ANNALES

DE

L'INSTITUT FOURIER

Luca F. DI CERBO & Matthew STOVER

Bielliptic ball quotient compactifications and lattices in PU$(2,1)$ with finitely generated commutator subgroup

Tome 67, nº 1 (2017), p. 315-328.

<http://aif.cedram.org/item?id=AIF_2017__67_1_315_0>

Ann. Inst. Fourier, Grenoble
67, 1 (2017) 315-328

BIELLIPTIC BALL QUOTIENT COMPACTIFICATIONS AND LATTICES IN PU$(2,1)$ WITH FINITELY GENERATED COMMUTATOR SUBGROUP

by Luca F. DI CERBO & Matthew STOVER (*)

Abstract. — We construct two infinite families of ball quotient compactifications birational to bielliptic surfaces. For each family, the volume spectrum of the associated noncompact finite volume ball quotient surfaces is the set of all positive integral multiples of $\frac{8}{3}\pi^2$, i.e., they attain all possible volumes of complex hyperbolic 2-manifolds. The surfaces in one of the two families all have 2-cusps, so that we can saturate the entire volume spectrum with 2-cusped manifolds. Finally, we show that the associated neat lattices have infinite abelianization and finitely generated commutator subgroup. These appear to be the first known nonuniform lattices in $\mathrm{PU}(2,1)$, and the first infinite tower, with this property.

Résumé. — Nous construisons deux familles infinies de quotients de la boule non-compacts de volume fini qui admettent une compactification birationnelle à une surface bi-elliptique. Pour chaque famille, l'ensemble des volumes consiste en tous les multiples entiers positifs de $\frac{8}{3}\pi^2$, donc il réalise tous les volumes possibles pour une variété hyperbolique complexe de dimension 2. Dans une des deux familles, toutes les surfaces ont exactement deux pointes, donc nous pouvons réaliser tout le spectre des volumes par des surfaces à deux pointes. Enfin, nous montrons que les réseaux associés (sans torsion, y compris á l'infini) ont un abélianisé infini, et un groupe dérivé de type fini. Ceux-ci semblent être les premiers réseaux non-uniformes connus dans $\mathrm{PU}(2,1)$ (ainsi que la première tour infinie) avec cette propriété.

Keywords: Ball quotients and their compactifications, volumes of complex hyperbolic manifolds.
Math. classification: 32Q45, 14M27, 57M50.
(*) The first author was supported by a Grant of the Max Planck Society: "Complex Hyperbolic Geometry and Toroidal Compactifications". The second author was supported by the National Science Foundation under Grant Number NSF DMS-1361000. The second author acknowledges support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network). Both authors thank the Max Planck Institute for Mathematics for its hospitality while this project was being completed. The first author thanks the MPIM for an ideal working environment throughout the whole duration of this project. The authors also thank Stefano Vidussi for comments on an early version of the paper.

1. Introduction

A complete complex hyperbolic surface, or a ball quotient surface, is a complete Hermitian surface of constant holomorphic sectional curvature -1. More precisely, if $\mathbb{B}^2$ denotes the unit ball in $\mathbb{C}^2$ equipped with the normalized Bergman metric, any ball quotient surface is of the form $Y = \mathbb{B}^2/\Gamma$ where $\Gamma \subset \mathrm{PU}(2,1)$ a torsion-free lattice. When Y is noncompact and of finite volume, then Y is a complex 2-manifold with a finite number of topological ends called the cusps of Y. The Baily–Borel or minimal compactification Y^* of Y is a normal projective surface with finitely many singular points in one-to-one correspondence with the cusps of Y. If Γ is neat (see §2.1), then Y^* admits a particularly nice minimal resolution of singularities X that is a smooth projective surface called the smooth toroidal compactification of Y [1, 14]. The exceptional divisors D in X over the singular points of Y^* are disjoint smooth elliptic curves with negative normal bundle in X. Moreover, it is well known that the pair (X, D) is of log-general type, again see §2.1.

Recall that the Kodaira–Enriques classification gives a satisfactory classification of smooth projective surfaces in terms of their Kodaira dimension $\kappa \in \{-\infty, 0, 1, 2\}$. Moreover, we have a quite complete description of surfaces that are not of general type, i.e., the class of surfaces with $\kappa \leqslant 1$. While most smooth toroidal compactifications of neat ball quotients are of general type with ample canonical class (Theorem A in [5]), it is an interesting and open question to decide which projective surfaces can arise as compactifications of ball quotients. More precisely, it would be interesting to classify all smooth projective surfaces that can be realized as smooth toroidal compactifications that are *not* of general type. The first explicit examples were constructed by Hirzebruch in [10], and the examples in his paper not of general type are all birational to a particular Abelian surface and hence have Kodaira dimension zero. Even though it has not been explicitly observed in the literature, it is known to experts that Hirzebruch's work implies that the volume spectrum of complex hyperbolic 2-manifolds is unobstructed, i.e., all possible values for the volume of a finite volume ball quotient manifold are achieved.

