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PURITY FOR FAMILIES OF GALOIS REPRESENTATIONS

by Jyoti Prakash SAHA

ABSTRACT. — We formulate a notion of purity for p-adic big Galois representations and pseudorepresentations of Weil groups of ℓ-adic number fields for $\ell \neq p$. This is obtained by showing that all powers of the monodromy of any big Galois representation stay "as large as possible" under pure specializations. Using purity for families, we improve a part of the local Langlands correspondence for GL_n in families formulated by Emerton and Helm. The role of purity for families in the study of variation of local Euler factors, local automorphic types along irreducible components, intersection points of irreducible components of p-adic families of automorphic Galois representations is illustrated using the examples of Hida families and eigenvarieties.

RÉSUMÉ. — Nous formulons une notion de pureté pour les familles p-adiques de représentations galoisiennes et pseudo-caractères du groupe de Weil d'un corps de nombres ℓ-adiques pour $\ell \neq p$. Ceci est obtenu en montrant que tous les puissances de la monodromie de toute représentation galoisienne restent aussi grandes que possible après spécialisations pures. En utilisant la pureté pour les familles, nous améliorons une partie de la correspondance de Langlands locale pour GL_n en familles formulée par Emerton et Helm. De plus, en utilisant les exemples de familles de Hida et variétés de Hecke, nous illustrons le rôle de pureté pour les familles dans l'étude de la variation des facteurs d'Euler locaux, types automorphes locaux le long des composantes irréductibles, les points d'intersection des composantes irréductibles de familles de représentations galoisiennes automorphes.

1. Introduction

1.1. Motivation

Let r be a geometric Galois representation of the absolute Galois group of a number field with coefficients in $\overline{\mathbb{Q}}_p$. Then the restriction r_v of r to

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the decomposition group at any finite place v not dividing p is potentially unipotent by Grothendieck's monodromy theorem (see [49, pp. 515–516]). Given a projective smooth variety X over a finite extension K of $\mathbb{Q}_\ell$, the weight-monodromy conjecture ([35, Conjecture 3.9]) says that for any prime $p \neq \ell$ and any integer $i \geqslant 0$, the $\mathrm{Gal}(\overline{K}/K)$-representation $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell}, \mathbb{Q}_p)$ is pure of weight i, i.e., the i-th shift of the associated monodromy filtration coincides with the associated weight filtration (see Definition 2.10). When r is irreducible, the representation r_v is expected to be pure. The Galois representations attached to cuspidal automorphic representations (which are algebraic in the sense of [13, Definition 1.8]) by the Langlands correspondence (which is often conjectural) provide ample examples of geometric representations. The purity of restrictions of p-adic automorphic Galois representations to decomposition groups at places outside p is known in many cases due to works of Carayol [9], Harris and Taylor [30], Blasius [4], Taylor and Yoshida [54], Shin [52], Caraiani [8], Scholze [47], Clozel [14] et al. Following works of Hida [31, 32, 34], Mazur [39], Coleman and Mazur [15], Chenevier [10], Bellaïche and Chenevier [2] et al., automorphic Galois representations are believed to live in p-adic families. Thus it is desirable to have a notion of purity for families. The goal of this article is to provide a formulation of this notion and to discuss its applications to p-adic families of Galois representations.

1.2. Purity for families

The most naive way to formulate purity for big Galois representations would be to relate the monodromy filtration with the weight filtration. However the Frobenius eigenvalues on a big Galois representation are elements of a ring of large Krull dimension and are not algebraic numbers in general, precluding the possibility of considering the weight filtration. Thus a formulation of purity for big Galois representations is not straightforward. On the other hand, it is natural to expect that such a formulation should include a compatibility statement at pure specializations.

This formulation is achieved in Theorem 3.1, which we call *purity for big Galois representations* because it says that the structures of Frobenius-semisimplifications of Weil–Deligne parametrizations of pure specializations of a (p-adic) big Galois representation (of the Weil group of an ℓ-adic number field with $\ell \neq p$) are "rigid". In other words, it says that given a pure Weil–Deligne representation, its lifts to Weil–Deligne representations over integral domains have the "same structure".

The eigenvarieties are an important source of examples of families of p-adic Galois representations. The traces of the Galois representations attached to the arithmetic points of an eigenvariety are interpolated by a pseudorepresentation defined over the global sections of the eigenvariety. Thus a notion of purity for pseudorepresentations is indispensable for the understanding of various local properties of the arithmetic points of eigenvarieties. This is provided by Theorem 4.3, which we call *purity for pseudorepresentations*. It says that given an $\mathscr{O}$-valued pseudorepresentation T of the Weil group of an ℓ-adic number field (where $\mathscr{O}$ is a domain over $\mathbb{Q}$), the Frobenius-semisimplification of two Weil–Deligne representations over two domains (containing $\mathscr{O}$ as a subalgebra) have the "same structure" if their traces are equal to T and each of them has a pure specialization. This is deduced using purity for big Galois representations.

By [3, Lemma 7.8.11], around each nonempty admissible open affinoid subset U, the pseudorepresentation defined over the global sections of an eigenvariety lifts to a Galois representation on a finite type module over some integral extension of the normalization of $\mathcal{O}(U)$. But this module is not known to be free over its coefficient ring. So Theorem 4.3 cannot be applied to eigenvarieties to study the local properties of all arithmetic points. However the local properties of certain type of arithmetic points can be described using Theorem 4.5. For example, if the Galois representations attached to the arithmetic points are absolutely irreducible and their restrictions to the decomposition group G_w at a finite place $w \nmid p$ are pure, then the structure of the Frobenius-semisimplification of the Weil–Deligne parametrization of $\rho_z|_{G_w}$ is "rigid" when z varies along an irreducible component of an eigenvariety (here ρ_z denotes the p-adic Galois representation attached to the arithmetic point z). Henceforth, by *purity for families*, we refer to Theorems 3.1, 4.3, 4.5.

In Theorem 1.1 below, we state a part of Theorem 3.1. Let p, ℓ be two distinct primes, K be a finite extension of $\mathbb{Q}_\ell$ and W_K denote the Weil group of K. The Frobenius-semisimplification of a Weil–Deligne representation V of W_K is denoted by $V^{\text{Fr-ss}}$. Let $\mathscr{O}$ be a domain over $\mathbb{Q}$ with fraction field $\mathscr{L}$. Suppose (ρ, N) is a Weil–Deligne representation of W_K on $\mathscr{O}^n$ where n is a positive integer. Such representations arise as Weil–Deligne parametrization (tensored with $\mathbb{Q}$) of representations of W_K on free modules of finite type over local rings with finite residue fields of characteristic p (resp. affinoid algebras) which are continuous with respect to the maximal ideal adic topology (resp. the Banach algebra topology). Note that these type of representations of W_K admit Weil–Deligne parametrization by Grothendieck's monodromy theorem, see [49, pp. 515–516] (resp. [3, Lemma 7.8.14]). Denote by $\mathcal{V}$ the Weil–Deligne representation $(\rho, N)\otimes_{\mathscr{O}}\overline{\mathscr{L}}$. Let $\mathcal{V}^{\text{Fr-ss}}$ be isomorphic to the direct sum $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(r_i)_{/\overline{\mathscr{L}}}$ of special representations (see Definition 2.3) where $m, t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m$ are positive integers, $r_1, \cdots, r_m$ are irreducible Frobenius-semisimple representations of W_K over $\mathscr{O}^{\text{intal}}$. Given a field E and a map $f : \mathscr{O} \to E$, the Weil–Deligne representation $(\rho, N) \otimes_{\mathscr{O},f} E$ is denoted by V_f. We fix an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$ and let rec denote the reciprocity map.

Theorem 1.1 (Purity for big Galois representations). — *Let $\lambda : \mathscr{O} \to \overline{\mathbb{Q}}_p$ be a map such that V_λ is pure. Then the following hold.*

(1) *The rank of no power of the monodromy N decreases after specializing at λ.*
(2) *The Weil–Deligne representations $V_\lambda^{\text{Fr-ss}}$ and $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(\lambda^{\text{intal}} \circ r_i)_{/\overline{\mathbb{Q}}_p}$ are isomorphic.*
(3) *The polynomial $\mathrm{Eul}(\mathcal{V}, X)^{-1}$ has coefficients in $\mathscr{O}^{\text{int}}$ and its specialization under λ^{intal} is $\mathrm{Eul}(V_\lambda, X)^{-1}$.*
(4) *If $\xi : \mathscr{O} \to \overline{\mathbb{Q}}_p$ is a map such that V_ξ is pure, then the automorphic types of $\mathrm{rec}^{-1}(\iota_p(V_\xi^{\text{Fr-ss}}))$ and $\mathrm{rec}^{-1}(\iota_p(V_\lambda^{\text{Fr-ss}}))$ are the same.*

Moreover, for any field $\mathcal{K}$ and any map $\mu : \mathscr{O} \to \mathcal{K}$ with $\lambda(\ker \mu) = 0$, the Weil–Deligne representation $(V_\mu \otimes_{\mathcal{K}} \overline{\mathcal{K}})^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(\mu^{\text{intal}} \circ r_i)_{/\overline{\mathcal{K}}}$.

1.3. Applications of purity for families

We explain the role of Theorems 3.1, 4.3, 4.5 in the study of some arithmetic aspects of p-adic families of Galois representations, for example, the local Langlands correspondence for GL_n in families, the local automorphic types of arithmetic points of p-adic families, the geometry of the underlying spaces of families etc. (see Theorems 5.3, 6.2, 6.4, 6.7).

1.3.1. Local Langlands correspondence for GL_n in families

The local Langlands correspondence was proved by Harris and Taylor [30]. It is extended to p-adic families of representations of $G_K = \mathrm{Gal}(\overline{K}/K)$ by Emerton and Helm in [23]. They show that given a continuous Galois representation $r : G_K \to \mathrm{GL}_n(A)$ (where A is a complete reduced p-torsion free Noetherian local ring with finite residue field of characteristic p), there exists at most one (up to isomorphism) admissible smooth

$\mathrm{GL}_n(K)$-representation V over A, which interpolates the Breuil–Schneider modified local Langlands correspondence along irreducible components of $\mathrm{Spec} A$ (see [23, Theorem 1.2.1] for the precise statement). Denote the residue field of a prime $\mathfrak{p}$ of A by $\kappa(\mathfrak{p})$. When V exists, it also interpolates the $\mathrm{GL}_n(K)$-representation attached to $\kappa(\mathfrak{p}) \otimes_A r$ via the Breuil–Schneider modified local Langlands correspondence whenever $\mathfrak{p}$ is a prime ideal of $A[1/p]$ containing only one minimal prime ideal of $A[1/p]$ and the rank of no power of the monodromy of r degenerates under mod $\mathfrak{p}$ reduction ([23, Theorem 6.2.5] gives the precise statement). Combining this result with Theorem 3.1, we prove in Theorem 5.3 that this interpolation property holds if $\mathfrak{p}$ is a prime ideal of $A[1/p]$ containing only one minimal prime ideal of $A[1/p]$ and $\mathfrak{p}$ is contained in a prime $\mathfrak{q}$ of $A[1/p]$ such that $\kappa(\mathfrak{q}) \otimes_A r$ is pure.

Hida's theory of ordinary automorphic representations provides continuous representations of absolute Galois groups of number fields with coefficients in rings of the form A. So their restrictions to decomposition groups at places not dividing p give representations of the form r, to which Theorem 5.3 apply. On the other hand, overconvergent forms also form families, although of rather different nature, for instance, there are examples of such families whose coefficient rings are not local (and there are also families of overconvergent forms defined over local rings, see [1]). The local Langlands correspondence is not yet extended to families defined over non-local rings or over affinoid algebras. However, the coefficient rings $\mathscr{O}, \mathcal{O}, \mathcal{O}'$ as in Theorems 3.1, 4.3, 4.5 are quite general, for instance, these are not assumed to be local. So once a notion of local Langlands correspondence for more general families is established, it is likely that one could use Theorems 3.1, 4.3, 4.5 to show that the extension (as in [23, §4.2]) of the Breuil–Schneider modified local Langlands correspondence is interpolated at the primes contained in kernels of pure specializations.

1.3.2. Hida families and eigenvarieties

Given a p-adic family of Galois representations of the absolute Galois group of a number field, the variation of the Frobenius-semisimplifications of the Weil–Deligne parametrizations of the local Galois representations attached to the members at the finite places outside p can be studied using Theorems 3.1, 4.3, 4.5. Thus purity for families illustrates the variation of local Euler factors of the arithmetic points of p-adic families of automorphic Galois representations and also the variation of local automorphic types of arithmetic points when local-global compatibility is known. In Section 6, we explain this variation using examples of Hida family of cusp forms, Hida family of ordinary automorphic representations of definite unitary groups and eigenvariety for definite unitary groups. We refer to Theorems 6.2, 6.4, 6.7 for the precise statements. Roughly speaking, these three results state that the structure of the local Galois representations attached to the arithmetic points of any given irreducible component of these families are constant (under some hypotheses). For related results, we refer to [24, Proposition 2.2.4], [40, §12.7.14], [43, Remark 2.4], [26, Lemma 2.14], [25, Lemma 3.9], [3, §7.5.3, 7.8.4], [11, Lemma 4.5], [45, Theorem A].

1.4. Notations

For every field F, we fix an algebraic closure $\overline{F}$ of it and denote by G_F the Galois group $\mathrm{Gal}(\overline{F}/F)$. For a finite place v of a number field E, the decomposition group $\mathrm{Gal}(\overline{E}_v/E_v)$ is denoted by G_v. Let $W_v \subset G_v$ (resp. $I_v \subset G_v$) denote the Weil group (resp. inertia group) and $\mathrm{Fr}_v \in G_v/I_v$ denote the geometric Frobenius element. We fix embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ once and for all. The fraction field of a domain A is denoted by $Q(A)$, the field $\overline{Q(A)}$ is denoted by $\overline{Q}(A)$. The integral closure of a domain R in $Q(R)$ (resp. $\overline{Q}(R)$) is denoted by R^{int} (resp. R^{intal}). If $f : R \to S$ is a map between two domains, then the map f has an extension to a map $R^{\mathrm{intal}} \to S^{\mathrm{intal}}$. We fix one such map and denote it by f^{intal}. The residue field of a prime $\mathfrak{p}$ of a ring R is denoted by $\kappa(\mathfrak{p})$ and the mod $\mathfrak{p}$ reduction map is denoted by $\pi_{\mathfrak{p}}$.

