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CONVERGENCE AND COUNTING IN INFINITE MEASURE

by Françoise DAL’BO, Marc PEIGNÉ, Jean-Claude PICAUD & Andrea SAMBUSETTI (*)

Abstract. — We construct non-uniform convergent lattices \( \Gamma \) of pinched, negatively curved Hadamard spaces, in any dimension \( N \geq 2 \). These lattices are exotic, by which we mean that they have a maximal parabolic subgroup \( P < \Gamma \) such that \( \delta(P) = \delta(\Gamma) \). We also give examples of divergent, non-uniform exotic lattices in dimension \( N = 2 \). Finally, we consider a particular class of such exotic lattices, with infinite Bowen–Margulis measure and whose cusps have a particular asymptotic profile (satisfying a “heavy tail condition”), and we give precise estimates of their orbital function; namely, we show that their orbital function is lower exponential with asymptotic behaviour \( \asymp e^{\delta_\Gamma R} R^{-1 - \kappa L(R)} \), for a slowly varying function \( L \).

Résumé. — Nous construisons des réseaux non uniformes et convergents \( \Gamma \) d’isométries d’une variété d’Hadamard à courbure strictement négative et pincée de dimension \( N \geq 2 \) quelconque. Ces réseaux sont dits exotiques, au sens où ils possèdent des sous-groupes paraboliques maximaux \( P < \Gamma \) d’exposant critique \( \delta(P) = \delta(\Gamma) \). Nous donnons aussi des exemples explicites de réseaux exotiques non uniformes et divergents en dimension \( N = 2 \). Enfin, nous étudions une classe particulières de tels réseaux exotiques non uniformes et divergents dont la mesure de Bowen–Margulis est infinie et dont les “cusps” présentent un profil asymptotique particulier, satisfaisant une propriété de “queue lourde”, et proposons une estimation précise du comportement asymptotique de leur fonction orbitale; plus précisément, nous montrons que leur fonction orbitale croît de façon sous-exponentielle avec un comportement à l’infini de la forme \( \asymp e^{\delta_\Gamma R} R^{-1 - \kappa L(R)} \), où \( L \) est une fonction à variations lentes.

Keywords: Poincaré exponent, convergent/divergent groups, Bowen–Margulis measure, orbital function.


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1. Introduction

Let $\Gamma$ be a Kleinian group, i.e. a discrete, torsionless group of isometries of a Hadamard space $X$ of negative, pinched curvature $-B^2 \leq K_X \leq -A^2 < 0$, with quotient $\tilde{X} = \Gamma \backslash X$. This paper is concerned with two mutually related problems:

Problem 1. — The description of the distribution of the orbits of $\Gamma$ on $X$, namely of fine asymptotic properties of the orbital function:

$$v_\Gamma(x, y; R) := \# \{ \gamma \in \Gamma / d(x, \gamma \cdot y) \leq R \}$$

for $x, y \in X$. This has been the subject of many investigations since Margulis' [24] (see Roblin's book [28] and Babillot's report on [1] for a clear overview). The motivations to understand the behaviour of the orbital function are numerous: for instance, a simple but important invariant is its exponential growth rate

$$\delta_\Gamma = \limsup_{R \to \infty} \frac{1}{R} \ln(v_\Gamma(x, y; R))$$

which has a major dynamical significance, since it coincides with the topological entropy of the geodesic flow when $\tilde{X}$ is compact, and is related to many interesting rigidity results and characterization of locally symmetric spaces, cf. [6, 8, 22].

Problem 2. — The pointwise behaviour of Poincaré series associated with $\Gamma$:

$$P_\Gamma(x, y, s) := \sum_{\gamma \in \Gamma} e^{-s d(x, \gamma \cdot y)}, \quad x, y \in X$$

at a neighborhood of $s = \delta_\Gamma$, which coincides with its exponent of convergence. The group $\Gamma$ is said to be convergent if $P_\Gamma(x, y, \delta_\Gamma) < \infty$, and divergent otherwise. Divergence can also be understood in terms of dynamics as, by Hopf–Tsuju–Sullivan theorem, it is equivalent to ergodicity and total conservativity of the geodesic flow with respect to the Bowen–Margulis measure $m_\Gamma$ on the unit tangent bundle $UX$ (see again [28] for a complete account and a definition of $m_\Gamma$).

The regularity of the asymptotic behaviour of $v_\Gamma$, in full generality, is well expressed in Roblin’s results, which trace back to Margulis’ work in the compact case:

Theorem 1.1 (Margulis [24] - Roblin [27, 28]). — Let $X$ be a Hadamard manifold with pinched negative curvature and $\Gamma$ a non elementary, discrete
subgroup of isometries of $X$ with non-arithmetic length spectrum\(^{(1)}\), then the exponential growth rate $\delta_\Gamma$ is a true limit and one gets:

1. If $\|m_\Gamma\| = \infty$ then $v_\Gamma(x, y; R) = o(e^{\delta_\Gamma R})$,
2. If $\|m_\Gamma\| < \infty$ then $v_\Gamma(x, y; R) \sim \|\mu_x\|\|\mu_y\| e^{\delta_\Gamma R}$,

where $(\mu_x)_{x \in X}$ denotes the family of Patterson $\delta_\Gamma$-conformal densities of $\Gamma$, and $m_\Gamma$ the Bowen–Margulis measure on $\bar{U}$.

Here, $f \sim g$ means that $f(t)/g(t) \to 1$ when $t \to \infty$; we will write $f \preceq g$ when $\frac{1}{c} \leq f(t)/g(t) \leq c$ for $c \geq 1$ and $t \gg 0$ (or just $f \asymp g$ when the constant $c$ is not specified). The best asymptotic regularity to be expected is the existence of an equivalent, as in (2); an explicit computation of the second term in the asymptotic development of $v_\Gamma$ is a difficult question for locally symmetric spaces (and almost a hopeless question in the general Riemannian setting).

Theorem 1.1 shows that the key assumption for a regular behaviour of $v_\Gamma$ is that the Bowen–Margulis measure $m_\Gamma$ is finite. This condition is clearly satisfied for uniform lattices $\Gamma$ (i.e. when $\bar{X} = X/\Gamma$ is compact), and more generally for groups $\Gamma$ such that $m_\Gamma$ has compact support (e.g., convex cocompact groups), but it may fail for nonuniform lattices, that is when $\bar{X} = X/\Gamma$ has finite volume but is not compact.

The finiteness of $m_\Gamma$ has a nice geometrical description in the case of geometrically finite groups. Recall that any orbit $\Gamma \cdot x$ accumulates on a closed subset $\Lambda_\Gamma$ of the geometric boundary $\partial X$ of $X$, called the limit set of $\Gamma$; the group $\Gamma$ (or the quotient manifold $\bar{X}$) is said to be geometrically finite if $\Lambda_\Gamma$ decomposes in the set of radial limit points (the limit points $\xi$ which are approached by orbit points in the $M$-neighborhood of any given ray issued from $\xi$, for some $M < \infty$) and the set of bounded parabolic points (those $\xi$ fixed by some parabolic subgroup $P$ acting cocompactly on $\partial X \setminus \{\xi\}$); for a complete study of geometrical finiteness in variable negative curvature see [10] and [28, Proposition 1.10], and for a description of their topology at infinity see [16]. A finite-volume manifold $\bar{X}$ is a particular case of geometrically finite manifold: it can be decomposed into a compact set and finitely many cusps $\bar{C}_i$, i.e. topological ends of $\bar{X}$ of finite volume which are quotients of a horoball $H_{\xi_i}$ centered at a bounded parabolic point $\xi_i \in \partial X$ by a maximal parabolic subgroup $P_i \subset \Gamma$ fixing $\xi_i$.

The principle ruling the regularity of the orbital function $v_\Gamma$ of nonuniform lattices, as pointed out in [11] and in the following papers of the

\(^{(1)}\) It means that the set $L(\bar{X}) = \{\ell(\gamma) : \gamma \in \Gamma\}$ of lengths of all closed geodesics of $\bar{X} = \Gamma \backslash X$ is not contained in a discrete subgroup of $\mathbb{R}$.
authors [13, 14, 15], is that the orbital functions $v_{P_i}$ (defined in a similar way as $v_{\Gamma}$) capture the relevant information on the wildness of the metric inside the cusps, which in turn may imply $\|m_{\Gamma}\| = \infty$ and the irregularity of $v_{\Gamma}$. In this regard, distinctive properties of the group $\Gamma$ and of its maximal parabolic subgroups $P_i$ are their type (convergent or divergent, as defined above) and the critical gap property (CGP), i.e. if $\delta_{P_i} < \delta_{\Gamma}$ for all $i$. Actually, in [11] it is proved that, for geometrically finite groups $\Gamma$, the divergence of $P_i$ implies $\delta_{P_i} < \delta_{\Gamma}$, and that the critical gap property implies that the group $\Gamma$ is divergent, with $\|m_{\Gamma}\| < \infty$. On the other hand there exist geometrically finite groups $\Gamma$ which do not satisfy the CGP: we call such groups exotic, and we say that a cusp is dominant if it is associated to a parabolic subgroup $P$ with $\delta_P = \delta_\Gamma$. Geometrically finite, exotic groups may as well be convergent or divergent: in the first case, they always have $\|m_{\Gamma}\| = \infty$ (by Poincaré recurrence and Hopf-Tsuji-Sullivan theorem, as $\|m_{\Gamma}\| < \infty$ implies total conservativity); in the second case, in [11] it is proved that the finiteness of $m_{\Gamma}$ depends on the convergence of the first moment series

$$\sum_{p \in P_i} d(x, p \cdot x)e^{-\delta_\Gamma d(x, p \cdot x)} < +\infty.$$ 

The main aim of this paper is to present examples of lattices $\Gamma$ for which $v_{\Gamma}$ has an irregular asymptotic behaviour. According to our discussion, we will then focus on exotic, non-uniform lattices.

The convergence property of exotic lattices is of interest to detect the precise asymptotic behaviour of the growth function and the existence of a Margulis’ function for the space $X$ (as explained in [13]). However, it is an interesting question on its own: while uniform lattices (as well as convex-cocompact groups) always are divergent, the only known examples of convergent groups, to the best of our knowledge, are given in [11] and have infinite covolume. The constraint of the finiteness of the volume of $X$ creates additional technical complications and needs more sophisticated tricks (namely, a sharp control of the spectral radius of a continuous family of transfer operators), as we will see in Section 5.

The first result of the paper is to show that both convergent and divergent exotic lattices do exist. Actually, in Section 3, by a variation of the construction in [11] we obtain:

**Theorem 1.2.** — For any $N \geq 2$, there exist $N$-dimensional, finite volume manifolds of pinched negative curvature whose fundamental group $\Gamma$ is (exotic and) convergent.
Constructing exotic, divergent lattices is more subtle. We prove in Section 5:

**Theorem 1.3.** — There exist non compact finite area surfaces of pinched negative curvature whose group $\Gamma$ is exotic and divergent.

We stress the fact that the examples of Theorem 1.2 have infinite Bowen–Margulis measure; on the other hand, the surfaces of Theorem 1.3 can have finite or infinite Bowen–Margulis measure, according to the prescribed behaviour of the metric in the cusps. Moreover, we believe that the assumption on the dimension in Theorem 1.3 is just technical, but at present we are not able to construct similar examples in dimension $N \geq 3$.

Finally, in Section 6 we address the initial question about how far from purely exponential the orbital function can be, giving estimates for the orbital function of a large family of exotic lattices with infinite Bowen–Margulis measure:

**Theorem 1.4.** — Let $\kappa \in ]1/2, 1[$. There exist non compact finite area surfaces with pinched negative curvature whose fundamental group $\Gamma$ satisfies the following asymptotic property: for any $x, y \in X$

$$v_\Gamma(x, y; R) \asymp \frac{e^{\delta_\Gamma R}}{R^{1-\kappa} L(R)}$$

as $R \to +\infty$

for some slowly varying function\(^{(2)}\) $L : \mathbb{R}^+ \to \mathbb{R}^+$.\(^{(2)}\)

We state in fact a general, but more technical, result on the orbital function of divergent exotic lattices whose cusps have a asymptotic profile satisfying a “heavy tail condition” (see Theorem 6.1 and conditions $H_0 - H_3$). As far as we know, except for some precise asymptotic formulas established by Pollicott and Sharp [26] for the orbital function of normal subgroups $\Gamma$ of a cocompact Kleinian group (hence, groups which are far from being geometrically finite or with finite Bowen–Margulis measure), these are the only examples of such precise asymptotic behaviour for the orbital function of Kleinian groups with infinite Bowen–Margulis measure.

The proof of Theorem 1.4 relies on the study of some singular integral, already done in [19] in the context of the renewal theory for non negative and infinite mean random walks, whose distribution satisfies some tail condition involving a certain parameter $\kappa \in ]0, 1[$. For technical reasons, K.B. Erickson is forced to assume $\kappa \in ]1/2, 1[$; it explains why this restrictive assumption also appears since we follow his proof in the analysis of

\(^{(2)}\) A function $L(t)$ is said to be “slowly varying” or “of slow growth” if it is positive, measurable and $L(\lambda t)/L(t) \to 1$ as $t \to +\infty$ for every $\lambda > 0$.\(^{(2)}\)
the singular integral which appears in our context. The cases $\kappa \in [0, 1/2]$ and $\kappa = 1$ may be studied, under additive conditions and using a different and very technical approach initiated recently by several authors (cite for instance [17], [21] and references therein); we refer to [30] for this counting problem and related questions for (infinite covolume) Schottky groups, where the coding is explicit and easier to work with.

Remark. — This work should be considered as a companion paper to [13] and [14], where we study the asymptotic properties of the integral version of $v_\Gamma$, i.e. the growth function of $X$:

$$v_X(x; R) := \text{vol}_X(B(x, R)).$$

In [13], we obtain optimal conditions on the geometry on the cusps in order that there exists a Margulis function, that is a $\Gamma$-invariant function $c : X \to \mathbb{R}^+$ such that

$$v_X(x; R) \sim c(x) e^{\omega_X R} \quad \text{for } R \to +\infty$$

where $\omega_X$ is the exponential growth rate of the function $v_X$ (the integral analogue of $\delta_\Gamma$). Notice that $\omega_X$ can be different from $\delta_\Gamma$, also for (non uniform) lattices, as we showed in [14].

2. Geometry of negatively curved manifolds with finite volume

2.1. Landscape

Additionally to those given in the introduction, we present here notations and familiar results about negatively curved manifolds. Amongst good references we suggest [4, 5, 18] and others specifically related to this work, [14, 23]. In the sequel, $\bar{X} = X/\Gamma$ is a $N$-dimensional complete connected Riemannian manifold with metric $g$ whose sectional curvatures satisfy:

$$-B^2 \leq K_X \leq -A^2 < 0$$

for fixed constants $A$ and $B$; we will assume $0 < A \leq 1 \leq B$ since in most examples $g$ will be obtained by perturbation of a hyperbolic one and the curvature will equal $-1$ on large subsets of $\bar{X}$.