In [15], it is claimed that an irregular smooth ball quotient compactification of nonpositive Kodaira dimension is necessarily birational to an Abelian surface. In particular, this result would imply the nonexistence of smooth ball quotient compactifications birational to *bielliptic* surfaces, or for simplicity bielliptic ball quotient compactifications. Unfortunately, the proof of the main result in [15] contains an error, and in fact the result is not true. The purpose of this paper is to produce two infinite families of explicit bielliptic ball quotient compactifications.

THEOREM 1.1. — *For any natural number n, there exists a smooth projective surface X_n birational to a bielliptic surface and neat lattice Γ_n in* $\mathrm{PU}(2,1)$ *of covolume $\frac{8}{3}\pi^2 n$ such that X_n is the smooth toroidal compactification of $\mathbb{B}^2/\Gamma_n$. In particular, the family $\{\mathbb{B}^2/\Gamma_n\}$ saturates the entire admissible volume spectrum of ball quotient surfaces with holomorphic sectional curvature -1.*

Furthermore, the associated smooth compactifications X_n have the property that their Albanese variety is always an elliptic curve. We will prove that this in fact gives a holomorphic fibration of the ball quotient with no multiple fibers. Moreover, we will show that the free rank of $\mathrm{H}_1(\mathbb{B}^2/\Gamma_n;\mathbb{Z})$ is always two. Using these facts along with a topological argument due to Nori [19], we obtain the following group theoretical result. For more on this circle of ideas, we refer to the results previously obtained by the second author [21].

THEOREM 1.2. — *There exists an infinite sequence of nested neat lattices Γ_n in* $\mathrm{PU}(2,1)$ *with infinite abelianization such that the commutator subgroup $[\Gamma_n,\Gamma_n]$ is finitely generated.*

The lattices Γ_n appear to be the first known examples of nonuniform lattices in $\mathrm{PU}(2,1)$ with infinite abelianization and finitely generated commutator subgroup, and the first infinite tower of lattices of any kind. It is well-known amongst experts that the so-called Cartwright–Steger lattice [4] provides a uniform lattice with infinite abelianization and finitely generated commutator subgroup. One can prove this directly using the fact that it fibers over an elliptic curve with no multiple fibers (e.g., see [3, Main Thm.]). Unlike that argument, our proof does not rely on computer calculations.

We prove finite generation by showing that $[\Gamma_n,\Gamma_n]$ is of finite index in the kernel of a homomorphism of Γ_n onto $\mathbb{Z}^2$, and we prove that this kernel is finitely generated. Recall that in [21], the second author constructed the first examples of lattices in $\mathrm{PU}(2,1)$ that maps onto $\mathbb{Z}$ with finitely generated kernel. In contrast, cocompact lattices in $\mathrm{PU}(2,1)$ that map onto $\mathbb{Z}$ with infinitely generated kernel must arise from ball quotients that are virtually fibered over a hyperbolic Riemann surface [18]. The examples presented here are quite different, not only from a group theoretical point of view, as the associated ball quotients fiber over an elliptic curve with as generic fiber either an elliptic curve with three punctures or an elliptic curve with four punctures.

An important tool in the proof is the following general result, first stated in recent work of Kasparian and Sankaran [12, Cor. 4.5], which allows us to conclude that the open ball quotient and its smooth toroidal compactification have the same first betti number.

Theorem 1.3. — *Let $M = \mathbb{B}^2/\Gamma$ be a noncompact ball quotient admitting a smooth toroidal compactification X. Then $b_1(M) = b_1(X)$. Equivalently, the first betti number of M equals its first L^2 betti number.*

Recall that the first betti number $b_1(M)$ is the rank of $H_1(M; \mathbb{Q})$. For the equality between $b_1(X)$ and the first L^2 betti number of M, see [17] (which also proves Theorem 1.3 for n-dimensional ball quotients, $n \geqslant 3$). We note that, while [17] only studies particular arithmetic lattices, the argument for this equality relies only on Hartogs' extension theorem, and is completely general. Kasparian and Sankaran proved Theorem 1.3 using the fundamental group, and we give an alternate very elementary proof using the Mayer–Vietoris sequence (cf. [24]).