2. Local Galois representations

Let q denote the cardinality of the residue field k of the ring of integers $\mathcal{O}_K$ of K. Fix an element $\phi \in G_K$ which lifts the geometric Frobenius $\mathrm{Fr}_k \in G_k$. Let ϖ denote a uniformizer of $\mathcal{O}_K$ and $\mathrm{val}_K : K^\times \twoheadrightarrow \mathbb{Z}$ be the ϖ-adic valuation. Let $|\cdot|_K := q^{-\mathrm{val}_K(\cdot)}$ be the corresponding norm. The Weil group W_K is defined as the subgroup of G_K consisting of elements which map to an integral power of Fr_k in G_k. Its topology is determined by decreeing that I_K is open and has its subspace topology induced from G_K. The Artin map $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ is normalized so that uniformizing parameters go to lifts of the geometric Frobenius element. Let I_K (resp. P_K) denote the inertia (resp. wild inertia) subgroup of G_K. Then given a compatible system $\zeta = (\zeta_n)_{\ell \nmid n}$ of primitive roots of unity, we have an isomorphism $t_\zeta : I_K/P_K \xrightarrow{\sim} \prod_{p \neq \ell} \mathbb{Z}_p$ such that $\sigma(\varpi^{1/n}) = \zeta_n^{(t_\zeta(\sigma) \bmod n)} \varpi^{1/n}$ for all $\sigma \in I_K/P_K$. By [41, Theorem 7.5.2], for all $\sigma \in W_K$ and $\tau \in I_K$, we have $t_\zeta(\sigma\tau\sigma^{-1}) = \varepsilon(\sigma)t_\zeta(\tau)$ where $\varepsilon := \prod_{p\neq\ell}\varepsilon_p : G_K \to \prod_{p\neq\ell}\mathbb{Z}_p^\times$ is the product of the cyclotomic characters. For a prime $p \neq \ell$, let $t_{\zeta,p} : I_K \to \mathbb{Z}_p$ denote the composition of the projection $I_K \to I_K/P_K$, the map t_ζ and the projection from $\prod_{p\neq\ell}\mathbb{Z}_p$ to $\mathbb{Z}_p$. Define $v_K : W_K \to \mathbb{Z}$ by $\sigma|_{K^{\mathrm{ur}}} = \mathrm{Fr}_k^{v_K(\sigma)}$ for all $\sigma \in W_K$. Denote by rec the reciprocity map of local Langlands correspondence for $\mathrm{GL}_n(K)$, which is known due to works of Harris and Taylor [30]. The map rec is chosen so that $\mathrm{rec}(\pi) = \pi \circ \mathrm{Art}_K^{-1}$ for any homomorphism $\pi : K^\times \to \mathbb{C}^\times$ which is continuous with respect to discrete topology on the target. The reciprocity map rec depends on the choice of a square root of q in $\overline{\mathbb{Q}}_p$, which we fix from now on.

Definition 2.1 ([19, 8.4.1]). — *Let A be a commutative domain of characteristic zero.*

(1) *A* Weil–Deligne representation *of W_K on a free A-module M of finite rank is a triple (r, M, N) consisting of a representation $r : W_K \to \mathrm{Aut}_A(M)$ with open kernel and a nilpotent endomorphism $N \in \mathrm{End}_A(M)$ such that*

$$r(\sigma)Nr(\sigma)^{-1} = q^{-v_K(\sigma)}N \quad \text{for all } \sigma \in W_K.$$

The operator N is called the monodromy *of (r, M, N) and the map* $\mathrm{tr}\, r : W_K \to A$ *is called the* trace *of (r, M, N).*

(2) *A representation ρ of W_K on a free A-module M of finite rank is said to be* irreducible Frobenius-semisimple *if $M \otimes \overline{Q}(A)$ is irreducible, the ϕ-action on $M \otimes \overline{Q}(A)$ is semisimple and $\ker\rho$ is open.*

The sum of Weil–Deligne representations are defined in the usual way (cf. [7, §31.2]).

Definition 2.2. — *Let A be a $\mathbb{Z}_p$-algebra of characteristic zero. Suppose M is an A-module equipped with an A-linear action ρ of W_K or of G_K. We say M is* monodromic *with* monodromy N over K' *if there exists a finite extension K'/K and a nilpotent element N of $\mathrm{End}_{A[1/p]}(M \otimes_A A[1/p])$ such that for all $\tau \in I_{K'}$*

$$\rho(\tau) = \exp(t_{\zeta,p}(\tau)N)$$

in $\mathrm{End}_{A[1/p]}(M \otimes_A A[1/p])$.

Given a $\mathbb{Z}_p$-algebra A of characteristic zero and a monodromic representation (ρ, M) of W_K or of G_K over A with monodromy N, (following [19, 8.4.2]) its *Weil–Deligne parametrization* is denoted by $\mathrm{WD}(M)$ and is defined to be the triple $(r, M[1/p], N)$ where r denotes the W_K-action on the

$A[1/p]$-module $M[1/p]$ given by

$$r(\sigma) = \rho(\sigma)\exp(-t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma)N) \quad \text{for all } \sigma \in W_K.$$

Suppose $(r, N) = (r, V, N)$ is a Weil–Deligne representation with coefficients in a field L of characteristic zero which contains the characteristic roots of all elements of $r(W_K)$. Let $r(\phi) = r(\phi)^{ss}u = ur(\phi)^{ss}$ be the Jordan decomposition of the operator $r(\phi)$ as the product of a diagonalizable operator $r(\phi)^{ss}$ and a unipotent operator u acting on V. Following [19, 8.5], define $\tilde{r}(\sigma) = r(\sigma)u^{-v_K(\sigma)}$ for all $\sigma \in W_K$. Then $(\tilde{r}, V, N)$ is a Weil–Deligne representation (by [19, 8.5] for example) and is called the *Frobenius-semisimplification* of (r, V, N) (cf. [19, 8.6]). It is denoted by $V^{\text{Fr-ss}}$. We say (r, V, N) is *Frobenius-semisimple* if $\tilde{r} = r$.

Definition 2.3. — *For an integer $t \geqslant 1$, a characteristic zero commutative domain A with $\ell \in A^\times$ and a representation (r, M) of W_K on a free module M of finite rank over A with* $\ker(r)$ *open in W_K, the* special representation $\mathrm{Sp}_t(r)_{/A}$ *(also denoted* $\mathrm{Sp}_t(r)$ *when A is understood from the context) is defined to be the Weil–Deligne representation with underlying module M^t on which W_K acts via*

$$r|\mathrm{Art}_K^{-1}|_K^{t-1} \oplus r|\mathrm{Art}_K^{-1}|_K^{t-2} \oplus \cdots \oplus r|\mathrm{Art}_K^{-1}|_K \oplus r$$

and the monodromy induces an isomorphism from $r|\mathrm{Art}_K^{-1}|_K^{i}$ to $r|\mathrm{Art}_K^{-1}|_K^{i+1}$ for all $0 \leqslant i \leqslant t-2$ and is zero on $r|\mathrm{Art}_K^{-1}|_K^{t-1}$.

Let Ω denote an algebraically closed field of characteristic zero.

Definition 2.4. — *A Weil–Deligne representation over Ω is said to be* indecomposable *if it is not isomorphic to a direct sum of two nonzero Weil–Deligne representations over Ω.*

Theorem 2.5. — *Let (ρ, V, N) be a Frobenius-semisimple Weil–Deligne representation over Ω. Then it is isomorphic to $\oplus_{i\in I}\mathrm{Sp}_{t_i}(r_i)_{/\Omega}$ for some irreducible Frobenius-semisimple representations $r_i : W_K \to \mathrm{GL}_{n_i}(\Omega)$ and positive integers t_i. This decomposition is unique up to reordering and replacing factors by isomorphic factors.*

Proof. — This follows from the proof of [18, Proposition 3.1.3(i)]. ☐

Definition 2.6. — *Given (ρ, V, N) as above, its* size *is defined to be the integer* $\max\{t_i | i \in I\}$.

Definition 2.7. — *An* indecomposable summand *of a Frobenius-semisimple Weil–Deligne representation V over Ω is a Weil–Deligne subrepresentation of V isomorphic to a summand $\mathrm{Sp}_{t_i}(r_i)_{/\Omega}$ via an isomorphism $V \simeq \oplus_{i\in I}\mathrm{Sp}_{t_i}(r_i)_{/\Omega}$ with t_i, r_i as in Theorem 2.5.*

Proposition 2.8. — *Let (r,N) be a Weil–Deligne representation over an integral domain A of characteristic zero. Let $\mathbb{Q}^{\mathrm{cl}}$ denote the algebraic closure of $\mathbb{Q}$ in $\overline{Q}(A)$. Let B be a subring of A such that the characteristic polynomial of $r(g)$ has coefficients in B for all $g \in W_K$. Then there exist positive integers $m, t_1 \leqslant \cdots \leqslant t_m$, $(B^{\mathrm{intal}})^\times$-valued unramified characters $\chi_1, \cdots, \chi_m$ of W_K and irreducible Frobenius-semisimple representations $\rho_1, \cdots, \rho_m$ of W_K with coefficients in $\mathbb{Q}^{\mathrm{cl}}$ with finite image such that $((r,N)\otimes_A \overline{Q}(A))^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(\chi_i \otimes \rho_i)$.*

Proof. — By Theorem 2.5, there exist positive integers $m, t_1 \leqslant \cdots \leqslant t_m$, irreducible Frobenius-semisimple representations $r_1, \cdots, r_m$ of W_K over $\overline{Q}(A)$ such that $((r,N)\otimes_A \overline{Q}(A))^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(r_i)$. From the proof of [7, 28.6 Proposition], it follows that for each $1 \leqslant i \leqslant m$, there exists an unramified character $\chi_i : W_K \to \overline{Q}(A)^\times$ such that the W_K-representation $\chi_i^{-1} \otimes r_i$ has finite image. So there exists an irreducible Frobenius-semisimple representation $\rho_i : W_K \to \mathrm{GL}_{d_i}(\mathbb{Q}^{\mathrm{cl}})$ with finite image such that $\chi_i^{-1} \otimes r_i$ and ρ_i are isomorphic over $\overline{Q}(A)$ (by [53, Theorem 1] for instance). So the product of $\chi_i(\phi)$ and a root of unity belongs to B^{intal}. Thus $\chi_i(\phi)$ belongs to B^{intal} and similarly, $\chi_i(\phi)^{-1}$ belongs to B^{intal}. Hence χ_i has values in $(B^{\mathrm{intal}})^\times$. This proves the result. □

Lemma 2.9. — *Let $r : W_K \to \mathrm{GL}_n(A)$ be an irreducible Frobenius-semisimple representation of W_K with coefficients in a domain A of characteristic zero. If B is a characteristic zero domain and $f : A \to B$ is a ring homomorphism, then $f \circ r$ is also an irreducible Frobenius-semisimple representation.*

Proof. — Let $\mathbb{Q}^{\mathrm{cl}}$ denote the algebraic closure of $\mathbb{Q}$ in $\overline{Q}(A)$. By Proposition 2.8, there exist an unramified character $\chi : W_K \to (A^{\mathrm{intal}})^\times$ and an irreducible Frobenius-semisimple representation $\rho : W_K \to \mathrm{GL}_n(\mathbb{Q}^{\mathrm{cl}})$ with finite image such that r is isomorphic to $\chi \otimes \rho$ over $\overline{Q}(A)$. So $f^{\mathrm{intal}}(\chi^{-1}\otimes r)$ has trace equal to $f^{\mathrm{intal}}(\mathrm{tr}(\chi^{-1}\otimes r)) = f^{\mathrm{intal}}(\mathrm{tr}\rho) = \mathrm{tr} f^{\mathrm{intal}}(\rho)$ and hence $f^{\mathrm{intal}}(\chi^{-1}\otimes r)$ is isomorphic to $f^{\mathrm{intal}}(\rho)$. Thus $f(r)$ is isomorphic to $f^{\mathrm{intal}}(\chi) \otimes f^{\mathrm{intal}}(\rho)$. This proves the lemma. □

Recall that a q-Weil number of weight $w \in \mathbb{Z}$ is an algebraic number $\alpha \in \overline{\mathbb{Q}}$ such that $q^n\alpha$ is an algebraic integer for some $n \in \mathbb{Z}$ and $|\sigma(\alpha)| = q^{w/2}$ for any embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Given a Weil–Deligne representation (r, V, N) on a vector space V, its associated monodromy filtration is an increasing filtration $M_\bullet$ on V where $M_k = \sum_{i+j=k} \ker N^{i+1} \cap N^{-j}V$ for $k \in \mathbb{Z}$ (see [20, 1.7.2]).

Definition 2.10. — *A Weil–Deligne representation V of W_K over $\overline{\mathbb{Q}}_p$ is said to be* pure of weight $w \in \mathbb{Z}$ *if the eigenvalues of one (and hence any) lift of the geometric Frobenius element on* $\mathrm{Gr}_i M_\bullet$ *are q-Weil numbers of weight $w+i$ where $M_\bullet$ denotes the monodromy filtration on $V^{\text{Fr-ss}}$.*

A finite dimensional representation V of G_K or of W_K over $\overline{\mathbb{Q}}_p$ is said to be pure of weight $w \in \mathbb{Z}$ *if it is monodromic and its Weil–Deligne parametrization with respect to one (and hence any) choice of ϕ and ζ is pure of weight w.*

Let Ω be an algebraically closed field of characteristic zero. For a Weil–Deligne representation (r, V, N) of W_K over Ω, its *Euler factor* $\mathrm{Eul}((r,N), X)$ is defined as the element $\det(1 - X\phi|_{V^{I_K, N=0}})^{-1}$ of $\Omega(X)$ where $V^{I_K,N=0}$ denotes the subspace of V on which I_K acts trivially and N is zero. For a representation $\rho : G_E \to \mathrm{GL}(V)$ of the absolute Galois group of a number field E on a finite dimensional vector space V over Ω, its *local Euler factor* $\mathrm{Eul}_v(\rho, X)$ at a finite place v of E not dividing p is defined to be the element $\mathrm{Eul}(\mathrm{WD}(V|_{G_v}), X)$ in $\Omega(X)$ if $V|_{G_v}$ is monodromic. Now we define the notion of automorphic types.

