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A SHARP LOWER BOUND FOR A RESONANCE-COUNTING FUNCTION IN EVEN DIMENSIONS

by T. J. CHRISTIANSEN

Abstract. — This paper proves sharp lower bounds on a resonance-counting function for obstacle scattering in even-dimensional Euclidean space without a need for trapping assumptions. Similar lower bounds are proved for some other compactly supported perturbations of $-\Delta$ on $\mathbb{R}^d$, for example, for the Laplacian for certain metric perturbations on $\mathbb{R}^d$. The proof uses a Poisson formula for resonances, complementary to one proved by Zworski in even dimensions.

Résumé. — L'objet de cette note est de montrer des bornes inférieures optimales pour la fonction de comptage des résonances, dans le cas d'obstacles sur l'espace euclidien en dimension paire; on ne fait aucune hypothèse de capture du flot de billard extérieur à l'obstacle. Des minorations similaires sont prouvées pour d'autres types de perturbations à support compact sur $\mathbb{R}^d$. La preuve utilise une formule de Poisson pour les résonances, complémentaire d'une formule montrée par Zworski en dimension paire.

1. Introduction

The purpose of this paper is to prove a sharp polynomial lower bound on a resonance-counting function for certain compactly supported perturbations of $-\Delta$ on $\mathbb{R}^d$ for even dimensions d. The operators we consider include, for example, the Laplacian on the exterior of a bounded obstacle in $\mathbb{R}^d$ and the Laplacian for many compactly supported metric perturbations of $\mathbb{R}^d$. These lower bounds complement the sharp upper bounds proved by Vodev [38, 39]. The lower bounds of this paper do not require any trapping conditions. In order to prove the result, we prove a Poisson formula for resonances in even dimensions, complementary to that proved by Zworski

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in [43]. The Poisson formula is valid for a large class of operators which are "black-box" perturbations of $-\Delta$ on $\mathbb{R}^d$ as defined in [31]. Here $\Delta \leqslant 0$ is the Laplacian on $\mathbb{R}^d$.

We begin with the classical problem of obstacle scattering. Let P denote $-\Delta$ with Dirichlet, Neumann, or Robin boundary conditions on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$, where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open set with smooth boundary. For $0 < \arg\lambda < \pi$ set $R(\lambda) = (P - \lambda^2)^{-1}$. Then it is well known that if $\chi \in C_c^\infty(\mathbb{R}^d)$ then $\chi R(\lambda)\chi$ has a meromorphic continuation to $\mathbb{C}$ if d is odd and to Λ, the logarithmic cover of $\mathbb{C} \setminus \{0\}$, if d is even. If χ is chosen to be 1 on an open set containing $\overline{\mathcal{O}}$, the location of the poles of $\chi R(\lambda)\chi$ is independent of the choice of χ. We denote the set of all such poles, repeated according to multiplicity, by $\mathcal{R}$.

We can describe a point $\lambda \in \Lambda$ by its norm $|\lambda|$ and argument $\arg\lambda$. In even dimensions we use the resonance counting functions defined as

$$n_k(r) \stackrel{\text{def}}{=} \#\{\lambda_j \in \mathcal{R} :\ |\lambda_j| < r \text{ and } k\pi < \arg\lambda_j < (k+1)\pi\}, \text{ for } k \in \mathbb{Z}.$$

By the symmetry of resonances for self-adjoint operators (cf. [7, Proposition 2.1]), $n_{-k}(r) = n_k(r)$.

Theorem 1.1. — *Let d be even and let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded open set with smooth compact boundary $\partial\mathcal{O}$. Assume in addition that $\mathcal{O} \neq \emptyset$ and $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ is connected. Let P be $-\Delta$ on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ with Dirichlet, Neumann, or Robin boundary conditions. Then there is a constant $C_0 > 0$ so that*

$$r^d/C_0 \leqslant n_{-1}(r) \leqslant C_0 r^d \text{ when } r \gg 1. \tag{1.1}$$

The upper bound $n_{-1}(r) \leqslant C_0 r^d$, $r \gg 1$, is due to Vodev [38, 39]. It is the lower bound which is new here. For the Robin boundary condition, a function v in the domain of P must satisfy $\partial_\nu v = \gamma v$ on $\partial(\mathbb{R}^d \setminus \overline{\mathcal{O}})$, where ∂_ν is the inward unit normal vector field and $\gamma \in C^\infty(\partial\mathcal{O}; \mathbb{R})$ is a fixed function. Choosing $\gamma \equiv 0$ gives the Neumann boundary condition. The condition that $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ be connected is not necessary, but makes the proof cleaner since we do not have infinitely many positive eigenvalues of P as we would if $\mathbb{R}^d \setminus \overline{\mathcal{O}}$ had a bounded component.

In *odd* dimension d, where $\mathcal{R} \subset \mathbb{C}$, we introduce the counting function

$$n_{odd}(r) = \#\{\lambda_j \in \mathcal{R} : |\lambda_j| < r\}.$$

In odd dimension $d \geqslant 3$, for obstacle scattering the analogous upper bound, $n_{odd}(r) \leqslant Cr^d$ for $r \gg 1$ of (1.1) is due to Melrose [24], but the known lower bound for general obstacles $\mathcal{O}$ is weaker: $n_{odd}(r) \geqslant cr^{d-1}$ for some $c > 0$ and for sufficiently large r [2, 18].

In either even or odd dimensions, Stefanov [34, Section 4] proved lower bounds on $n_{odd}(r)$ and $n_{-1}(r)$ proportional to r^d under certain trapping assumptions on the geometry of $\mathbb{R}^d \setminus \mathcal{O}$. On the other hand, again in either even or odd dimensions, for a class of strictly convex $\mathcal{O}$ asymptotics of the number of resonances (of order r^{d-1}) in certain regions are known, [17, 33]. In odd dimensions asymptotics of the resonance counting function have been proved in the special case of $\mathcal{O}$ equal to a ball [35, 41].

The primary result of [6] is that in even dimension d, under the hypotheses of Theorem 1.1 (with some additional restrictions for the Robin boundary condition), if $k \in \mathbb{Z} \setminus \{0\}$, then $\limsup_{r\to\infty} \log n_k(r)/\log r = d$. For $k = -1$ this is weaker than the lower bound of Theorem 1.1. Moreover, the proofs in [6] use results of [2, 19] for the particular case of the operator P of Theorem 1.1, and those results do not obviously generalize to the setting of Theorem 1.2 below. However, the techniques of this paper do not seem to give results for $n_k(r)$, $k \neq \pm 1$.

Obstacle scattering, as considered in Theorem 1.1, forms a canonical class of scattering problems. However, our result and its proof can easily be extended to a larger class of operators. In even dimension d, for the operator P defined below, and for the much larger class of operators of the "black-box" type of [31], the resolvent $(P - \lambda^2)^{-1}$ has a meromorphic continuation to Λ, [31]. Hence one can define resonances, the set $\mathcal{R}$, and the resonance counting functions $n_k(r)$ just as for the Laplacian on the exterior of a compact set.

Theorem 1.2. — *Let d be even, and let (M, g) be a smooth, connected, noncompact d-dimensional Riemannian manifold, perhaps with smooth compact boundary ∂M. Suppose there is a compact set $K \subset M$ and an $R_0 > 0$ so that $M \setminus K$ is diffeomorphic to $\mathbb{R}^d \setminus \overline{B}(0; R_0)$, and that g restricted to $M \setminus K$ agrees with the flat metric on $\mathbb{R}^d$. Then let $P = -\Delta_g$ on M, with Dirichlet or Robin boundary conditions if $\partial M \neq \emptyset$. In the special case of $M = \mathbb{R}^d \setminus \mathcal{O}$ with $\mathcal{O}$ as in Theorem 1.1 and metric agreeing with the Euclidean metric outside a compact set, we may choose $P = -\Delta_g + V$ with Dirichlet or Neumann boundary conditions, for some $V \in C_c^\infty(\mathbb{R}^d \setminus \mathcal{O}; \mathbb{R})$. Then if vol$(K) \neq$ vol$(B(0; R_0))$, then there is a constant $C_0 > 0$ so that*

$$r^d/C_0 \leqslant n_{-1}(r) \leqslant C_0 r^d \text{ for } r \gg 1.$$

In the statement of the theorem, $B(a; R) = \{x \in \mathbb{R}^d : |x - a| < R\}$, and $\Delta_g \leqslant 0$ is the Laplacian on (M, g). The operators P defined in Theorem 1.2 are examples of "black box" operators as defined by Sjöstrand–Zworski [31], as recalled in Section 1.1.

Again, it is the lower bound of Theorem 1.2 which is new, as the upper bound is due to [38, 39]. We shall prove a more general result, Theorem 4.1, from which Theorems 1.1 and 1.2 follow.

Tang [36] showed that for non-flat, compactly supported perturbations of the Euclidean metric on $\mathbb{R}^d$, $d = 4,\ 6$, the associated Laplacian has infinitely many resonances. Under certain geometric conditions one can prove the existence of many resonances for operators of the type considered in Theorem 1.2. In addition to the references already cited, we mention [29, 43] and references therein.

The proof of the lower bound of Theorems 1.1 and 1.2 uses the wave trace, a distribution informally given by

$$u(t) = 2\,\mathrm{tr}(\cos(t\sqrt{P}) - \cos(t\sqrt{-\Delta})), \tag{1.2}$$

and more formally defined in (1.3). For the operators of Theorem 1.2, because we are in even dimension, the leading order singularity of u at 0 "spreads out." This "spreading out" of the leading order singularity at $t = 0$ does not happen in the analogous odd-dimensional scattering problems (for example, in odd-dimensional obstacle scattering). Thus, this particular technique does not give sharp lower bounds in odd-dimensional Euclidean scattering.

That the "spreading" of the leading singularity of the wave trace at $t = 0$ can, when combined with a Poisson-type formula, give good lower bounds on similar resonance-counting functions was proved in [32] and has been used, for example, in [3, 12, 32]. In particular, the papers [3, 12] prove lower bounds analogous to (1.1) for certain even-dimensional manifolds hyperbolic near infinity.