The paper is organized as follows. Section 2 collects preliminary facts about the ball and its Bergman metric, smooth toroidal compactifications, the Bogomolov–Miyaoka–Yau inequality, and the basic theory of bielliptic surfaces. In Sections 3 and 4 we give the arguments of the proofs of Theorems 1.1 and 1.2. Finally, in the last section, we construct a different family of ball quotients $\mathbb{B}^2/\Lambda_n$ which can be alternatively used in the proofs of Theorems 1.1 and 1.2. These surfaces have smooth toroidal compactification again biholomorphic to X_n, but they are quite different from the ball quotients $\mathbb{B}^2/\Gamma_n$. In particular, all surfaces in the family $\mathbb{B}^2/\Lambda_n$ have exactly two cusps, while for any $n \geqslant 1$ the surface $\mathbb{B}^2/\Gamma_n$ always has $n+1$ cusps. It follows that we can saturate the whole volume spectrum with 2-cusped ball quotient surfaces (it follows from [21] that one can do this with 4-cusped ball quotients). This is the last result presented in this paper.

Theorem 1.4. — *For any natural number n, there exists a neat lattice Λ_n in $\mathrm{PU}(2,1)$ of covolume $\frac{8}{3}\pi^2 n$ such that the associated finite volume ball quotient $\mathbb{B}^2/\Lambda_n$ has exactly two cusps.*

2. Preliminaries

2.1. Smooth toroidal compactifications, their volumes and the Bogomolov–Miyaoka–Yau inequality

Let $\mathbb{B}^2$ be the unit ball in $\mathbb{C}^2$ with its Bergman metric. The group of biholomorphic isometries of $\mathbb{B}^2$ is isomorphic to $\mathrm{PU}(2,1)$. See the [8] for more on its geometry. Let $\Gamma \subset \mathrm{PU}(2,1)$ be a nonuniform torsion-free lattice, so $\mathbb{B}^2/\Gamma$ is a noncompact finite volume complex hyperbolic manifold. Suppose further that Γ is *neat*, i.e., that the subgroup of $\mathbb{C}$ generated by the eigenvalues (of an appropriate lift to $\mathrm{SU}(2,1)$) of any fixed nontrivial element of Γ is torsion-free. In particular, any neat lattice is automatically torsion-free. This implies that Y admits a particularly nice smooth toroidal compactification X by adding a collection of disjoint elliptic curves D. Given the pair (X, D), the line bundle $K_X + D$ is big and nef, hence (X, D) is of log-general type, i.e., the Kodaira dimension of $K_X + D$ is maximal and the intersection number $(K_X + D)^2$ is strictly positive. See [1] and [14] for the explicit construction and more details.

Let Y be the quotient of $\mathbb{B}^2$ by a neat lattice and X its smooth toroidal compactification. Then $X \setminus Y$ consists of a finite union of disjoint elliptic curves T_i, each having negative normal bundle in X, i.e., $T_i^2 < 0$ for all i. Let $D = \sum T_i$. Hirzebruch–Mumford proportionality [16] implies that

$$\bar{c}_1^2(X, D) = 3\bar{c}_2(X, D), \tag{2.1}$$

where $\bar{c}_1^2$ and $\bar{c}_2$ are the log-Chern numbers of the pair (X, D). Recall that $\bar{c}_2(X, D)$ is the topological Euler number of $X \setminus D$ and $\bar{c}_1^2(X, D)$ is the self-intersection of the log-canonical divisor $K_X + D$.

Next, let X be a smooth projective surface and let D be a normal crossings divisor on X such that $K_X + D$ is big and nef. Then, the logarithmic version of Yau's solution to the Calabi conjecture [23] implies that the log-Chern numbers of the pair (X, D) satisfy the so-called logarithmic Bogomolov–Miyaoka–Yau inequality

$$\bar{c}_1^2(X, D) \leqslant 3\bar{c}_2(X, D).$$

Furthermore, in the case of equality

$$Y = X \setminus D = \mathbb{B}^2/\Gamma$$

for some torsion-free lattice $\Gamma \subset \mathrm{PU}(2,1)$.

Summarizing, a pair (X, D) with X smooth and D a simple normal crossings divisor with $K_X + D$ big and nef, hence of log-general type, achieves equality in the logarithmic Bogomolov–Miyaoka–Yau inequality if and only if it is a smooth toroidal compactification of a ball quotient surface.

We conclude this section by recalling Harder's generalization of the Gauss–Bonnet formula for noncompact finite volume arithmetically defined locally symmetric varieties [9]. This formula is important for us because it gives the basic structure of the volume spectrum of ball quotients. If the holomorphic sectional curvature of a nonuniform neat ball quotient surface is normalized to be -1, the generalized Gauss–Bonnet formula implies that the Euler characteristic of $\mathbb{B}^2/\Gamma$ is proportional to its Riemannian volume. More precisely,

$$\chi(\mathbb{B}^2/\Gamma) = \frac{3}{8\pi^2}\mathrm{Vol}_{-1}(\mathbb{B}^2/\Gamma),$$

where $\chi(\mathbb{B}^2/\Gamma)$ is the topological Euler characteristic of the ball quotient $\mathbb{B}^2/\Gamma$. We therefore conclude that the normalized volume spectrum can at most be the set of all positive integral multiples of $\frac{8}{3}\pi^2$. The volume spectrum then coincides with the set of all positive integral multiples of $\frac{8}{3}\pi^2$ if and only if we can find ball quotient surfaces with topological Euler number n for any $n \geqslant 1$. If this is the case it is natural to say that the volume spectrum is unobstructed, as there are no constraints other than the obvious restriction coming from the fact that the Euler number is a positive integer.