Definition 2.11. — *Let (ρ, N) be a Frobenius-semisimple Weil–Deligne representation of W_K over $\overline{\mathbb{Q}}_p$. Let $m, t_1, \cdots, t_m$ be positive integers and $r_1, \cdots, r_m$ be irreducible Frobenius-semisimple representations of W_K over $\overline{\mathbb{Q}}_p$ such that (ρ, N) is isomorphic to $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(r_i)$. We define the* automorphic representation type $\mathrm{AT}^{\mathrm{rep}}(\mathrm{rec}^{-1}(\iota_p(\rho, N)))$ of $\mathrm{rec}^{-1}(\iota_p(\rho,N))$ *to be the unordered tuple*

$$\mathrm{AT}^{\mathrm{rep}}(\mathrm{rec}^{-1}(\iota_p(\rho, N))) = ((\mathrm{rec}^{-1}(\iota_p(r_1)), t_1), \cdots, (\mathrm{rec}^{-1}(\iota_p(r_m)), t_m))$$

and the automorphic type $\mathrm{AT}(\mathrm{rec}^{-1}(\iota_p(\rho, N)))$ of $\mathrm{rec}^{-1}(\iota_p(\rho,N))$ *to be the unordered tuple*

$$\mathrm{AT}(\mathrm{rec}^{-1}(\iota_p(\rho, N))) = ((\dim r_1, t_1), \cdots, (\dim r_m, t_m)).$$

3. Purity for big Galois representations

Recall from §1.2 that (ρ, N) is a Weil–Deligne representation of W_K on $\mathscr{O}^n$, where $\mathscr{O}$ is an integral domain over $\mathbb{Q}$ with fraction field $\mathscr{L}$. The Weil–Deligne representation $(\rho, N) \otimes_{\mathscr{O}} \overline{\mathscr{L}}$ is denoted by $\mathcal{V}$. For a map $\lambda : \mathscr{O} \to \overline{\mathbb{Q}}_p$, denote by π_λ the automorphic representation $\mathrm{rec}^{-1}(\iota_p(V_\lambda^{\text{Fr-ss}}))$ where $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$ denotes the isomorphism fixed in §1.2.

THEOREM 3.1 (Purity for big Galois representations). — *Let $m, t_1 \leqslant \cdots \leqslant t_m$ be positive integers, $r_1, \cdots, r_m$ be irreducible Frobenius-semi-simple representations of W_K over $\mathscr{O}^{\mathrm{intal}}$ such that*

$$\mathcal{V}^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{m} \mathrm{Sp}_{t_i}(r_i)_{/\overline{\mathscr{L}}}. \tag{3.1}$$

Suppose $\lambda : \mathscr{O} \to \overline{\mathbb{Q}}_p$ is a map such that V_λ is pure. Then the following hold.

(1) *The Weil–Deligne representations $V_\lambda^{\text{Fr-ss}}$ and $\oplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\lambda^{\mathrm{intal}} \circ r_i)_{/\overline{\mathbb{Q}}_p}$ are isomorphic.*
(2) *The rank of no power of the monodromy N decreases after specializing at λ.*
(3) *The polynomial $\mathrm{Eul}(\mathcal{V}, X)^{-1}$ has coefficients in $\mathscr{O}^{\mathrm{int}}$ and its specialization under λ^{intal} is $\mathrm{Eul}(V_\lambda, X)^{-1}$.*
(4) *The automorphic representation type $\mathrm{AT}^{\mathrm{rep}}(\pi_\lambda)$ of π_λ is equal to the unordered tuple*

$$\left((\mathrm{rec}^{-1}(\iota_p(\lambda^{\mathrm{intal}} \circ r_1)), t_1), \cdots, (\mathrm{rec}^{-1}(\iota_p(\lambda^{\mathrm{intal}} \circ r_m)), t_m)\right).$$

(5) *The automorphic type $\mathrm{AT}(\pi_\lambda)$ of π_λ is equal to the unordered tuple $((\dim \rho_1, t_1), \cdots, (\dim \rho_m, t_m))$.*

Moreover, for any field $\mathcal{K}$ and any map $\mu : \mathscr{O} \to \mathcal{K}$ with $\lambda(\ker \mu) = 0$, the Weil–Deligne representation $(V_\mu \otimes_{\mathcal{K}} \overline{\mathcal{K}})^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)_{/\overline{\mathcal{K}}}$. Furthermore, there exist m, t_i, r_i with the above-mentioned properties such that equation (3.1) holds.

Note that the above theorem can also be stated in terms of monodromic representations of the Weil group W_K or the Galois group G_K and thus it can be considered as a statement about big Galois representations.

Proof. — Suppose V_λ is pure of weight w. For $1 \leqslant j \leqslant m$, denote the multiset

$$\bigcup_{i=1}^{j} \{[\lambda^{\mathrm{intal}} \circ r_i], [\lambda^{\mathrm{intal}} \circ (|\mathrm{Art}_K^{-1}|_K r_i)], \cdots, [\lambda^{\mathrm{intal}} \circ (|\mathrm{Art}_K^{-1}|_K^{t_i - 1} r_i)]\}$$

by S_j (where $[r]$ denotes the isomorphism class of a representation r). Note that conditions (A), (B), (C) below hold.

(A) $V_\lambda^{\text{Fr-ss}}$ is pure of weight w,
(B) $\lambda^{\mathrm{intal}} \circ \mathrm{tr}\mathcal{V}^{\text{Fr-ss}} = \mathrm{tr}V_\lambda^{\text{Fr-ss}}$,
(C) $V_\lambda^{\text{Fr-ss}}$ is annihilated by the t_m-th power of its monodromy.

By [20, 1.6.7], the monodromy filtration associated to a direct sum of Weil–Deligne representations is equal to the direct sum of the monodromy filtrations associated to its summands. So the indecomposable summands of $V_\lambda^{\text{Fr-ss}}$ are pure of weight w (by condition (A)). Moreover, they are of size (see Definition 2.6) at most t_m by condition (C). Since the elements of S_m are (isomorphism classes of) irreducible Frobenius-semisimple W_K-representations (by Lemma 2.9) and the sum of their traces is equal to $\mathrm{tr} V_\lambda^{\text{Fr-ss}}$ (by condition (B)), the difference of the weights of any two elements of the multiset S_m is at most $2(t_m-1)$. Note that the difference of the weights of $\lambda^{\text{intal}}(r_m), \lambda^{\text{intal}}(|\mathrm{Art}_K^{-1}|_K^{t_m-1} r_m)$ is $2(t_m-1)$. So these are a highest weight and a lowest weight element of S_m respectively. By condition (A), w is equal to the average of the weights of a highest weight and a lowest weight element of S_m, i.e., the average of the weights of $\lambda^{\text{intal}}(r_m)$ and $\lambda^{\text{intal}}(|\mathrm{Art}_K^{-1}|_K^{t_m-1} r_m)$. So $\lambda^{\text{intal}}(r_m)$ has weight $w+t_m-1$. Since $\lambda^{\text{intal}}(r_m)$ is a highest weight element of S_m and $V_\lambda^{\text{Fr-ss}}$ is pure of weight w (by condition (A)), the Weil–Deligne representation $\mathrm{Sp}_{t_m}(\lambda^{\text{intal}}(r_m))$ is a direct summand of $V_\lambda^{\text{Fr-ss}}$.

Now assume that for an integer $1 \leqslant m' < m$, the representation $\mathrm{Sp}_{t_{m'+1}}(\lambda^{\text{intal}} \circ r_{m'+1}) \oplus \cdots \oplus \mathrm{Sp}_{t_m}(\lambda^{\text{intal}} \circ r_m)$ is a direct summand of $V_\lambda^{\text{Fr-ss}}$, i.e., there is an isomorphism

$$V_\lambda^{\text{Fr-ss}} \simeq W \oplus \bigoplus_{i=m'+1}^{m} \mathrm{Sp}_{t_i}(\lambda^{\text{intal}} \circ r_i). \tag{3.2}$$

Let $\mathcal{W}$ denote the Weil–Deligne representation $\oplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(r_i)$. Then the sum $\sum_{i=m'+1}^{m}(t_i - t_{m'}) \dim r_i$ is equal to the integer $\dim_{\overline{\mathscr{L}}} N^{t_{m'}}(\mathcal{V}^{\text{Fr-ss}})$ (by equation (3.1)), which is greater than or equal to $\dim_{\overline{\mathbb{Q}}_p} \lambda(N)^{t_{m'}}(V_\lambda^{\text{Fr-ss}})$ and this is equal to $\dim_{\overline{\mathbb{Q}}_p} \lambda(N)^{t_{m'}} W + \sum_{i=m'+1}^{m}(t_i - t_{m'}) \dim r_i$ (by equation (3.2)). So $\lambda(N)^{t_{m'}}(W) = 0$. Thus conditions (A'), (B'), (C') below hold.

(A') W is pure of weight w,
(B') $\lambda^{\text{intal}} \circ \mathrm{tr}\mathcal{W} = \mathrm{tr} W$,
(C') W is annihilated by the $t_{m'}$-th power of its monodromy.

In other words, the conditions (A), (B), (C) also hold when $V_\lambda^{\text{Fr-ss}}, \mathcal{V}^{\text{Fr-ss}}, m$ are replaced by $W, \mathcal{W}, m'$ respectively. So the argument in the paragraph next to conditions (A), (B), (C) also hold when $V_\lambda^{\text{Fr-ss}}, \mathcal{V}^{\text{Fr-ss}}, m$ are replaced by $W, \mathcal{W}, m'$ respectively (and conditions (A'), (B'), (C') are used instead of (A), (B), (C)). So it follows that the Weil–Deligne representation $\mathrm{Sp}_{t_{m'}}(\lambda^{\text{intal}} \circ r_{m'})$ is a direct summand of W. Then equation (3.2) shows that $\mathrm{Sp}_{t_{m'}}(\lambda^{\mathrm{intal}} \circ r_{m'}) \oplus \mathrm{Sp}_{t_{m'+1}}(\lambda^{\mathrm{intal}} \circ r_{m'+1}) \oplus \cdots \oplus \mathrm{Sp}_{t_m}(\lambda^{\mathrm{intal}} \circ r_m)$ is a direct summand of $V_\lambda^{\mathrm{Fr\text{-}ss}}$. This proves part (1) by induction.

Note that part (2) follows from part (1) and equation (3.1). Since the characteristic polynomial of $\rho(g)$ has coefficients in $\mathscr{O}$ for all $g \in W_K$, the roots of the characteristic polynomial of $\rho(g)$ are elements of $\mathscr{O}^{\mathrm{intal}}$. So the coefficients of $\mathrm{Eul}(\mathcal{V}, X)^{-1} = \det(1 - X\phi|_{\mathcal{V}^{I_K, N=0}})$ are elements of $\mathscr{O}^{\mathrm{intal}}$. However this polynomial has coefficients in the fraction field $\mathscr{L}$ of $\mathscr{O}$. Hence $\mathrm{Eul}(\mathcal{V}, X)^{-1}$ has coefficients in $\mathscr{O}^{\mathrm{int}}$. To complete the proof of part (3), note that $\mathcal{V}^{I_K,N=0}$ is isomorphic to $\oplus_{i=1}^m (r_i|\mathrm{Art}_K^{-1}|_K^{t_i-1})^{I_K}$ over $\overline{\mathscr{L}}$, and by part (1), the representation $V_\lambda^{I_K,N=0}$ is isomorphic to $\oplus_{i=1}^m (\lambda^{\mathrm{intal}}(r_i|\mathrm{Art}_K^{-1}|_K^{t_i-1}))^{I_K}$ over $\overline{\mathbb{Q}}_p$. Denote by ρ_i the representation $r_i|\mathrm{Art}_K^{-1}|_K^{t_i-1}$ of W_K over $\mathscr{O}^{\mathrm{intal}}$. Since $\rho_i(I_K)$ is finite, the map λ^{intal} induces an isomorphism

$$\rho_i^{I_K} \otimes_{\mathscr{O}^{\mathrm{intal}}, \lambda^{\mathrm{intal}}} \overline{\mathbb{Q}}_p \simeq (\lambda^{\mathrm{intal}} \circ \rho_i)^{I_K}$$

of W_K-representations. So under the map λ^{intal}, the polynomial $\det(1 - X\phi|_{\rho_i^{I_K}})$ specializes to $\det(1 - X\phi|_{(\lambda^{\mathrm{intal}} \circ \rho_i)^{I_K}})$, i.e., the specialization of $\mathrm{Eul}(\mathcal{V}, X)^{-1}$ under λ^{intal} is equal to $\mathrm{Eul}(V_\lambda, X)^{-1}$. This proves part (3). Note that part (1) implies parts (4) and (5).