To use the result of [32] requires a Poisson formula for resonances which is valid in any sufficiently small deleted neighborhood of $t = 0$. Thus one of the main results of this paper is Theorem 3.6, a Poisson formula for resonances in even dimensions. This result holds for a large class of "black-box" perturbations (in the sense of [31]) of $-\Delta$ on $\mathbb{R}^d$, d even. Our result is complementary to the results of [43]. See Section 3.1 for further discussion and references for Poisson formulae in both even and odd dimensions.

We comment briefly on the structure of the paper. Section 2 proves some bounds on the scattering matrix which are needed later in the proof to control a term appearing in the Poisson formula. In Section 3 we state and prove the Poisson formula, Theorem 3.6. Theorem 4.1 is a more general version of the lower bound than Theorem 1.2. In Section 4 we prove Theorem 4.1, using the Poisson formula and results from [32] and Section 2. We finish the proof of Theorems 1.1 and 1.2 by using results on the singularity at 0 of $u(t)$, e.g. [13, 16, 25], and estimates on the cut-off resolvent on the positive real axis due to Burq [4] and Cardoso–Vodev [5].

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1.1. Set-up and notation

We briefly introduce some notation which we shall use throughout the paper.

We recall the black-box perturbations of [31], using notation as in that paper. Let $\mathcal{H}$ be a complex Hilbert space with orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^d \setminus B(0;R_0))$$

where $B(0;R_0) \subset \mathbb{R}^d$ is the ball of radius R_0 centered at the orgin. Let $\mathbb{1}_{\mathbb{R}^n\setminus B(0;R_0)} : \mathcal{H} \to L^2(\mathbb{R}^d \setminus B(0;R_0))$ denote orthogonal projection. Let $P : \mathcal{H} \to \mathcal{H}$ be a linear self-adjoint operator with domain $\mathcal{D} \subset \mathcal{H}$. We assume $\mathbb{1}_{\mathbb{R}^n\setminus B(0;R_0)}\mathcal{D} = H^2(\mathbb{R}^d \setminus B(0;R_0))$ and $\mathbb{1}_{\mathbb{R}^n\setminus B(0;R_0)}P = -\Delta \restriction_{\mathbb{R}^n\setminus B(0;R_0)}$. Moreover, P is lower semi-bounded and there is a $k_0 \in \mathbb{N}$ so that $(1 - \mathbb{1}_{\mathbb{R}^n\setminus B(0;R_0)})(P+i)^{-k_0}$ is trace class.

Under these assumptions on P, we may carefully define the wave trace given informally by (1.2). Thus

$$\begin{aligned}(1.3)\quad u(t) \stackrel{\text{def}}{=} 2\Big[\operatorname{tr}_{\mathcal{H}}\Big(\cos(t\sqrt{P}) - \mathbb{1}_{\mathbb{R}^n\setminus B(0;R_0)}\cos(t\sqrt{-\Delta})\mathbb{1}_{\mathbb{R}^n\setminus B(0;R_0)}\Big)\\ - \operatorname{tr}_{L^2(\mathbb{R}^d)}\Big(\mathbb{1}_{B(0;R_0)}\cos(t\sqrt{-\Delta})\mathbb{1}_{B(0;R_0)}\Big)\Big]\end{aligned}$$

where we understand this distributionally. The factor of 2 is included to be consistent with the definition of u given via the wave group as in, for example, [22, 43].

Let P be a black-box perturbation of $-\Delta$ on $\mathbb{R}^d$, and, for $0 < \arg\lambda < \pi$, set $R(\lambda) = (P-\lambda^2)^{-1}$. We denote by $R(\lambda)$ the meromorphic continuation to Λ, the logarithmic cover of $\mathbb{C}\setminus\{0\}$, if d is even. If d is odd, the continuation is to $\mathbb{C}$, and again we denote it $R(\lambda)$. We use $\mathcal{R}$ to denote all the poles of the continuation of the resolvent $R(\lambda)$, repeated according to multiplicity.

We explicitly include both those poles corresponding to eigenvalues and those which do not.

A point $\lambda \in \Lambda$ can be identified by specifying both its norm $|\lambda|$ and argument $\arg \lambda$ where we do not identify points in Λ whose arguments differ by nonzero integral multiples of 2π. For $\lambda \in \Lambda$, we denote $\overline{\lambda} = \overline{|\lambda| e^{i \arg \lambda}} = |\lambda| e^{-i \arg \lambda}$. For $k \in \mathbb{N}$, set

$$\Lambda_k = \{\lambda \in \Lambda : k\pi < \arg \lambda < (k+1)\pi\}.$$

Throughout this paper, C denotes a positive constant, the value of which may change from line to line without comment.

2. Preliminary estimates on the scattering matrix and related quantities

We shall need some estimates on the determinant of the scattering matrix and on $\|S(\lambda) - I\|_{\mathrm{tr}}$. The proof uses a representation for the scattering matrix from [28] which we recall for the reader's convenience. We remark that there are a number of related representations in the literature; see, for example, [27, Section 2] or [43, Section 3].

Proposition 2.1 ([28, Proposition 2.1]). — *For $\phi \in C_c^\infty(\mathbb{R}^d)$, let us denote by*

$$\mathbb{E}^{\phi}_{\pm}(\lambda) : L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$$

the operator

$$\left(\mathbb{E}^{\phi}_{\pm}(\lambda) f\right)(\theta) = \int_{\mathbb{R}^d} f(x)\phi(x) \exp(\pm i\lambda\langle x, \omega\rangle) dx. \tag{2.1}$$

Let $\chi_i \in C_c^\infty(\mathbb{R}^d)$, $i = 1,\ 2,\ 3$ be such that $\chi_i \equiv 1$ near $\overline{B}(0; R_0)$ and $\chi_{i+1} \equiv 1$ on $\operatorname{supp} \chi_i$.

Then for $0 < \arg \lambda < \pi$, $S(\lambda) = I + A(\lambda)$, where

$$A(\lambda) = i\pi(2\pi)^{-d}\lambda^{d-2}\mathbb{E}^{\chi_3}_{+}(\lambda)[\Delta, \chi_1]R(\lambda)[\Delta, \chi_2]\,{}^t\mathbb{E}^{\chi_3}_{-}(\lambda)$$

where ${}^t\mathbb{E}$ denotes the transpose of $\mathbb{E}$. The identity holds for $\lambda \in \Lambda$ by analytic continuation.

Maciej Zworski suggested the proof of the following lemma. This follows techniques of [24, 37, 40]. We note that results of Burq [4] and Cardoso–Vodev [5] show that there are a large class of examples of black-box operators P for which (2.2) holds.

Lemma 2.2. — *Let the dimension $d \geqslant 2$ be even or odd, and let P denote an operator satisfying the general black-box conditions of* [31] *recalled in Section 1.1. Assume that there is an $R_1 > R_0$ so that for all $b > a > R_1$, if $\chi \in C_c^\infty(\mathbb{R}^d)$ has support in $\{x \in \mathbb{R}^d : a < |x| < b\}$ then there are constants m_0 and C_0 depending on P and χ so that*

$$\|\chi R(\tau)\chi\| \leqslant C\tau^{m_0} \quad \text{for } \tau \in (1,\infty) \ (\text{that is, } \arg\tau = 0,\ |\tau| > 1). \tag{2.2}$$

Let $S(\lambda)$ denote the scattering matrix for P, unitary when $\arg\lambda = 0$. Then, for $\tau \in (1,\infty)$, there is a constant C so that $\|I - S(\tau)\|_{\mathrm{tr}} \leqslant C(1+\tau)^{d-1}$ where $\|\cdot\|_{\mathrm{tr}}$ is the trace class norm.

Proof. — We note first that this is a result about large τ behavior, as $\lim_{\tau\downarrow 0} S(\tau) = I$, see [6, Section 6].

For a bounded linear operator B, we denote by $s_1(B) \geqslant s_2(B) \geqslant \ldots$ the singular values of B.

We use the representation of $S(\tau)$ recalled in Proposition 2.1. We have

$$\begin{aligned} &s_j(S(\tau) - I) \\ &\leqslant \|i\pi(2\pi)^{-d}\tau^{d-2}\mathbb{E}_+^{\chi_3}(\tau)\|_{L^2\to L^2}\|[\Delta,\chi_1]R(\tau)[\Delta,\chi_2]\|_{L^2\to L^2} s_j({}^t\mathbb{E}_-^{\chi_3}(\tau)). \end{aligned}$$

We may choose $\chi_1,\ \chi_2$ so that by (2.2),

$$\|[\Delta,\chi_1]R(\tau)[\Delta,\chi_2]\|_{L^2\to L^2} \leqslant C(1+\tau^{m_0+2}).$$

Moreover, $\|i\pi(2\pi)^{-d}\tau^{d-2}\mathbb{E}_+^{\chi_3}(\tau)\|_{L^2\to L^2} \leqslant C\tau^{d-2}$ since $\tau \in (0,\infty)$. From [8, (3.4.14)], which follows [40],

$$s_j({}^t\mathbb{E}_-^{\chi_3}(\tau)) \leqslant C_1 \exp(C_1\tau - j^{\frac{1}{d-1}}/C_1).$$

Summarizing,

$$s_j(S(\tau) - I) \leqslant C(\tau^{d+m_0})\exp(C_1\tau - j^{\frac{1}{d-1}}/C_1),\ \tau > 1. \tag{2.3}$$

However, since for $\tau \in (0,\infty)$, $S(\tau)$ is unitary we have that $s_j(S(\tau)-I) \leqslant 2$ for all j. Using this and (2.3), by choosing C_2 sufficiently large for $\tau \in (0,\infty)$

$$s_j(S(\tau) - I) \leqslant \begin{cases} 2 & \text{if } j \leqslant C_2\tau^{d-1} \\ Cj^{-\frac{d+1}{d}} & \text{if } j \geqslant C_2\tau^{d-1}. \end{cases} \tag{2.4}$$

Thus, for $\tau \in (1,\infty)$

$$\begin{aligned} \|S(\tau) - I\|_{\mathrm{tr}} &= \sum s_j(S(\tau) - I) \\ &= \sum_{j\leqslant C_2\tau^{d-1}} s_j(S(\tau) - I) + \sum_{j > C_2\tau^{d-1}} s_j(S(\tau) - I) \\ &\leqslant 2C_2\tau^{d-1} + \sum_{j\geqslant C_2\tau^{d-1}} Cj^{-\frac{d+1}{d}} \leqslant 2C_2\tau^{d-1} + C_3. \end{aligned}$$ □

In this paper we use Lemma 2.2 to prove Lemma 2.3. This argument has the advantage of working for a large class of perturbations P of $-\Delta$. However, for the special case of the Laplacian on the exterior of an obstacle, a portion of Lemma 2.3 has been proved in [6] using instead of Lemma 2.2 "inside-outside duality" results of Eckmann–Pillet [9] and Lechleiter–Peters [20]. In this special case, the remainder of Lemma 2.3 could be proved in a similar way.