2.2. Bielliptic Surfaces and their basic properties

In this section, we give the definition of *bielliptic* surface, recall their place in the Kodaira–Enriques classification, and give some of their topological properties. By definition, a bielliptic surface is a minimal surface of Kodaira dimension zero and irregularity one. As shown by Bagnera and de Franchis more than a century ago, all such surfaces can be obtained as quotients of products of elliptic curves. More precisely, we have the following result for which we refer to [2, Ch. VI].

THEOREM 2.1 (Bagnera–de Franchis, 1907). — *Let E_λ and E_τ denote elliptic curves associated with the respective lattices $\mathbb{Z}[1,\lambda]$ and $\mathbb{Z}[1,\tau]$ in $\mathbb{C}$ and K be a group of translations of E_τ acting on E_λ such that $E_\lambda/K = \mathbb{P}^1$. Then every bielliptic surface is of the form $(E_\lambda \times E_\tau)/K$ where K has one of the following types:*

(1) *$K = \mathbb{Z}/2\mathbb{Z}$ acting on E_λ by $x \to -x$;*

(2) *$K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acting on E_λ by*

$$x \to -x \quad \text{and} \quad x \to x + \alpha_2,$$

where α_2 is a 2-torsion point;

(3) *$K = \mathbb{Z}/4\mathbb{Z}$ acting on E_λ by $x \to \lambda x$ with $\lambda = i$;*

(4) *$K = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acting on E_λ by*

$$x \to \lambda x \quad \text{and} \quad x \to x + \frac{1+\lambda}{2},$$

with $\lambda = i$;

(5) *$K = \mathbb{Z}/3\mathbb{Z}$ acting on E_λ by $x \to \lambda x$ with $\lambda = e^{\frac{2\pi i}{3}}$;*

(6) *$K = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ acting on E_λ by*

$$x \to \lambda x \quad \text{and} \quad x \to x + \frac{1-\lambda}{3},$$

with $\lambda = e^{\frac{2\pi i}{3}}$;

(7) *$K = \mathbb{Z}/6\mathbb{Z}$ acting on E_λ by $x \to \zeta x$ with $\lambda = e^{\frac{2\pi i}{3}}$ and $\zeta = e^{\frac{\pi i}{3}}$.*

Given E_λ, E_τ and K as in Theorem 2.1, let $B_{\lambda,\tau} = (E_\lambda \times E_\tau)/K$ denote the associated bielliptic surface. The natural projections from $E_\lambda \times E_\tau$ onto its factors give maps

$$\psi_1 : B_{\lambda,\tau} \to E_\lambda/K \cong \mathbb{P}^1, \quad \psi_2 : B_{\lambda,\tau} \to E_\tau/K.$$

Clearly, the map ψ_1 must have multiple fibers, while E_τ/K is an elliptic curve and the map ψ_2 is precisely the Albanese map for the surface $B_{\lambda,\tau}$. In particular, the $\mathbb{Z}$-rank of the homology group $\mathrm{H}_1(B_{\lambda,\tau};\mathbb{Z})$ is always two. Depending on the group K, the associated bielliptic surface $B_{\lambda,\tau}$ may or may not have torsion in $\mathrm{H}_1(B_{\lambda,\tau};\mathbb{Z})$. For the computation of these torsion groups we refer to [20]. Finally, we note that, since the fundamental group is a birational invariant, we have that the first $\mathbb{Z}$-homology group of a blown-up bielliptic surface is the same as the first $\mathbb{Z}$-homology group of its minimal model.

3. Proof of Theorem 1.1

In this section we explicitly construct the surfaces X_n. Let $\rho = e^{2\pi i/3}$ and n be any positive natural number. Then $\Delta_n = \mathbb{Z}[n, 1-\rho]$ is a lattice in $\mathbb{C}$ for each $n \geqslant 1$, with $\Delta = \Delta_1 = \mathbb{Z}[\rho]$. We then have elliptic curves $G_n = \mathbb{C}/\Delta_n$, and let $G = G_1$. Define $a = \frac{1-\rho}{3}$, set $A_n = G \times G_n$, let $[w,z]$ be coordinates on A, and consider the curves

$$E_1 = [z,z], \quad E_2 = [\rho z - a, z], \quad E_3 = [\rho^2 z - 2a, z].$$

Then, for any $i \neq j$ we have

$$E_i \cap E_j = \bigcup_{\substack{0 \leqslant l \leqslant 2 \\ 0 \leqslant m \leqslant n-1}} \left[\frac{2}{3} + la, \frac{2}{3} + la + m\right].$$