To prove the statement about $V_\mu \otimes \overline{\mathcal{K}}$, we assume that $\mathcal{K}$ is algebraically closed (to simplify notations). Let $\mathcal{O}_\mu$ (resp. $\mathcal{O}_\lambda$) denote the image of μ (resp. λ) and $\eta : \mathcal{O}_\mu \to \mathcal{O}_\lambda$ denote the map such that $\lambda = \eta \circ \mu$. Let $\lambda^\dagger$ denote the map $\eta^{\mathrm{intal}} \circ \mu^{\mathrm{intal}}$. By Proposition 2.8, there exist positive integers $M, t'_1 \leqslant \cdots \leqslant t'_M$ and irreducible Frobenius-semisimple representations $s_1, \cdots, s_M$ over $\mathcal{O}_\mu^{\mathrm{intal}}$ such that $V_\mu^{\mathrm{Fr\text{-}ss}}$ is isomorphic to $\oplus_{i=1}^M \mathrm{Sp}_{t'_i}(s_i)$. By part (1), $V_\lambda^{\mathrm{Fr\text{-}ss}}$ is isomorphic to $\oplus_{i=1}^M \mathrm{Sp}_{t'_i}(\eta^{\mathrm{intal}} \circ s_i)$. Hence $M = m$ and $t'_i = t_i$ for all $1 \leqslant i \leqslant M$. So $\eta^{\mathrm{intal}} \circ s_i, \lambda^\dagger \circ r_i$ are of weight $w + t_i - 1$ for all $1 \leqslant i \leqslant m$. Note that for some integers $1 \leqslant j \leqslant m$, $0 \leqslant a \leqslant t_j - 1$, the representations $\mu^{\mathrm{intal}} \circ r_m$ and $s_j|\mathrm{Art}_K^{-1}|_K^a$ are isomorphic. So the representations $\lambda^\dagger \circ r_m, \eta^{\mathrm{intal}} \circ (s_j|\mathrm{Art}_K^{-1}|_K^a)$ are of equal weight. This shows $t_m = t_j - 2a$, hence $a = 0, t_j = t_m$. Thus $\mathrm{Sp}_{t_m}(\mu^{\mathrm{intal}} \circ r_m)$ is a direct summand of $V_\mu^{\mathrm{Fr\text{-}ss}}$. Suppose for an integer $1 \leqslant m' < m$, the representation $\oplus_{i=m'+1}^m \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)$ is a direct summand of $V_\mu^{\mathrm{Fr\text{-}ss}}$. By Proposition 2.8, there exist irreducible Frobenius-semisimple representations $s'_1, \cdots, s'_{m'}$ over $\mathcal{O}_\mu^{\mathrm{intal}}$ such that $V_\mu^{\mathrm{Fr\text{-}ss}}$ is isomorphic to $\bigoplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(s'_i) \oplus \bigoplus_{i=m'+1}^m \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)$. By part (1), $V_\lambda^{\mathrm{Fr\text{-}ss}}$ is isomorphic to $\bigoplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(\eta^{\mathrm{intal}} \circ s'_i) \oplus \bigoplus_{i=m'+1}^m \mathrm{Sp}_{t_i}(\eta^{\mathrm{intal}} \circ \mu^{\mathrm{intal}} \circ r_i)$. So $\eta^{\mathrm{intal}} \circ s'_i, \lambda^\dagger \circ r_i$ are of weight $w + t_i - 1$ for all $1 \leqslant i \leqslant m'$. Note that for some integers $1 \leqslant k \leqslant m'$ and $0 \leqslant b \leqslant t_k - 1$, the representations $\mu^{\mathrm{intal}} \circ r_{m'}$ and $s'_k|\mathrm{Art}_K^{-1}|_K^b$ are isomorphic. So the representations $\lambda^\dagger \circ r_{m'}, \eta^{\mathrm{intal}} \circ (s'_k|\mathrm{Art}_K^{-1}|_K^b)$ are of equal weight. This shows $t_{m'} = t_k - 2b$, hence $b = 0, t_k = t_{m'}$. Thus $\mathrm{Sp}_{t_{m'}}(\mu^{\mathrm{intal}} \circ r_{m'})$ is a direct summand of $\oplus_{i=1}^{m'} \mathrm{Sp}_{t_i}(s'_i)$ and hence $\oplus_{i=m'}^{m} \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)$ is a direct summand of $V_\mu^{\text{Fr-ss}}$. By induction, it follows that $(V_\mu \otimes_{\mathcal{K}} \overline{\mathcal{K}})^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\mu^{\mathrm{intal}} \circ r_i)_{/\overline{\mathcal{K}}}$. Finally, the existence of m, t_i, r_i is guaranteed by Proposition 2.8. □

Note that when the cofficient ring $\mathscr{O}$ (as in §1.2) of the representation (ρ, N) is replaced by a more general ring (for instance, a ring which is not an integral domain), no analogue of Theorem 3.1 seems to exist. In fact a crucial step in its proof is to express the trace of $\mathcal{V}$ as a sum of traces of irreducible Frobenius-semisimple representations over $\mathscr{O}^{\mathrm{intal}}$ and then to pin down the factors of powers of the character $|\mathrm{Art}_K^{-1}|_K$ in them. The amount of these factors is governed by the size of the Jordan blocks of the monodromy of $\mathcal{V}$. When the coefficient ring $\mathscr{O}$ of (ρ, N) is not a domain, then the shapes of the Jordan blocks of the images of N in the residue fields of $\mathscr{O}_{\mathfrak{a}}$ for minimal primes $\mathfrak{a}$ of $\mathscr{O}$ need not be independent of $\mathfrak{a}$. Thereby, in no reasonable manner, it is possible to pin down the factors of powers of $|\mathrm{Art}_K^{-1}|_K$ present in the representations stated above. Even in the very simple case where $\mathscr{O} = \mathbb{Z}_p[\![X]\!] \times \mathbb{Z}_p[\![X]\!] \times \mathbb{Z}_p[\![X]\!]$, $\mathcal{V}$ is semistable and $N \in M_3(\mathscr{O})$ is the strictly upper triangular matrix with $N_{12} = (X, 0, 0), N_{13} = 0, N_{23} = (0, X, 0)$, we cannot track the 'right' factors of powers of q in the characteristic roots of ϕ on $\mathcal{V}$. Thus it seems hard to have a reasonable analogue of equation (3.1) that could lead to an analogue of Theorem 3.1 when $\mathscr{O}$ is a more general ring than a domain (for example, a non-integral domain). So we are compelled to assume that $\mathscr{O}$ is a domain.

4. Purity for pseudorepresentations

In this section, we prove Theorem 4.3 which is an analogue of Theorem 3.1 in the context of pseudorepresentations of Weil groups. We refer to Wiles [55] and Taylor [53] for the notion of pseudorepresentations. They are defined by abstracting the crucial properties of the trace of a group representation.

4.1. Preliminaries

Let $\mathcal{O}_1, \mathcal{O}_2$ be integral domains with fraction fields $\mathcal{L}_1, \mathcal{L}_2$ respectively. Let $\mathrm{res}_1 : \mathscr{O} \hookrightarrow \mathcal{O}_1$, $\mathrm{res}_2 : \mathscr{O} \hookrightarrow \mathcal{O}_2$ be injective maps. Let $T_0 : W_K \to$

$\mathscr{O}^{\text{intal}}$ be a pseudorepresentation of dimension $d \geqslant 1$ and $(r_1, N_1) : W_K \to \mathrm{GL}_d(\mathcal{O}_1^{\text{intal}})$, $(r_2, N_2) : W_K \to \mathrm{GL}_d(\mathcal{O}_2^{\text{intal}})$ be Weil–Deligne representations such that

$$\mathrm{res}_1^{\dagger} \circ T_0 = \mathrm{tr}(r_1), \quad \mathrm{res}_2^{\ddagger} \circ T_0 = \mathrm{tr}(r_2)$$

for some lifts $\mathrm{res}_1^{\dagger} : \mathscr{O}^{\text{intal}} \to \mathcal{O}_1^{\text{intal}}, \mathrm{res}_2^{\ddagger} : \mathscr{O}^{\text{intal}} \to \mathcal{O}_2^{\text{intal}}$ of $\mathrm{res}_1, \mathrm{res}_2$ respectively. Suppose that there exist maps $f_1 : \mathcal{O}_1^{\text{intal}} \to \overline{\mathbb{Q}}_p$, $f_2 : \mathcal{O}_2^{\text{intal}} \to \overline{\mathbb{Q}}_p$ such that $f_1 \circ (r_1, N_1), f_2 \circ (r_2, N_2)$ are pure. We first state a proposition, which plays a crucial role in the proof of Theorem 4.3. Next, we prove a lemma which will be used to establish this proposition.

Proposition 4.1. — *The size of $(f_1 \circ (r_1, N_1))^{\text{Fr-ss}}$ is less than or equal to the size of $(f_2 \circ (r_2, N_2))^{\text{Fr-ss}}$. Consequently, these two representations have the same size. Suppose there is an isomorphism*

$$((r_1, N_1) \otimes_{\mathcal{O}_1} \overline{\mathcal{L}}_1)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{\kappa} \mathrm{Sp}_{t_i}(\mathrm{res}_1^{\dagger} \circ \theta_i) \tag{4.1}$$

where $\kappa, t_1 \leqslant \cdots \leqslant t_\kappa$ are positive integers and $\theta_1, \cdots, \theta_\kappa$ are irreducible Frobenius-semisimple representations of W_K over $\mathscr{O}^{\text{intal}}$. Then the representation $\mathrm{Sp}_{t_\kappa}(\mathrm{res}_2^{\ddagger} \circ \theta_\kappa)$ is a direct summand of $((r_2, N_2) \otimes_{\mathcal{O}_2} \overline{\mathcal{L}}_2)^{\text{Fr-ss}}$.

Lemma 4.2. — *Let $k, s_1 \leqslant \cdots \leqslant s_k$ be positive integers and $\vartheta_1, \cdots, \vartheta_k$ be irreducible Frobenius-semisimple representations of W_K over $\mathcal{O}_2^{\text{intal}}$ such that*

$$((r_2, N_2) \otimes_{\mathcal{O}_2} \overline{\mathcal{L}}_2)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{k} \mathrm{Sp}_{s_i}(\vartheta_i). \tag{4.2}$$

Then for some integers $1 \leqslant a, b \leqslant k$, we have

$$\begin{gathered}(\mathrm{res}_2^{\ddagger} \circ (\theta_\kappa |\mathrm{Art}_K^{-1}|_K^{t_\kappa - 1}))_{/\overline{\mathcal{L}}_2} \simeq (\vartheta_a |\mathrm{Art}_K^{-1}|_K^{s_a - 1})_{/\overline{\mathcal{L}}_2}, \\ (\mathrm{res}_2^{\ddagger} \circ \theta_\kappa)_{/\overline{\mathcal{L}}_2} \simeq (\vartheta_b)_{/\overline{\mathcal{L}}_2},\end{gathered} \tag{4.3}$$

$$2t_\kappa = s_a + s_b \leqslant 2s_k. \tag{4.4}$$

Proof. — By Lemma 2.9, $\mathrm{res}_1^{\dagger} \circ \theta_i$ is an irreducible Frobenius-semisimple representation of W_K over $\mathcal{O}_1^{\text{intal}}$. Since $f_1 \circ (r_1, N_1)$ is pure, Theorem 3.1 and equation (4.1) give

$$(f_1 \circ (r_1, N_1))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{\kappa} \mathrm{Sp}_{t_i}(f_1 \circ \mathrm{res}_1^{\dagger} \circ \theta_i). \tag{4.5}$$

Since $t_1 \leqslant \cdots \leqslant t_\kappa$ and $f_1 \circ (r_1, N_1)$ is pure, by equation (4.5), no eigenvalue of ϕ on $f_1 \circ (r_1, N_1)$ has weight strictly more (resp. less) than the weight of the ϕ-eigenvalues on $f_1 \circ \mathrm{res}_1^{\dagger} \circ \theta_\kappa$ (resp. $f_1 \circ \mathrm{res}_1^{\dagger} \circ (\theta_\kappa |\mathrm{Art}_K^{-1}|_K^{t_\kappa - 1})$). So there are no integers i, j with $1 \leqslant i \leqslant \kappa, 1 \leqslant j \leqslant t_i$ such that $\theta_i|\mathrm{Art}_K^{-1}|_K^{j-1}$ is isomorphic to $\theta_\kappa|\mathrm{Art}_K^{-1}|_K^{-\nu}$ or $\theta_\kappa|\mathrm{Art}_K^{-1}|_K^{t_\kappa-1+\nu}$ for some integer $\nu \geqslant 1$. Note that by Lemma 2.9, there exist integers $1 \leqslant a, b \leqslant k$ such that the W_K-representation $\mathrm{res}_2^\ddagger \circ (\theta_\kappa|\mathrm{Art}_K^{-1}|_K^{t_\kappa-1})$ (resp. $\mathrm{res}_2^\ddagger \circ \theta_\kappa$) is isomorphic to $\vartheta_a|\mathrm{Art}_K^{-1}|_K^{j_1}$ (resp. $\vartheta_b|\mathrm{Art}_K^{-1}|_K^{j_2}$) over $\overline{\mathcal{L}}_2$ where $0 \leqslant j_1 \leqslant s_a - 1$ (resp. $0 \leqslant j_2 \leqslant s_b - 1$). Now for some $1 \leqslant i \leqslant \kappa, 1 \leqslant j \leqslant t_i$, the W_K-representations $\mathrm{res}_2^\ddagger \circ \theta_i|\mathrm{Art}_K^{-1}|_K^{j-1}, \vartheta_a|\mathrm{Art}_K^{-1}|_K^{s_a-1} \simeq \mathrm{res}_2^\ddagger \circ (\theta_\kappa|\mathrm{Art}_K^{-1}|_K^{t_\kappa-1-j_1+s_a-1})$ are isomorphic over $\overline{\mathcal{L}}_2$. As res_2 is injective and the traces of the representations $\theta_i|\mathrm{Art}_K^{-1}|_K^{j-1}$ and $\theta_\kappa|\mathrm{Art}_K^{-1}|_K^{t_\kappa-1-j_1+s_a-1}$ coincide after composing them with $\mathrm{res}_2^\ddagger$, these representations are isomorphic over $\overline{\mathscr{L}}$ (by [48, Chapter 1, §2] for instance). As noted before, $s_a - 1 - j_1$ cannot be positive. So j_1 is equal to $s_a - 1$. Similarly j_2 is zero. Thus equation (4.3) holds. Using Theorem 3.1 and equation (4.2), we get

$$(f_2 \circ (r_2, N_2))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{k} \mathrm{Sp}_{s_i}(f_2 \circ \vartheta_i). \tag{4.6}$$

Let w denote the weight of $f_2 \circ (r_2, N_2)$. So the weight of any ϕ-eigenvalue on $f_2 \circ \vartheta_b$ (resp. $f_2 \circ \vartheta_a|\mathrm{Art}_K^{-1}|_K^{s_a-1}$) is equal to $w + (s_b - 1)$ (resp. $w - (s_a - 1)$). Thus their difference is equal to $s_a + s_b - 2$, and this difference is also equal to $2(t_\kappa - 1)$ by equation (4.3). Since s_a, s_b are less than or equal to s_k, we get equation (4.4). □