The family of normalized distance functions:

$$d(x_0, x) - d(x, \cdot)$$

converges uniformly on compact subsets to the Busemann function $B_\xi(x_0, \cdot)$ for $x \to \xi \in \partial X$. The horoballs $H_\xi$ (resp. the horospheres $\partial H_\xi$) centered at
\( \xi \) are the sup-level sets (resp. the level sets) of the function \( B_\xi(x_0, \cdot) \). Given a horosphere \( \partial H_\xi \) passing through a point \( x \), we also set, for all \( t \in \mathbb{R} \),

\[
\mathcal{H}_\xi(t) := \{ y / B_\xi(x_0, y) \geq B_\xi(x_0, x) + t \}
\]

resp. \( \partial H_\xi(t) := \{ y / B_\xi(x_0, y) = B_\xi(x_0, x) + t \} \). We will refer to \( t = B_\xi(x_0, y) - B_\xi(x_0, x) \) as to the height of \( y \) (or of the horosphere \( \partial H_\xi(t) \)) in \( \mathcal{H}_\xi \). Also, when no confusion is possible, we will drop the index \( \xi \in \partial X \) denoting the center of the horoball. Recall that the Busemann function satisfies the fundamental cocycle relation

\[
B_\xi(x, z) = B_\xi(x, y) + B_\xi(y, z)
\]

which will be crucial in the following.

An origin \( x_0 \in X \) being fixed, the Gromov product between \( x, y \in \partial X \cong S^{n-1}, x \neq y \), is defined as

\[
(x \mid y)_{x_0} = \frac{B_x(x_0, z) + B_y(x_0, z)}{2}
\]

where \( z \) is any point on the geodesic \( (x, y) \) joining \( x \) to \( y \); then, for any \( 0 < \kappa^2 \leq A^2 \) the expression

\[
D(x, y) = e^{-\kappa(x \mid y)_{x_0}}
\]

defines a distance on \( \partial X \), cp [9]. Recall that for any \( \gamma \in \Gamma \) one gets

\[
D(\gamma \cdot x, \gamma \cdot y) = e^{-\frac{B_x(\gamma^{-1} \cdot x_0, x_0)}{2}} e^{-\frac{B_y(\gamma^{-1} \cdot x_0, x_0)}{2}} D(x, y).
\]

In other words, the isometry \( \gamma \) acts on \( (\partial X, D) \) as a conformal transformation with coefficient of conformality \( |\gamma'(x)| = e^{-\kappa B_x(\gamma^{-1} \cdot x_0, x_0)} \) at \( x \), since equality (2.1) may be rewritten

\[
D(\gamma \cdot x, \gamma \cdot y) = \sqrt{|\gamma'(x)| |\gamma'(y)|} D(x, y).
\]

Recall that \( \Gamma \) is a torsion free nonuniform lattice acting on \( X \) by hyperbolic or parabolic isometries. For any \( \xi \in \partial X \), denote by \( (\psi_{\xi, t})_{t \geq 0} \) the radial semi-flow defined as follows: for any \( x \in X \), the point \( \psi_{\xi, t}(x) \) lies on the geodesic ray \( [x, \xi] \) at distance \( t \) from \( x \). By classical comparison theorems on Jacobi fields (see for instance [23]), the differential of \( \psi_{\xi, t} : \partial H_\xi \to \partial H_\xi(t) \) satisfies \( e^{-B t \|v\|} \leq \|d\psi_{\xi, t}(v)\| \leq e^{-A t \|v\|} \) for any \( t \geq 0 \) and any vector \( v \) in the tangent space \( T(\partial H_\xi) \); consequently, if \( \mu_t \) is the Riemannian measure induced on \( \partial H_\xi(t) \) by the metric of \( X \), we have, for any Borel set \( F \subset \partial H_\xi \)

\[
e^{-B(N-1)t} \mu_0(F) \leq \mu_t(\psi_{\xi, t}(F)) = \int_F |Jac(\psi_{\xi, t})(x)| d\mu_0(x)
\]

\[
\leq e^{-A(N-1)t} \mu_0(F).
\]
As $\tilde{X} = \Gamma \backslash X$ is non compact and $\text{vol}(\tilde{X}) < \infty$, the manifold $\tilde{X}$ can be decomposed into a disjoint union of a relatively compact subset $\tilde{K}$ and finitely many cusps $\tilde{C}_1, \ldots, \tilde{C}_l$, each of which is a quotient of a horoball $\mathcal{H}_{\xi_i}$, centered at some boundary point $\xi_i$, by a maximal parabolic subgroup $P_i$. As a consequence of Margulis’ lemma, we can choose the family $(\mathcal{H}_{\xi_i})_{1 \leq i \leq l}$ so that any two $\Gamma$-translates of the $\mathcal{H}_{\xi_i}$ are either disjoint or coincide (cf. [28, Proposition 1.10]); we call $(\mathcal{H}_{\xi_i})_{1 \leq i \leq l}$ a fundamental system of horospheres for $X$. Accordingly, the Dirichlet domain $D$ of $\Gamma$ centered at the base point $x_0$ can be decomposed into a disjoint union $D = K \cup C_1 \cup \cdots \cup C_l$, where $K$ is a convex, relatively compact set containing $x_0$ in its interior and projecting to $\tilde{K}$, and the $C_i$ are connected fundamental domains for the action of $P_i$ on $\mathcal{H}_{\xi_i}$, projecting to $\tilde{C}_i$. We let $S_i = D \cap \partial \mathcal{H}_{\xi_i}$ be the corresponding, relatively compact fundamental domain for the action of $P_i$ on $\partial \mathcal{H}_{\xi_i}$, so that $C_i = D \cap \mathcal{H}_{\xi_i} \simeq S_i \times \mathbb{R}_+$. Fixing an end $\tilde{C}$, and omitting in what follows the index $i$, let $\mu_t$ be the Riemannian measure induced by the Riemannian metric on the horosphere $\partial \mathcal{H}_\xi(t)$ corresponding to $\tilde{C}$. In [14] we defined the horospherical area function associated with the cusp $\tilde{C}$ as:

$$A(t) = \mu_t(P\backslash \partial \mathcal{H}_\xi(t)) = \mu_t(\psi_{\xi,t}(S)).$$

This function depends on the choice of the initial horosphere $\partial \mathcal{H}_\xi$ for the end $\tilde{C}$, and the following result shows that this dependence is unessential for our counting problem:

**Proposition 2.1** ([14]). — There exists a constant $c = c(A, B, \text{diam}(S))$ such that

$$v_P(R) \lesssim \frac{1}{A(R/2)}.$$

This weak equivalence is the key to relate the irregularity of the metric in the cusp to the irregular asymptotic behaviour of the orbital function of $P$. The second crucial step of our work will then to describe precisely the contribution of $v_P$ in the asymptotic behaviour of $v_\Gamma$ assuming $\delta_P = \delta_\Gamma$.

### 2.2. Cuspidal geometry

The strategy to construct examples with irregular orbital functions as in Theorems 1.2, 1.3 and 1.4 is to perturb in a suitable manner the metric of a finite volume hyperbolic manifold $\Gamma \backslash \mathbb{H}^N$ in one cuspidal end $P\backslash \mathcal{H}$. If $\mathcal{H} = \{y/B_\xi(x_0, y) \geq t_0\}$, the hyperbolic metric writes on $\mathcal{H} \simeq \partial \mathcal{H} \times \mathbb{R}_+$.
Let \( \mathbb{R}_+ \equiv \mathbb{R}^N \times \mathbb{R}_+ \) as \( g = e^{-(t-t_0)}dx^2 + dt^2 \) in the horospherical coordinates \( y = (x, t) \), where \( dx^2 \) denotes the induced flat Riemannian metric of \( \partial \mathcal{H} \) and \( t = B_\xi(x_0, y) - t_0 \). We will consider a new metric \( g \) in \( P \setminus \mathcal{H}_\xi \) whose lift to \( \mathcal{H} \) writes, in the same coordinates, as
\[
g = \tau^2(t)dx^2 + dt^2.
\]
We extend this metric by \( \Gamma \)-invariance to \( \Gamma \mathcal{H} \) and produce a new Hadamard space \((X, g)\) with quotient \( \bar{X} = \Gamma \setminus X \). The new manifold \( \bar{X} \) has again finite volume, provided that
\[
\int_0^{+\infty} \tau^{N-1}(t)dt < \infty
\]
and the end \( \bar{C} = P \setminus \mathcal{H} \) is a new cusp; we call the function \( \tau \) the analytic profile of the cusp \( \bar{C} \).

The horospherical area function \( A \) associated with the profile \( \tau \) satisfies \( A \asymp \tau^{N-1} \); by Proposition 2.1, it implies that:

(a) the parabolic group \( P \) has critical exponent \( \delta_P = \frac{(N-1)\omega}{2} \), for \( \omega := \limsup_{R \to +\infty} \frac{1}{R} |\ln(\tau(R))| \);
(b) \( P \) is convergent if and only if \( \int_0^{+\infty} \frac{e^{-\omega T N^{-1}}(N-1)}{\tau N^{-1}}(N-1)^t dt < \infty \).

Also, notice that the sectional curvatures at \((x, t)\) are given by
\[
K_{(x, t)}( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} ) = -\left( \frac{\tau'}{\tau} \right)^2
\]
and \( K_{(x, t)}( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t} ) = - \frac{\tau''}{\tau} \) (see [7]).

In Sections 4, 5 and 6 we will apply this strategy to prescribe bounds on curvature and analytic profiles \( \tau \) at certain depths, depending on additional real parameters \( a, b, \eta \), defined as follows.

For any convex function \( \tau : \mathbb{R} \to \mathbb{R}_+ \) with \( \int_0^{+\infty} \tau^{N-1}(t)dt < \infty \) and satisfying the conditions
\begin{align}
\forall \ t \leq t_0 \quad & \tau(t) = e^{-(t-t_0)} \\
A^2 < \tau''/\tau < B^2 \\
\omega_\tau = \limsup_{t \to +\infty} \frac{|\ln(\tau(t))|}{t} < B
\end{align}
we will set, \( a \geq t_0 \)
\[
\tau_a(t) = e^{-a}\tau(t-a) \quad \text{for} \  t \in \mathbb{R}.
\]
This profile defines a manifold \( \bar{X} \) with a cusp \( \bar{C} \) which is hyperbolic at height less than \( a \) and then has (renormalized) profile equal to \( \tau \).
Moreover, given parameters \( a \geq t_0, b \geq 0 \) and \( \eta \in ]0, A[ \), a straightforward calculus proves the existence of a profile \( \tau_{a,b,\eta} \) such that

\[
\begin{align*}
A^2 - \eta &< \frac{\tau''_{a,b,\eta}(t)}{\tau_{a,b,\eta}(t)} < B^2 + \eta & \text{for all } t, \\
\tau_{a,b,\eta}(t) &= e^{-t} & \text{for } t \leq a, \\
\tau_{a,b,\eta}(t) &= e^{-\omega_\tau t} & \text{for } t \in [\Delta + a, \Delta + a + b], \\
\tau_{a,b,\eta}(t) &= e^{-(2\Delta + a + b) \tau(t - (2\Delta + a + b))} & \text{for } t \geq 2\Delta + a + b.
\end{align*}
\]

for some constant \( \Delta = \Delta(A,B,\eta) \gg 0 \). The profile \( \tau_{a,b,\eta} \) induces the geometry of the cusp \( \bar{C} \) which is hyperbolic till height \( a \), with constant curvature \( -\omega^2_\tau \) in a band of width \( b \) at height \( \Delta + a \), and which then has the same asymptotic invariants as the initial profile \( \tau \) after height \( 2\Delta + a + b \).

3. Proof of Theorem 1.2: construction of convergent lattices

An example of manifold of negative curvature with infinite volume and whose fundamental group is convergent is due to Dal’Bo–Otal–Peigné [11]. We propose in this section a variation of their argument to produce a convergent nonuniform lattice. We will consider a finite volume hyperbolic manifold \( \Gamma \backslash \mathbb{H}^N \) with one cuspidal end \( P \backslash \mathcal{H} \) and deform the metric in this end as explained before to obtain a new Hadamard space \( (X,g) \) such that the quotient \( \bar{X} = \Gamma \backslash X \) has finite volume and a dominant cusp \( \bar{C} = P \backslash \mathcal{H} \) with a convergent parabolic group \( P \), whose exponent \( \delta_P \):

(1) is greater than the Poincaré exponent of \( \Gamma \) acting on \( \mathbb{H}^N \), that is \( N - 1 \);

(2) equals the Poincaré exponent \( \delta_\Gamma \) of \( \Gamma \) corresponding to the new metric.

For this, we choose \( \tau \) satisfying the conditions (2.3), (2.4), (2.5) with \( \delta_P = \frac{(N-1)\omega_\tau}{2} > (N-1) \) and \( \int_0^{+\infty} \frac{e^{-\omega_\tau t^{(N-1)t}}}{t}\,dt < \infty \), and we consider the profile \( \tau_a \) for some \( a > 0 \) to be precised.

Remark that the first condition can be satisfied only if \( \omega_\tau > 2 \) which requires \( B^2/A^2 > 4 \). We will denote by \( d \) the distance on \( (X,g) \) corresponding to this new metric, and by \( d_0 \) the hyperbolic distance. We emphasize that the perturbation of the metric will not change neither the algebraic structure of the groups \( \Gamma \) and \( P \), nor the horospheres \( \mathcal{H}(t) \) (which are only modified in size and not as subsets of \( \mathbb{H}^n \)) and their radial flow; however, the orbital functions \( v_P \) and \( v_\Gamma \) have different behaviour before and after perturbation.
Now, we need to introduce a natural decomposition of geodesic segments according to their excursions in the cusp, which will enable us to encode elements of $\Gamma$ by sequences of parabolic elements travelling far in the cusp and elements staying in a fixed, compact subset. We use the same notations as in 2.1 for the compact subset $\mathcal{K}$, the fundamental horosphere $\mathcal{H} = \mathcal{H}_{\xi_i}$ of $X$, and the Borel fundamental domain $\mathcal{S}$ for the action of $P$ on $\partial \mathcal{H}$. Let $h > 0$: for every $\gamma \in \Gamma$, the geodesic segment $[x_0, \gamma \cdot x_0]$ intersects $r = r(\gamma)$ disjoint translates $g \cdot \mathcal{H}(h)$ (with the convention $r = 0$ if the intersection with $\bigcup_{g \in \Gamma} g \cdot \mathcal{H}(h)$ is empty). In case $r \geq 1$, denote by $z_1^-, \ldots, z_r^-$ (resp. $z_1^+, \ldots, z_r^+$) the hitting (resp. exit) points of the oriented geodesic segment $[x_0, \gamma \cdot x_0]$ with translates of $\mathcal{H}(h)$ in this order. Hence we get

$$[x_0, \gamma \cdot x_0] \cap \left( \bigcup_{g \in \Gamma} g \cdot \mathcal{H}(h) \right) = [z_1^-, z_1^+] \cup \cdots \cup [z_r^-, z_r^+] .$$

Accordingly, when $r \geq 1$ we can define the points $y_i^-, y_i^+ \in [x_0, g \cdot x_0]$ such that, for any $1 \leq i < r$, the geodesic segment $[y_i^-, y_i^+]$ is the connected component of $[x_0, \gamma \cdot x_0] \cap \left( \bigcup_{g \in \Gamma} g \cdot \mathcal{H} \right)$ containing $[z_i^-, z_i^+]$. We also set $y_0^+ = x_0$ and $y_{r+1}^- = \gamma \cdot x_0$.