Next, consider the degree three automorphism $\varphi : A_n \to A_n$ given by

$$\varphi([w, z]) = [\rho w, z + a],$$

and let $\pi_n : A_n \to B_n$ be the associated degree three étale cover. Then B_n is a bielliptic surface with Albanese map $\mathrm{Alb}_n : B_n \to \mathbb{C}/\mathbb{Z}[n, a]$. Next, we observe that

$$\varphi(E_1) = E_2, \quad \varphi(E_2) = E_3, \quad \varphi(E_3) = E_1,$$

so that the image in B_n of the curves E_1, E_2 and E_3 is a singular irreducible curve C_n with exactly n regular singular points of degree three. For $1 \leqslant j \leqslant n$, let F_j denote the fiber of Alb_n over the point $[\frac{2}{3} + j - 1] \in \mathbb{C}/\mathbb{Z}[n, a]$. The fibers F_j intersect the curve C_n in its n singular points.

We now claim that by blowing up these n points we obtain a smooth toroidal compactification. To see this, let X_n be the blowup of B_n at the n singular points of the curve C_n. Then $K^2_{X_n} = -n$ and $\chi(X_n) = n$. Let T_0 be the proper transform of C_n in X_n and for $i = 1, ..., n$ let T_i be the proper transform of F_i in X_n. We have $T_0^2 = -3n$ and $T_i^2 = -1$ for $1 \leqslant i \leqslant n$. Consider $D_n = \sum_{i=0}^n T_i$. Then, the pair (X_n, D_n) satisfies

$$\begin{aligned} \bar{c}_1^2(X_n, D_n) &= (K_{X_n} + D_n)^2 \\ &= K^2_{X_n} - T_0^2 - \cdots - T_n^2 \\ &= -n + 3n + n \\ &= 3\bar{c}_2(X_n, D_n). \end{aligned}$$

The construction of D_n implies immediately that $K_{X_n} + D_n$ is big and nef, hence (X_n, D_n) is of log-general type. We already saw that $(K_{X_n} + D_n)^2 > 0$. To see that $K_{X_n} + D_n$ is nef, one uses the above calculations to show that every curve not contained in the support of D_n intersects $K_{X_n} + D_n$ positively. Therefore every irreducible curve on X_n intersects $K_{X_n} + D_n$ nonnegatively. Since any nef divisor with positive self-intersection is big, we conclude that $K_{X_n} + D_n$ is a big and nef divisor. It follows immediately that $X_n \smallsetminus D_n$ is biholomorphic to $\mathbb{B}^2/\Gamma_n$ for some nonuniform torsion-free lattice $\Gamma_n \in \mathrm{PU}(2,1)$.

Next, we must show that the lattices Γ_n are indeed *neat*. To see this, first notice that the surfaces X_n form a tower of coverings

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1,$$

which induces a tower of coverings of the associated ball quotients

$$\cdots \to \mathbb{B}^2/\Gamma_n \to \mathbb{B}^2/\Gamma_{n-1} \to \cdots \to \mathbb{B}^2/\Gamma_1.$$

In particular, the lattices Γ_n are nested in a sequence of subgroups of Γ_1

$$\cdots \subset \Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1.$$

As shown in [6], the lattice Γ_1 is neat, and this suffices to imply neatness of all the Γ_n. More precisely, the surface $\mathbb{B}^2/\Gamma_1$ corresponds to Example 2 in Section 6 of [6].

It remains to compute the volumes of the surfaces $\mathbb{B}^2/\Gamma_n$. Since $\chi(\mathbb{B}^2/\Gamma_n) = \chi(X_n) = n$, we conclude that the ball quotient surfaces $\mathbb{B}^2/\Gamma_n$ saturate the whole volume spectrum since, as recalled in §2, we have

$$\mathrm{Vol}_{-1}(\mathbb{B}^2/\Gamma_n) = \frac{8}{3}\pi^2\chi(\mathbb{B}^2/\Gamma_n) = \frac{8}{3}\pi^2 n.$$

The proof of Theorem 1.1 is therefore complete.

4. Proof of Theorems 1.2 and 1.3

It has been noticed many times in the literature that if X is a smooth toroidal compactification of the ball quotient manifold $M = \mathbb{B}^n/\Gamma$, then $b_1(M) \geqslant b_1(X)$, where b_1 denotes the first betti number, i.e., the rank of H_1 with $\mathbb{Q}$ coefficients. For a general result, applicable not only to ball quotient compactifications, we refer to [13, Prop. 2.10]. Further, Murty and Ramakrishnan showed that $b_1(X)$ equals the first L^2 betti number of M, and that $b_1(M) = b_1(X)$ when $n \geqslant 3$ [17]. Kasparian and Sankaran [12, Cor. 4.5] proved that $b_1(M) = b_1(X)$ when $n = 2$ using the fundamental group, and we now give a very elementary proof of that result (cf. [24]).