Proof of Proposition 4.1. — Equations (4.4), (4.5), (4.6) together imply that the size of $(f_1 \circ (r_1, N_1))^{\text{Fr-ss}}$ is less than or equal to the size of $(f_2 \circ (r_2, N_2))^{\text{Fr-ss}}$. Similarly, the size of $(f_2 \circ (r_2, N_2))^{\text{Fr-ss}}$ is less than or equal to the size of $(f_1 \circ (r_1, N_1))^{\text{Fr-ss}}$. So, these two representations have the same size. Thus t_κ is equal to s_k. Then equation (4.4) gives $s_a = s_b = s_k$. So s_b is equal to t_κ and ϑ_b is isomorphic to $\mathrm{res}_2^\ddagger \circ \theta_\kappa$ over $\overline{\mathcal{L}}_2$ (by equation (4.3)). Thus $\mathrm{Sp}_{t_\kappa}(\mathrm{res}_2^\ddagger \circ \theta_\kappa)$ is isomorphic to $\mathrm{Sp}_{s_b}(\vartheta_b)$ and hence it is a direct summand of $((r_2, N_2) \otimes_{\mathcal{O}_2} \overline{\mathcal{L}}_2)^{\text{Fr-ss}}$ by equation (4.2). □

4.2. Pseudorepresentations of Weil groups

THEOREM 4.3 (Purity for pseudorepresentations). — *Let $T : W_K \to \mathscr{O}$ be a pseudorepresentation of dimension $n \geqslant 1$. Let $\mathcal{O}$ be an integral domain,* $\mathrm{res} : \mathscr{O} \hookrightarrow \mathcal{O}$ *be an injective map and $(r, N) : W_K \to \mathrm{GL}_n(\mathcal{O})$ be a Weil–Deligne representation such that* $\mathrm{res} \circ T = \mathrm{tr} r$ *and $f \circ (r, N)$ is pure for some map $f : \mathcal{O} \to \overline{\mathbb{Q}}_p$. Suppose*

$$\big((r, N) \otimes_{\mathcal{O}} \overline{Q}(\mathcal{O})\big)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\mathrm{res}^{\mathrm{intal}} \circ r_i) \tag{4.7}$$

where $m, t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m$ are positive integers and $r_1, \cdots, r_m$ are irreducible Frobenius-semisimple representations of W_K with coefficients in $\mathscr{O}^{\mathrm{intal}}$. Then for any Weil–Deligne representation $(r', N') : W_K \to \mathrm{GL}_n(\mathcal{O}')$ over a domain $\mathcal{O}'$ satisfying

- $\mathrm{res}' \circ T = \mathrm{tr} r'$ *for some injective map* $\mathrm{res}' : \mathscr{O} \hookrightarrow \mathcal{O}'$,
- $f' \circ (r', N')$ *is pure for some map* $f' : \mathcal{O}' \to \overline{\mathbb{Q}}_p$,

there are isomorphisms

$$((r', N') \otimes_{\mathcal{O}'} \overline{Q}(\mathcal{O}'))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\mathrm{res}'^{\dagger} \circ r_i), \tag{4.8}$$

$$(f' \circ (r', N'))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{m} \mathrm{Sp}_{t_i}(f'^{\dagger} \circ \mathrm{res}'^{\dagger} \circ r_i) \tag{4.9}$$

for any lift $\mathrm{res}'^{\dagger} : \mathscr{O}^{\mathrm{intal}} \to \mathcal{O}'^{\mathrm{intal}}$ of res' and any lift $f'^{\dagger} : \mathcal{O}'^{\mathrm{intal}} \to \overline{\mathbb{Q}}_p$ of f'. Moreover, there exist m, t_i, r_i with the above-mentioned properties such that equation (4.7) holds.

Proof. — Let $\mathcal{L}'$ denote the fraction field of $\mathcal{O}'$. By Proposition 4.1, $\mathrm{Sp}_{t_m}(\mathrm{res}'^{\dagger} \circ r_m)$ is a direct summand of $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\text{Fr-ss}}$. Suppose for some $1 \leqslant k < m$, the representation $\oplus_{i=k+1}^{m} \mathrm{Sp}_{t_i}(\mathrm{res}'^{\dagger} \circ r_i)$ is a direct summand of $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\text{Fr-ss}}$. By Proposition 2.8, there exist positive integers $Q, s_1 \leqslant \cdots \leqslant s_Q$ and irreducible Frobenius-semisimple representations $\eta_1, \cdots, \eta_Q$ of W_K with coefficients in $\mathcal{O}'^{\mathrm{intal}}$ such that $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\text{Fr-ss}}$ is isomorphic to $\bigoplus_{i=1}^{Q} \mathrm{Sp}_{s_i}(\eta_i) \oplus \bigoplus_{i=k+1}^{m} \mathrm{Sp}_{t_i}(\mathrm{res}'^{\dagger} \circ r_i)$. Note that the specialization of the pseudorepresentation $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \mathrm{tr} r_i |\mathrm{Art}_K^{-1}|_K^{j-1} : W_K \to \mathscr{O}^{\mathrm{intal}}$ under $\mathrm{res}^{\mathrm{intal}}$ (resp. $\mathrm{res}'^{\dagger}$) is equal to the trace of the Weil–Deligne representation $\oplus_{i=1}^{k} \mathrm{Sp}_{t_i}(\mathrm{res}^{\mathrm{intal}} \circ r_i)$ (resp. $\oplus_{i=1}^{Q} \mathrm{Sp}_{s_i}(\eta_i)$) of W_K with coefficients in $\mathcal{O}^{\mathrm{intal}}$ (resp. $\mathcal{O}'^{\mathrm{intal}}$). So by Proposition 4.1, the representation $\mathrm{Sp}_{t_k}(\mathrm{res}'^{\dagger} \circ r_k)$ is a direct summand of $\oplus_{i=1}^{Q} \mathrm{Sp}_{s_i}(\eta_i)$. This shows $\oplus_{i=k}^{m} \mathrm{Sp}_{t_i}(\mathrm{res}'^{\dagger} \circ r_i)$ is a direct summand of $((r', N') \otimes_{\mathcal{O}'} \overline{\mathcal{L}}')^{\text{Fr-ss}}$. So we obtain equation (4.8) by induction. Then Theorem 3.1 gives equation (4.9). Now let $\mathcal{L}$ denote the fraction field of $\mathcal{O}$. By Proposition 2.8, there exist positive integers m, $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m$ and irreducible Frobenius-semisimple representations $\tau_1, \cdots, \tau_m$ of W_K with coefficients in $\mathcal{O}^{\mathrm{intal}}$ such that $((r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}})^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\tau_i)$. Since $\mathrm{tr} r = \mathrm{res} \circ T$, the characteristic polynomial of τ_i has coefficients in $(\mathrm{res}\mathscr{O})^{\mathrm{intal}}$ (we consider $Q(\mathrm{res}\mathscr{O})$ as a subfield of $\mathcal{L}$ and thus $(\mathrm{res}\mathscr{O})^{\mathrm{intal}}$ is a subring of $\mathcal{O}^{\mathrm{intal}}$). So by Proposition 2.8, we may (and do) assume that τ_i has coefficients in

$(\mathrm{res}\mathscr{O})^{\mathrm{intal}}$. The map res being injective, induces an isomorphism between $\mathscr{O}$ and its image $\mathrm{res}\mathscr{O}$. Thus $\mathrm{res}^{\mathrm{intal}}$ is an isomorphism between $\mathscr{O}^{\mathrm{intal}}$ and $(\mathrm{res}\mathscr{O})^{\mathrm{intal}}$. Since τ_i has coefficients in $(\mathrm{res}\mathscr{O})^{\mathrm{intal}}$, there exist irreducible Frobenius-semisimple representations $r_1, \cdots, r_m$ of W_K with coefficients in $\mathscr{O}^{\mathrm{intal}}$ such that $\mathrm{res}^{\mathrm{intal}} \circ r_1 = \tau_1, \cdots, \mathrm{res}^{\mathrm{intal}} \circ r_m = \tau_m$, and hence equation (4.7) holds. □

4.3. Pure specializations of pseudorepresentations of global Galois groups

Let F be a number field and $T : G_F \to \mathscr{O}$ be a pseudorepresentation such that $T = T_1 + \cdots + T_n$ where $T_1 : G_F \to \mathscr{O}, \cdots, T_n : G_F \to \mathscr{O}$ are pseudorepresentations. Fix a finite place w of F not dividing p and assume that $\mathscr{O}$ is a $\mathbb{Z}_p$-algebra.

Definition 4.4. — *The* irreducibility and purity locus *of $T_1, \cdots, T_n$ is defined to be the collection of all tuples of the form $(\mathcal{O}, \mathfrak{m}, \kappa, \mathrm{loc}, \rho_1, \cdots, \rho_n)$ where $\mathcal{O}$ is a $\mathbb{Z}_p$-algebra and it is a Henselian Hausdorff domain with maximal ideal $\mathfrak{m}$, κ denotes the residue field of $\mathcal{O}$ and is an algebraic extension of $\mathbb{Q}_p$, $\mathrm{loc} : \mathscr{O} \hookrightarrow \mathcal{O}$ is an injective $\mathbb{Z}_p$-algebra homomorphism and for each $1 \leqslant i \leqslant n$, ρ_i is an irreducible G_F-representation over $\overline{\kappa}$ such that the trace of ρ_i is equal to $\mathrm{loc} \circ T_i \bmod \mathfrak{m}$ and $\rho_i|_{G_w}$ is pure.*

For each element $(\mathcal{O}, \mathfrak{m}, \kappa, \mathrm{loc}, \rho_1, \cdots, \rho_n)$ of this locus, by [42, Théorème 1], there exist semisimple G_F-representations $\widetilde{\rho}_1, \cdots, \widetilde{\rho}_n$ over $\mathcal{O}$ such that $\mathrm{tr}\widetilde{\rho}_i = \mathrm{loc} \circ T_i$ for all $1 \leqslant i \leqslant n$. Using [53, Theorem 1], choose semisimple representations $\sigma_1, \cdots, \sigma_n$ of G_F over $\overline{\mathscr{L}}$ such that $\mathrm{tr}\sigma_i = T_i$ for all $1 \leqslant i \leqslant n$.

Theorem 4.5. — *Suppose the irreducibility and purity locus of $T_1, \cdots, T_n$ is nonempty and the restrictions of $\sigma_1, \cdots, \sigma_n$ to W_w are monodromic (see Definition 2.2). Then there exist positive integers $m, t_1, t_2, \cdots, t_m$ and irreducible Frobenius-semisimple representations $r_1, \cdots, r_m$ of W_w with coefficients in $\mathscr{O}^{\mathrm{intal}}$ such that*

$$\mathrm{WD}\left(\bigoplus_{i=1}^{n} \sigma_i|_{W_w}\right)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{m} \mathrm{Sp}_{t_i}(r_i)_{/\overline{\mathscr{L}}} \tag{4.10}$$

and for any element $(\mathcal{O}, \mathfrak{m}, \kappa, \mathrm{loc}, \rho_1, \cdots, \rho_n)$ *in the irreducibility and purity locus of* $T_1, \cdots, T_n$*, there are isomorphisms*

$$\mathrm{WD}\left(\bigoplus_{i=1}^{n} \rho_i|_{W_w}\right)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\pi_{\mathfrak{m}}^{\text{intal}} \circ \mathrm{loc}^{\text{intal}} \circ r_i)_{\overline{\kappa}}, \tag{4.11}$$

$$\mathrm{WD}\left(\bigoplus_{i=1}^{n} \widetilde{\rho}_i|_{W_w} \otimes \overline{Q}(\mathcal{O})\right)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{m} \mathrm{Sp}_{t_i}(\mathrm{loc}^{\text{intal}} \circ r_i)_{/\overline{Q}(\mathcal{O})}. \tag{4.12}$$

Proof. — Since $\widetilde{\rho}_i$ has coefficients in $\mathcal{O}$, $\mathrm{tr}\widetilde{\rho}_i$ mod $\mathfrak{m}$ is equal to $\mathrm{tr}\rho_i$ and ρ_i is irreducible, the representation $\widetilde{\rho}_i$ is irreducible. Note that the G_F-representations $\sigma_i \otimes \overline{Q}(\mathcal{O}), \widetilde{\rho}_i$ have same traces. So they are isomorphic. Since $\sigma_i|_{W_w}$ is monodromic, $\widetilde{\rho}_i|_{W_w}$ is also monodromic. So its Weil–Deligne parametrization $\mathrm{WD}(\widetilde{\rho}_i|_{W_w})$ is defined, it has coefficients in $\mathcal{O}$, its trace is equal to $\mathrm{loc} \circ T_i|_{W_w}$ and its mod $\mathfrak{m}$ reduction is isomorphic to the pure representation $\mathrm{WD}(\rho_i|_{W_w})$. Then Theorem 4.3 gives equations (4.11) and (4.12). Since $\widetilde{\rho}_i$ is isomorphic to $\sigma_i \otimes \overline{Q}(\mathcal{O})$, we get equation (4.10) from equation (4.12). □

5. Local Langlands correspondence for GL_n in families

Let (ρ, N) be a Frobenius-semisimple Weil–Deligne representation of W_K over a finite extension L of $\mathbb{Q}_p$. Let $\pi(\rho, N)$ denote the indecomposable admissible representation of $\mathrm{GL}_n(K)$ over L attached to (ρ, N) via the Breuil–Schneider modified local Langlands correspondence (see [6, pp. 161–164]). To define the representation $\pi(\rho, N)$, one needs to choose a square root of q. However the representation $\pi(\rho, N)$ is independent of this choice. In [23], this modified correspondence is extended to Frobenius-semisimple Weil–Deligne representations over arbitrary field extensions of $\mathbb{Q}_p$. For a Frobenius-semisimple Weil–Deligne representation (ρ, N) of W_K over an extension L of $\mathbb{Q}_p$, let $\pi(\rho, N)$ denote the indecomposable admissible representation of $\mathrm{GL}_n(K)$ over L attached to (ρ, N) (see §4.2 of loc. cit.). The smooth contragredient of $\pi(\rho, N)$ is denoted by $\widetilde{\pi}(\rho, N)$. If r is a monodromic representation of W_K on a finite dimensional vector space over a field extension L of $\mathbb{Q}_p$ and L contains the characteristic roots of all elements of $r(W_K)$, then we denote by $\widetilde{\pi}(r)$ the representation $\widetilde{\pi}(\mathrm{WD}(r)^{\text{Fr-ss}})$. Let $(A, \mathfrak{m})$ be a complete reduced p-torsion free Noetherian local ring with finite residue field of characteristic p. For example, $\mathbb{Z}_p[\![X]\!][Y]/(Y^2 + XY)$ is such a ring and it has two minimal primes, which are generated by Y and $Y + X$. The typical examples of A are p-adically completed Hecke algebras (see [22, Definition 5.2.5]). For a prime ideal $\mathfrak{p}$ of A, the mod $\mathfrak{p}$ reduction of a representation ρ on a free A-module is denoted by $\rho_{\mathfrak{p}}$. We refer to [23] for unfamiliar notations and terminologies used below.