Lemma 2.3. — *Let the dimension d be even, and let P denote an operator satisfying the hypotheses of Lemma 2.2 on P. Then there is a constant $C > 0$ so that for $\tau > 0$ (i.e., $\arg \tau = 0$),*

$$1 \leqslant |\det S(\tau e^{i\pi})| \leqslant C \exp(C\tau^{d-1})$$

and

$$|\arg \det S(\tau e^{i\pi}) - \arg \det S(e^{i\pi})| \leqslant C(\tau^{d-1} + 1)$$

where the argument is chosen to depend continuously on $\tau \in (0, \infty)$.

Proof. — From [7, Proposition 2.1], $S(e^{i\pi}\tau) = 2I - JS^*(\tau)J$, where $(Jf)(\omega) = f(-\omega)$. Hence

$$\det S(e^{i\pi}\tau) = \det(2I - S^*(\tau)). \tag{2.5}$$

Using that $S(\tau)$ is unitary, we can write $\det S(e^{i\pi}\tau) = \prod(2 - e^{-i\theta_j(\tau)})$, where $\{e^{i\theta_j(\tau)}\}$ are the eigenvalues of $S(\tau)$ other than 1, repeated with multiplicity, and $\theta_j(\tau) \in \mathbb{R}$. Since $|2 - e^{-i\theta_j(\tau)}| \geqslant 1$, we see immediately that $|\det S(\tau e^{i\pi})| \geqslant 1$. Moreover, from Lemma 2.2, (2.5) and the estimate $|\det(I + B)| \leqslant \exp(\|B\|_{\mathrm{tr}})$ we obtain the upper bound on $|\det S(\tau e^{i\pi})|$.

Now we turn to the question of bounding $\arg S(\tau e^{i\pi})$. Note first that θ_j can be chosen so that $e^{i\theta_j(\tau)}$ depends continuously on τ, except, perhaps, at points where $e^{i\theta_j(\tau)}$ approaches 1. Let log denote the principal branch of the logarithm. Since $\mathrm{Re}(2 - e^{-i\theta_j(\tau)}) \geqslant 1$, $\log(2 - e^{-i\theta_j(\tau)})$ is well-defined and is a continuous function of τ if $e^{i\theta_j(\tau)}$ is a continuous function of τ. Now

$$\det S(\tau e^{i\pi}) = \exp\left(\sum \log(2 - e^{-i\theta_j(\tau)})\right). \tag{2.6}$$

Let τ_0 be a possible point of discontinuity of $\sum \log(2 - e^{-i\theta_j(\tau)})$; that is, a value of τ for which for some j_0 $\lim_{\tau\to\tau_0\pm} e^{i\theta_{j_0}(\tau)} = 1$ for some choice of sign $\pm$. Since if $\lim_{\tau\to\tau_0\pm} e^{i\theta_{j_0}(\tau)} = 1$, then $\lim_{\tau\to\tau_0^\pm} \log(2 - e^{i\theta_{j_0}(\tau)}) = 0$, the sum $\sum \log(2 - e^{-i\theta_j(\tau)})$ depends continuously on $\tau \in (0, \infty)$, so that

$$\arg S(\tau e^{i\pi}) - \arg S(e^{i\pi}) = \sum \mathrm{Im} \log(2 - e^{-i\theta_j(\tau)}) - \sum \mathrm{Im} \log(2 - e^{-i\theta_j(1)}).$$

But
$$\left|\sum \operatorname{Im}\log(2-e^{-i\theta_j(\tau)})\right| \leqslant \sum |\operatorname{Im}\log(2-e^{-i\theta_j(\tau)})| \leqslant C\sum |(1-e^{-i\theta_j(\tau)})|$$
using that $|\log(1+z)| \leqslant C|z|$ when $\operatorname{Re} z \geqslant 0$. Then
$$\begin{aligned}\left|\sum \operatorname{Im}\log(2-e^{-i\theta_j(\tau)})\right| &\leqslant C\sum |(1-e^{-i\theta_j(\tau)})| \\ &\leqslant C\|S(\tau)-I\|_{\mathrm{tr}} \leqslant C(1+\tau^{d-1})\end{aligned}$$
by Lemma 2.2. □

The result about the argument of $\det S(\tau e^{i\pi})$ in Lemma 2.3 is particular to even dimensions. In odd dimensions d, if $\tau > 0$, then $\det S(\tau e^{i\pi}) = \overline{\det S(\tau)}$. To be specific, consider the case of scattering by an obstacle $\mathcal{O}$ with Dirichlet or Neumann boundary conditions. In that case the scattering phase has Weyl asymptotics: $\frac{1}{i}\det S(\tau) = c_d \mathrm{vol}(\mathcal{O})\tau^d + O(\tau^{d-1})$ as $\tau \to \infty$, for a nonzero constant c_d, e.g. [26, 30]. Hence for obstacle scattering in *odd* dimensions d, $\arg S(\tau e^{i\pi}) = -c_d \mathrm{vol}(\mathcal{O})\tau^d + O(\tau^{d-1})$ as $\tau \to \infty$. Similar results hold for many other classes of black-box scattering in odd dimensions.

3. The Poisson formula

We begin with several complex-analytic results which will be helpful in proving the estimates we need on $\det S(\lambda)$ and related quantities.

We shall use the following lemma when working with functions holomorphic or meromorphic in a sector.

Lemma 3.1. — *Set $U = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Suppose f is analytic on U and that there exist constants C, $p \geqslant 1$, so that $|f(z)| \leqslant C\exp(C|z|^p)$ for any $z \in U$. Suppose in addition that f is nowhere vanishing on U. Then there is a function g analytic on U so that $\exp g(z) = f(z)$, $z \in U$. Moreover, given $\epsilon > 0$ there is a constant C_ϵ so that*
$$|g(z)| \leqslant C_\epsilon |z|^{p+\epsilon} \text{ if } |\arg z| < \pi/2 - \epsilon,\ |z| > 1.$$

Proof. — The existence of an analytic g so that $\exp(g) = f$ is immediate. The bound on $|g|$ can be proved, for example, by using the representation [11, Theorem 3.3] of a function of finite order analytic in an angle. □

Note that the hypothesis $p \geqslant 1$ in Lemma 3.1 is necessary. For example, the function $f(z) = e^{-z}$ satisfies the other hypotheses of Lemma 3.1 with $p = 0$, but does not satisfy the conclusion of the lemma with $p = 0$. For more about related questions, see [11, Chapter 1].

We recall that the canonical factors E_p are given by

$$E_0(z) = 1 - z \text{ and } E_p(z) = (1-z)\exp(z + z^2/2 + \cdots + z^p/p) \text{ for } p \in \mathbb{N}.$$

For entire functions f, a stronger result than the following lemma holds (compare [21, Theorem I.9]). The following lemma, while likely not sharp, will suffice for our needs.

Lemma 3.2. — *Set $U = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Suppose f is analytic on U and that there exist constants C, $p \geqslant 1$, so that $|f(z)| \leqslant C\exp(C|z|^p)$ for any $z \in U$. Suppose in addition that there is a constant C so that*

$$\#\{z_j :\ z_j \in U; f(z_j) = 0 \text{ and } |z_j| < r\} \leqslant C(1 + r^p)$$

where the zeros z_j are repeated with multiplicity. Then for any $\epsilon, \eta > 0$ there is a constant $C_{\epsilon,\eta}$ so that for any $R > 2$

$$\ln|f(z)| \geqslant -C_{\eta,\epsilon}(1 + R^{p+\epsilon}),\ 1 \leqslant |z| \leqslant R,\ |\arg z| < \pi/2 - \epsilon$$

if z lies outside a family of excluded disks, the sum of whose radii does not exceed ηR. Moreover, if $\varphi(z) = \prod E_{\lfloor p \rfloor}(z/z_j)$, then

$$|f(z)/\varphi(z)| \leqslant \exp(C_\epsilon(1 + R^{p+\epsilon})),\ 1 \leqslant |z| \leqslant R,\ |\arg z| < \pi/2 - \epsilon/2.$$

Proof. — By standard estimates on canonical products, for $\epsilon > 0$ the entire function φ satisfies $|\varphi(z)| \leqslant \exp(C_\epsilon(1 + |z|^{p+\epsilon/2}))$. Hence, by [21, Theorem I.9], for $R > 1$ and any $\eta > 0$, there is a constant $C_{\eta,\epsilon}$ so that

$$\ln|\varphi(z)| \geqslant -C_{\eta,\epsilon}(1 + R^{p+\epsilon/2}),\ |z| \leqslant R$$

where z in addition lies outside a family of disks the sum of whose radii does not exceed ηR.

Now consider the function $f(z)/\varphi(z)$, which is a nonvanishing analytic function on U. Moreover, with perhaps a new constant $C_{\eta,\epsilon}$

$$|f(z)/\varphi(z)| \leqslant \exp(C_{\eta,\epsilon}(1 + R^{p+\epsilon/2})),\ |z| \leqslant R,\ z \in U$$

if z lies outside a family of disks the sum of whose radii does not exceed ηR. Since f/φ is analytic, by the maximum principle if $R \gg 1$ and $\eta > 0$ is chosen sufficiently small

$$|f(z)/\varphi(z)| \leqslant \exp(C_{\eta,\epsilon}(1 + R^{p+\epsilon/2})),\ |z| \leqslant R,\ |\arg z| < \pi/2 - \epsilon/2.$$

The reason for shrinking the size of the sector is the need, for every previously excluded point z' (ie, every point in one of the originally excluded disks) to have bounds on $|f/\varphi|$ on a closed curve lying inside U and containing z' in its interior. Now since f/φ is nonvanishing on U we can apply Lemma 3.1 to the function $(f/\varphi)(z^{1/(1-\epsilon/2)})$ to complete the proof. $\square$

The proof of Lemma 3.4 will use the following result from [43, Section 2]. Here, for the convenience of the reader, we show how this expression for $S(\lambda) - I$ can be derived from Proposition 2.1 using also [31, Section 3]. We note that a similar result is proved in [12, Section 3] in a different setting and by somewhat different methods. In the statement of the lemma and subsequent proof, $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ when $0 < \arg\lambda < \pi$ and is the holomorphic extension otherwise.