Proof of Theorem 1.3. — Consider a smooth toroidal compactification (X, D) of the ball quotient manifold M, where D consists of k disjoint elliptic curves. Choose an open neighborhood U of D consisting of k mutually disjoint open sets, one for each irreducible component of D. In what follows, we use H_i to denote homology with $\mathbb{Q}$ coefficients, since we do not care about torsion.

Then U deformation retracts on k disjoint 2-tori. Thus, we have:

$$H_i(U) = \begin{cases} \mathbb{Q}^k & i = 0, 2 \\ \mathbb{Q}^{2k} & i = 1 \\ \{0\} & i = 3, 4 \end{cases}$$

Now define $V = U \smallsetminus D$, so V deformation retracts on the disjoint union of k closed Nil 3-manifolds. We then have:

$$H_i(V) = \begin{cases} \mathbb{Q}^k & i = 0, 3 \\ \mathbb{Q}^{2k} & i = 1, 2 \\ \{0\} & i = 4 \end{cases}$$

Recall that any closed Nil 3-manifold N arising as the cusp cross section in a smooth toroidal compactification is a circle bundle over a 2-torus satisfying

$$H_1(N^3; \mathbb{Z}) = \mathbb{Z}^2 \oplus \text{Torsion},$$

with the torsion part depending on the specific nilmanifolds, while $H_2(N; \mathbb{Z})$ is always torsion-free equal to $\mathbb{Z}^2$ by duality. For these facts we refer to [22].

Next, we apply the Mayer–Vietoris sequence to $X = M \cup U$. First, we consider

$$\cdots \to H_1(X) \to H_0(V) \to H_0(M) \oplus H_0(U) \to H_0(X) \to \{0\}$$

which gives

$$\cdots \to H_1(X) \to \mathbb{Q}^k \to \mathbb{Q} \oplus \mathbb{Q}^k \to \mathbb{Q} \to \{0\}$$

and it follows that $H_1(X) \to H_0(V)$ is zero. This gives an exact sequence

$$\cdots \to H_2(X) \to H_1(V) \to H_1(M) \oplus H_1(U) \to H_1(X) \to \{0\}$$

that becomes

$$\cdots \to H_2(X) \to \mathbb{Q}^{2k} \to H_1(M) \oplus \mathbb{Q}^{2k} \to H_1(X) \to \{0\}$$

It follows immediately that $b_1(M) + 2k = 2k - \ell + b_1(X)$, where ℓ is the dimension of the image of the map $H_2(X) \to H_1(V)$. In other words, $b_1(M) = b_1(X) - \ell$. However, as discussed above, it is well-known that $b_1(M) \geqslant b_1(X)$, so $\ell = 0$ and the theorem follows. $\square$

Remark 4.1. — It is not necessarily the case that $H_1(M; \mathbb{Z}) \cong H_1(X; \mathbb{Z})$. Indeed, we can construct examples, closely related to the examples in this paper, where $H_1(M; \mathbb{Z})$ has torsion but $H_1(X; \mathbb{Z})$ is torsion-free.

Remark 4.2. — The remainder of the Mayer–Vietoris sequence reduces to an exact sequence:

$$\{0\} \to \mathbb{Q}^{k-1} \to H_3(M) \to H_3(X) \to \mathbb{Q}^{2k} \to H_2(M) \oplus \mathbb{Q}^k \to H_2(X) \to \{0\}$$

The image in $H_3(M)$ of $\mathbb{Q}^{k-1}$ is generated by any $k - 1$ of the cusp cross-sections of M (the k^{th} is clearly linearly dependent, since the union of all the cusp cross-sections obviously bounds). For example, one can then conclude that the betti numbers of M satisfy:

$$b_3(M) \geqslant k-1$$
$$b_2(M) - b_3(M) = 1 - b_3(X) + b_2(X)$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. — Recall the surfaces X_n from Theorem 1.1. For any n, let $Alb_n : B_n \to E_n = \mathbb{C}/\mathbb{Z}[n,a]$ be the Albanese map and let $\pi_n : X_n \to B_n$ be the blowup map. Consider $\psi_n : M_n \to E_n$, which is the composition $\psi_n = Alb_n \circ \pi_n \circ i_n$, where $i_n : M_n \to X_n$ is the inclusion. We then obtain a surjective morphism

$$(\psi_n)_* : \pi_1(M_n) \to \pi_1(E_n) \cong \mathbb{Z}^2.$$

The generic fiber F_n of the surjective fibration $\psi_n : M_n \to E_n$ is a reduced torus with three punctures. The singular fibers of this fibrations are reduced smooth rational curves with four punctures. Note that there are exactly n singular fibers corresponding with each of the n exceptional divisors in X_n.