Theorem 5.1. — *Let E be a number field and S denote a finite set of non-archimedean places of E, none of which divides p. For each $v \in S$, let $r_v : G_v \to \mathrm{GL}_n(A)$ be a continuous representation. Write $G = \prod_{v\in S}\mathrm{GL}_n(E_v)$. Then there exists at most one (up to isomorphism) admissible smooth representation V of G over A satisfying the conditions below.*

(1) *The module V is A-torsion free, i.e., all associated primes of V are minimal prime ideals of A.*
(2) *For every minimal prime $\mathfrak{a}$ of A, there exists a G-equivariant isomorphism*

$$\bigotimes_{v\in S}\widetilde{\pi}(r_{v,\mathfrak{a}}) \xrightarrow{\sim} \kappa(\mathfrak{a})\otimes_A V. \tag{5.1}$$

(3) *The G-cosocle $\mathrm{cosoc}(V/\mathfrak{m}V)$ of $V/\mathfrak{m}V$ is absolutely irreducible and generic, the kernel of the natural surjection $V/\mathfrak{m}V \to \mathrm{cosoc}(V/\mathfrak{m}V)$ contains no generic subrepresentations.*

Proof. — It is a part of [23, Theorem 6.2.1]. □

When V exists, we denote it by $\widetilde{\pi}(\{r_v\}_{v\in S})$. If S contains only one place, we denote V by $\widetilde{\pi}(r_v)$. By [23, Proposition 6.2.4], the $A[G]$-module $\widetilde{\pi}(\{r_v\}_{v\in S})$ exists if and only if the $A[\mathrm{GL}_n(E_v)]$-module $\widetilde{\pi}(r_v)$ exists for any $v \in S$. For a minimal prime $\mathfrak{a}$ of $A[1/p]$, the monodromy of $r_{v,\mathfrak{a}}$ is denoted by $N_v(\mathfrak{a})$ (which exists by [23, Proposition 4.1.6]).

Theorem 5.2. — *Let S be as in Theorem 5.1 and $\mathfrak{p}$ be a prime ideal of $A[1/p]$. Suppose that the $A[G]$-module $\widetilde{\pi}(\{r_v\}_{v\in S})$ exists. If $\mathfrak{p}$ lies on exactly one irreducible component of $\mathrm{Spec}A[\frac{1}{p}]$, then there exists a $\kappa(\mathfrak{p})$-linear G-equivariant surjection*

$$\gamma_{\mathfrak{p}} : \bigotimes_{v\in S}\widetilde{\pi}(r_{v,\mathfrak{p}}) \to \kappa(\mathfrak{p})\otimes_A \widetilde{\pi}(\{r_v\}_{v\in S}),$$

which is an isomorphism if for some minimal prime $\mathfrak{a}$ of A contained in $\mathfrak{p}$, the rank of $N_v(\mathfrak{a})^j$ is equal to the rank of $(N_v(\mathfrak{a})\otimes_{A/\mathfrak{a}}\kappa(\mathfrak{p}))^j$ for all $j \geqslant 1$ and for any $v \in S$. Suppose $\mathfrak{a}_1, \cdots, \mathfrak{a}_s$ are the minimal primes of A contained in $\mathfrak{p}$. Let V_i denote the maximal A-torsion free quotient of $\widetilde{\pi}(\{r_v\}_{v\in S})\otimes_A A/\mathfrak{a}_i$. Let $W_{\mathfrak{p}}$ denote the image of the diagonal map

$$\kappa(\mathfrak{p})\otimes_A \widetilde{\pi}(\{r_v\}_{v\in S}) \to \prod_{i=1}^{s}\kappa(\mathfrak{p})\otimes_{A/\mathfrak{a}_i} V_i.$$

Then there exists a $\kappa(\mathfrak{p})$-linear G-equivariant surjection

$$\varsigma_{\mathfrak{p}} : \bigotimes_{v\in S} \widetilde{\pi}(r_{v,\mathfrak{p}}) \to W_{\mathfrak{p}},$$

which is an isomorphism if for some $1 \leqslant i \leqslant s$, the rank of $N_v(\mathfrak{a}_i)^j$ is equal to the rank of $(N_v(\mathfrak{a}_i) \otimes_{A/\mathfrak{a}_i} \kappa(\mathfrak{p}))^j$ for all $j \geqslant 1$ and for any $v \in S$.

Proof. — The statement about the map $\gamma_{\mathfrak{p}}$ (resp. $\varsigma_{\mathfrak{p}}$) is the content of [23, Theorem 6.2.5] (resp. [23, Theorem 6.2.6]). □

The results stated above are proved by Emerton and Helm in [23], which provides a formulation of the local Langlands correspondence (LLC, for short) for families. Recall that LLC, proved by Harris and Taylor [30], asserts that there is a canonical bijection between the isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ and the isomorphism classes of n-dimensional Frobenius-semisimple complex Weil–Deligne representations of W_K. So roughly speaking, given a continuous representation $r : G_K \to \mathrm{GL}_n(A)$, an extension of LLC to families is expected to provide a unique (up to isomorphism) admissible smooth representation V of $\mathrm{GL}_n(K)$ over A whose specializations at the primes of A would be related to the representations of $\mathrm{GL}_n(K)$ attached via LLC to the specializations of r at the primes of A. Indeed, Theorem 5.1 says that given a continuous representation $r_v : G_v \to \mathrm{GL}_n(A)$ of a decomposition group of a number field E (we assume for simplicity that S contains a single place v of E not dividing p), there exists at most one (up to isomorphism) admissible smooth representation V of $\mathrm{GL}_n(E_v)$ over A such that conditions (1) and (3) of Theorem 5.1 hold and for each minimal prime ideal $\mathfrak{a}$ of A, there is a G_v-equivariant isomorphism (as in equation (5.1)) between $\kappa(\mathfrak{a}) \otimes_A V$ and the representation $\widetilde{\pi}(r_{v,\mathfrak{a}})$ associated to the reduction $r_{v,\mathfrak{a}}$ of r_v modulo $\mathfrak{a}$ via the Breuil–Schneider modified LLC. Let us assume that V exists and denote it by $\widetilde{\pi}(r_v)$. Moreover, by Theorem 5.2, for any prime $\mathfrak{p}$ of $A[\frac{1}{p}]$ contained in exactly one irreducible component of $\mathrm{Spec} A[\frac{1}{p}]$, there exists a $\kappa(\mathfrak{p})$-linear G_v-equivariant surjection $\gamma_{\mathfrak{p}} : \widetilde{\pi}(r_{v,\mathfrak{p}}) \to \kappa(\mathfrak{p}) \otimes_A \widetilde{\pi}(r_v)$, which is an isomorphism when the rank of no power of the monodromy of $r_{v,\mathfrak{a}}$ decreases under reduction modulo $\mathfrak{p}$ for some minimal prime $\mathfrak{a}$ of $A[\frac{1}{p}]$ contained in $\mathfrak{p}$. This describes the sense in which $\widetilde{\pi}(r_v)$ interpolates the Breuil–Schneider modified LLC over $\mathrm{Spec} A[\frac{1}{p}]$. Furthermore, even when $\mathfrak{p}$ is contained in multiple irreducible components of $\mathrm{Spec} A[\frac{1}{p}]$, then the statement in Theorem 5.2 about the map $\gamma_{\mathfrak{p}}$ is conjectured to hold (see [23, Conjecture 6.2.7]) and is known to be true if $n = 2$ or $n = 3$ (by [23, Proposition 6.2.8]). For a prime $\mathfrak{p}$ of $A[\frac{1}{p}]$ contained in multiple irreducible components of $\mathrm{Spec} A[\frac{1}{p}]$, the existence of a map $\varsigma_{\mathfrak{p}}$ from $\widetilde{\pi}(r_{v,\mathfrak{p}})$ to $W_{\mathfrak{p}}$ is established (i.e., instead of having a map from $\widetilde{\pi}(r_{v,\mathfrak{p}})$ to $\kappa(\mathfrak{p}) \otimes_A \widetilde{\pi}(r_v)$, we have a map from $\widetilde{\pi}(r_{v,\mathfrak{p}})$ to $W_\mathfrak{p}$, which is the image of $\kappa(\mathfrak{p}) \otimes_A \widetilde{\pi}(r_v)$ in a space mentioned in Theorem 5.2). Moreover, the map $\varsigma_\mathfrak{p}$ is a surjection and is an isomorphism when the rank of no power of the monodromy of $r_{v,\mathfrak{a}}$ decreases under reduction modulo $\mathfrak{p}$ for some minimal prime $\mathfrak{a}$ of $A[\frac{1}{p}]$ contained in $\mathfrak{p}$. On the other hand, Theorem 3.1 gives information about non-degeneracy of monodromy under pure specializations. Thus combining the above result with Theorem 3.1, we obtain the result below.

Theorem 5.3. — *Let S, G, r_v be as in Theorem 5.1 and suppose that the $A[G]$-module $\widetilde{\pi}(\{r_v\}_{v\in S})$ exists. Let $\mathfrak{p}$ be a prime of $A[1/p]$. Suppose there exists a $\mathbb{Z}_p$-algebra homomorphism $i_\mathfrak{p} : A \to \overline{\mathbb{Q}}_p$ such that $\mathfrak{p}$ is contained inside the kernel of $i_\mathfrak{p}$ and $r_v \otimes_{A,i_\mathfrak{p}} \overline{\mathbb{Q}}_p$ is pure for all $v \in S$. Then the surjection $\varsigma_\mathfrak{p}$ is an isomorphism. Moreover, if $\mathfrak{p}$ lies on only one irreducible component of* $\mathrm{Spec} A[1/p]$, *then the surjection $\gamma_\mathfrak{p}$ is also an isomorphism.*

Proof. — Let $\mathfrak{a}$ denote a minimal prime of A contained in $\mathfrak{p}$. By Theorem 3.1, the rank of the j-th power of monodromy of $r_{v,\mathfrak{a}}$ is equal to the rank of the j-th power of the monodromy of $r_{v,\mathfrak{p}}$ for any $j \geqslant 1$ and any $v \in S$. So the result follows from Theorem 5.2. □

6. Families of Galois representations

This section illustrates the role of purity for families (Theorems 3.1, 4.3, 4.5) in the study of variation of local Euler factors, local automorphic types, intersection points of irreducible components etc. for families of Galois representations.

6.1. Hida families

For Hida theory of ordinary cusp forms, we follow [33] and refer to the references [31, 32] contained therein. We follow [28] for Hida theory for definite unitary groups.

6.1.1. Cusp forms

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized eigen cusp form of weight $k \geqslant 2$. Then by the works of Eichler [21], Shimura [50], [51, 68c], Deligne [17], Ribet [46, Theorem 2.3], there exists a unique (up to equivalence) continuous

Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that $\mathrm{tr}\rho_f(\mathrm{Fr}_\ell) = a_\ell$ for any prime ℓ not dividing p and the level of f. Let $\pi(f) = \otimes'_{\ell \leqslant \infty} \pi(f)_\ell$ denote the irreducible unitary representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to f (see [27, Theorems 5.19, 4.30]).

Let N be a positive integer and p be an odd prime with $p \nmid N$ and $Np \geqslant 4$. Let h^{ord} be the universal p-ordinary Hecke algebra of tame level N (denoted $h^{\mathrm{ord}}(N; \mathbb{Z}_p)$ in [33]). It is a $\mathbb{Z}_p[\![X]\!]$-algebra. Let $\mathfrak{a}$ be a minimal prime of h^{ord}. Let $\mathcal{R}(\mathfrak{a})$ denote the ring $h^{\mathrm{ord}}/\mathfrak{a}$ and $\mathcal{Q}(\mathfrak{a})$ denote the fraction field of $\mathcal{R}(\mathfrak{a})$. Let $\overline{\mathcal{Q}}(\mathfrak{a})$ be an algebraic closure of $\mathcal{Q}(\mathfrak{a})$. Let S denote the set of places of $\mathbb{Q}$ dividing $Np\infty$. By [33, Theorem 3.1], there exists a unique (up to equivalence) continuous (in the sense of [33, §3]) absolutely irreducible Galois representation $\rho_{\mathfrak{a}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathcal{Q}}(\mathfrak{a}))$ such that $\rho_{\mathfrak{a}}$ has traces in $\mathcal{R}(\mathfrak{a})$ and $\mathrm{tr}(\rho_{\mathfrak{a}}(\mathrm{Fr}_\ell)) = T_\ell \bmod \mathfrak{a}$ for all prime $\ell \nmid Np$ where $T_\ell \in h^{\mathrm{ord}}$ denotes the Hecke operator associated to ℓ. A $\mathbb{Z}_p$-algebra homomorphism $\lambda : h^{\mathrm{ord}} \to \overline{\mathbb{Q}}_p$ is said to be an *arithmetic specialization* if $\lambda((1+X)^{p^r} - (1+p)^{(k-2)p^r}) = 0$ for some integers $k \geqslant 2$ and $r \geqslant 0$. The arithmetic specializations of h^{ord} are in one-to-one correspondence (by the isomorphism of [33, Theorem 2.2]) with the p-ordinary p-stabilized (in the sense of [55, p. 538]) normalized eigen cusp forms of tame level a divisor of N and weight at least 2. Moreover, the trace of ρ_{f_λ} is equal to $\lambda \circ \mathrm{tr}\rho_{\mathfrak{a}}$ for any arithmetic specialization λ of h^{ord} with $\lambda(\mathfrak{a}) = 0$. For an arithmetic specialization λ of h^{ord}, denote the corresponding ordinary form by f_λ and the kernel of λ by $\mathfrak{p}_\lambda$.