Lemma 3.3 ([43, Section 2]). — *The scattering matrix $S(\lambda) = I + A(\lambda)$, where*

$$A(\lambda) = i\pi(2\pi)^{-d}\lambda^{d-2}\mathbb{E}_+^{\phi_1}(\lambda)(I + K(\lambda,\lambda_0))^{-1}[\Delta,\psi_3]^t\mathbb{E}_-^{\phi_2}(\lambda) \tag{3.1}$$

with $\mathbb{E}_\pm^{\phi_j}$ as defined in (2.1)*. The operator $K(\lambda,\lambda_0)$ is a compact operator, analytic on Λ, defined in* [31, Section 3]*:*

$$K(\lambda,\lambda_0) = [\Delta,\psi_0]R_0(\lambda)(1-\psi_1)\psi_4 - [\Delta,\psi_2]R(\lambda_0)\psi_1 + \psi_2(\lambda_0^2 - \lambda^2)R(\lambda_0)\psi_1$$

$$\phi_i,\psi_i,\chi_1 \in C_c^\infty(\mathbb{R}^d),\ \psi_i \equiv 1 \text{ on } \overline{B}(0;R_0),\ \psi_i\psi_{i-1} = \psi_{i-1},$$

$$\chi_1\psi_2 = \psi_2,\ \chi_1\psi_3 = \chi_1,\ \phi_1 = (1-\psi_1)\chi_1, \phi_2 = (1-\psi_1)\psi_4.$$

Here λ_0 is a point in Λ with $0 < \arg\lambda_0 < \pi$ and is chosen to ensure the invertibility of $I + K(\lambda_0,\lambda_0)$.

Proof. — We begin by noting that by analytic Fredholm theory $(I + K(\lambda,\lambda_0))^{-1}$ is a meromorphic function on Λ. The operator K arises in the construction of the meromorphic continuation of the resolvent in [31, Section 3] and

$$\begin{aligned}\psi_4 R(\lambda)\psi_4 \\ = \psi_4\left\{(1-\psi_0)R_0(\lambda)(1-\psi_1)\psi_4 + \psi_2 R(\lambda_0)\psi_1\right\}(I + K(\lambda,\lambda_0))^{-1}\psi_4.\end{aligned} \tag{3.2}$$

Now we use Proposition 2.1, choosing the χ_j so that $\chi_1 = 1$ on support of ψ_2 and choosing $\chi_2 = \psi_3$. Then from (3.2) and the support properties of χ_1, ψ_2,

$$[\Delta,\chi_1]R(\lambda)\psi_4 = [\Delta,\chi_1]R_0(\lambda)(1-\psi_1)\psi_4(I + K(\lambda,\lambda_0))^{-1}\psi_4.$$

Now if $g \in H_0^2(\mathbb{R}^d)$ and $\chi_3 = 1$ on the support of g, then

$$\mathbb{E}_+^{\chi_3}(\lambda)(\Delta g) = -\lambda^2\mathbb{E}_+^{\chi_3}(\lambda)g.$$

Hence

$$\mathbb{E}_+^{\chi_3}(\lambda)[\Delta,\chi_1]R_0(\lambda)(1-\psi_1)\psi_4 = \mathbb{E}_+^{\chi_3}(\lambda)\chi_1(1-\psi_1)\psi_4 = \mathbb{E}_+^{\chi_3}(\lambda)\chi_1(1-\psi_1)$$

Using this and Proposition 2.1 we see that

$$A(\lambda) = i\pi(2\pi)^{-d}\lambda^{d-2}\mathbb{E}_+^{\chi_3}(\lambda)\chi_1(1-\psi_1)(I + K(\lambda,\lambda_0))^{-1}[\Delta,\chi_2]^t\mathbb{E}_-^{\chi_3}(\lambda).$$

Since we chose $\chi_2 = \psi_3$, $[\Delta, \chi_2]^t \mathbb{E}_-^{\chi_3}(\lambda) = [\Delta, \psi_3]^t \mathbb{E}_-^{\phi_2}(\lambda)$ using in addition $\psi_1\psi_3 = \psi_1$, $\psi_3\psi_4 = \psi_3$. Moreover, from our definition of ϕ_1 and the properties of χ_3 we find $\mathbb{E}_+^{\chi_3}(\lambda)\chi_1(1-\psi_1) = \mathbb{E}_+^{\phi_1}(\lambda)$. □

We continue to denote by S the scattering matrix, unitary for $\arg\lambda = 0$, associated to a self-adjoint black-box type operator P. Then define

$$f(\lambda) = \frac{\det S(\lambda)}{\det S(\lambda e^{i\pi})}. \tag{3.3}$$

The proof of the following lemma closely resembles that of [42, Proposition 6] and of a related result of [43, Section 2]. We include the proof here for the convenience of the reader and because the need for working in sectors of $\mathbb{C}$ rather than in balls in $\mathbb{C}$ means that Lemmas 3.1 and 3.2 substitute for what can be done with Cartan's lemma and Caratheodory's inequality for the disk. The proof mostly follows [43], highlighting the points at which Lemmas 3.1 and 3.2 are needed.

Lemma 3.4. — *Let d be even, and let f be as in (3.3). Let $p \in \mathbb{N}$ be such that for the operator P the counting functions n_{-1}, n_{-2} satisfy $n_{-1}(r) + n_{-2}(r) \leqslant Cr^p$ for some constant C, and set*

$$P_1(\lambda) = \prod_{\substack{\lambda_j \in \mathcal{R} \\ -\pi/2 < \arg\lambda_j < 3\pi/2}} E_p(\lambda/\overline{\lambda_j}),\ P_2(\lambda) = \prod_{\substack{\lambda_j \in \mathcal{R} \\ -3\pi/2 < \arg\lambda_j < \pi/2}} E_p(\lambda e^{i\pi}/\overline{\lambda_j})$$

$$Q_1(\lambda) = \prod_{\substack{\lambda_j \in \mathcal{R} \\ -3\pi/2 < \arg\lambda_j < \pi/2}} E_p(\lambda/\lambda_j),\ Q_2(\lambda) = \prod_{\substack{\lambda_j \in \mathcal{R} \\ -\pi/2 < \arg\lambda_j < 3\pi/2}} E_p(\lambda e^{i\pi}/\lambda_j)$$

Then for $-3\pi/2 < \arg\lambda < \pi/2$,

$$f(\lambda) = e^{g(\lambda)} \frac{P_1(\lambda)}{Q_1(\lambda)} \frac{Q_2(\lambda)}{P_2(\lambda)}$$

where $g(\lambda)$ is analytic in $-3\pi/2 < \arg\lambda < \pi/2$ and for some $p' \geqslant p, C > 0$ we have

$$|g(\lambda)| \leqslant C(1 + |\lambda|^{p'}),\ -4\pi/3 < \arg\lambda < \pi/3.$$

Proof. — We identify the region $\{\lambda \in \Lambda :\ -3\pi/2 < \arg\lambda < \pi/2\}$ with $\mathbb{C} \setminus i[0,\infty)$.

The results of [38, 39] show that with our assumptions on P, there is a $p \in \mathbb{N}$ so that $n_{-1}(r) + n_{-2}(r) = O(r^p)$ as $r \to \infty$. Using this and the fact that $R(\lambda)$ has at most finitely many poles in Λ_0, the functions P_j, Q_j, $j = 1, 2$, are well-defined entire functions. Since $S(\lambda)S^*(\overline{\lambda}) = I$, λ_j is a pole of $\det S(\lambda)$ if and only if $\overline{\lambda_j}$ is a zero of $\det S(\lambda)$. The function

$$h(\lambda) \stackrel{\text{def}}{=} f(\lambda)\frac{Q_1(\lambda)}{P_1(\lambda)}\frac{P_2(\lambda)}{Q_2(\lambda)}$$

is an analytic, nowhere vanishing function if $-3\pi/2 < \arg\lambda < \pi/2$, see [7, Theorem 4.5], so the existence of a function g so that $\exp(g) = h$ in this region is immediate. What needs to be proved is the polynomial bound on g in this region.

We use the representation of the scattering matrix recalled in Lemma 3.3, along with that notation. The assumptions made on P ensure that the operator $K(\lambda,\lambda_0)^{2k_0}$ is trace class. We remark that we make no effort here to find the optimal value of p' so that the statement of the lemma holds. By techniques of [37, 41] (see also [38]), if $|\arg\lambda| \leqslant \theta_0$ there is a natural number m and a constant C (depending on θ_0, but not on $|\lambda|$), so that

$$|\det(I + K(\lambda,\lambda_0)^{2k_0})| \leqslant \det(I + |K(\lambda,\lambda_0)|^{2k_0}) \leqslant C\exp(C|\lambda|^m). \tag{3.4}$$

Moreover, the number of zeros of $|\det(I + K(\lambda,\lambda_0)^{2k_0})$ (counted with multiplicity) in the region $\{\lambda \in \Lambda : \ |\arg\lambda| \leqslant \theta_0;\ |\lambda| \leqslant r\}$ is $O(r^m)$ ([38, 39]). We apply this to the inequality ([10, Theorem 5.1])

$$\|(I + K(\lambda,\lambda_0))^{-1}\| \leqslant \|(I + \|K(\lambda,\lambda_0)\|)^{2k_0-1}\ \frac{\det(I + |K(\lambda,\lambda_0)|^{2k_0})}{|\det(I + K(\lambda,\lambda_0)^{2k_0})|}.$$

It is here we can see the need of Lemmas 3.1 and 3.2. Using these lemmas and the estimate (3.4), we see that given $\eta,\epsilon > 0$, there is a constant $C_{\eta,\epsilon}$ so that for $|\arg\lambda| < \theta_0 - \epsilon$ and $1 \leqslant |\lambda| \leqslant R$, outside a family of excluded disks, the sum of whose radii does not exceed ηR,

$$\|(I + K(\lambda,\lambda_0))^{-1}\| \leqslant C_{\eta,\epsilon}\exp(C(1 + R^{m+\epsilon})).$$

Using this and Lemma 3.3 as in [42], one can show that, perhaps with new constant $C_{\eta,\epsilon}$

$$|\det S(\lambda)| \leqslant C_{\eta,\epsilon}\exp C_{\eta,\epsilon}(R^{(m+\epsilon)d}) \tag{3.5}$$

for λ in the same region.