With these notations, there exist uniquely determined isometries $g_1, \ldots, g_r \in \Gamma$ and $p_1, \ldots, p_r \in P$ such that $y_i^- \in g_1 \cdot \mathcal{S}$, $y_i^+ \in g_1 p_1 \cdot \mathcal{S}$, $\ldots$, $y_r^+ \in g_1 p_1 \ldots g_r p_r \cdot \mathcal{S}$. Finally, we define $g_{r+1}$ by the relation

$$\gamma = g_1 p_1 \cdots g_r p_r g_{r+1}$$

which we call the horospherical decomposition of $\gamma$ at height $h$. Notice that this decomposition depends only on the initial hyperbolic metric. We also set $x_0^+ = x_0, x_{r+1}^- = \gamma \cdot x_0$ and $x_i^- = g_1 p_1 \cdots g_i \cdot x_0, x_i^+ = g_1 p_1 \cdots g_i p_i \cdot x_0$ for $1 \leq i \leq r$. We then have:

**Lemma 3.1.** Let $\gamma = g_1 p_1 \cdots p_r g_{r+1}$ be the horospherical decomposition of $\gamma$ at height $h$:

\begin{enumerate}
  \item for every $i \in \{1, \ldots, r + 1\}$ the geodesic segments $[x_{i-1}^+, x_i^-]$ and $[x_0, g_i \cdot x_0]$ lie outside the set $\bigcup_{g \in \Gamma} g \cdot \mathcal{H}(c)$, for $c = \text{diam}(\mathcal{K})$;
  \item for every $i \in \{1, \ldots, r\}$ the geodesic segments $[x_0, p_i \cdot x_0]$ have length greater than $2(h - c)$ and lie outside the set $\bigcup_{g \in \Gamma} g \cdot \mathcal{H}(h - c)$.
\end{enumerate}

**Proof.** Assume $r \geq 1$ and fix $1 \leq i \leq r + 1$. By construction, each geodesic segment $[y_{i-1}^+, y_i^-]$ lies outside $\bigcup_{g \in \Gamma} g \cdot \mathcal{H}$. Then, each segment $[x_{i-1}^+, x_i^-]$ lies outside $\bigcup_{g \in \Gamma} g \cdot \mathcal{H}(c)$ since $d(x_{i-1}^+, y_{i-1}^+)$ and $d(x_i^-, y_i^-)$ are both smaller than $c$. Since $[x_{i-1}^+, x_i^-] = g_1 p_1 \cdots g_{i-1} p_{i-1} \cdot [x_0, g_i \cdot x_0]$ and the set $\bigcup_{g \in \Gamma} g \cdot \mathcal{H}(c)$ is $\Gamma$-invariant, the same holds for the segment $[x_0, g_i \cdot x_0]$. 

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To prove statement (ii), notice that the segment \([y_i^- , y_i^+]\) intersects the set \(\bigcup_{g \in \Gamma} g . H (h)\) and has endpoints in \(\partial H\), so that \(d(y_i^-, y_i^+) \geq 2h\); one concludes using the facts that \(d(x_i^-, y_i^-)\) and \(d(x_i^+, y_i^+)\) are both smaller than \(c\) and that \([x_i^-, x_i^+] = g_1 p_1 \ldots p_{i-1} g_i \cdot [x_0 , p_i \cdot x_0] \).

Moreover, the distance function is almost additive, with respect to this decomposition:

**Lemma 3.2.** There exists a constant \(C > 0\) such that for all \(\gamma \in \Gamma\) with horospherical decomposition \(\gamma = g_1 p_1 \ldots g_r p_r g_{r+1}\) at height \(h\):

\[
d(x_0 , \gamma \cdot x_0) \geq \sum_{i=1}^{r+1} d(x_0 , g_i \cdot x_0) + \sum_{i=1}^{r} d(x_0 , p_i \cdot x_0) - r C.
\]

**Proof.** With the above notations one gets, for \(C := 4 \text{diam}(K)\)

\[
d(x_0 , \gamma \cdot x_0) = \sum_{i=1}^{r+1} d(y_{i-1}^+ , y_i^-) + \sum_{i=1}^{r} d(y_i^- , y_i^+) \\
\geq \sum_{i=1}^{r+1} d(x_{i-1}^+ , x_i^-) + \sum_{i=1}^{r} d(x_i^- , x_i^+) - 4r \text{diam}(K) \\
= \sum_{i=1}^{r+1} d(x_0 , g_i \cdot x_0) + \sum_{i=1}^{r} d(x_0 , p_i \cdot x_0) r C.
\]

Now assume \(a > \max(h , \text{diam}(K))\) and let \(P_a = \{ p \in P / d_0 (x_0 , p \cdot x_0) \geq 2a \}\) and \(\Gamma_a = \{ \gamma \in \Gamma / [x_0 , \gamma \cdot x_0] \cap g . H (a) = \emptyset \text{ for all } g \in \Gamma\}\). Let \(\gamma \in \Gamma\) with horospherical decomposition \(\gamma = g_1 p_1 \ldots g_r p_r g_{r+1}\) at height \(h\). By Lemma 3.1, the geodesic segments \([x_0 , g_i \cdot x_0]\), \(1 \leq i \leq r+1\), stay outside the perturbed set \(\bigcup_{g \in \Gamma} g . H (a)\), so that \(g_i \in \Gamma_a\) and \(d(x_0 , g_i \cdot x_0) = d_0 (x_0 , g_i \cdot x_0)\).

Consequently, the Poincaré series of \(\Gamma\) for the perturbed metric is

\[
P_\Gamma (x_0 , \delta) \leq 1 + \sum_{\Gamma_a \setminus \{ 1 \} d} e^{- \delta d (x_0 , \gamma \cdot x_0)} + \sum_{p \in P_a} e^{- \delta d (x_0 , p \cdot x_0)} \\
+ \sum_{r \geq 1} \sum_{g_1 , \ldots , g_{r+1} \in \Gamma_a} e^{- \delta d (x_0 , g_1 \ldots g_r p_{r+1} \cdot x_0)} \\
\leq \sum_{\gamma \in \Gamma} e^{- \delta d_0 (x_0 , \gamma \cdot x_0)} + \sum_{p \in P_a} e^{- \delta d (x_0 , p \cdot x_0)} \\
+ \sum_{r \geq 1} \left( e^{C \delta} \sum_{g \in \Gamma_a} e^{- \delta d (x_0 , g \cdot x_0)} \sum_{p \in P_a} e^{- \delta d (x_0 , p \cdot x_0)} \right)^r.
\]
\[ \leq \sum_{\gamma \in \Gamma} e^{-\delta d_0(x_0, \gamma \cdot x_0)} + \sum_{p \in P_a} e^{-\delta d(x_0, p \cdot x_0)} + \sum_{r \geq 1} \left( e^{C\delta} \sum_{\gamma \in \Gamma} e^{-\delta d_0(x_0, \gamma \cdot x_0)} \sum_{p \in P_a} e^{-\delta d(x_0, p \cdot x_0)} \right)^r \]

For \( \delta = \delta_P \) the term \( \sum_{\gamma \in \Gamma} e^{-\delta d_0(x_0, \gamma \cdot x_0)} < +\infty \) since \( \delta_P > N-1 \). On the other hand, the group \( P \) being convergent with respect to the new metric \( g \), we deduce that \( \sum_{p \in P_a} e^{-\delta_P d(x_0, p \cdot x_0))} \to 0 \) as \( a \to +\infty \); we may thus choose \( a \) large enough so that

\[ e^{C\delta} \sum_{\gamma \in \Gamma} e^{-\delta_P d_0(x_0, \gamma \cdot x_0)} \sum_{p \in P_a} e^{-\delta_P d(x_0, p \cdot x_0)} < 1 \]

which readily implies \( P_1(x_0, \delta_P) < +\infty \), hence \( \delta_P \geq \delta_\Gamma \). As \( P \) is a subgroup of \( \Gamma \), this implies that \( \delta_P = \delta_\Gamma \), hence \( \Gamma \) is a convergent group. \( \square \)

4. Critical gap property versus divergence

In this section we start constructing a hyperbolic lattice \( \Gamma \) of \( \mathbb{H}^2 \) which is generated by suitable parabolic isometries, so that the resulting surface \( \Gamma \backslash \mathbb{H}^2 \) has finite volume. In the disk model of the hyperbolic plane, we choose \( r \geq 2 \) and \( 2r \) boundary points \( \xi_0 = \eta_0, \xi_1, \eta_1, \ldots, \xi_r, \eta_r = \xi_0 \) of \( S^1 = \partial \mathbb{B}^2 \) in cyclic order, and consider (uniquely determined) parabolic isometries \( p_1, \ldots, p_r \) such that for \( 1 \leq i \leq r \) we have \( p_i \cdot \xi_i = \xi_i \) and \( p_i \cdot \eta_{i-1} = \eta_i \). We remark that all \( \eta_i \) belong to the \( \Gamma \) orbit of the point \( \eta_0 \), which is a parabolic fixed point of the isometry \( \rho_0 := p_r p_{r-1} \ldots p_1 \).

**Proposition.** — The group \( \Gamma = \langle p_1, \ldots, p_r \rangle \) is a free non abelian group over \( p_1, \ldots, p_r \). The quotient \( \Gamma \backslash \mathbb{B}^2 \) is a finite surface with \( r + 1 \) cuspidal ends, with a cusp \( \mathcal{C}_i \) for each parabolic subgroup \( \mathcal{P}_i = \langle p_i \rangle \) for \( i = 1, \ldots, r \), and another cusp \( \mathcal{C}_0 \) corresponding to the parabolic subgroup \( \mathcal{P}_0 = \langle \rho_0 \rangle \) fixing \( \xi_0 \).

Each element \( \gamma \in \Gamma \backslash \{ \text{Id} \} \) can be written in a unique way as a word with letters in the alphabet \( \mathcal{A} := \{ p_1^{\pm 1}, \ldots, p_r^{\pm 1} \} \); namely, one gets

\[ \gamma = p_{j_1}^{\epsilon_1} \ldots p_{j_n}^{\epsilon_n} \]

with \( p_{j_1}^{\epsilon_1}, \ldots, p_{j_n}^{\epsilon_n} \in \mathcal{A}, n \geq 1 \) and with adjacent letters which are not inverse to each other. Such expression with respect to the natural (but not canonical) choice of the alphabet \( \mathcal{A} \) is called a coding of elements of \( \Gamma \). We will call \( j_1 \) the first index of \( \gamma \), denoted by \( i_\gamma \); similarly \( j_n \) the last index and is denoted by \( l_\gamma \).
4.1. A new coding for elements of $\Gamma$

We code here the elements of $\Gamma$ by blocks, with some admissibility rules to be precised. This new coding is designed to obtain a contraction property for an operator that will be introduced and studied in the next sections whose restriction to some suitable space of functions present remarkable spectral properties.

We first rewrite the decomposition (4.1) as follows

$$\gamma = p_{i_1}^{\ell_1} p_{i_2}^{\ell_2} \ldots p_{i_m}^{\ell_m}$$

with $m \geq 1, \ell_1, \ldots, \ell_m \in \mathbb{Z}^*$ and $i_j \neq i_{j+1}$ for $1 \leq j < m$. When all the $\ell_j, 1 \leq j \leq m$, belong to $\{\pm 1\}$, one says that $\gamma$ is a level 1 word; the set of such words is denoted by $W_1$. Then, we select all the $\ell_j, 1 \leq j \leq m$, with $|\ell_j| \geq 2$ and write $\gamma$ as

$$\gamma = p_{j_0}^{l_0} Q_{j_1}^{l_1} p_{j_2}^{l_2} \ldots p_{j_k}^{l_k} Q_{j_{k+1}}^{l_{k+1}}$$

for $k \geq 0$, with:

- $|l_1|, \ldots, |l_k| \geq 2$,
- $|l_0|, |l_k| \neq 1$,
- each $Q_i$ is either the identity element or a level 1 word, with $i_{Q_i} \neq j_i-1$ and $l_{Q_i} \neq j_i$.

The decomposition by blocks (4.3) is still unique ; it only uses letters from the new alphabet

$${\mathfrak{B}} := \widehat{\mathcal{P}}_1 \cup \cdots \cup \widehat{\mathcal{P}}_r \cup \mathcal{W}_1$$

where $\widehat{\mathcal{P}}_i := \{p_{i}^n / |n| \geq 2\}$ for $1 \leq i \leq r$, possibly with $p_{i_0}^{l_0} = 1$ or $p_{i_r}^{l_r} = 1$. Notice that this decomposition does not depend on the metric obtained by perturbation inside the cusps, but only on the presentation chosen for $\Gamma$.

We will call blocks the letters of this new alphabet, and say that a word $\beta_1 \ldots \beta_m$ in the alphabet $\mathfrak{B}$ is admissible if the last letter (for the original alphabet $\mathfrak{A}$) of any block $\beta_i$ is different from the first one of $\beta_{i+1}$ for $1 \leq i \leq m-1$. So, any $\gamma \in \Gamma \setminus \{\text{Id}\}$ can be written as a finite, admissible word $\beta_1 \ldots \beta_m$ on $\mathfrak{B}$; the ordered sequence of the $\beta_i$’s is called the $\mathfrak{B}$-decomposition of $\gamma$ and the number $m$ of blocks is denoted by $|\gamma|_{\mathfrak{B}}$. Finally, we denote by $\Sigma_{\mathfrak{B}}$ the set of all finite admissible words with respect to $\mathfrak{B}$.