Definition 6.1. — *The* automorphic type *of a minimal prime $\mathfrak{a}$ of h^{ord} at a prime $\ell \neq p$ is defined to be the unordered tuple $\mathrm{AT}_\ell(\mathfrak{a})$ if the automorphic types of $\pi(f_\lambda)_\ell$ are equal to $\mathrm{AT}_\ell(\mathfrak{a})$ for all arithmetic specialization λ of h^{ord} with $\lambda(\mathfrak{a}) = 0$.*

Theorem 6.2. — *Let $\mathfrak{a}$ be a minimal prime of h^{ord} and $\ell \neq p$ be a prime. Then the following hold.*

(1) *If $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_\ell})^{\text{Fr-ss}}$ is indecomposable and has no monodromy, then there exists an irreducible Frobenius-semisimple representation r over $\mathcal{R}(\mathfrak{a})^{\mathrm{intal}}[1/p]$ such that $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_\ell})^{\text{Fr-ss}}$ is isomorphic to r over $\overline{\mathcal{Q}}(\mathfrak{a})$ and $\mathrm{WD}(\rho_{f_\lambda}|_{W_\ell})^{\text{Fr-ss}}$ is isomorphic to $\lambda^{\mathrm{intal}} \circ r$ for any arithmetic specialization λ of h^{ord} with $\lambda(\mathfrak{a}) = 0$.*

(2) *If $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_\ell})^{\text{Fr-ss}}$ is indecomposable and has nonzero monodromy, then there exists an $\mathcal{R}(\mathfrak{a})^{\mathrm{intal}}$-valued character χ of W_ℓ such that $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_\ell})^{\text{Fr-ss}}$ is isomorphic to $\mathrm{Sp}_2(\chi)$ over $\overline{\mathcal{Q}}(\mathfrak{a})$ and $\mathrm{WD}(\rho_{f_\lambda}|_{W_\ell})^{\text{Fr-ss}}$ is isomorphic to $\lambda^{\mathrm{intal}} \circ \mathrm{Sp}_2(\chi)$ for any arithmetic specialization λ of h^{ord} with $\lambda(\mathfrak{a}) = 0$.*

(3) *If* $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_\ell})^{\text{Fr-ss}}$ *is decomposable, then there exist* $\mathcal{R}(\mathfrak{a})^{\text{intal}}$*-valued characters* χ_1, χ_2 *of* W_ℓ *such that* $\mathrm{WD}(\rho_{\mathfrak{a}}|_{W_\ell})^{\text{Fr-ss}}$ *is isomorphic to* $\chi_1 \oplus \chi_2$ *over* $\overline{\mathcal{Q}}(\mathfrak{a})$ *and* $\mathrm{WD}(\rho_{f_\lambda}|_{W_\ell})^{\text{Fr-ss}}$ *is isomorphic to* $\lambda^{\text{intal}} \circ (\chi_1 \oplus \chi_2)$ *for any arithmetic specialization* λ *of* h^{ord} *with* $\lambda(\mathfrak{a}) = 0$.

Consequently, the notion of automorphic types of minimal prime ideals of h^{ord} *is well-defined.*

The constancy of local automorphic types of arithmetic specializations is also established in [40, §12.7.14], [43, Remark 2.4], [26, Lemma 2.14], [25, Lemma 3.9] (see also the proof of [24, Proposition 2.2.4]).

Proof. — Note that $\mathrm{tr}\rho_{\mathfrak{a}}$ is a pseudorepresentation of $G_{\mathbb{Q}}$ with values in $\mathcal{R}(\mathfrak{a})$ and $\rho_{\mathfrak{a}}$ is irreducible. For any prime $\mathfrak{p}$ of $\mathcal{R}(\mathfrak{a})$, the ring $\mathcal{R}(\mathfrak{a})_{\mathfrak{p}}$ is Noetherian. So its Henselization $\mathcal{R}(\mathfrak{a})^h_{\mathfrak{p}}$ is also Noetherian (see [29, Théorème 18.6.6(v)]). Moreover $\mathcal{R}(\mathfrak{a})^h_{\mathfrak{p}}$ is Hausdorff by Krull's intersection theorem (see [38, Theorem 8.10]). Fix a minimal prime $\mathfrak{n}_{\mathfrak{p}}$ of $\mathcal{R}(\mathfrak{a})^h_{\mathfrak{p}}$. For each arithmetic specialization λ of h^{ord} with $\lambda(\mathfrak{a}) = 0$, ρ_{f_λ} is an irreducible $G_{\mathbb{Q}}$-representation (by [46, Theorem 2.3]) over an algebraic closure of the residue field of $\mathcal{R}(\mathfrak{a})^h_{\mathfrak{p}_\lambda}/\mathfrak{n}_{\mathfrak{p}_\lambda}$. So, by [42, Théorème 1], ρ_{f_λ} lifts to a representation of $G_{\mathbb{Q}}$ over $\mathcal{R}(\mathfrak{a})^h_{\mathfrak{p}_\lambda}/\mathfrak{n}_{\mathfrak{p}_\lambda}$ and the trace of this lift coincides with the trace of $\rho_{\mathfrak{a}}$. Note that the map $\mathcal{R}(\mathfrak{a}) \to \mathcal{R}(\mathfrak{a})^h_{\mathfrak{p}}/\mathfrak{n}_{\mathfrak{p}}$ is injective (since the map $\mathcal{R}(\mathfrak{a})_{\mathfrak{p}} \to \mathcal{R}(\mathfrak{a})^h_{\mathfrak{p}}$ is flat (by [29, Théorème 18.6.6(iii)]) and flat maps satisfy going down property by [37, (5.D) Theorem 4]) and $\rho_{\mathfrak{a}}|_{G_\ell}$ is monodromic by Grothendieck's monodromy theorem (see the proof of [3, Lemma 7.8.14]). Moreover the G_ℓ-representation $\rho_{f_\lambda}|_{G_\ell}$ is pure (by [9]). So Theorem 4.3 gives parts (1), (2), (3). Since local-global compatibility holds for cusp forms (by [9]) and each minimal prime ideal of h^{ord} is contained in the kernel of some arithmetic specialization of h^{ord} (as h^{ord} is free of finite rank over $\mathbb{Z}_p[\![X]\!]$), the final part follows. □

6.1.2. Automorphic representations for definite unitary groups

Let F be a CM field, F^+ be its maximal totally real subfield. Let $n \geqslant 2$ be an integer and assume that if n is even, then $n[F^+ : \mathbb{Q}]$ is divisible by 4. Let $\ell > n$ be a rational prime and assume that every prime of F^+ lying above ℓ splits in F. Let K be a finite extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ that contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_\ell$. Let S_ℓ denote the set of places of F^+ lying above ℓ. Let R denote a finite set of finite places of F^+ disjoint from S_ℓ and consisting of places that split in F. For each place $v \in S_\ell \cup R$, choose once and for all a place $\widetilde{v}$ of F lying above v. For $v \in R$, let $\mathrm{Iw}(\widetilde{v})$

be the compact open subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ and χ_v be the character as in [28, §2.1, 2.2].

Let G be the reductive algebraic group over F^+ as in [28, §2.1]. For each dominant weight λ (as in [28, Definition 2.2.3]) for G, the group $G(\mathbb{A}_{F^+}^{\infty,R}) \times \prod_{v\in R} \mathrm{Iw}(\widetilde{v})$ acts on the spaces $S_{\lambda,\{\chi_v\}}(\overline{\mathbb{Q}}_\ell)$, $S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(\mathcal{O}_K)$ (as in [28, Definitions 2.2.4, 2.4.2]). For an irreducible constituent π of the $G(\mathbb{A}_{F^+}^{\infty,R}) \times \prod_{v\in R} \mathrm{Iw}(\widetilde{v})$-representation $S_{\lambda,\{\chi_v\}}(\overline{\mathbb{Q}}_\ell)$, let $\mathrm{WBC}(\pi)$ denote the weak base change of π to $\mathrm{GL}_n(\mathbb{A}_F)$ (which exists by [36, Corollaire 5.3]) and let $r_\pi : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ (as in [28, Proposition 2.7.2]) denote the unique (up to equivalence) continuous semisimple representation attached to $\mathrm{WBC}(\pi)$ via [12, Theorem 3.2.3].

An irreducible constituent π of the $G(\mathbb{A}_{F^+}^{\infty,R}) \times \prod_{v\in R} \mathrm{Iw}(\widetilde{v})$-representation $S_{\lambda,\{\chi_v\}}(\overline{\mathbb{Q}}_\ell)$ is said to be an *ordinary automorphic representation for G* if $\pi^{U(\mathfrak{l}^{b,c})} \cap S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}), \mathcal{O}_K) \neq 0$ for some integers $0 \leqslant b \leqslant c$ (see [28, Definition 2.2.4, §2.3] for details). Let U be a compact open subgroup of $G(\mathbb{A}^\infty_{F^+})$, T be a finite set of finite places of F^+ containing $R \cup S_\ell$ and such that every place in T splits in F (see [28, §2.3]). Let $\mathbb{T}^{\mathrm{ord}}$ denote the universal ordinary Hecke algebra $\mathbb{T}^{T,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{l}^\infty), \mathcal{O}_K)$ (as in [28, Definition 2.6.2]). Let Λ be the completed group algebra as in [28, Definition 2.5.1]. By definition of $\mathbb{T}^{\mathrm{ord}}$, it is equipped with a Λ-algebra structure and is finite over Λ. An $\mathcal{O}_K$-algebra homomorphism $f : A \to \overline{\mathbb{Q}}_\ell$ is said to be an *arithmetic specialization* of a finite Λ-algebra A if $\ker(f|_\Lambda)$ is equal to the prime ideal $\wp_{\lambda,\alpha}$ (as in [28, Definition 2.6.3]) of Λ for some dominant weight λ for G and a finite order character $\alpha : T_n(\mathfrak{l}) \to \mathcal{O}_K^\times$. By [28, Lemma 2.6.4], each arithmetic specialization η of $\mathbb{T}^{\mathrm{ord}}$ determines an ordinary automorphic representation π_η for G. An arithmetic specialization η of $\mathbb{T}^{\mathrm{ord}}$ is said to be *stable* if $\mathrm{WBC}(\pi_\eta)$ is cuspidal.

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^{\mathrm{ord}}$ (in the sense of [28, §2.7]). Let $r_\mathfrak{m}$ denote the representation of G_{F^+} as in [28, Proposition 2.7.4]. Then by restricting it to G_F and then composing with the projection $\mathrm{GL}_n(\mathbb{T}^{\mathrm{ord}}_\mathfrak{m}) \times \mathrm{GL}_1(\mathbb{T}^{\mathrm{ord}}_\mathfrak{m}) \to \mathrm{GL}_n(\mathbb{T}^{\mathrm{ord}}_\mathfrak{m})$, we get a continuous representation $G_F \to \mathrm{GL}_n(\mathbb{T}^{\mathrm{ord}}_\mathfrak{m})$ which is denoted by $r_\mathfrak{m}$ by abuse of notation. Since $\mathfrak{m}$ is non-Eisenstein, the G_F-representations $\eta \circ r_\mathfrak{m}$ and r_{π_η} are isomorphic for any arithmetic specialization η of $\mathbb{T}^{\mathrm{ord}}_\mathfrak{m}$ (by [28, Propositions 2.7.2, 2.7.4]).

Definition 6.3. — *Let w be a finite place of F not lying above ℓ and $\mathfrak{a}$ be a minimal prime of $\mathbb{T}^{\mathrm{ord}}$. If the maximal ideal of $\mathbb{T}^{\mathrm{ord}}$ containing $\mathfrak{a}$ is non-Eisenstein and some stable arithmetic specialization of $\mathbb{T}^{\mathrm{ord}}$ vanishes on $\mathfrak{a}$, then the* automorphic type *of $\mathfrak{a}$ at w is defined to be the unordered* *tuple* $\mathrm{AT}_w(\mathfrak{a})$ *if the automorphic types of* $\mathrm{WBC}(\pi_\eta)_w$ *are equal to* $\mathrm{AT}_w(\mathfrak{a})$ *for all stable arithmetic specialization* η *of* $\mathbb{T}^{\mathrm{ord}}$ *with* $\eta(\mathfrak{a}) = 0$.