Since $(\det S(\lambda))\,Q_1(\lambda)$ is analytic for $-3\pi/2 < \arg\lambda < \pi/2$, using (3.5) and the maximum principle gives

$$|Q_1(\lambda)\det S(\lambda)| \leqslant C_\epsilon \exp C_{\eta,\epsilon}(|\lambda|^{(m+\epsilon)d})\ |\lambda| > 1,\ -\frac{3\pi}{2} + \epsilon < \arg\lambda < \frac{\pi}{2} - \epsilon.$$

Since $Q_1(\lambda)\det S(\lambda)/P_1(\lambda)$ is a nowhere vanishing analytic function in $-3\pi/2 < \arg\lambda < \pi/2$, applying Lemmas 3.2 and 3.1 we find that

$$Q_1(\lambda)(\det S(\lambda))/P_1(\lambda) = \exp g_1(\lambda)$$

here, with g_1 polynomially bounded in the sector $-3\pi/2+\epsilon < \arg\lambda < \pi/2-\epsilon$. The same argument gives $Q_2(\lambda)\det S(\lambda e^{i\pi})/P_2(\lambda) = \exp g_2(\lambda)$, with g_2 polynomially bounded in the same region. Since $g = g_1 - g_2$, we are done. □

We continue to use the function f defined in (3.3). From the fact that $S(\lambda)$ is unitary when $\arg\lambda = 0$ and from Lemma 2.3, we see that $f(\lambda)$ has neither zeros nor poles with $\arg\lambda = 0$. Since

$$\det S(\tau e^{i\pi})\,\overline{\det S(\tau e^{-i\pi})} = 1, \text{ if } \tau > 0,$$

from Lemma 2.3 the function $f(\lambda)$ has neither poles nor zeros with $\arg\lambda = -\pi$.

We identify $\{\lambda \in \Lambda : \ -3\pi/2 < \arg\lambda < \pi/2\}$ with $\mathbb{C} \setminus i[0,\infty)$ and consider the function f defined in this region, as studied in Lemma 3.4. Define the *distribution*

$$v(t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\,\frac{f'(\lambda)}{f(\lambda)}d\lambda. \tag{3.6}$$

We clarify that in this integral for $\lambda \in \mathbb{R}_+$ we understand $\arg\lambda = 0$, and for $\lambda \in \mathbb{R}_-$ we understand $\arg\lambda = -\pi$. Note that this is well-defined as a distribution, as we describe below. The function f has neither zeros nor poles with $\arg\lambda = 0$ or $\arg\lambda = \pm\pi$. Moreover, $f(\lambda) \to 1$ as $\lambda \to 0$ with $\arg\lambda = 0$ or $\arg\lambda = \pm\pi$, see [6, Section 6]. Hence (identifying $\arg\lambda = -\pi$ with $(-\infty,0)$ and $\arg\lambda = 0$ with $(0,\infty)$), we can find a continuous function $\ell(f(\lambda))$ on $\mathbb{R}$ so that $\exp(\ell(f(\lambda))) = f(\lambda)$ for $\lambda \in \mathbb{R}$. Moreover, $\ell(f(\lambda))$ is in fact smooth on $\mathbb{R} \setminus \{0\}$, and has an expansion in powers of λ and $\log\lambda$ at 0, see [6, Section 6]. Using Lemma 3.4 we see that $\ell(f)$ is a tempered distribution on $\mathbb{R}$. Hence its derivative, f'/f, is also a tempered distribution.

Lemma 3.5. — *The distribution $v(t)$ defined by (3.6) is given by*

$$v(t) = 2\pi i \sum_{\lambda_j \in \Lambda_{-1}\cap\mathcal{R}} \left(e^{-i\lambda_j t} + e^{i\overline{\lambda}_j t}\right) - 2\pi i \sum_{\lambda_j \in \Lambda_0\cap\mathcal{R}} \left(e^{-i\overline{\lambda_j} t} + e^{i\lambda_j t}\right) \text{ if } t > 0.$$

Proof. — We use the representation for f from Lemma 3.4. Hence

$$\begin{aligned}\frac{f'(\lambda)}{f(\lambda)} &= g'(\lambda) + \frac{P_1'(\lambda)}{P_1(\lambda)} - \frac{Q_1'(\lambda)}{Q_1(\lambda)} + \frac{Q_2'(\lambda)}{Q_2(\lambda)} - \frac{P_2'(\lambda)}{P_2(\lambda)} \\ &= g'(\lambda) + \sum_{\substack{\lambda_j\in\mathcal{R} \\ -\pi/2<\arg\lambda_j<3\pi/2}} \frac{(\lambda/\overline{\lambda_j})^p}{\lambda-\overline{\lambda_j}} - \sum_{\substack{\lambda_j\in\mathcal{R} \\ -3\pi/2<\arg\lambda_j<\pi/2}} \frac{(\lambda/\lambda_j)^p}{\lambda-\lambda_j} \\ &\quad - \sum_{\substack{\lambda_j\in\mathcal{R} \\ -\pi/2<\arg\lambda_j<3\pi/2}} \frac{(e^{i\pi}\lambda/\lambda_j)^p}{e^{i\pi}\lambda-\lambda_j} + \sum_{\substack{\lambda_j\in\mathcal{R} \\ -3\pi/2<\arg\lambda_j<\pi/2}} \frac{(e^{i\pi}\lambda/\overline{\lambda_j})^p}{e^{i\pi}\lambda-\overline{\lambda_j}}. \end{aligned} \tag{3.7}$$

Let q be a polynomial. For $t>0$, $a,b\in\mathbb{R}$ and $b\neq 0$,

$$\mathcal{F}\left\{\frac{q(\xi)}{\xi-(a+ib)}\right\}(t) = \begin{cases} -2\pi i e^{-i(a+ib)t} q(a+ib) & \text{if } b<0,\ t>0 \\ 0 & \text{if } b>0,\ t>0\end{cases}$$

as a distribution. Applying this to (3.7), we find that as a distribution

$$\begin{aligned} v(t) = 2\pi i &\left(- \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_0} e^{-i\overline{\lambda_j}t} + \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_{-1}} e^{-i\lambda_j t} - \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_0} e^{i\lambda_j t}\right) \\ &+ 2\pi i \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_{-1}} e^{i\overline{\lambda_j}t} + \int_{-\infty}^{\infty} e^{-i\lambda t} g'(\lambda)d\lambda, \quad \text{if } t>0. \end{aligned} \tag{3.8}$$

Now consider $\int_{-\infty}^{\infty} e^{-i\lambda t} g'(\lambda)d\lambda$, which is well-defined as a distribution on $\mathbb{R}$. By Lemma 3.4 the distribution $\int_{-\infty}^{\infty} e^{-it\lambda} g(\lambda)d\lambda$ has inverse Fourier transform which is analytic and polynomially bounded in the open lower half plane. Thus, by a version of the Paley–Weiner–Schwartz Theorem (e.g. [15, Theorem 7.4.3]), the distribution $\int_{-\infty}^{\infty} e^{-it\lambda} g(\lambda)d\lambda$ is supported in $t\leqslant 0$. Since, in the sense of distributions,

$$\int_{-\infty}^{\infty} e^{-i\lambda t} g'(\lambda)d\lambda = it\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda)d\lambda,$$

the distribution $\int_{-\infty}^{\infty} e^{-i\lambda t} g'(\lambda)d\lambda$ is supported in $t\leqslant 0$ as well. Hence, for $t>0$

$$v(t) = 2\pi i\left(- \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_0} (e^{-i\overline{\lambda_j}t} + e^{i\lambda_j t}) + \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_{-1}} (e^{-i\lambda_j t} + e^{i\overline{\lambda_j}t})\right), \ t>0.$$

□

The following theorem gives a Poisson formula for resonances in even dimensions, complementary to that of [43]. The integral appearing here may be thought of as an error or remainder term. Lemma 4.2 uses Lemma 2.3 to bound its contribution in our application, the proof of Theorem 4.1.

Theorem 3.6. — *Let d be even, and u denote the distribution defined by* (1.3) *for a self-adjoint black-box type perturbation. Set $s(\lambda) = \det S(\lambda)$. Then, for $t \neq 0$,*

$$u(t) = \sum_{\lambda_j \in \mathcal{R} \cap \Lambda_{-1}} \left(e^{-i\lambda_j |t|} + e^{i\overline{\lambda_j}|t|} \right) + \sum_{\substack{-\sigma_j^2 \in \sigma_p(P) \cap (-\infty,0) \\ \sigma_j > 0}} \left(e^{\sigma_j |t|} - e^{-\sigma_j |t|} \right)$$
$$+ \sum_{\substack{\mu_l^2 \in \sigma_p(P) \cap (0,\infty) \\ \mu_l > 0}} \left(e^{i\mu_l t} + e^{-i\mu_l t} \right) + m(0)$$
$$- \frac{1}{2\pi i} \int_0^\infty \left(e^{-i\lambda |t|} \frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})} + e^{i\lambda |t|} \frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})} \right) d\lambda.$$

Here $m(0)$ is the multiplicity of 0 *as a pole of the resolvent of P, chosen to make the Birman–Krein formula* (3.9) *correct, and $\sigma_p(P)$ is the point spectrum of P.*

Before proving the theorem, we note that alternatively we could write

$$\sum_{\lambda_j \in \mathcal{R} \cap \Lambda_{-1}} \left(e^{-i\lambda_j |t|} + e^{i\overline{\lambda_j}|t|} \right) + \sum_{\substack{\mu_l^2 \in \sigma_p(P) \cap (0,\infty) \\ \mu_l > 0}} \left(e^{i\mu_l t} + e^{-i\mu_l t} \right)$$
$$= \sum_{\lambda_j \in \mathcal{R}, -\pi < \arg \lambda_j \leqslant 0} \left(e^{-i\lambda_j |t|} + e^{i\overline{\lambda_j}|t|} \right)$$

using that if $\mu_l^2 > 0$ is an eigenvalue of P, then $|\mu_l|$ is a pole of $R(\lambda)$ and hence, by our convention, an element of $\mathcal{R}$.

