kostenko@univie.ac.at

Gerald TESCHL
Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1
1090 Wien (Austria)
International Erwin Schrödinger Institute for
Mathematical Physics
Boltzmanngasse 9
1090 Wien (Austria)
Gerald.Teschl@univie.ac.at