Theorem 6.4. — *Let* $w \nmid \ell$ *be a finite place of* F, $\mathfrak{a}$ *be a minimal prime of* $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ *and* $\mathfrak{m}$ *be the maximal ideal of* $\mathbb{T}^{\mathrm{ord}}$ *containing* $\mathfrak{a}$. *Suppose* $\mathfrak{m}$ *is non-Eisenstein. Denote the quotient ring* $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}/\mathfrak{a}$ *by* $\mathcal{R}(\mathfrak{a})$ *and the representation* $r_{\mathfrak{m}} \bmod \mathfrak{a}$ *by* $r_{\mathfrak{a}}$. *Then there exist positive integers* $m, t_1 \leqslant \cdots \leqslant t_m$ *and irreducible Frobenius-semisimple representations* $r_1, \cdots, r_m$ *of* W_w *over* $\mathcal{R}(\mathfrak{a})^{\mathrm{intal}}[1/\ell]$ *such that* $\mathrm{WD}(r_{\mathfrak{a}}|_{W_w})^{\text{Fr-ss}}$ *is isomorphic to* $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(r_i)$ *over* $\overline{Q}(\mathcal{R}(\mathfrak{a}))$ *and*

$$\mathrm{WD}(r_{\pi_\eta}|_{W_w})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^m \mathrm{Sp}_{t_i}(\eta^{\mathrm{intal}} \circ r_i) \tag{6.1}$$

for any stable arithmetic specialization η *of* $\mathcal{R}(\mathfrak{a})$. *Consequently, the notion of local automorphic types of minimal prime ideals of* $\mathbb{T}^{\mathrm{ord}}$ *is well-defined. Moreover, two minimal prime ideals of* $\mathbb{T}^{\mathrm{ord}}$ *are contained in two non-Eisenstein maximal ideals and are both contained in the kernel of a stable arithmetic specialization of* $\mathbb{T}^{\mathrm{ord}}$ *only if their automorphic types at any finite place* $v \nmid \ell$ *of* F *are the same.*

Proof. — If π is an irreducible constituent of the $G(\mathbb{A}_{F^+}^{\infty,R}) \times \prod_{v\in R} \mathrm{Iw}(\widetilde{v})$-representation $S_{\lambda,\{\chi_v\}}(\overline{\mathbb{Q}}_\ell)$ such that $\mathrm{WBC}(\pi)$ is cuspidal, then for any finite place w of F not dividing ℓ, $r_\pi|_{G_w}$ is pure by [8, Theorems 1.1, 1.2] and the proofs of Theorem 5.8, Corollary 5.9 of loc. cit. Note that $r_{\mathfrak{a}}|_{W_w}$ is monodromic by Grothendieck's monodromy theorem (see [49, pp. 515–516]). So by Theorem 4.5, we obtain integers $m, t_1, \cdots, t_m$ and representation $r_1, \cdots, r_m$ with the prescribed properties such that $\mathrm{WD}(r_{\mathfrak{a}}|_{W_w})^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i=1}^m \mathrm{Sp}_{t_i}(r_i)$ over $\overline{Q}(\mathcal{R}(\mathfrak{a}))$ and equation (6.1) holds for any stable arithmetic specialization η of $\mathcal{R}(\mathfrak{a})$. By [8, Theorem 1.1] on local-global compatibility of cuspidal automorphic representations for GL_n, the notion of local automorphic types is well-defined. Finally, note that if a minimal prime ideal $\mathfrak{b}$ of $\mathbb{T}^{\mathrm{ord}}$ is contained in some non-Eisenstein maximal ideal of $\mathbb{T}^{\mathrm{ord}}$, then the local automorphic type of $\mathfrak{b}$ at w is equal to the automorphic type of $\mathrm{WBC}(\pi_\eta)_w$ for any stable arithmetic specialization η of $\mathbb{T}^{\mathrm{ord}}$ with $\eta(\mathfrak{b}) = 0$. So the last statement follows. □

6.2. Eigenvarieties

Let X be a rigid analytic space over a finite extension of $\mathbb{Q}_p$. The restriction map between the global sections of two admissible open subsets $U \supset V$

of X is denoted by res_{UV}. For a point x of X, denote the rigid analytic local ring of X at x by $\mathcal{O}_{X,x}$, the map $\mathcal{O}(X) \to \mathcal{O}_{X,x}$ by $\mathrm{loc}_{X,x}$, the maximal ideal of $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$, the residue field of $\mathcal{O}_{X,x}$ by $k(x)$. Note that $\pi_{\mathfrak{m}_x}$ denotes the map $\mathcal{O}_{X,x} \to k(x)$. If x is an element of $X(\overline{\mathbb{Q}}_p)$, then the map $\mathcal{O}(X) \to \overline{\mathbb{Q}}_p$ is denoted by $\mathrm{ev}_{X,x}$. For any admissible open subset U of X, the ring $\mathcal{O}(U)$ is equipped with the coarsest locally convex topology such that the restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is continuous for any open affinoid $V \subset U$ (where $\mathcal{O}(V)$ is equipped with its Banach algebra topology).

Let $E/\mathbb{Q}$ be an imaginary quadratic field and G denote the definite unitary group $U(m)$ (as in [3, §6.2.2]) in $m \geqslant 1$ variables. We assume that p splits in E. Let $\mathcal{H}$ denote the Hecke algebra as in [3, §7.2.1]. Let $\mathcal{Z}_0 \subset \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^m$ be the set of pairs $(\psi_{(\pi,\mathcal{R})}, \underline{k})$ associated to the p-refined automorphic representations $(\pi, \mathcal{R})$ of any weight $\underline{k}$ (see [3, §7.2.2, 7.2.3]). Let e be the idempotent as in [3, §7.3.1] and let $\mathcal{Z}_e \subset \mathcal{Z}_0$ denote the subset of $(\psi_{(\pi,\mathcal{R})}, \underline{k})$ such that $e(\pi^p) \neq 0$. We assume that $\mathcal{Z}_e$ is nonempty. Then by [3, §7.3], there exists an eigenvariety for $\mathcal{Z}_e$, i.e., there exist a reduced rigid analytic space X over a finite extension L of $\mathbb{Q}_p$, a ring homomorphism $\psi : \mathcal{H} \to \mathcal{O}(X)$, an analytic map $\omega : X \to \mathrm{Hom}((\mathbb{Z}_p^\times)^m, \mathbb{G}_m^{\mathrm{rig}}) \times_{\mathbb{Q}_p} L$ and an accumulation and Zariski-dense subset Z of $X(\overline{\mathbb{Q}}_p)$ such that conditions (i), (ii), (iii) of [3, Definition 7.2.5] hold. In particular, $z \mapsto (\mathrm{ev}_{X,z} \circ \psi, \omega(z))$ induces a bijection $Z \xrightarrow{\sim} \mathcal{Z}_e$. The set Z is called the set of *arithmetic points* of X. Let $Z_{\mathrm{reg}} \subset Z$ be the subset of points parametrizing the p-refined automorphic representations $(\pi, \mathcal{R})$ such that π_∞ is regular and the semisimple conjugacy class of π_p has m distinct eigenvalues (see [3, §7.5.1]). By [3, Lemma 7.5.3], Z_{reg} is a Zariski-dense subset of X. For each $z \in Z$, we fix a p-refined automorphic representation π_z of $U(m)$ such that z corresponds to π_z under the bijection $Z \xrightarrow{\sim} \mathcal{Z}_e$. For each $z \in Z_{\mathrm{reg}}$, let $\rho_{z,p} : G_E \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ denote the unique (up to equivalence) continuous semisimple representation attached to $\mathrm{WBC}(\pi_z)$ via [12, Theorem 3.2.3]. By [3, Proposition 7.5.4], there exists an m-dimensional continuous pseudorepresentation $T : G_E \to \mathcal{O}(X)$ such that $\mathrm{ev}_{X,z} \circ T = \mathrm{tr}\rho_{z,p}$ for all $z \in Z_{\mathrm{reg}}$. Let $Z_{\mathrm{reg}}^{\mathrm{st}}$ denote the set of points $z \in Z_{\mathrm{reg}}$ such that $\mathrm{WBC}(\pi_z)$ is cuspidal. Note that by [8, Theorems 1.1, 1.2], the representation $\rho_{z,p}|_{W_w}$ is pure for any $z \in Z_{\mathrm{reg}}^{\mathrm{st}}$. For $z \in Z_{\mathrm{reg}}^{\mathrm{st}}$, the Galois representation $\rho_{z,p}$ is expected to be irreducible. It is known when $m \leqslant 3$ by [5] and in many cases when $m = 4$ by an unpublished work of Ramakrishnan. By [44, Theorem D], it is known for infinitely many primes p.

Definition 6.5. — *Let w be a finite place of E not lying above p and Y_0 be an irreducible component of X such that $Z_{\mathrm{reg}}^{\mathrm{st}} \cap Y_0$ is nonempty.*

The automorphic type *of Y_0 at w is defined to be the unordered tuple* $\mathrm{AT}_w(Y_0)$ *if the automorphic types of* $\mathrm{WBC}(\pi_z)_w$ *are equal to* $\mathrm{AT}_w(Y_0)$ *for all $z \in Z^{\mathrm{st}}_{\mathrm{reg}} \cap Y_0$.*

Let $\xi : \widetilde{X} \to X$ be a normalization of X. Let C be a connected component of $\widetilde{X}$ and Y be the irreducible component $\xi(C)$ (together with its canonical structure of reduced rigid space) of X. By [16, Lemma 2.2.1 (2)], the map $\xi|_C : C \to Y$ is a normalization. For each $x \in X(\overline{\mathbb{Q}}_p) \cap Y$, we fix a point $\widetilde{x}$ in $C(\overline{\mathbb{Q}}_p)$ which goes to x under the map $C(\overline{\mathbb{Q}}_p) \to Y(\overline{\mathbb{Q}}_p)$. Note that $\mathcal{O}(C)$ is a domain by [16, Lemma 2.1.4]. So by [53, Theorem 1], there exists a unique (up to equivalence) semisimple representation σ_C of G_E over $\overline{Q}(\mathcal{O}(C))$ such that $\mathrm{tr}\sigma_C = \mathrm{res}_{\widetilde{X}C} \circ \xi \circ T$.

Lemma 6.6. — *For any finite place w of E not lying above p, the restriction of σ_C to W_w is monodromic.*

Proof. — Let $U \subset C$ be a nonempty open affinoid. Let σ_U be a semisimple representation of G_E over $\overline{Q}(\mathcal{O}(U))$ such that $\mathrm{tr}\sigma_U = \mathrm{res}_{CU} \circ \mathrm{res}_{\widetilde{X}C} \circ \xi \circ T$. Note that the pseudorepresentation $T : G_E \to \mathcal{O}(X)$ (introduced before Definition 6.5) is continuous by [3, Proposition 7.5.4] where G_E has the profinite topology and $\mathcal{O}(X)$ has the topology mentioned in the beginning of §6.2. So the pseudorepresentation $\mathrm{res}_{CU} \circ \mathrm{res}_{\widetilde{X}C} \circ \xi \circ T$ is also continuous. Hence the restriction of σ_U to W_w is monodromic by [3, Lemmas 7.8.11(i), 7.8.14]. Note that the map res_{CU} is injective by [16, Lemma 2.1.4]. Since the semisimple representations $\sigma_C \otimes \overline{Q}(\mathcal{O}(U))$ and σ_U have same traces, they are isomorphic. So $\sigma_C|_{W_w}$ is monodromic. □

Theorem 6.7. — *Let $w \nmid p$ be a finite place of E. Suppose that the intersection of $Z^{\mathrm{st}}_{\mathrm{reg}}$ with any irreducible component of X is nonempty and for any $z \in Z^{\mathrm{st}}_{\mathrm{reg}}$, the Galois representation $\rho_{z,p}$ is irreducible. Then there exist positive integers $n, t_1, \cdots, t_n$, irreducible Frobenius-semisimple representations $r_1, \cdots, r_n$ of W_w over $\mathcal{O}(C)^{\mathrm{intal}}$ such that the following hold.*

(1) *There is an isomorphism*

$$\mathrm{WD}(\sigma_C|_{W_w})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{n} \mathrm{Sp}_{t_i}(r_i).$$

(2) *If $z \in Z^{\mathrm{st}}_{\mathrm{reg}} \cap Y$, or more generally if $z \in Y(\overline{\mathbb{Q}}_p)$ such that $\mathrm{ev}_{X,z} \circ T$ is the trace of an irreducible representation $\rho_{z,p} : G_E \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ and $\rho_{z,p}|_{W_w}$ is pure, then there is an isomorphism*

$$\mathrm{WD}(\rho_{z,p}|_{W_w})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{n} \mathrm{Sp}_{t_i}(\pi^{\mathrm{intal}}_{\mathfrak{m}_{\widetilde{z}}} \circ \mathrm{loc}^{\mathrm{intal}}_{C,\widetilde{z}} \circ r_i).$$

(3) *Let V be a nonempty connected admissible open subset of C and $(\rho_V, N_V) : W_w \to \mathrm{GL}_m(\mathcal{O}(V)^{\mathrm{intal}})$ be a Weil–Deligne representation such that $\mathrm{res}_{CV} \circ \mathrm{res}_{\widetilde{X}C} \circ \xi \circ T = \mathrm{tr}\rho_V$ and $f_V \circ (\rho_V, N_V)$ is pure for some map $f_V : \mathcal{O}(V)^{\mathrm{intal}} \to \overline{\mathbb{Q}}_p$. Then there is an isomorphism*

$$((\rho_V, N_V) \otimes_{\mathcal{O}(V)} \overline{Q}(\mathcal{O}(V)))^{\text{Fr-ss}} \simeq (\mathrm{res}_{CV}^{\mathrm{intal}} \circ (\rho_C, N_C)) \otimes_{\mathcal{O}(V)^{\mathrm{intal}}} \overline{Q}(\mathcal{O}(V)).$$

Consequently, the notion of local automorphic types of irreducible components of X is well-defined. Moreover, two irreducible components of X intersect at a point of $Z_{\mathrm{reg}}^{\mathrm{st}}$ only if their local automorphic types at any finite place of E outside p are the same.

Proof. — Note that $\mathrm{res}_{\widetilde{X}C} \circ \xi \circ T$ is a pseudorepresentation of G_E with values in $\mathcal{O}(C)$. It is equal to the trace of the semisimple representation σ_C, whose restriction to W_w is monodromic by Lemma 6.6. Let z be a point as in part (2). Then the tuple $(\mathcal{O}_{C,\widetilde{z}}, \mathfrak{m}_{\widetilde{z}}, k(\widetilde{z}), \mathrm{loc}_{C,\widetilde{z}}, \rho_{z,p})$ lies in the irreducibility and purity locus of $\mathrm{res}_{\widetilde{X}C} \circ \xi \circ T$. So parts (1), (2) follow from Theorem 4.5 and part (3) follows from Theorem 4.3. By [8, Theorem 1.1] on local-global compatibility of cuspidal automorphic representations for GL_n, the notion of local automorphic types is well-defined. Finally, note that if an irreducible component Y_0 of X has nonempty intersection with $Z_{\mathrm{reg}}^{\mathrm{st}}$, then its automorphic type at a finite place w of E outside p is equal to the automorphic type of $\mathrm{WBC}(\pi_z)_w$ for any $z \in Z_{\mathrm{reg}}^{\mathrm{st}} \cap Y_0$. Consequently, if two irreducible components of X intersect at a point of $Z_{\mathrm{reg}}^{\mathrm{st}}$, then their local automorphic types at any finite place of E outside p are the same. This completes the proof. □

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