Proof. — By the Birman–Krein formula,

(3.9)
$$u(t) = \frac{1}{2\pi i} \int_0^\infty (e^{-i\lambda t} + e^{-i\lambda t}) \frac{s'(\lambda)}{s(\lambda)} d\lambda + \sum_{\substack{-\sigma_j^2 \in \sigma_p(P) \cap (-\infty,0) \\ \sigma_j > 0}} \left(e^{\sigma_j |t|} + e^{-\sigma_j |t|} \right)$$
$$+ \sum_{\substack{\mu_l^2 \in \sigma_p(P) \cap (0,\infty) \\ \mu_l > 0}} \left(e^{i\mu_l t} + e^{-i\mu_l t} \right) + m(0).$$

Using the same convention as discussed after (3.6) and the definition of f (3.3), we write the distribution $v(t)$ from (3.6) as

$$\begin{aligned} v(t) &= \int_{-\infty}^{\infty} e^{-i\lambda t}\left(\frac{s'(\lambda)}{s(\lambda)} + \frac{s'(e^{i\pi}\lambda)}{s(e^{i\pi}\lambda)}\right) d\lambda \\ &= \int_0^\infty (e^{-i\lambda t}+e^{i\lambda t})\frac{s'(\lambda)}{s(\lambda)}d\lambda + \int_0^\infty\left(e^{-i\lambda t}\frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})}+e^{i\lambda t}\frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})}\right)d\lambda \end{aligned} \tag{3.10}$$

where the second equality follows by a change of variable for the integral over $(-\infty, 0)$. The first integral on the right hand side is $2\pi i$ times the first term on the right hand side in (3.9). Solving (3.10) for the integral in (3.9) and using Lemma 3.5 gives, for $t > 0$,

$$\begin{aligned} &\frac{1}{2\pi i}\int_0^\infty (e^{-i\lambda t} + e^{i\lambda t})\frac{s'(\lambda)}{s(\lambda)}d\lambda \\ &\qquad = \frac{-1}{2\pi i}\int_0^\infty\left(e^{-i\lambda t}\frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})} + e^{i\lambda t}\frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})}\right)d\lambda \\ &\qquad\quad + \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_{-1}}\left(e^{-i\lambda_j t}+e^{i\overline{\lambda_j}t}\right) - \sum_{\lambda_j\in\mathcal{R}\cap\Lambda_0}\left(e^{-i\overline{\lambda_j}t}+e^{i\lambda_j t}\right). \end{aligned}$$

If $\lambda_j \in \Lambda_0 \cap \mathcal{R}$, then $\lambda_j^2 \in \sigma_p(P) \cap (-\infty, 0)$. Using this proves the theorem for $t > 0$. To prove the theorem for $t < 0$, we note that u is a distribution which is even in t. □

3.1. Comparison of Theorem 3.6 to other Poisson formulae for resonances

We briefly compare the result of Theorem 3.6 to earlier Poisson formulae, both in odd and even dimensions.

We note that the proof of Theorem 3.6 can, with a small modification, be adapted to prove the *odd*-dimensional Poisson formula. In the generality of the black-box setting we consider here, this is due to Sjöstrand–Zworski [32], but it follows earlier work for obstacle scattering by Lax–Phillips [19], Bardos–Guillot–Ralston [1], and Melrose [22, 23] for increasingly large sets of $t \in \mathbb{R}$. The proof we describe here is not very different from, but a bit less direct than, that given in [42]. The value of including this particular variant of the proof of the odd-dimensional result is that it shows the consistency of our methods with the trace formula of [1, 19, 22, 23, 32] in odd dimensions.

Most of our proof of Theorem 3.6 is not dimension-dependent. In fact, the Birman–Krein formula holds in both even and odd dimensions. We use the distribution v defined in the proof of Theorem 3.6, and note that the computation of $v(t)$, $t>0$ in Lemma 3.5 holds in odd dimensions as well, where we make the (natural) identification of Λ_0 with the complex upper half plane, and Λ_{-1} with the lower half plane, and use the bound on g proved in [42]. In *odd* dimension d,

$$\begin{aligned} v(t) &= \int_{-\infty}^{\infty} e^{-i\lambda t}\left(\frac{s'(\lambda)}{s(\lambda)}+\frac{s'(e^{i\pi}\lambda)}{s(e^{i\pi}\lambda)}\right)d\lambda \\ &= \int_0^{\infty}(e^{-i\lambda t}+e^{i\lambda t})\frac{s'(\lambda)}{s(\lambda)}d\lambda+\int_0^{\infty}\left(e^{-i\lambda t}\frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})}+e^{i\lambda t}\frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})}\right)d\lambda \\ &= \int_0^{\infty}(e^{-i\lambda t}+e^{i\lambda t})\frac{s'(\lambda)}{s(\lambda)}d\lambda+\int_0^{\infty}(e^{-i\lambda t}+e^{i\lambda t})\frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})}d\lambda \end{aligned} \tag{3.11}$$

where for the last equality we used $s(\lambda e^{i\pi})=s(\lambda e^{-i\pi})$ in odd dimensions. But in odd dimensions, $s(\lambda e^{i\pi})s(\lambda)=1$, so that

$$s'(\lambda)/s(\lambda)=s'(\lambda e^{\pm i\pi})/s(\lambda e^{\pm i\pi}).$$

Hence for odd dimensions

$$v(t)=2\int_0^{\infty}(e^{-i\lambda t}+e^{i\lambda t})\frac{s'(\lambda)}{s(\lambda)}d\lambda.$$

Dividing both sides by 2 and then continuing to follow the proof from the even-dimensional case gives, for $t\neq 0$,

$$\begin{aligned} u(t) = \frac{1}{2}\sum_{\lambda_j\in\mathcal{R},-\pi<\arg\lambda_j<0}(e^{-i\lambda_j|t|}+e^{i\overline{\lambda}_j|t|})+\sum_{\substack{-\sigma_j^2\in\sigma_p(P)\cap(-\infty,0)\\ \sigma_j>0}}e^{\sigma_j|t|} \\ +\sum_{\substack{\mu_l^2\in\sigma_p(P)\cap(0,\infty)\\ \mu_l>0}}(e^{i\mu_l t}+e^{-i\mu_l t})+m(0),\ t\neq 0,\ d \text{ odd.} \end{aligned} \tag{3.12}$$

Noting that in odd dimensions λ_j is a resonance of a self-adjoint operator if and only if $-\overline{\lambda}_j$ is a resonance gives

$$u(t)=\sum_{\lambda_j\in\mathcal{R},\ \lambda_j\neq 0}e^{-i\lambda_j|t|}+m(0),\ t\neq 0$$

showing the consistency with the odd-dimensional Poisson formula.

Now we return to the case of even dimension d. Theorem 1 of [43] is, when stated using our notion

Theorem 3.7. — *([43, Theorem 1] adapted) Let d be even, P be a self-adjoint operator satisfying the black-box conditions, u the wave trace defined in* (1.3)*, and $0<\rho<\pi$. Let $s(\lambda)=\det S(\lambda)$, and $\psi\in C_c^\infty(\mathbb{R};[0,1])$ be equal to* 1 *near* 0*. Then*

$$u(t)=\sum_{\lambda_j\in\mathcal{R},-\rho/2<\arg\lambda\leqslant 0}(e^{-i\lambda_j|t|}+e^{i\overline{\lambda_j}|t|})+\sum_{-\sigma_j^2\in\sigma_p(P)\cap(-\infty,0),\sigma_j>0}e^{\sigma_j|t|}$$
$$+m(0)+\frac{1}{\pi i}\int_0^\infty\psi(\lambda)\frac{s'(\lambda)}{s(\lambda)}\cos(t\lambda)d\lambda+v_{\rho,\psi}(t),\ t\neq 0$$

with

$$v_{\rho,\psi}\in C^\infty(\mathbb{R}\setminus\{0\}),\ \partial_t^k v_{\rho,\psi}(t)=O(t^{-N})\ \forall k,N,\ |t|\to\infty.$$

For the reader comparing the statement of [43, Theorem 1] with this statement, we note that there are several differences. One is caused by the convention of the location of the physical half plane (here, $0<\arg\lambda<\pi$; in [43], $-\pi<\arg\lambda<0$) and the resulting difference in the location of the resonances. Another is caused Zworski's convention (see the first paragraph of [43, Section 2]); the diagram should have a cut extending along the entire imaginary axis) *defining*, for $\operatorname{Re}\lambda<0$, $s'(\lambda)/s(\lambda)=-\overline{s'(-\overline{\lambda})/s(-\overline{\lambda})}$. This means that each resonance with $-\rho/2<\arg\lambda_j\leqslant 0$ actually contributes twice to the sum which appears in [43, Theorem 1] (once $e^{-i|t|\lambda_j}$, and then again $e^{-(-i|t|\overline{\lambda_j})}$).

In each of Theorems 3.6 and 3.7, any term which does not arise from an eigenvalue or resonance may be considered part of a "remainder." The remainder terms from Theorem 3.7 are smooth away from $t=0$; and one ($v_{\rho,\psi}$) is well-controlled when $|t|\to\infty$. The smoothness of the remainder in [43, Theorem 1] means that Zworski's Poisson formula can be used to show that if the wave trace has singularities at a *nonzero* time then there is a lower bound on the number of resonances in sectors near the real axis, see [43]. See [44] for another application of the Poisson formula of [43], also related to the singularities of the wave trace away from 0.

The integral appearing in Theorem 3.6 does not appear to generally yield a term which is smooth in t, even away from $t=0$. However, the remainder term has the advantage of being in some sense more explicit than that of Theorem 3.7. As we shall see in the next section, Lemma 2.3 provides enough information about the remainder in Theorem 3.6 to use the singularity of $u(t)$ at 0 to prove Theorem 4.1 below, and hence Theorems 1.1 and 1.2.