4.2. A new metric in the cusps

We consider a fundamental system of horoballs $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_r$ centered respectively at the parabolic points $\xi_0, \xi_1, \xi_2, \ldots, \xi_r$ and such that all
the horoballs $\gamma \cdot \mathcal{H}_i$, for $\gamma \in \Gamma, 0 \leq i \leq r$, are disjoint or coincide, as in Subsection 2.1. Then, we modify the hyperbolic metric in the cuspidal ends $\bar{C}_i = P_i \setminus \mathcal{H}_i$ as follows. We choose positive constants $a_0, a_1, \ldots, a_{r-1}, a_r, b$ and $\eta$, functions $\tau_0, \ldots, \tau_{r-1}$ and $\tau_r$ as in Subsection 2.2 such that

$$\omega_{\tau_r} = \max(\omega_{\tau_0}, \ldots, \omega_{\tau_r}) > 1$$

and we prescribe the profile $\tau_{i,a_i}$ for the $i$-th cusp $\bar{C}_i$ for $0 \leq i \leq r - 1$, and the profile $\tau_{r,a_r,b}$ on the last dominant cusp $\bar{C}_r = \bar{C}$. This yields a new surface $X = (\mathbb{H}^2, g_{a_0, \ldots, a_r, b})$, with quotient $\bar{X} = \Gamma \setminus X$ of finite area. Since the metric on $X$ depends, in particular, on the value of the parameter $b$, which will play a role in what follows, we shall denote the induced distances on $X$, $\partial X \cong S^1$ and the conformal factor respectively by $d_b, D_b$ and $|\cdot|_b$; on the other hand, we shall omit the index $b$ in the Busemann function and in the Gromov product, to simplify notations. The dependence of $D_b$ on the parameter $b$ is described by the following lemma, whose proof can be found in [25]:

**Lemma 4.1.** — Let $b_0 > 0$ be fixed. There exists $c \geq 1$ and $\alpha \in [0, 1]$ such that the family of distances $(D_b)_{0 \leq b \leq b_0}$, are Hölder equivalent; namely for all $b \in [0, b_0]$ we have

$$\frac{1}{c} D_0^{1/\alpha} \leq D_b \leq c D_0^{\alpha}.$$  

**4.3. Ping-pong by blocks.**

For any $1 \leq i \leq r$, we consider the sub-arcs $I_i := [\eta_{i-1}, \eta_i]$ and $I'_i := [p_i^{-1} \cdot \eta_{i-1}, p_i \cdot \eta_i]$ of $S^1$ containing $\xi_i$. There exists a ping-pong dynamic between these intervals: namely, for any block $\beta \in \mathcal{B}$, we have

- if $\beta \in \mathcal{W}_1$, then $\beta \cdot I'_i \subset I_{i,\beta}$ for any $i \neq l_\beta$
- if $\beta \in \bar{\mathcal{P}}_l$ with $l = i_\beta = l_\beta$, then $\beta \cdot I_i \subset I'_i$ for any $i \neq l$.

Moreover, for any $\gamma$ with $\mathcal{B}$-decomposition $\beta_1 \ldots \beta_m$, we define a compact subset $K_\gamma \subset S^1$ as follows:

- $K_\gamma = \cup_{i \neq l_\gamma} I'_i$, if $\beta_m \in \mathcal{W}_1$
- $K_\gamma = \cup_{i \neq l_\gamma} I_{i,\gamma}$, if $\beta_m \in \bar{\mathcal{P}}_l$ with $l = l_\gamma$.

Then, using the fact that the closure of the sets $I'_i$ and $\partial X \setminus I_i$ are disjoint, one gets:

**Lemma 4.2.** — There exists a constant $C = C(A, \eta) > 0$ such that

$$d_b(x_0, \gamma \cdot x_0) - C \leq B_x(\gamma^{-1} \cdot x_0, x_0) \leq d_b(x_0, \gamma \cdot x_0)$$

for any $\gamma \in \Gamma$ and any $x \in K_\gamma$. 

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Which yield to:

**Corollary.** — There exist a real number \( r \in ]0,1[ \) and a constant \( C > 0 \) such that for any \( \gamma \in \Gamma \) with length \( |\gamma|_\mathcal{B} = k \), one gets

\[
\forall x \in K_\gamma \quad |\gamma'(x)|_0 \leq Cr^k.
\]

**Proof.** — See [3, Prop. 2.2]. \( \square \)

4.4. Coding for the limit points.

An infinite word on the alphabet \( \mathcal{B} \), i.e. an infinite sequence \( \bar{\beta} = (\beta_n)_{n \geq 1} \) of elements of \( \mathcal{B} \) is called *admissible* if any finite subword \( \beta_1 \ldots \beta_m \) is admissible ; the set of such words is denoted by \( \Sigma^+_{\mathcal{B}} \). Corollary 4.3 implies in particular the following fundamental fact:

**Lemma 4.3.** — For any \( \bar{\beta} \in \Sigma^+_{\mathcal{B}} \), the sequence \((\beta_1 \ldots \beta_n \cdot x_0)_{n \geq 0}\) converges to some point \( \pi(\bar{\beta}) \in \partial X \); the map \( \pi : \Sigma^+_{\mathcal{B}} \to \partial X \) is one-to-one, and its image \( \pi(\Sigma^+_{\mathcal{B}}) \) coincides with the subset \( \partial_0 X := \partial X \setminus \left( \bigcup_{i=1}^r \Gamma \cdot \xi_i \cup \Gamma \cdot W_1 \right) \) where \( W_1 := \overline{W_1 \cdot x_0} \cap \partial X \).

Notice that, if \( \gamma \) has \( \mathcal{B} \)-decomposition \( \beta_1 \ldots \beta_m \), then the subset \( K_\gamma \) defined in 4.3 is the closure of the subset corresponding, via the coding map \( \pi \), to the infinite sequences \( \bar{\beta} = (\beta_n)_{n \geq 1} \) such that the concatenation \( \gamma \ast \bar{\beta} = (\beta_1 \ldots \beta_m \beta_1 \ldots \beta_i \ldots) \) is admissible. We also set \( J_\gamma := \text{Cl}\{ \pi(\gamma \ast \bar{\beta}) \mid \bar{\beta} \in K_\gamma \} \), that is the closure of points corresponding to admissible sequences obtained by concatenation with the \( \mathcal{B} \)-decomposition of \( \gamma \).

As indicated previously, this coding by blocks is of interest since the classical shift operator on \( \Sigma^+_{\mathcal{B}} \) induces locally, exponentially expanding maps \( T^n \) on \( \partial_0 X \); the map \( T \), described for instance in details in [12], has countably many inverse branches, each of them acting by contraction on some subset of \( \partial X \). Namely, we consider on \( \Sigma^+_{\mathcal{B}} \) the natural shift \( \theta \) defined by

\[
\theta(\bar{\beta}) := (\beta_{k+1})_{k \geq 1}, \quad \forall \bar{\beta} = (\beta_k)_{k \geq 1} \in \Sigma^+_{\mathcal{B}}
\]

This map induces a transformation \( T : \partial_0 X \to \partial_0 X \) via the coding \( \pi \); moreover, \( T \) can be extended to the whole \( \partial X \) by setting, for any \( \gamma \) with \( \mathcal{B} \)-decomposition \( \gamma = \gamma_1 \gamma_2 \ldots \gamma_n \) and \( \xi \in \{\xi_1, \ldots, \xi_r\} \cup W_1 \)

\[
T(\gamma, \xi) := \gamma_2 \ldots \gamma_n \cdot \xi
\]

and \( T(\xi) := \xi \). Then, for every block \( \beta \in \mathcal{B} \), the restriction of \( T \) to \( J_\beta \) is the action by \( \beta^{-1} \); by the dynamic described above, the inverse branches of the map \( T \) have the following property:
**Proposition.** — There exist $0 < r < 1$ and a constant $C > 0$ such that, for any $\gamma \in \Gamma$ with $|\gamma|_B = k$ and for $x, y \in K_\Gamma$ we have

$$D_0(\gamma \cdot x, \gamma \cdot y) \leq Cr^k D_0(x, y).$$

This property is crucial for the investigation of the spectral properties of the transfer operator, which will be introduced in the next section.

### 5. Existence of divergent exotic lattices

This section is devoted to prove Theorem 1.3: there exist two dimensional exotic and divergent lattices. Existence of infinite covolume exotic and divergent discrete groups has already been provided in [25], by the following procedure.

Given a Schottky group $\Gamma = <h, p> \subset Isom^+(\mathbb{H}^n)$ with $h$ hyperbolic and $p$ parabolic, one fixes an horoball $\mathcal{H}_0$ centered at the fixed point of $p$, hence a neighborhood $<p>\backslash \mathcal{H}_0$ of the cusp associated with $p$. Then, one chooses a profile $\tau$ which does not modify the critical exponent $\delta_p$ of $\langle p \rangle$ and makes the group $\Gamma$ a convergent lattice of a new manifold $X$. It can then be shown that there exists a critical value $a^*$ such that

1. for any $a > a^*$ the metric perturbed by $\tau_a$ in the cusp (more explicitly the metric whose profile is $\tau$ beyond the height $a$ - see again Subsection 2.2) makes $\Gamma$ non exotic, hence divergent,
2. the one given by $\tau_{a^*}$ makes $\Gamma$ exotic and divergent.

In the first case, the divergence comes from the contribution of elements of $\Gamma_a \subset \Gamma$ corresponding to geodesic loops staying at height less than $a$ in the cusp, which is preponderant in the value of the Poincaré series of $\Gamma$ since $\delta_p$ is strictly less than the critical exponent of $\Gamma \subset Isom^+(\mathbb{H}^n)$.

Here, we adapt this approach to obtain a discrete group $\Gamma$ with finite covolume, in dimension 2. We start from the surface $\bar{X} = \Gamma \backslash X$ with $r + 1$ cusps described in 4.2, with a dominant cusp $\bar{C}_r = P_r \backslash \mathcal{H}_r$ and make $\Gamma$ convergent by choosing $a_0, \ldots, a_r \gg 0$, as in Theorem 1.2. Besides a different coding by blocks (due to the generators which are all parabolic) which gives a slightly different expression for the transfer operator associated to $\Gamma$, the main difficulty here is to show that $\Gamma$ can be made divergent. This cannot be achieved now by simply pushing the perturbation far away in the cusp, since in our case $\delta_{P_r}$ is strictly greater than the critical exponent of the subset $\Gamma_a$ of elements staying in the compact, non-perturbed part; so, even choosing the $a_i \gg 0$, the group $\Gamma$ remains convergent! To obtain the divergence we rather modify $\bar{C}_r$ with a profile $\tau_{r,a_r,b}$ which equals the
profile of a cusp with constant curvature metric $-\omega_2^2 \tau_r$ on a sufficiently large band of width $b \gg 0$. With those conventions, we will prove the existence of $b^*$ such that:

1. $\Gamma$ with the metric perturbed by $\tau_{a,b}$ is non exotic and divergent for every $b > b^*$,

2. $\Gamma$ with the metric perturbed by $\tau_{a,b^*}$ is exotic and still divergent.

The approach to obtain the existence of the critical value $b^*$ is to consider a family of transfer operators $(\mathcal{L}_{b,z})_{b,z}$ associated with the transformation $T$, the latter being described at the end of Section 4 and depending on $b$. The continuity of $b \mapsto L_{b,z}$ and $b \mapsto \rho(L_{b,z})$ where $\rho(L_{b,z})$ is the spectral radius of $L_{b,z}$ in some suitable functional Banach space is a key point of the approach.

5.1. On the spectrum of transfer operators

The map $T$ encodes a large part of the action of the group $\Gamma$ on $\partial_0 X$; from an analytic point of view, this dynamic may be described throughout the family $(\mathcal{L}_{b,z})_{b,z}$ of transfer operators associated with $T$, which takes into account the different inverse branches of $T$ and is defined formally by:

for any Borel bounded function $\varphi : \partial X \to \mathbb{R}$ and any $x \in \partial_0 X$ and $z \in \mathbb{C}$

\begin{equation}
\mathcal{L}_{b,z} \varphi(x) = \sum_{y/Ty=x} e^{-z\mathfrak{C}(y)} \varphi(y),
\end{equation}

for a “ceiling” function $\mathfrak{C}$ to be defined. The function $\mathfrak{C}$ will depend on the metric $g_{a_0,\ldots,a_r,b}$ and especially on the width $b$ of the band inside the cusp $\mathcal{C}_r$ where the curvature is prescribed to be $-\omega_2^2$; we will need to understand precisely the dependence of $\mathcal{L}_{z,b}$ with respect to $b$ and thus notice the ceiling function $\mathfrak{C}_b$ and its corresponding transfer operator $\mathcal{L}_{b,z}$.

The alphabet $\mathfrak{B}$ being countable, the pre-images of $x \in \partial_0 X$ by $T$ are the points $y = \beta \cdot x$, for those blocks $\beta \in \mathfrak{B}$ such that $x$ belongs to $K_\beta$, that is:

- if $x \in I_i'$ then $\beta \in \bigcup_{j \neq i} \bar{P}_j$ or $\beta \in \mathcal{W}_1$ with $l_\beta \neq i$,
- if $x \in I_i \setminus I_i'$ then $\beta \in \bigcup_{j \neq i} \bar{P}_j$.

For such $y = \beta \cdot x$, the quantity $\mathfrak{C}_b(y)$ is given by

\begin{equation}
\mathfrak{C}_b(y) = B_{\beta^{-1} \cdot y}(\beta^{-1} \cdot x_0, x_0) = B_x(\beta^{-1} \cdot x_0, x_0)
\end{equation}

(\text{where $B$ is the Busemann function with respect to the metric $g = g_{a_0,\ldots,a_r,b}$}). By Lemma 4.2, the quantity $B_x(\beta^{-1} \cdot x_0, x_0)$ differs from
d(β⁻¹, x₀, x₀) by a term which is bounded uniformly in β ∈ B and x ∈ Kβ, this will allow us to compare the sum \( \sum_{k \geq 0} \mathcal{L}_b^k 1(x) \) with the Poincaré series \( P_\Gamma(x₀, x₀, s) = \sum_{\gamma \in \Gamma} e^{-sd(x₀, \gamma x₀)} \approx \sum_{k \geq 0} \sum_{|\gamma|_B = k} e^{-sd(x₀, \gamma x₀)} \).