By Lemma 1.5 in [19], the sequence

$$\pi_1(F_n) \to \pi(M_n) \to \pi(E_n) \cong \mathbb{Z}^2 \to 1$$

is exact. We therefore conclude that $\mathrm{Ker}((\psi_n)_*)$ is finitely generated for any n. Also, the free rank of $\mathrm{H}_1(M_n;\mathbb{Z})$ is always two by Theorem 1.3. On the other hand, we have that $\Gamma_n/\mathrm{Ker}((\psi_n)_*) \cong \mathbb{Z}^2$ for any n. It follows that the commutator subgroup $[\Gamma_n, \Gamma_n]$ is of finite index in $\mathrm{Ker}((\psi_n)_*)$. Since finite index subgroups of finitely generated groups are finitely generated, the proof is complete. □

Remark 4.3. — It follows from arguments in [11] that $[\Gamma_n, \Gamma_n]$, while finitely generated, cannot be finitely presented.

5. Two-cusped examples

In this section, we explicitly construct a second distinct family of ball quotients $\mathbb{B}^2/\Lambda_n$ which can be alternatively used in the proofs of the main theorems presented in this paper. These ball quotients have toroidal compactifications biholomorphic to the nonminimal bielliptic surfaces X_n constructed above. In other words, for any n, even if the ball quotients $\mathbb{B}^2/\Gamma_n$ and $\mathbb{B}^2/\Lambda_n$ are not biholomorphic, they nevertheless have biholomorphic smooth toroidal compactifications. For many more examples and a detailed study of the multiple realizations problem of varieties as ball quotient compactifications, we refer to [7].

The most natural way to distinguish the surfaces $\mathbb{B}^2/\Lambda_n$ from the surfaces $\mathbb{B}^2/\Gamma_n$ is to look at their cusps. In particular, we will show that all of surfaces in $\mathbb{B}^2/\Lambda_n$ have exactly two cusps, while we already know that for any $n \geqslant 1$ the surface $\mathbb{B}^2/\Gamma_n$ has $n+1$ cusps. This is clearly enough to show that the two families are distinct for $n \geqslant 2$. For $n = 1$ this is still the case but the argument is different and we refer to end of this section for details.

As before, let $\rho = e^{2\pi i/3}$ and let n be a any positive natural number. Let the lattices Δ_n in $\mathbb{C}$ be as above with $G_n = \mathbb{C}/\Delta_n$ the associated elliptic curves. Again set $a = \frac{1-\rho}{3}$, $A_n = G \times G_n$, let $[w, z]$ be coordinates on A, and consider the curves:

$$E_1 = [z, z], \quad E_2 = [\rho z - a, z], \quad E_3 = [\rho^2 z - 2a, z].$$

Recall that if $i \neq j$, then

$$E_i \cap E_j = \bigcup_{\substack{0 \leqslant m \leqslant n-1 \\ 0 \leqslant l \leqslant 2}} \left[\frac{2}{3} + la, \frac{2}{3} + la + m\right].$$

We again consider $\varphi : A_n \to A_n$ given by

$$\varphi([w, z]) = [\rho w, z + a],$$

which satisfies

$$\varphi(E_1) = E_2, \quad \varphi(E_2) = E_3, \quad \varphi(E_3) = E_1,$$

and let $\pi_n : A_n \to B_n$ the associated degree three étale cover. Then B_n is a bielliptic surface with Albanese map $\mathrm{Alb}_n : B_n \to \mathbb{C}/\mathbb{Z}[n, a]$. The image in B_n of the curves E_1, E_2 and E_3 is a singular irreducible curve C_n with exactly n regular singular points of degree three.

Now we diverge from the previous construction. Consider

$$H_1 = \left[\frac{2}{3}, z\right], \quad H_2 = \left[\frac{2}{3} + a, z\right], \quad H_3 = \left[\frac{2}{3} + 2a, z\right],$$

and observe that

$$\varphi(H_1) = H_2, \quad \varphi(H_2) = H_3, \quad \varphi(H_3) = H_1.$$

To prove this, it suffices to compute that $\frac{2}{3} + a = \frac{2\rho}{3}$ and $\frac{2}{3} + 2a = \frac{2\rho^2}{3}$ modulo Δ. In particular, the image in B_n of the curves H_1, H_2 and H_3 is a single smooth elliptic curve F_n that is a smooth fiber of the map $\phi_n : B_n \to G/\mathbb{Z}_3 \cong \mathbb{P}^1$.