4. Proof of Theorems 1.1 and 1.2

We first prove a more general result, Theorem 4.1, and then show that Theorems 1.1 and 1.2 satisfy the hypotheses of Theorem 4.1.

Theorem 4.1. — *Let the dimension d be even and let P be a black-box operator satisfying the conditions of* [31]*; see Section 1.1. Suppose there is an $R_1 > R_0$ so that for all $b > a > R_1$, if $\chi \in C_c^\infty(\mathbb{R}^d)$ has support in $\{x \in \mathbb{R}^d : a < |x| < b\}$ then there are constants C_0, m_0 depending on χ so that*

$$\|\chi R(\lambda)\chi\| \leqslant C_0 \lambda^{m_0}, \quad \lambda \in (1, \infty). \tag{4.1}$$

Let $n_e(r)$ denote the eigenvalues of P of norm at most r^2, counted with multiplicity, and assume that

$$n_e(r) + n_{-1}(r) \leqslant C'(1 + r^d) \text{ for some } C' > 0.$$

Let u be the distribution defined in (1.3)*. Suppose there is a constant $\alpha \neq 0$ and $\epsilon_1,\ \epsilon_2 > 0$ so that*

$$t^{d-\epsilon_1}\left(u(t) - \alpha |D_t|^{d-1}\delta_0(t)\right) \in C^0([0, \epsilon_2]). \tag{4.2}$$

Then there is a constant $C_0 > 0$ so that

$$r^d / C_0 \leqslant n_{-1}(r) + n_e(r) \text{ for } r \gg 1.$$

Now we specialize to the case of P as in the statement of Theorem 4.1.

Set, for $t \neq 0$,

$$w(t) = u(t) - \sum_{\lambda_j \in \mathcal{R} \cap \Lambda_{-1}} \left(e^{-i\lambda_j |t|} + e^{i\overline{\lambda_j}|t|}\right) - \sum_{\substack{\mu_l^2 \in \sigma_p(P) \cap (0,\infty) \\ \mu_l > 0}} (e^{i\mu_l t} + e^{-i\mu_l t}). \tag{4.3}$$

Lemma 4.2. — *Let $\phi \in C_c^\infty((0,\infty))$ and set $\phi_\gamma(t) = \frac{1}{\gamma}\phi\left(\frac{t}{\gamma}\right)$, $\gamma > 0$. Then for P satisfying the conditions of Theorem 4.1 and with w as defined by* (4.3) *there is a constant $C > 0$ so that*

$$\left|\int w(t)\phi_\gamma(t)dt\right| \leqslant C\gamma^{-(d-1)} \text{ for } \gamma \in (0, 1].$$

Proof. — We use Theorem 3.6, and note that according to the theorem there are three terms to bound. We begin with the simplest to bound:

$\int \phi_\gamma(t)m(0)dt = m(0)\int \phi(t)dt$ is independent of γ. Moreover, recalling that there are at most finitely many negative eigenvalues

$$\begin{aligned}
&\left| \int \sum_{\substack{-\sigma_j^2\in\sigma_p(P)\cap(-\infty,0)\\ \sigma_j>0}} (e^{\sigma_j|t|} - e^{-\sigma_j|t|})\phi_\gamma(t)dt \right| \\
&\qquad = \left| \sum_{\substack{-\sigma_j^2\in\sigma_p(P)\cap(-\infty,0)\\ \sigma_j>0}} \int_0^\infty \phi(t)(e^{\gamma\sigma_j t} - e^{-\gamma\sigma_j t})dt \right| \\
&\qquad \leqslant \sum_{\substack{-\sigma_j^2\in\sigma_p(P)\cap(-\infty,0)\\ \sigma_j>0}} \int_0^\infty \phi(t)|e^{\gamma\sigma_j t} - e^{-\gamma\sigma_j t})|dt \\
&\qquad \leqslant C \text{ for } \gamma\in(0,1]. \qquad (4.4)
\end{aligned}$$

It remains to bound the term corresponding to the integral appearing in the Poisson formula of Theorem 3.6. Since for $\arg\lambda = 0$, $\det S(\lambda e^{i\pi}) \neq 0$, there is a differentiable function g_1 defined on $(0,\infty)$ so that $s(\lambda e^{i\pi}) = e^{g_1(\lambda)}$ when $\arg\lambda = 0$. There is still some flexibility in defining g_1. Since $\lim_{\lambda\downarrow 0} s(\lambda e^{i\pi}) = 1$, we may choose g_1 to satisfy $\lim_{\lambda\downarrow 0} g_1(\lambda) = 0$. Using the relation $S^*(\overline{\lambda})S(\lambda) = I$,

$$\frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})} = -\overline{\left(\frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})}\right)} \text{ if } \arg\lambda = 0.$$

Hence

$$\frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})} = -g_1'(\lambda), \text{ if } \arg\lambda = 0 \text{ and } \frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})} = \overline{g}_1'(\lambda), \text{ if } \arg\lambda = 0.$$

Thus, for $t > 0$

$$\begin{aligned}
\int_0^\infty \left(e^{-i\lambda t}\frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})} + e^{i\lambda t}\frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})} \right) d\lambda \qquad (4.5)\\
= -\int_0^\infty \left(e^{-i\lambda t} g_1'(\lambda) - e^{i\lambda t}\overline{g}_1'(\lambda)\right) d\lambda
\end{aligned}$$

For $\tau\in\mathbb{R}$, set

$$g_2(\tau) = \begin{cases} g_1(\tau) & \text{if } \tau > 0 \\ \overline{g}_1(-\tau) & \text{if } \tau < 0 \\ 0 & \text{if } \tau = 0. \end{cases}$$

Note that g_2 is continuous, with

$$g_2'(\tau) = \begin{cases} g_1'(\tau) & \text{if } \tau > 0 \\ -\overline{g}_1'(-\tau) & \text{if } \tau < 0. \end{cases}$$

From (4.5) and using the continuity of g_2,

$$\begin{aligned}\int_0^\infty \left(e^{-i\lambda t} \frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})} + e^{i\lambda t} \frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})} \right) d\lambda &= -\int_{-\infty}^\infty e^{-i\tau t} g_2'(\tau) d\tau \\ &= -\widehat{g_2'}(t),\ t > 0.\end{aligned}$$

Thus

$$\begin{aligned}\int_{-\infty}^\infty \phi_\gamma(t) \int_0^\infty \left(e^{-i\lambda t} \frac{s'(\lambda e^{i\pi})}{s(\lambda e^{i\pi})} + e^{i\lambda t} \frac{s'(\lambda e^{-i\pi})}{s(\lambda e^{-i\pi})} \right) d\lambda dt \\ = -\int_{-\infty}^\infty \phi_\gamma(t) \widehat{g_2'}(t) dt \\ = -\int_{-\infty}^\infty \hat{\phi}(\gamma\tau) g_2'(\tau) d\tau \\ = \int_{-\infty}^\infty \gamma \hat{\phi}'(\gamma\tau) g_2(\tau) d\tau.\end{aligned}$$

By Lemma 2.3, there is a constant C so that $|g_1(\lambda)| \leqslant C(1+|\lambda|^{d-1})$ for $\arg\lambda = 0$. Thus

$$\begin{aligned}\left| \int_{-\infty}^\infty \gamma \hat{\phi}'(\gamma\tau) g_2(\tau) d\tau \right| &= \left| \int_{-\infty}^\infty \hat{\phi}'(\tau) g_2(\tau/\gamma) d\tau \right| \\ &\leqslant \int_{-\infty}^\infty C(1+|\tau|)^{-d-1}(1+|\tau/\gamma|)^{d-1} d\tau \\ &\leqslant C\gamma^{-(d-1)} \int_{-\infty}^\infty (1+|\tau|)^{-2} d\tau \leqslant C\gamma^{-(d-1)}. \qquad (4.6)\end{aligned}$$

Together with Theorem 3.6, (4.4), and the boundedness of the contribution of $m(0)$, this proves the lemma. ☐

Lemma 4.3. — *Let d be even, and let P and u satisfy the hypotheses of Theorem 4.1. Let $\phi \in C_c^\infty((0,\infty))$, $\phi \geqslant 0$, $\phi(1) \neq 0$. Then there are constants $c > 0$ and $\gamma_0 > 0$ so that*

$$\left| \int \phi_\gamma(t) u(t) dt \right| \geqslant c\, |\alpha| \gamma^{-d} \text{ if } \gamma \in (0, \gamma_0]$$

where α is as in (4.2).

Proof. — We recall the assumption that there are $\epsilon_1,\ \epsilon_2 > 0$ so that

$$t^{d-\epsilon_1} \left(u(t) - \alpha |D_t|^{d-1} \delta_0(t) \right) \in C^0([0, \epsilon_2]). \qquad (4.7)$$

There is a constant $b \neq 0$ so that for $t > 0$, $|D_t|^{d-1}\delta_0(t) = bt^{-d}$. Thus

$$(4.8)\quad \left|\int_{-\infty}^{\infty} \phi_\gamma(t)|D_t|^{d-1}\delta_0(t)dt\right| = \left|b\int_0^\infty \frac{1}{\gamma}\phi(t/\gamma)t^{-d}dt\right| = \gamma^{-d}\left|b\int_0^\infty \phi(t)t^{-d}dt\right| \geqslant c'\gamma^{-d}$$

with $c' > 0$.