We can now make explicit the definition of transfer operators \( \mathcal{L}_{b,z} \), extending formula (5.1) for any \( x ∈ \partial X \): for \( b \geq 0, z ∈ C \), any bounded Borel function \( \varphi : \partial X \to C \) and any \( x ∈ \partial X \) we set

\[
\mathcal{L}_{b,z} \varphi(x) = \sum_{\beta ∈ B} \mathcal{K}_b(x)e^{-zB_x(\beta^{-1}\cdot x₀,x₀)} \varphi(\beta \cdot x).
\]

In other words, \( \mathcal{L}_{b,z} \varphi(x) = \sum_{\beta ∈ B} w_{b,z}(\beta, x)\varphi(\beta \cdot x) \) where the \( w_{b,z}(\gamma, \cdot) : \partial X \to C, z ∈ C \) and \( \gamma ∈ \Gamma \) are defined by

\[
w_{b,z}(\gamma, x) = K_x(\gamma)(x)e^{-zB_x(\gamma^{-1}\cdot x₀,x₀)}
\]

and called weight functions. Observe now that these functions satisfy the cocycle relation: if the \( B \)-decomposition of \( \gamma = \gamma_1 \gamma_2 \) is given by the simple concatenation \( \gamma_1 \ast \gamma_2 \) of the \( \gamma_i \), i.e. \( |\gamma_1 \gamma_2|_B = |\gamma_1|_B + |\gamma_2|_B \), then

\[
w_{b,z}(\gamma_1 \gamma_2, x) = w_{b,z}(\gamma_1, \gamma_2 \cdot x) \cdot w_{b,z}(\gamma_2, x).
\]

This equality leads to the following simple expression of the iterates of the transfer operators: for any \( k \geq 1 \) and \( x ∈ \partial X \)

\[
\mathcal{L}_{b,z}^k \varphi(x) = \sum_{|\gamma|_B = k} w_{b,z}(\gamma, x)\varphi(\gamma \cdot x)
\]

(5.4)

The operator \( \mathcal{L}_{b,z} \) is well defined when \( \Re(z) > \delta \), and also for \( \Re(z) = \delta = \delta_\Gamma \) if \( \Gamma \) is convergent. It acts on the space \( C(\partial X) \) of \( C \)-valued continuous functions on \( \partial X \) endowed with the norm \( |\cdot|_\infty \) of the uniform convergence; however, to obtain a quasi-compact operator with good spectral properties, we will consider its restriction to a subspace \( \mathbb{L}_\alpha \subset C(\partial X) \) of Hölder continuous functions with respect to \( D_0 \), for \( \alpha \) given by Lemma 4.1. Namely we let

\[
\mathbb{L}_\alpha := \{ \varphi ∈ C(\partial X) : \|\varphi\| = |\varphi|_\infty + |\varphi|_\alpha < +\infty \}
\]

where \( |\varphi|_\alpha := \sup_{x, y ∈ \partial X} \frac{|\varphi(x) - \varphi(y)|}{D_0^{(x,y)}} \); then, \( \mathcal{L}_{b,z} \) acts on \( \mathbb{L}_\alpha \) because of the following

**Lemma 5.1.** — Each weight \( w_{b,z}(\gamma, \cdot) \) belongs to \( \mathbb{L}_\alpha \) and for any \( z ∈ C \), there exists \( C = C(z) > 0 \) such that for any \( \gamma ∈ \Gamma \)

\[
\|w_{b,z}(\gamma, \cdot)\| \leq Ce^{-\Re(z)d_0(x₀,\gamma x₀)}.
\]
Proof. — By Lemma 4.2, the family \( \{ e^{R(z)}d_b(\alpha,\gamma,\mathbf{o}) | w_{b,z}(\gamma,\cdot) \}_{\infty} \), \( \gamma \in \Gamma \) is bounded. The control of the Lipschitz-coefficient of \( w_{b,z}(\gamma,\cdot) \) is more delicate, we first recall briefly the proof in the constant curvature case and refer [3, Lemma III.3] in variable curvature. To estimate \( w_{b,z}(\gamma,x) - w_{b,z}(\gamma,y) \) for any points \( x, y \) belonging to the same subset \( K_{\gamma} \), note that there exists a constant \( A > 0 \) such that \( |B_{x}(\gamma^{-1} \cdot x_{0}, x_{0}) - B_{y}(\gamma^{-1} \cdot x_{0}, x_{0})| \leq A |x - y| \). The inequality \( |e^{Z} - 1| \leq 2|Z| e^{|R(Z)}| \) readily implies

\[
|e^{-zB_{x}(\gamma^{-1} \cdot x_{0}, x_{0})} - e^{-zB_{y}(\gamma^{-1} \cdot x_{0}, x_{0})}| \leq 2A|z| e^{-|z| |x-y|} e^{-R(z)|B_{x}(\gamma^{-1} \cdot x_{0}, x_{0})| |x-y|}.
\]

So, the Hölder coefficient of \( w_{z}(\gamma,\cdot) \) satisfies \( [w_{z}(\gamma,\cdot)]_{\alpha} \leq C e^{-R(z)d(\alpha,\gamma,\mathbf{o})} \) for some constant \( C = C(z) \). \( \square \)

The following theorem plays a key role in the sequel; it allows us to control the spectrum of the operators \( L_{b,s} \) for real parameters \( b \geq 0 \) and \( s \geq \delta := \omega_{\tau}/2 \).

**Theorem 5.2.** — For any \( b \geq 0 \) and \( s \geq \delta = \omega_{\tau}/2 \), the operator \( L_{b,s} \) acts both on \( (C(\partial X), |\cdot|_{\infty}) \) and \( (L_{\alpha}, \| \cdot \|) \), with respective spectral radius \( \rho_{\infty}(L_{b,s}) \) and \( \rho_{\alpha}(L_{b,s}) \). Furthermore, the operator \( L_{b,s} \) is quasi-compact\(^{(3)}\) on \( L_{\alpha} \), and:

1. \( \rho_{\alpha}(L_{b,s}) = \rho_{\infty}(L_{b,s}) \),
2. \( \rho_{\alpha}(L_{b,s}) \) is a simple, isolated eigenvalue of \( L_{b,s} \),
3. the eigenfunction \( h_{b,s} \) associated with \( \rho_{\alpha}(L_{b,s}) \) is non negative on \( \partial X \),
4. For any \( s \geq \delta \), the map \( b \mapsto L_{b,s} \) is continuous from \( \mathbb{R}^{+} \) to the space of continuous linear operators on \( L_{\alpha} \),
5. The map \( s \mapsto \rho_{\infty}(L_{b,s}) \) is decreasing on \( [\delta, +\infty] \).

**Sketch of the proof.** — We follow [25, Sections 4.3 and 4.4]. The key argument is the following inequality: for any \( \beta \in \mathfrak{B}, x, y \in K_{\beta}, s \geq \delta \) and \( k \geq 1 \)

\[
|L_{b,s}^{k}\varphi(x) - L_{b,s}^{k}\varphi(y)| \leq \sum_{\gamma \in \Gamma, |\gamma|_{\infty} = k} w_{b,s}(\gamma, x) |\varphi(\gamma \cdot x) - \varphi(\gamma \cdot y)|
\]

\[
+ \sum_{\gamma \in \Gamma, |\gamma|_{\infty} = k} |w_{b,s}(\gamma, x) - w_{b,s}(\gamma, y)| \times |\varphi|_{\infty},
\]

\(^{(3)}\)In other words its essential spectral radius on this space is less than \( \rho_{\alpha}(L_{b,s}) \)
so that, by Corollary 4.3, there exists sequences \((r_k)_k\) and \((R_k)_k\) such that
\[
(5.5) \quad \|L_{b,s}^k \varphi\| \leq r_k \|\varphi\| + R_k |\varphi|_\infty
\]
with \(\limsup_{k \to +\infty} r_k^{1/k} \leq r |L_{b,s}|_\infty\). Using a version due to H. Hennion of
the Ionescu–Tulcea–Marinescu’s theorem concerning quasi-compact operators, we conclude
that the essential spectral radius of \(L_{b,s}\) on \(L_\alpha\) is not greater than \(r \rho_\infty(L_{b,s})\). The control of the peripheral spectrum of \(L_{b,s}\) in
\(L_\alpha\) is based on the positivity of this family of operators, as in [25]; similarly,
one adapts the proof of Proposition 4.7 in [25] to establish assertion (2).
Proof of (3) is direct. \(\square\)

Remark. — Let \(\rho_{b,s} = \rho_\infty(L_{b,s})\). Then, \(L_{b,s} h_{b,s} = \rho_{b,s} h_{b,s}\), the function
\(h_{b,s}\) being unique up to a multiplicative constant. By duality, there also exists
a unique probability measure \(\sigma_{b,s}\) on \(\partial X\) such that \(\sigma_{b,s} L_{b,s} = \rho_{b,s} \sigma_{b,s}\);
the function \(h_{b,s}\) becomes uniquely determined imposing the condition
\(\sigma_{b,s}(h_{b,s}) = 1\), which we will assume from now on.

5.2. From convergence to divergence: Proof of Theorem 1.3

Combining expression (5.4) with Lemma 4.2, one gets for any \(s, b > 0\) and \(k \geq 0\)
\[
(5.6) \quad |L_{b,s}^k 1|_\infty \asymp \sum_{\gamma \in \Gamma, m = k} \exp(-s d_b(x_0, \gamma \cdot x_0))
\]
Consequently, the Poincaré series \(P_\Gamma(s)\) of \(\Gamma\) relatively to \(d_b\) and the series
\(\sum_{k \geq 0} |L_{b,s}^k 1|_\infty\) converge or diverge simultaneously. Following [25], we see
that the function \(s \mapsto \rho_\infty(L_{b,s})\) is strictly decreasing on \(\mathbb{R}^+\); the Poincaré
exponent of \(\Gamma\) relatively to \(d_b\) is then equal to
\[
\delta_\Gamma = \sup \left\{ s \geq 0 : \rho_\infty(L_{b,s}) \geq 1 \right\} = \inf \left\{ s \geq 0 : \rho_\infty(L_{b,s}) \leq 1 \right\}
\]
and the latter expression will be useful to prove Theorem 1.3. We first get the

Lemma 5.3. — Assume that the profiles \(\tau_0, \ldots, \tau_1, \ldots, \tau_r = \tau\) are convergent and satisfy the condition \(\omega_\tau = \max(\omega_{\tau_0}, \ldots, \omega_{\tau_r}) > 2\). Then there exist non negative reals \(a_0, \ldots, a_r\) and \(b_0 > 0\) such that

- The group \(\Gamma\) has exponent \(\frac{\omega_\tau}{2}\) and is convergent with respect to
  \(g_{a_0, \ldots, a_r, 0}\);
- The group \(\Gamma\) has exponent greater than \(\frac{\omega_\tau}{2}\) and is divergent with
  respect to \(g_{a_0, \ldots, a_r, b}\) for any \(b \geq b_0\).
Proof. — By the previous section and the choice of the profiles $\tau_i, 0 \leq i \leq r$ (with the additional notation $\tau = \tau_r$), we may fix the constants $a_0, \ldots, a_r$ large enough, in order that the group $\Gamma$ acting on $(X, g_{a_0, \ldots, a_r, 0})$ is a convergent lattice with exponent $\delta = \frac{\omega}{2\tau}$.

To prove the second point we will take into account only the contribution of words containing powers of $p_r$; we get by the triangle inequality

$$\sum_{\gamma \in \Gamma} e^{-\delta d_b(x_0, \gamma x_0)} \geq \sum_{k \geq 1} \sum_{\left| q_1 \right|, \ldots, \left| q_k \right| \in W_1} e^{-\delta d_b(x_0, p_{r, k}^{q_1} \cdots p_{r, k}^{q_k} x_0)}$$

$$\geq \sum_{k \geq 1} \left( \sum_{\left| q \right| \geq 2} e^{-\delta d_b(x_0, p_{r, l}^{q} x_0)} \sum_{Q \in W_1} e^{-\delta d_b(x_0, Q x_0)} \right)^k.$$

We now use the following fact:

**Lemma 5.4.** — There exist $b_0 > 0$ and, for any $b \geq b_0$, integers $n_{a_r} \leq n_b$ depending respectively on $a_r$ and $b \geq b_0$, such that:

- $n_b \to \infty$ when $b \to \infty$,
- for any $l$ satisfying $n_{a_r} \leq |l| \leq n_b$, there exists a constant $C$ depending only on $a_r$ and the bounds of the curvature such that:

$$d_b(x_0, p_{r, l}^{l} \cdot x_0) \geq \frac{2}{\omega_r} \ln |l| - C.$$

**Proof.** — Proof of Lemma 5.4 When $|l|$ is large enough, say $|l| \geq n_{a_r} \geq 2$, the geodesic segments $[x_0, p_{r, l}^{l} \cdot x_0]$ intersect the horoball $H_{x_r}(a_r + \Delta)$ and when $|l|$ is not too big the same geodesics do not intersect $H_{x_r}(a_r + b + \Delta)$. If we set $n_b := \max\{n \in \mathbb{N} : |k| \leq n \Rightarrow [x_0, p_{r, b}^{k} x_0] \cap H_{x_r}(a_r + b + \Delta) = \emptyset\}$, the latter is well defined and satisfies $n_b \to \infty$ when $b \to \infty$. Fix $l$ satisfying $n_{a_r} \leq |l| \leq n_b$ and define respectively by $x_1$ and $x_2$ the enter and exit point of the oriented geodesic segment $[x_0, p_{r, l}^{l} \cdot x_0]$ in $H_{x_r}(a_r + \Delta)$. Using a comparison argument with the geodesic from $x_0$ tangent to $\partial H_{x_r}(a_r + \Delta)$, we can observe that

$$|d_b(x_0, x_1) - d_0(x_0, x_1)| < c \quad \text{and} \quad |d_0(x_0, x_1) - a_r| < c$$

for a constant $c$ only depending on the bounds of the curvature. In the horospherical annulus $H_{x_r}(a_r + \Delta) \setminus H_{x_r}(a_r + b + \Delta) \setminus H_{x_r}(a_r + \Delta)$ the curvature is $-\omega_r^2$ so that $d_b(x_1, x_2) = \frac{d_{hyp}(x_1, x_2)}{\omega_r}$. Combining this with the above inequalities we get the existence of a constant $C$ depending on $a_r, r$ and on the bounds of the curvature such that for $l$ satisfying $n_{a_r} \leq |l| \leq n_b$:

$$d_b(x_0, p_{r, l}^{l} x_0) \geq \frac{d_{hyp}(x_0, p_{r, l}^{l} x_0)}{\omega_r} - C = \frac{2}{\omega_r} \ln(|l|) - C. \quad \square$$
To conclude the proof of Lemma 5.3, notice that for $\delta = \frac{\omega_r}{2}$ we have:

$$\sum_{\{l : |l| \geq 2\}} e^{-\delta d_l(x_0,p^l \cdot x_0)} \geq \sum_{|l| = n_{a_r}} e^{-\delta d_b(x_0,p^l \cdot x_0)} \geq \sum_{|l| = n_{a_r}} \left( \frac{1}{|l|^{2/\omega_r}} \right)^{\omega_r/2} \to +\infty \text{ as } b \to +\infty.$$  

So, there exists $b_0 \geq 0$ such that

$$\left( \sum_{|l| \geq 2} e^{-\delta d_l(x_0,p^l \cdot x_0)} \sum_{Q \in W_1} e^{-\delta d_b(x_0,Q \cdot x_0)} \right) > 1$$

as soon as $b \geq b_0$; in particular, continuity and monotonicity in $s$ gives:

$$\left( \sum_{|l| \geq 2} e^{-s d_{b_0}(x_0,p^l \cdot x_0)} \sum_{Q \in W_1} e^{-s d_{b_0}(x_0,Q \cdot x_0)} \right) > 1$$

for some $s > \frac{\omega_r}{2}$. This ensures that $\delta_1 \geq s \geq \frac{\omega_r}{2} = \delta_{p_r} = \max\{\delta_{p_i} \mid 0 \leq i \leq r\}$, and $\Gamma$ is divergent with respect to the metric $g_{a_0,\ldots,a_r,b}$ by the critical gap property recalled in the introduction. 

End of proof of Theorem 1.3. — Recall that $\delta = \omega_r/2$. Since $\rho_\alpha(L_{b,\delta})$ is an eigenvalue of $L_{b,\delta}$ which is isolated in the spectrum of $L_{b,\delta}$, the function $b \mapsto \rho_\alpha(L_{b,\delta})$ (which is a priori semi-continuous) has the same regularity as $b \mapsto L_{b,\delta}$. For $b_0$ given by Lemma 5.3, we have $\rho_\alpha(L_{0,\delta}) = \rho_\infty(L_{0,\delta}) \leq 1$ and $\rho_\alpha(L_{b_0,\delta}) = \rho_\infty(L_{b_0,\delta}) \geq 1$; thus, there exists $b_* \in [0,b_0]$ such that $\rho_\alpha(L_{b_*,\delta}) = \rho_\infty(L_{b_*,\delta}) = 1$. Since the function $s \mapsto \rho_\infty(L_{b_*,s})$ is strictly decreasing on $[\delta, +\infty[$, one gets $\rho_\infty(L_{b_*,s}) < 1$ as soon as $s > \delta$. For such values of $s$, the Poincaré series $P_\Gamma(s)$ of $\Gamma$ relatively to the metric $g_{b_*}$ thus converges, this implies that its Poincaré exponent $\delta_{\Gamma,b_*}$ is not greater than $\delta$ ; actually we have $\delta_{\Gamma,b_*} = \delta$ since $\delta_{(p_r)} = \delta$ and $p_r \in \Gamma$. Finally, the eigenfunction $h_{b_*,\delta}$ of $L_{b_*,\delta}$ associated with $\rho_\alpha(L_{b_*,\delta})$ being non negative on $\partial X$, one gets $h_{b_*,\delta} \geq 1$ and so

$$\sum_{k \geq 0} |L_{b_*,\delta}^k 1|_\infty \leq \sum_{k \geq 0} |L_{b_*,\delta}^k h_{b_*,\delta}|_\infty = \sum_{k \geq 0} |h_{b_*,\delta}|_\infty = \infty$$

which implies, by (5.6), that $\Gamma$ is divergent with respect to the metric $g_{b_*}$. 

6. Counting for some divergent exotic lattices

We prove here the following general result, which implies Theorem 1.4 given in the Introduction:
THEOREM 6.1. — Let $\bar{X}$ be a $(r+1)$-punctured sphere, endowed with a metric of finite area which has a cusp $\mathcal{C}_i$ with profile $\tau_i$ for each puncture and is hyperbolic outside $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \mathcal{C}_r$. Let $p_0, \ldots, p_r$ be parabolic isometries, each one generating a parabolic subgroup $\mathcal{P}_i$ associated to $\mathcal{C}_i$, such that $\Gamma = \pi_1(\bar{X})$ is a free group over $p_1, \ldots, p_r$ and $p_0 = p_r \ldots p_1$. Let $X$ be the universal cover of $\bar{X}$, let $x_0 \in X$, and assume that the lifted metric $g$ on $X$ satisfies the following assumptions:

- $\mathbf{H}_1$: the group $\Gamma$ is exotic and divergent with Poincaré exponent $\delta = \delta_\Gamma = \frac{\omega}{2}$, where $\omega = \omega_{\tau_r} = \max(\omega_{\tau_0}, \ldots, \omega_{\tau_r}) > 2$;
- $\mathbf{H}_2$: there exist $\kappa \in [1/2, 1]$ and a slowly varying function\(^{(4)}\) $L$ such that

$$\sum_{p \in \mathcal{P}_i / d(x_0, p \cdot x_0) > t} e^{-\delta d(x_0, p \cdot x_0)} \xrightarrow{t \to +\infty} L(t) t^\kappa.$$  

- $\mathbf{H}_3$: the parabolic groups $\mathcal{P}_i$, for $1 \leq l \leq r - 1$, satisfy the condition

$$\sum_{p \in \mathcal{P}_i / d(x_0, p \cdot x_0) > t} e^{-\delta d(x_0, \gamma \cdot x_0)} = o\left(\frac{L(t)}{t^\kappa}\right).$$

- $\mathbf{H}_4$: the set $\mathcal{W}_1$ of level 1 words satisfy the condition

$$\sum_{Q \in \mathcal{W}_1 / d(x_0, Q \cdot x_0) > t} e^{-\delta d(x_0, Q \cdot x_0)} = o\left(\frac{L(t)}{t^\kappa}\right).$$

Then, for any $1 \leq j \leq r$ and any fixed $x_j \in \partial X$ far enough from the fixed point of $p_j$, there exists $C_j > 0$ such that

$$\sharp\{\gamma \in \Gamma_j / B_{x_j}(\gamma^{-1} \cdot x_0, x_0) \leq R\} \xrightarrow{R \to +\infty} C_j \frac{e^{\delta_j R}}{R^{1-\kappa}L(R)},$$

where $\Gamma_j$ is the set of $\gamma \in \Gamma$ with last letter $j$, with respect to the alphabet $\mathcal{A}$. As a consequence, we have

$$\sharp\{\gamma \in \Gamma / d(x_0, \gamma \cdot x_0) \leq R\} \xrightarrow{R \to +\infty} \frac{e^{\delta_j R}}{R^{1-\kappa}L(R)}.$$  

Notice that (6.5) easily follows from (6.4) summing over $j \in \{1, \ldots, r\}$, as for each $j$ there exists $c = c(x_j) > 0$ such that $d(x_0, \gamma \cdot x_0) - c \leq B_{x_j}(\gamma^{-1} \cdot x_0, x_0) \leq d(x_0, \gamma \cdot x_0)$.

\(^{(4)}\) A function $L(t)$ is said to be “slowly varying” or “of slow growth” if it is positive, measurable and $L(\lambda t)/L(t) \to 1$ as $t \to +\infty$ for every $\lambda > 0$. 

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Let us make some remarks:

1. We have seen in the previous sections how to perturb a hyperbolic metric to obtain a negatively curved metric $g = g_{a_0,\ldots,a_r,b^*}$ on $\bar{X}$ with profiles $\tau_0,\ldots,\tau_r$ and parameters $a_0,\ldots,a_r$ and $b^*$ so that the hypothesis $H_1$ holds.

2. Hypothesis $Hb_2$ is inspired by probability theory and it corresponds to a heavy tail condition satisfied by random walks, which have been intensively investigated [19]. It holds in particular when

$$d(x_0, p^n \cdot x_0) = \frac{2 \ln n + 2(1 + \kappa)(\ln \ln n + O(1))}{\omega}$$

for some $\omega > 2$ (the prescription of the profile $\tau_r$ in dimension 2 is equivalent to the prescription of the sequence $(d(x_0, p^n \cdot x_0))_{n \geq 1}$, we refer to [11, Section 3] for a detailed construction of a metric in the cusp leading to this asymptotic behaviour). This equality concerns only the asymptotic geometry on the cusp $\bar{C}_r$ as it is equivalent to prescribe a profile without modifying its exponential growth rate. Hence it is compatible with any choice of the parameter $b$. The critical exponent of $\mathcal{P}_r$ is thus $\delta = \omega/2$ and one gets, as $t \to +\infty$, $

\sum_{p \in \mathcal{P}_r, d(x_0, p \cdot x_0) > t} e^{-\delta d(x_0, p \cdot x_0)} = \sum_{n \in \mathbb{N}, d(x_0, p^n \cdot x_0) > t} e^{-\delta d(x_0, p^n \cdot x_0)}$

$$\asymp \sum_{n > e^{\omega t/2}} \frac{1}{n (\ln n)^{1+\kappa}}$$

$$\asymp \int_{e^{\omega t/2}}^{+\infty} \frac{du}{u (\ln u)^{1+\kappa}}$$

$$\asymp \frac{1}{t^\kappa}.$$ 

3. The condition $\kappa \in ]1/2, 1]$ readily implies

$$\sum_{p \in \mathcal{P}_r} d(x_0, p \cdot x_0) e^{-\delta d(x_0, p \cdot x_0)} \asymp \sum_{N \geq 1} N \left( \sum_{p \in \mathcal{P}_r} \sum_{N < d(x_0, p \cdot x_0) \leq N+1} e^{-\delta d(x_0, p \cdot x_0)} \right)$$

$$\asymp \sum_{N \geq 1} \left( \sum_{p \in \mathcal{P}_r, d(x_0, p \cdot x_0) > N} e^{-\delta d(x_0, p \cdot x_0)} \right)$$

$$\asymp \sum_{N \geq 1} \frac{L(N)}{N^\kappa} = +\infty.$$
By Theorem B in [11], it readily follows that the Bowen–Margulis measure associated with $\Gamma$ is infinite.

(4) Notice that hypothesis $H_3$ is satisfied in particular when the gap property $\delta_{P_l} < \delta$ holds for any $0 < l < r - 1$.

(5) Similarly, hypothesis $H_4$ is satisfied in particular when the exponential growth rate $\delta_{W_1} := \limsup_{R \to +\infty} \frac{1}{R} \ln \sharp \{Q \in W_1 / d(x_0, Q \cdot x_0) \leq R\}$ of the level 1 words set is strictly less than $\delta$.

(6) Since $P_0 = \langle p_0 \rangle \subset W_1$, hypothesis $H_4$ readily implies that equality (6.2) holds also for $P_0$; nevertheless, it is a strictly stronger assumption, taking into account:

- the level 1 words $w$ corresponding to geodesic loops staying in a compact subset of the surface $\overline{X}$ (and whose exponential growth rate is strictly less than $\delta$, by a straightforward subadditivity argument)
- the words constructed by combination of the $w$’s and the powers of $p_0$.

Under these assumptions, we will see that the subgroup $P_r$ corresponding to the cusp $C_r$ has a dominant influence on the behaviour of the orbital function of $\Gamma$.

We present here the main steps of the proof of Theorem 6.1, and refer to [12] for details. Let $\overline{X} = \Gamma \setminus X$, where $X$ is its universal cover. First notice that, under the assumption of the theorem, we can assume (by replacing the cusps $\overline{C}_i$ with purely hyperbolic cusps) that $\Gamma$ acts on the disk model of $\mathbb{H}^2$ with a fundamental polygon $D$ as described at the beginning of Section 4, with $2r$ boundary points $\xi_0 = \eta_0, \xi_1, \eta_1, \ldots, \xi_r, \eta_r = \xi_0$, enumerated in cyclic order, with $p_i \cdot \xi_i = \xi_i$ and $p_i \cdot \eta_{i-1} = \eta_i$ for $1 \leq i \leq r$, and all points $\eta_i$ belonging to the $\Gamma$-orbit of the fixed point $\xi_0$ of $p_0$; so, we can assume that the metric $g$ is a modification of the hyperbolic one on the fundamental system of horospheres $H_{\xi_i}$, with $\overline{C}_i = P_i \setminus H_{\xi_i}$. We therefore have a coding of elements of $\Gamma$ by blocks $\beta \in \mathcal{B}$ and a ping-pong dynamic, for the alphabet $\mathcal{B}$ as described in Section 4.

For any $R \geq 0$, let us denote $W_j (R, \cdot)$ the measure on $\mathbb{R}$ defined by: for any Borel non negative function $\psi : \mathbb{R} \to \mathbb{R}$

$$W_j (R, \psi) := \sum_{\gamma \in \Gamma_j} e^{-\delta B_j (\gamma^{-1} \cdot x_0, x_0)} \psi (B_j (\gamma^{-1} \cdot x_0, x_0) - R).$$

One gets $0 \leq W_j (R, \psi) < +\infty$ when $\psi$ has a compact support in $\mathbb{R}$ since the group $\Gamma$ is discrete, furthermore $\sum_{j=1}^{r} W_j (R, \psi) \asymp e^{-\delta R v_\Gamma (R)}$ when $\psi(t) = e^{\delta t} 1_{t \leq 0}$. If one proves that for any non negative and continuous
function \( \psi \) with compact support and such that \( \int_{\mathbb{R}^+} \psi(x)dx > 0 \)

\[
W_j(R, \psi) \xrightarrow{R \to +\infty} \frac{C_j}{R^{1-\alpha}L(R)} \int_{\mathbb{R}} \psi(x)dx,
\]

this convergence will also hold for non-negative functions with compact support in \( \mathbb{R} \) and whose discontinuity set has 0 Lebesgue measure; Theorem 6.1 follows since \( v_T(R) = e^{\delta R} \sum_{j=1}^r \sum_{n \geq 0} W_j(R, e^{\delta t}1_{-(n+1),-n]}(t)) \).