Set $w_r(t) = u(t) - \alpha|D_t|^{d-1}\delta_0(t)$. Since by (4.7) $t^{d-\epsilon_1}w_r(t)$ is continuous for $t \in [0, \epsilon_2]$, there is a constant C so that $|\phi_\gamma(t)w_r(t)| \leqslant Ct^{-(d-\epsilon_1)}\phi_\gamma(t)$ when $\gamma > 0$ is sufficiently small. Thus for $\gamma > 0$ sufficiently small,

$$(4.9)\quad \left|\int \phi_\gamma(t)w_r(t)dt\right| \leqslant \int Ct^{-(d-\epsilon_1)}\phi_\gamma(t)dt \leqslant C\gamma^{-(d-\epsilon_1)}\int \phi_\gamma(t)dt \leqslant C\gamma^{-d+\epsilon_1}.$$

The lemma follows from (4.7), (4.8), and (4.9), by choosing $\gamma_0 > 0$ sufficiently small. □

Proof of Theorem 4.1. — Theorem 3.6 together with Lemmas 4.2 and 4.3 have shown that if

$$\tilde{w}(t) = \sum_{\lambda_j \in \mathcal{R}\cap\Lambda_{-1}} \left(e^{-i\lambda_j|t|} + e^{i\overline{\lambda_j}|t|}\right) + \sum_{\substack{\mu_l^2 \in \sigma_p(P)\cap(0,\infty)\\ \mu_l > 0}} (e^{i\mu_l t} + e^{-i\mu_l t}),\ t \neq 0$$

then

$$\left|\int \phi_\gamma(t)\tilde{w}(t)dt\right| \geqslant c|\alpha|\gamma^{-d}/2 > 0 \text{ if } \gamma \in (0, \gamma_0]$$

for some $c,\ \gamma_0 > 0$. Here ϕ_γ is as in Lemma 4.3. The theorem now follows almost immediately from an application of [32, Proposition 4.2]. Here we use, in the notation of that proposition, $V(r) = r^d$. (This V has no relation to the potential V in the statement of Theorem 1.2.) We use our assumption that $n_{-1}(r) = O(r^d)$ and $n_e(r) = O(r^d)$ as $r \to \infty$. □

Proof of Theorems 1.1 and 1.2. — If M has no boundary, then there is an $\epsilon > 0$ and a constant $\tilde{c}_d \neq 0$ so that

$$(4.10)\quad t^{d-1}\left(u(t) - \tilde{c}_d(\mathrm{vol}K - \mathrm{vol}B(0;R_0))|D_t|^{d-1}\delta_0(t)\right) \in C^\infty([0,\epsilon]),$$

see [13]. In case M has a boundary (in particular, if $M = \mathbb{R}^d \setminus \mathcal{O}$), that the distribution in (4.10) is continuous follows from [16, 25] or [14, Prop. 29.1.2 and the proof of Prop. 29.3.3].

By results of Burq [4, Theorem 4] for the case of $M = \mathbb{R}^d \setminus \mathcal{O}$ or Cardoso–Vodev [5] for $P = -\Delta_g$ on more general manifolds M, we have that (4.1) holds. We note that with the assumptions we have made on $\mathcal{O}$ and M

the operator P has no positive eigenvalues, and only finitely many negative eigenvalues. Using the upper bound of [38, 39], $n_{-1}(r) = O(r^d)$. Then Theorem 1.2 (which implies Theorem 1.1) follows immediately from Theorem 4.1. $\square$

BIBLIOGRAPHY

[1] C. Bardos, J.-C. Guillot & J. Ralston, "La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion", *Comm. Partial Differential Equations* **7** (1982), no. 8, p. 905-958.

[2] J. T. Beale, "Purely imaginary scattering frequencies for exterior domains", *Duke Math. J.* **41** (1974), p. 607-637.

[3] D. Borthwick, T. J. Christiansen, P. D. Hislop & P. A. Perry, "Resonances for manifolds hyperbolic near infinity: optimal lower bounds on order of growth", *Int. Math. Res. Not. IMRN* **2011** (2011), no. 19, p. 4431-4470.

[4] N. Burq, "Lower bounds for shape resonances widths of long range Schrödinger operators", *Amer. J. Math.* **124** (2002), no. 4, p. 677-735.

[5] F. Cardoso & G. Vodev, "Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II", *Ann. Henri Poincaré* **3** (2002), no. 4, p. 673-691.

[6] T. J. Christiansen, "Lower bounds for resonance counting functions for obstacle scattering in even dimensions", to appear in *Amer. J. Math.*

[7] T. J. Christiansen & P. D. Hislop, "Some remarks on resonances in even-dimensional Euclidean scattering", *Trans. Amer. Math. Soc.* **368** (2016), no. 2, p. 1361-1385.

[8] S. Dyatlov & M. Zworski, "Mathematical theory of scattering resonances", Book in preparation. `http://math.mit.edu/~dyatlov/res/res.pdf`.

[9] J.-P. Eckmann & C.-A. Pillet, "Spectral duality for planar billiards", *Comm. Math. Phys.* **170** (1995), no. 2, p. 283-313.

[10] I. C. Gohberg & M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, R.I., 1969, xv+378 pages.

[11] N. V. Govorov, *Riemann's boundary problem with infinite index*, Operator Theory: Advances and Applications, vol. 67, Birkhäuser Verlag, Basel, 1994, Edited and with an introduction and an appendix by I. V. Ostrovskiĭ, Translated from the 1986 Russian original by Yu. I. Lyubarskiĭ, xii+252 pages.

[12] L. Guillopé & M. Zworski, "Scattering asymptotics for Riemann surfaces", *Ann. of Math.* **145** (1997), no. 3, p. 597-660.

[13] L. Hörmander, "The spectral function of an elliptic operator", *Acta Math.* **121** (1968), p. 193-218.

[14] ———, *The analysis of linear partial differential operators. IV*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 275, Springer-Verlag, Berlin, 1985, Fourier integral operators, vii+352 pages.

[15] ———, *The analysis of linear partial differential operators. I*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1990, Distribution theory and Fourier analysis, xii+440 pages.

[16] V. J. Ivriĭ, "The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary", *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 2, p. 25-34.

[17] L. Jin, "Scattering resonances of convex obstacles for general boundary conditions", *Comm. Math. Phys.* **335** (2015), no. 2, p. 759-807.

[18] P. D. Lax & R. S. Phillips, "Decaying modes for the wave equation in the exterior of an obstacle.", *Comm. Pure Appl. Math.* **22** (1969), p. 737-787.

[19] ———, "The time delay operator and a related trace formula", in *Topics in functional analysis (essays dedicated to M. G. Kreĭn on the occasion of his 70th birthday)*, Adv. in Math. Suppl. Stud., vol. 3, Academic Press, New York-London, 1978, p. 197-215.

[20] A. Lechleiter & S. Peters, "Analytical characterization and numerical approximation of interior eigenvalues for impenetrable scatterers from far fields", *Inverse Problems* **30** (2014), no. 4, p. 045006, 22.

[21] B. Y. Levin, *Distribution of zeros of entire functions*, revised ed., Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman, xii+523 pages.

[22] R. B. Melrose, "Scattering theory and the trace of the wave group", *J. Funct. Anal.* **45** (1982), no. 1, p. 29-40.

[23] ———, "Polynomial bound on the number of scattering poles", *J. Funct. Anal.* **53** (1983), no. 3, p. 287-303.

[24] ———, "Polynomial bound on the distribution of poles in scattering by an obstacle", *Journées équations aux dérivées partielles* (1984), p. 1-8.

[25] ———, "The trace of the wave group", in *Microlocal analysis (Boulder, Colo., 1983)*, Contemp. Math., vol. 27, Amer. Math. Soc., Providence, RI, 1984, p. 127-167.

[26] ———, "Weyl asymptotics for the phase in obstacle scattering", *Comm. Partial Differential Equations* **13** (1988), no. 11, p. 1431-1439.

[27] V. Petkov & L. Stoyanov, "Sojourn times of trapping rays and the behavior of the modified resolvent of the Laplacian", *Ann. Inst. H. Poincaré Phys. Théor.* **62** (1995), no. 1, p. 17-45.

[28] V. Petkov & M. Zworski, "Semi-classical estimates on the scattering determinant", *Ann. Henri Poincaré* **2** (2001), no. 4, p. 675-711.

[29] G. Popov, "On the contribution of degenerate periodic trajectories to the wave-trace", *Comm. Math. Phys.* **196** (1998), no. 2, p. 363-383.

[30] D. Robert, "On the Weyl formula for obstacles", in *Partial differential equations and mathematical physics (Copenhagen, 1995; Lund, 1995)*, Progr. Nonlinear Differential Equations Appl., vol. 21, Birkhäuser Boston, Boston, MA, 1996, p. 264-285.

[31] J. Sjöstrand & M. Zworski, "Complex scaling and the distribution of scattering poles", *J. Amer. Math. Soc.* **4** (1991), no. 4, p. 729-769.

[32] ———, "Lower bounds on the number of scattering poles. II", *J. Funct. Anal.* **123** (1994), no. 2, p. 336-367.

[33] ———, "Asymptotic distribution of resonances for convex obstacles", *Acta Math.* **183** (1999), no. 2, p. 191-253.

[34] P. Stefanov, "Quasimodes and resonances: sharp lower bounds", *Duke Math. J.* **99** (1999), no. 1, p. 75-92.

[35] ———, "Sharp upper bounds on the number of the scattering poles", *J. Funct. Anal.* **231** (2006), no. 1, p. 111-142.

[36] S.-H. Tang, "Existence of resonances for metric scattering in even dimensions", *Lett. Math. Phys.* **52** (2000), no. 3, p. 211-223.

[37] G. Vodev, "Sharp bounds on the number of scattering poles for perturbations of the Laplacian", *Comm. Math. Phys.* **146** (1992), no. 1, p. 205-216.

[38] ———, "Sharp bounds on the number of scattering poles in even-dimensional spaces", *Duke Math. J.* **74** (1994), no. 1, p. 1-17.

[39] ———, "Sharp bounds on the number of scattering poles in the two-dimensional case", *Math. Nachr.* **170** (1994), p. 287-297.

[40] M. Zworski, "Sharp polynomial bounds on the number of scattering poles", *Duke Math. J.* **59** (1989), no. 2, p. 311-323.

[41] ———, "Sharp polynomial bounds on the number of scattering poles of radial potentials", *J. Funct. Anal.* **82** (1989), no. 2, p. 370-403.

[42] ———, "Poisson formulae for resonances", in *Séminaire sur les Équations aux Dérivées Partielles, 1996–1997*, École Polytech., Palaiseau, 1997, Exp. No. XIII, 14 p.

[43] ———, "Poisson formula for resonances in even dimensions", *Asian J. Math.* **2** (1998), no. 3, p. 609-617.

[44] ———, "A remark on: "Inverse resonance problem for $\mathbb{Z}_2$-symmetric analytic obstacles in the plane" [in Geometric methods in inverse problems and PDE control, 289–321, Springer, New York, 2004] by S. Zelditch", *Inverse Probl. Imaging* **1** (2007), no. 1, p. 225-227.

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