From now on, we fix a continuous function \( \psi : \mathbb{R} \to \mathbb{R}^+ \) with compact support; one gets, for any \( 1 \leq j \leq r \) fixed

\[
W_j(R, \psi) = \sum_{k \geq 0} \left( \sum_{\gamma \in \Gamma_j/\gamma|\gamma = k} e^{-\delta B_{x_j}((\gamma^{-1} \cdot x_0), x_0)} \psi(B_{x_j}(\gamma^{-1} \cdot x_0, x_0) - R) \right).
\]

Notice that for \( \gamma \in \Gamma_j \) with \( \mathcal{B} \)-decomposition \( \gamma = \beta_1 \ldots \beta_k \), one gets, setting \( y := \gamma \cdot x_j \),

\[
B_{x_j}(\gamma^{-1} \cdot x_0, x_0) = \mathcal{C}(\beta_1 \ldots \beta_k \cdot x_j) + \mathcal{C}(\beta_2 \ldots \beta_k \cdot x_j) + \cdots + \mathcal{C}(\beta_k \cdot x_j)
\]

\[
= \mathcal{C}(y) + \mathcal{C}(T \cdot y) + \cdots + \mathcal{C}(T^{k-1} \cdot y) = S_k \mathcal{C}(y)
\]

where \( \mathcal{C} \) is the “ceiling” function defined as in (5.2), so that

\[
W_j(R, \psi) = \sum_{k \geq 0} \sum_{y \in \partial X/T^k \cdot y = x_j} e^{-\delta S_k \mathcal{C}(y)} \psi(S_k \mathcal{C}(y) - R).
\]

By a classical argument in probability theory due to Stone (see for instance [19]), it suffices to check that the convergence (6.6) holds when \( \psi \) has a \( C^\infty \) Fourier transform \( \hat{\psi} : t \mapsto \int_{\mathbb{R}} e^{itx} \psi(x)dx \) with compact support: indeed, the test function \( \psi \) may vary in the set \( \mathcal{H} \) of functions of the form \( \psi(x) = e^{itx} \psi_0(x) \) where \( \psi_0 \) is an integrable and strictly positive function on \( \mathbb{R} \) whose Fourier transform is \( C^\infty \) with compact support. When \( \psi \in \mathcal{H} \), one can use the inversion Fourier formula \( \psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \hat{\psi}(t)dt \); this yields, for any \( 0 < s < 1 \)

\[
W_j(s, R, \psi) := \sum_{k \geq 0} s^k \sum_{y \in \partial X/T^k \cdot y = x_j} e^{-\delta S_k \mathcal{C}(y)} \psi(S_k \mathcal{C}(y) - R)
\]

\[
= \sum_{k \geq 0} s^k \sum_{y \in \partial X/T^k \cdot y = x_j} \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} e^{-(\delta + it)S_k \mathcal{C}(y)} \hat{\psi}(t)dt
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \left( \sum_{k \geq 0} s^k \sum_{y \in \partial X/T^k \cdot y = x_j} e^{-(\delta + it)S_k \mathcal{C}(y)} \right) dt
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \left( \sum_{k \geq 0} s^k \mathcal{L}_{\delta + it}^k 1(x_j) \right) dt
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) (I - s\mathcal{L}_{\delta + it})^{-1} 1(x_j) dt
\]
where $L_z, z \in \mathbb{C}$, is the transfer operator associated to the function $C$ and the metric $g$, formally defined in Section 5.

When $\text{Re}(z) \geq \delta$, we know that the $L_z$ are bounded and quasi-compact on the space $\mathbb{L}_\omega(\partial X)$ of Hölder continuous function on $(\partial X, D_0)$. The Subsection 6.1 is devoted to the control of the peripherical spectrum of $L_{\delta + it}$ on $\mathbb{L}_\omega$. In Subsection 6.2, we describe the local expansion of the dominant eigenvalue. At last we achieve the proof using arguments coming from renewal theory (Subsection 6.3).

### 6.1. The spectrum of $L_{\delta + it}$ on $\mathbb{L}_\omega$

First we need to control the spectrum of $L_z$ when $z = \delta + it, t \in \mathbb{R}$. By Lemma 5.1, the operators $L_z$ are bounded on $\mathbb{L}_\omega$ when $\text{Re}(z) \geq \delta$; the spectral radius of $L_z$ will be denoted $\rho_{\omega}(z)$ throughout this section. In the following Proposition, we describe its spectrum on $\mathbb{L}_\omega$ when $\text{Re}(z) = \delta$.

**Proposition 6.2.** — There exist $\epsilon_0 > 0$ and $\rho_0 \in ]0,1[$ such that, for any $t \in \mathbb{R}$ with modulus less than $\epsilon_0$, the spectral radius $\rho_{\omega}(\delta + it)$ of $L_{\delta + it}$ is $> \rho_0$ and the operator $L_{\delta + it}$ has a unique eigenvalue $\lambda_t$ of modulus $\rho(\delta + it)$, which is simple and closed to 1, the rest of the spectrum being included in a disc of radius $\rho_0$.

Furthermore, for any $A > 0$, there exists $\rho_A \in ]0,1[$ such that $\rho_{\omega}(\delta + it) \leq \rho_A$ for any $t \in \mathbb{R}$ such that $\epsilon_0 \leq |t| \leq A$.

**Notation.** — We denote $\sigma$ the unique probability measure on $\partial X$ such that $\sigma L_\delta = \sigma$ and $h$ the element of $\mathbb{L}_\omega$ such that $L_\delta h = h$ and $\sigma(h) = 1$.

**Proof.** — This is exactly the same proof that the one presented in [2, Proposition 2.2] and [12]: the operators $L_{\delta + it}$ are quasi-compact on $\mathbb{L}_\omega$ and it is sufficient to control their peripherical spectrum. When $t$ is closed to 0, we use the perturbation theory to conclude that the spectrum of $L_{\delta + it}$ is closed to the one of $L_\delta$: it is thus necessary to prove that the map $t \mapsto L_{\delta + it}$ is continuous on $\mathbb{R}$. The following Lemma is devoted to precise the type of continuity of this function.

**Lemma 6.3.** — Under the hypotheses $H_1 - H_4$, there exists a constant $C > 0$ such that

$$\|L_{\delta + it'} - L_{\delta + it}\| \leq C|t' - t|^{\kappa} L\left(\frac{1}{|t' - t|}\right)$$

**Proof of Lemma 6.3.** — We will use the following classical fact ([20, p. 272] and [19, Lemmas 1 and 2]):
**Lemma 6.4.** — Let $\mu$ be a probability measure on $\mathbb{R}^+$, set $F_\mu(t) := \mu(0,t]$ and $m(t) := \int_0^t (1 - F_\mu(s)) ds$ and assume that there exist $\kappa \in \mathbb{R}$ and a slowly varying function $L$ such that $1 - F_\mu(t) \sim \frac{L(t)}{t^\kappa}$ as $t \to +\infty$. One thus gets

$$\lim_{t \to +\infty} \frac{t(1 - F_\mu(t))}{m(t)} = 1 - \kappa$$

and the characteristic function $\hat{\mu}(t) := \int_0^{+\infty} e^{itx} \mu(dx)$ of $\mu$ has the following local expansion as $t \to 0^+$

$$\hat{\mu}(t) = 1 - e^{-i\pi \frac{\kappa}{2}} \Gamma(1 - \kappa) t^\kappa L(\frac{1}{t})(1 + o(t)).$$

Noticing that $\hat{m}(t) := \int_0^t x \mu(dx) = m(t) - t(1 - F_\mu(t))$ one also gets $\hat{m}(t) \sim \kappa m(t)$ as $t \to +\infty$; furthermore, decomposing $\int_0^{+\infty} |e^{itx} - 1| \mu(dx)$ as $\int_0^{1/t} |e^{itx} - 1| \mu(dx) + \int_{1/t}^{+\infty} |e^{itx} - 1| \mu(dx)$ and applying the previous estimations, one gets, for any $t \in \mathbb{R}$

$$\int_0^{+\infty} |e^{itx} - 1| \mu(dx) \leq t^\kappa L(1/t).$$

We now apply (6.9) with the probability measures $\mu_l, 1 \leq l \leq r$, on $\mathbb{R}^+$ defined by $\mu_l := c_l \sum_{p \in \mathcal{P}_l} \delta_{d(x_0,p \cdot x_0)}$ where $c_l > 0$ is some normalizing constant. As a direct consequence, under hypotheses $H_1 - H_4$, one gets (up to a modification of $L$ by multiplicative constant)

$$\sum_{p \in \mathcal{P}_l, d(x_0,p \cdot x_0) \leq t} d(x_0,p \cdot x_0) e^{-\delta d(x_0,p \cdot x_0)} \to_{t \to +\infty} t^{1-\kappa} L(t),$$

and

$$\sum_{p \in \mathcal{P}_l} |e^{itd(x_0,p \cdot x_0)} - 1| \times e^{-\delta d(x_0,p \cdot x_0)} \leq t^\kappa L(1/t);$$

similarly, for $1 \leq l < r$, one has

$$\sum_{p \in \mathcal{P}_l} |e^{itd(x_0,p \cdot x_0)} - 1| \times e^{-\delta d(x_0,p \cdot x_0)} = t^\kappa L(1/t)o(t).$$

In the same way, by Lemma 4.2, for any $x \in \partial X \setminus I_r$, one gets

$$\sum_{p \in \mathcal{P}_r/\mathcal{C}(p \cdot x) \leq t} e^{-\delta \mathcal{C}(p \cdot x)} \asymp \frac{L(t)}{t^\kappa}, \quad \sum_{p \in \mathcal{P}_r/\mathcal{C}(p \cdot x) > t} \mathcal{C}(p \cdot x) e^{-\delta \mathcal{C}(p \cdot x)} \asymp t^{1-\kappa} L(t),$$

$$\sum_{p \in \mathcal{P}_r} |e^{it\mathcal{C}(p \cdot x)} - 1| \times e^{-\delta \mathcal{C}(p \cdot x)} \leq t^\kappa L(1/t).$$
and, for $1 \leq l \leq r - 1$ and $x \in \partial X \setminus I_l$

\begin{equation}
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\sum_{p \in \mathcal{P}_l} |e^{it\xi(p \cdot x)} - 1| \times e^{-\delta \xi(p \cdot x)} = t^\kappa L(1/t) o(t).
\end{equation}

In the same way, using Hypothesis $\textbf{H4}$, we may write, for any $1 \leq l \leq r$ and $x \in \bigcup_{l \neq l'} I_{l'}$

\begin{equation}
\sum_{\substack{Q \in \mathcal{W}_l \, l_Q = l}} |e^{it\xi(\beta \cdot x)} - 1| \times e^{-\delta \xi(Q \cdot x)} = t^\kappa L(1/t) o(t).
\end{equation}

Noticing that for any $\beta \in \mathcal{B}$ and $x \in K_\beta$

\[ |w_{\delta + it}(\beta, x) - w_{\delta + it'}(\beta, x)| \leq |e^{i(t-t') \cdot x})| - 1| \times e^{-\delta \xi(\beta \cdot x)} \]

one readily gets, combining the above estimations (6.14) to (6.17) all together

\begin{equation}
|L_{\delta + it} - L_{\delta + it'}| \leq \sum_{\beta \in \mathcal{B}} |w_{\delta + it}(\beta, \cdot) - w_{\delta + it'}(\beta, \cdot)| \leq |t - t'|^\kappa L \left( \frac{1}{|t - t'|} \right).
\end{equation}

Now, for any $\beta \in \mathcal{B}$ and $x, y \in K_\beta$, one gets

\[
(w_{\delta + it}(\beta, x) - w_{\delta + it'}(\beta, x)) - (w_{\delta + it}(\beta, y) - w_{\delta + it'}(\beta, y)) \\
\leq e^{-\delta \xi(\beta \cdot x)} \times \left( e^{it\xi(\beta \cdot y)} - e^{it'\xi(\beta \cdot y)} \right) \times \left( e^{it(\xi(\beta \cdot x) - \xi(\beta \cdot y))} - 1 \right) \\
+ e^{-\delta \xi(\beta \cdot x)} e^{it'\xi(\beta \cdot y)} \\
\times \left( \left( e^{it(\xi(\beta \cdot x) - \xi(\beta \cdot y))} - 1 \right) - \left( e^{it'\xi(\beta \cdot x) - \xi(\beta \cdot y)} \right) \right) \\
+ \left( e^{-\delta \xi(\beta \cdot x)} - e^{-\delta \xi(\beta \cdot y)} \right) \left( e^{it\xi(\beta \cdot y)} - e^{it'\xi(\beta \cdot y)} \right) \\
\leq e^{-\delta \xi(\beta \cdot x)} \left( e^{it\xi(\beta \cdot y)} - e^{it'\xi(\beta \cdot y)} \right) \left( e^{it(\xi(\beta \cdot x) - \xi(\beta \cdot y))} - 1 \right) \\
+ e^{-\delta \xi(\beta \cdot x)} e^{it'\xi(\beta \cdot y)} \left( e^{it(\xi(\beta \cdot x) - \xi(\beta \cdot y))} - 1 \right) \\
+ \left( e^{-\delta \xi(\beta \cdot x)} - e^{-\delta \xi(\beta \cdot y)} \right) \left( e^{it\xi(\beta \cdot y)} - e^{it'\xi(\beta \cdot y)} \right)
\]
which readily implies
\[
\left| \left( w_{\delta + it}(\beta, x) - w_{\delta + it'}(\beta, x) \right) - \left( w_{\delta + it}(\beta, y) - w_{\delta + it'}(\beta, y) \right) \right| \\
\leq e^{-\delta |\text{C} \cdot x|} \left| e^{i(t-t')|\text{C} \cdot x|} - 1 \right| \\
+ e^{-\delta |\text{C} \cdot y|} \left| e^{i(t-t')|\text{C} \cdot y|} - 1 \right| \\
\leq e^{-\delta |\text{C} \cdot x|} \left| e^{i(t-t')|\text{C} \cdot x|} - 1 \right| \\
+ e^{-\delta |\text{C} \cdot y|} \left| e^{i(t-t')|\text{C} \cdot y|} - 1 \right| \\
\leq e^{-\delta |\text{C} \cdot x|} \left| e^{i(t-t')|\text{C} \cdot x|} - 1 \right| |t||\text{C} \circ \beta| \|D(x, y)\| \\
+ e^{-\delta |\text{C} \cdot y|} \left| e^{i(t-t')|\text{C} \cdot y|} - 1 \right| |t||\text{C} \circ \beta| \|D(x, y)\|.
\]

We achieve the proof of the Lemma combining this last inequality with (6.18). □

6.2. On the local expansion of the dominant eigenvalue \( \lambda_t \)

We explicit here the local expansion near 0 of the map \( t \mapsto \lambda_t \):

**Proposition 6.5.** — Under the hypotheses \( H_1 - H_4 \), there exists \( C_\Gamma > 0 \) such that

\[
\lambda_t = 1 - C_\Gamma e^{-i\pi^{\frac{\text{C}}{2}} t^{\gamma} L(1/t)} (1 + o(t)).
\]

**Proof.** — Recall first that 1 is a simple eigenvalue of \( \mathcal{L}_\delta \) with \( \mathcal{L}_\delta h = h \) and \( \sigma(h) = 1 \); since \( t \mapsto \mathcal{L}_\delta + it \) is continuous on \( \mathbb{R} \), for \( t \) closed to 0 there exists a function \( h_t \in \mathbb{L}_\omega \) such that \( \mathcal{L}_\delta + it h_t = \lambda_t h_t \), this function being unique if we impose the normalization condition \( \sigma(h_t) = 1 \). The maps \( t \mapsto \lambda_t \) and \( t \mapsto h_t \) have the same type of continuity than \( t \mapsto \mathcal{L}_\delta + it \) and one gets the identity

\[
\lambda_t = \sigma(\mathcal{L}_t h_t) = \sigma(\mathcal{L}_t h) + \sigma((\mathcal{L}_\delta + it - \mathcal{L}_\delta)(h_t - h)).
\]

By the previous subsection, the second term of this last expression is \( \lesssim \left( t^{\gamma} L(1/t) \right)^2 \). It remains to precise the local behaviour of the map \( t \mapsto \sigma(\mathcal{L}_t h) \); one gets

\[
\sigma(\mathcal{L}_t h) = 1 + \sum_{l=0}^{r} \sigma_l
\]
with
\[
\sigma_0 := \sum_{\beta \in \mathcal{W}_1} \int_{K_\beta} h(\beta \cdot x) e^{-\delta \mathcal{C}(\beta \cdot x)} \left( e^{-it \mathcal{C}(\beta \cdot x)} - 1 \right) \sigma(dx)
\]
and, for \(1 \leq l \leq r\)
\[
\sigma_l := \sum_{\beta \in \widehat{\mathcal{P}}_l} \int_{K_\beta} h(\beta \cdot x) e^{-\delta \mathcal{C}(\beta \cdot x)} \left( e^{-it \mathcal{C}(\beta \cdot x)} - 1 \right) \sigma(dx).
\]
By (6.17), the terms \(\sigma_l, l \neq r\), are of the form \(t^k L(1/t)o(t)\). To control the term \(\sigma_r\), one sets \(\Delta(n, x) := C(p_n \cdot x) - d(x_0, p_n \cdot x_0)\) for any \(x \in \partial X \setminus I_r\) and \(n \in \mathbb{Z}\). The following lemma readily implies that the quantity \(\Delta(n, x)\) tends to \(-(-x|\xi_r)_{x_0}\) as \(|n| \to +\infty\).

**Lemma 6.6.** — For any parabolic group \(\mathcal{P} := <p>\) with fixed point \(\xi\), we have
\[
\mathcal{B}_x(p^{\pm n} \cdot x_0, x_0) = d_b(p^{\pm n} \cdot x_0, x_0) - 2(\xi|x)_{x_0} + \epsilon_x(n)
\]
with \(\lim_{n \to +\infty} \epsilon_x(n) = 0\), the convergence being uniform on compact sets of \(\partial X \setminus \{\xi\}\).

**Proof of Lemma 6.6.** — Let \((x_m)\) be a sequence of elements of \(X\) converging to \(x\). We have
\[
\mathcal{B}_x(p^{\pm n} \cdot x_0, x_0) = \lim_m d(p^{\pm n} \cdot x_0, x_m) - d(x_0, x_m)
\]
\[
= d(p^{\pm n} \cdot x_0, x_0) - \lim_m \left( d(x_0, x_m) + d(p^{\pm n} \cdot x_0, x_0) - d(p^{\pm n} \cdot x_0, x_m) \right)
\]
with
\[
\lim_n \left( \lim_m d(x_0, x_m) + d(p^{\pm n} \cdot x_0, x_0) - d(p^{\pm n} \cdot x_0, x_m) \right)
\]
\[
= 2 \lim_n (p^{\pm n} \cdot x_0|x)_{x_0}
\]
\[
= 2(\xi|x)_{x_0}
\]
and the conclusion follows as the Gromov product \((p^{\pm n} \cdot x_0|x)_{x_0}\) tends uniformly to \((\xi|x)_{x_0}\) on compacts of \(S^1 \setminus \{\xi\}\). \(\square\)
We write
\[
\sigma_r = \sum_{|n| \geq 2} e^{-\delta d(x_0, p^n_r \cdot x_0)} \left( e^{-itd(x_0, p^n_r \cdot x_0)} - 1 \right) \int_{\partial X \setminus I_r} h(p^n_r \cdot x) e^{-\delta \Delta(n,x)} \sigma(dx)
\]
\[\quad + \sum_{|n| \geq 2} e^{-\delta d(x_0, p^n_r \cdot x_0)} \times \int_{\partial X \setminus I_r} h(p^n_r \cdot x) e^{-\delta \Delta(n,x)} \left( e^{-it \mathcal{C}(p^n_r \cdot x)} - e^{-itd(x_0, p^n_r \cdot x_0)} \right) \sigma(dx)
\]
\[\quad = \sigma_{r1} + \sigma_{r2}.
\]
One gets \( \int_{\partial X \setminus I_r} h(p^n_r \cdot x) e^{-\delta \Delta(n,x)} \sigma(dx) > 0 \) for any \( |n| \geq 2 \); by (6.10), there exists \( c > 0 \) such that \( \sigma_{r1} = -ce^{-i\pi \frac{\kappa}{2}} \Gamma(1 - \kappa) t^\kappa L(\frac{1}{t})(1 + o(t)) \).

On the other hand
\[
|\sigma_{r2}| \leq |t| \sum_{|n| \geq 2} e^{-\delta d(x_0, p^n_r \cdot x_0)} \times \int_{\partial X \setminus I_r} h(p^n_r \cdot x) e^{-\delta \Delta(n,x)} |\mathcal{C}(p^n_r \cdot x) - d(x_0, p^n_r \cdot x_0)| \sigma(dx)
\]
\[\quad = O(t).
\]
Equality (6.19) follows immediately. \( \square \)

### 6.3. Renewal theory and proof of Theorem 6.1

For technical reasons (see for instance [19]) which will appear in the control of the term \( \widetilde{W}_j^{(3)}(R, \psi) \) we need to symmetrize the quantities \( W_j(R, \psi) \) and \( W_j(s, R, \psi) \) setting
\[
\widetilde{W}_j(R, \psi) := \sum_{k \geq 0} \sum_{y \in \partial X/T^k \cdot y = x_j} e^{-\delta S_k \mathcal{C}(y)} \left( \psi(S_k \mathcal{C}(y) - R) + \psi(-S_k \mathcal{C}(y) - R) \right)
\]
and
\[
\widetilde{W}_j(s, R, \psi) := \sum_{k \geq 0} s^k \sum_{y \in \partial X/T^k \cdot y = x_j} e^{-\delta S_k \mathcal{C}(y)} \left( \psi(S_k \mathcal{C}(y) - R) + \psi(-S_k \mathcal{C}(y) - R) \right).
\]
Notice that, when \( \psi \) is a continuous function with compact support in \( \mathbb{R}^+ \), the terms \( \psi(S_k \mathcal{C}(y) - R) \) of these sums vanish for \( R \) large enough, so that
\( \widetilde{W}_j(R, \psi) = W_j(R, \psi) \) and \( \widetilde{W}_j(s, R, \psi) = W_j(s, R, \psi) \) in this case. By (6.8), for any \( 0 < s < 1 \), one gets,

\[
\widetilde{W}_j(s, R, \psi) := \sum_{k \geq 0} s^k \sum_{y \in \partial X} e^{-\delta S_k \mathcal{C}(y)} \left( \psi(S_k \mathcal{C}(y) - R) + \psi(-S_k \mathcal{C}(y) - R) \right) \\
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \left( (I - s\mathcal{L}_{\delta+it})^{-1} - (I - s\mathcal{L}_{\delta-it})^{-1} \right) 1(x_j) dt.
\]

Fix \( \epsilon_0 > 0 \) and let \( \rho(t) \) be a symmetric \( C^\infty \)-function on \( \mathbb{R} \) which is equal to 1 on a \( [-\epsilon_0, \epsilon_0] \) and which vanishes outside \( [-2\epsilon_0, 2\epsilon_0] \); one thus decomposes \( \widetilde{W}_j(s, R, \psi) \) as

\[
\widetilde{W}_j(s, R, \psi) = \widetilde{W}_j^{(1)}(s, R, \psi) + \widetilde{W}_j^{(2)}(s, R, \psi) + \widetilde{W}_j^{(3)}(s, R, \psi)
\]

with

\[
\widetilde{W}_j^{(1)}(s, R, \psi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t)(1 - \rho(t)) (I - s\mathcal{L}_{\delta+it})^{-1} 1(x_j) dt \\
+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t)(1 - \rho(t)) (I - s\mathcal{L}_{\delta-it})^{-1} 1(x_j) dt,
\]

\[
\widetilde{W}_j^{(2)}(s, R, \psi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \rho(t) \\
\times \left( (I - s\mathcal{L}_{\delta+it})^{-1} 1(x_j) - \frac{\sigma(\partial X \setminus I_j) h(x_j)}{1 - s\lambda_{\tau}} \right) dt \\
+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \rho(t) \\
\times \left( (I - s\mathcal{L}_{\delta-it})^{-1} 1(x_j) - \frac{\sigma(\partial X \setminus I_j) h(x_j)}{1 - s\lambda_{-\tau}} \right) dt
\]

and

\[
\widetilde{W}_j^{(3)}(s, R, \psi) = \frac{\sigma(\partial X \setminus I_j) h(x_j)}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \rho(t) \left( \frac{1}{1 - s\lambda_{\tau}} - \frac{1}{1 - s\lambda_{-\tau}} \right) dt.
\]

Using Proposition 6.5, letting \( s \to 1 \), one gets

\[
\widetilde{W}_j(R, \psi) = \widetilde{W}_j^{(1)}(R, \psi) + \widetilde{W}_j^{(2)}(R, \psi) + \widetilde{W}_j^{(3)}(R, \psi)
\]
with
\[
\tilde{W}_j^{(1)}(R, \psi) = \lim_{s \searrow 1} \tilde{W}_j^{(1)}(s, R, \psi)
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t)(1 - \rho(t)) \times \left( (I - \mathcal{L}_{\delta+it})^{-1} 1(x_j) + (I - \mathcal{L}_{\delta-it})^{-1} 1(x_j) \right) dt,
\]
\[
\tilde{W}_j^{(2)}(R, \psi) = \lim_{s \searrow 1} W_j^{(2)}(s, R, \psi)
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \rho(t) \times \left( (I - \mathcal{L}_{\delta+it})^{-1} 1(x_j) - \frac{\sigma(\partial X \setminus I_j)h(x_j)}{1 - \lambda_t} \right) dt
+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \rho(t) \times \left( (I - \mathcal{L}_{\delta-it})^{-1} 1(x_j) - \frac{\sigma(\partial X \setminus I_j)h(x_j)}{1 - \lambda_{-t}} \right) dt
\]
and
\[
\tilde{W}_j^{(3)}(R, \psi) = \lim_{s \searrow 1} \tilde{W}_j^{(3)}(s, R, \psi)
= \frac{\sigma(\partial X \setminus I_j)h(x_j)}{2\pi} \int_{\mathbb{R}} e^{itR} \hat{\psi}(t) \rho(t) \text{Re} \left( \frac{1}{1 - \lambda_t} \right) dt.
\]
The functions \(t \mapsto \lambda_t\) has the same regularity as \(t \mapsto L_{\delta+it}\); by Lemma 6.3 one can thus check that
\[
\psi_1 : t \mapsto \hat{\psi}(t)(1 - \rho(t)) \left( (I - \mathcal{L}_{\delta+it})^{-1} 1(x_j) + (I - \mathcal{L}_{\delta-it})^{-1} 1(x_j) \right)
\]
and
\[
\psi_2 : t \mapsto \hat{\psi}(t) \rho(t) \left( (I - \mathcal{L}_{\delta+it})^{-1} 1(x_j) - \frac{\sigma(\partial X \setminus I_j)h(x_j)}{1 - \lambda_t} \right)
+ \left( (I - \mathcal{L}_{\delta-it})^{-1} 1(x_j) - \frac{\sigma(\partial X \setminus I_j)h(x_j)}{1 - \lambda_{-t}} \right)
\]
satisfy the inequality \(|\psi_k(s) - \psi_k(t)| \leq |s - t|^\kappa L \left( \frac{1}{|s-t|} \right), k = 1, 2\). This yields some information on the speed of convergence to 0 of their Fourier transform: indeed, for any \(\theta < \kappa\), there exists \(C_\theta > 0\) such that
\[
(6.20) \quad \left| \tilde{W}_j^{(1)}(R, \psi) \right| \leq \frac{C_\theta}{R^\theta} \quad \text{and} \quad \left| \tilde{W}_j^{(2)}(R, \psi) \right| \leq \frac{C_\theta}{R^\theta}.
\]
Since \(\kappa > 1/2\), one may choose \(\theta \in ]1 - \kappa, \kappa[\) so that
\[
\lim_{R \to +\infty} R^{1-\kappa} L(R) \tilde{W}_j^{(1)}(R, \psi) = \lim_{R \to +\infty} R^{1-\kappa} L(R) \tilde{W}_j^{(2)}(R, \psi) = 0.
\]
On the other hand, by [19, Section 5], one gets for \( 1/2 < \kappa < 1 \)

\[
\lim_{R \to +\infty} R^{1-\kappa} L(R) \tilde{W}^{(3)}_j(R, \psi) = C_j \hat{\psi}(0) = C_j \int_{\mathbb{R}} \psi(x) dx
\]

with \( C_j = \sigma(\partial X \setminus I_j) \frac{\sin \pi \kappa}{\pi \kappa} \); notice that the value \( h(x_j) \) is uniquely determined by the normalization \( \sigma(h) = 1 \). This achieves the proof of Theorem 6.1. \( \square \)

Concluding remarks.

(1) As it is clear from the proof, we do not really need that the metric \( g \) is hyperbolic outside the cusps \( \tilde{C}_i \); hyperbolicity is only needed to describe the initial fundamental polygon \( D \) with a fundamental system of horospheres. As far as the metric \( g \) on \( \tilde{X} \) has negative curvature bounded from above by a negative constant (hence, the conformal structure of \( \partial X \) is well defined and the related properties as in 4.1, 4.3, 4.4 are satisfied) and satisfies the conditions \( H_1 - H_4 \), all the arguments in the proof of Theorem continue to hold.

(2) The same estimates for the orbital function can be deduced for surfaces with \( r + 1 \) punctures and genus \( g > 0 \), which have essentially free fundamental group \( \Gamma \), according to the definition in [29]. This can be achieved by using a different coding as given in [29] (which cannot be used in our case, where adjacent sides are paired by a parabolic element). We limited ourselves to genus 0 surfaces as our coding is particularly simple and explicit in this case.

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Françoise DAL’BO
IRMAR, Université de Rennes-I
Campus de Beaulieu
35042 Rennes Cedex (France)
francoise.dalbo@univ-rennes1.fr

Marc PEIGNÉ
LMPT, UMR 6083
Faculté des Sciences et Techniques
Parc de Grandmont
37200 Tours (France)
peigne@univ-tours.fr

Jean-Claude PICAUD
LMPT, UMR 6083
Faculté des Sciences et Techniques
Parc de Grandmont
37200 Tours (France)
jean-claude.picaud@lmpt.univ-tours.fr

Andrea SAMBUSETTI
Istituto di Matematica G. Castelnuovo
Sapienza Università di Roma
P.le Aldo Moro 5
00185 Roma (Italy)
sambuset@mat.uniroma1.it