The curves C_n and F_n meet transversally exactly in the n singular points of C_n. Thus, let X_n be the blowup of B_n at the n singular points of C_n, T_0 denote the proper transform of C_n in X_n, and T_1 be the proper transform of F_n in X_n. Set $D_n = \sum_{i=0}^{1} T_i$. Again, one checks that $\overline{c}_1^2(X_n, D_n) = 3\overline{c}_2(X_n, D_n)$ with $K_{X_n} + D_n$ big and nef, so the pair (X_n, D_n) is a smooth toroidal compactification. Thus, $X_n \smallsetminus D_n$ is biholomorphic to $\mathbb{B}^2/\Lambda_n$ for some nonuniform torsion-free lattice in $\Lambda_n \in \mathrm{PU}(2,1)$. It follows immediately from the construction that $\mathbb{B}^2/\Lambda_n$ has exactly two cusps.

The surface $\mathbb{B}^2/\Lambda_1$ corresponds to Example 3 in Section 6 of [6], and it follows as with the previous examples that the lattices Λ_n are neat. The volume calculations are also exactly the same. The family $\mathbb{B}^2/\Lambda_n$ appears to be the first family of 2-cusped ball quotients that saturate the entire volume spectrum (see [21] for 4-cusped examples). This completes the proof of Theorem 1.4.

BIBLIOGRAPHY

[1] A. Ash, D. Mumford, M. Rapoport & Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, 2010.

[2] A. Beauville, *Complex algebraic surfaces*, second ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, 1996.

[3] D. Cartwright, V. Koziarz & S.-K. Yeung, "On the Cartwright–Steger surface", https://arxiv.org/abs/1412.4137.

[4] D. Cartwright & T. Steger, "Enumeration of the 50 fake projective planes", *C. R. Math. Acad. Sci. Paris* **348** (2010), no. 1-2, p. 11-13.

[5] L. F. Di Cerbo, "Finite-volume complex-hyperbolic surfaces, their toroidal compactifications, and geometric applications", *Pacific J. Math.* **255** (2012), no. 2, p. 305-315.

[6] L. F. Di Cerbo & M. Stover, "Classification and arithmeticity of toroidal compactifications with $3\overline{c}_2 = \overline{c}_1^2 = 3$", https://arxiv.org/abs/1505.01414v2.

[7] ———, "Multiple realizations of varieties as ball quotient compactifications", *Michigan Math. J.* **65** (2016), no. 2, p. 441-447.

[8] W. Goldman, *Complex hyperbolic geometry*, Oxford Mathematical Monographs, Oxford University Press, 1999.

[9] G. Harder, "A Gauss-Bonnet formula for discrete arithmetically defined groups", *Ann. Sci. École Norm. Sup.* **4** (1971), p. 409-455.

[10] F. Hirzebruch, "Chern numbers of algebraic surfaces: an example", *Math. Ann.* **266** (1984), no. 3, p. 351-356.

[11] M. Kapovich, "On normal subgroups of the fundamental groups of complex surfaces", Preprint, 1998.

[12] A. Kasparian & G. Sankaran, "Fundamental groups of toroidal compactifications", https://arxiv.org/abs/1501.00053v2.

[13] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton University Press, 1995.

[14] N. Mok, "Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume", in *Perspectives in analysis, geometry, and topology*, Progr. Math., vol. 296, Birkhäuser/Springer, 2012, p. 331-354.

[15] A. Momot, "Irregular ball-quotient surfaces with non-positive Kodaira dimension", *Math. Res. Lett.* **15** (2008), no. 6, p. 1187-1195.

[16] D. Mumford, "Hirzebruch's proportionality theorem in the noncompact case", *Invent. Math.* **42** (1977), p. 239-272.

[17] V. K. Murty & D. Ramakrishnan, "The Albanese of unitary Shimura varieties", in *The zeta functions of Picard modular surfaces*, Univ. Montréal, 1992, p. 445-464.

[18] T. Napier & M. Ramachandran, "Hyperbolic Kähler manifolds and proper holomorphic mappings to Riemann surfaces", *Geom. Funct. Anal.* **11** (2001), no. 2, p. 382-406.

[19] M. V. Nori, "Zariski's conjecture and related problems", *Ann. Sci. École Norm. Sup.* **16** (1983), no. 2, p. 305-344.

[20] F. Serrano, "Divisors of bielliptic surfaces and embeddings in $\mathbf{P}^4$", *Math. Z.* **203** (1990), no. 3, p. 527-533.

[21] M. Stover, "Cusps and b_1 growth for ball quotients and maps onto $\mathbb{Z}$ with finitely generated kernel", https://arxiv.org/abs/1506.06126v2.

[22] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, 1997, Edited by Silvio Levy.

[23] G. Tian & S.-T. Yau, "Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry", in *Mathematical aspects of string theory (San Diego, Calif., 1986)*, Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, 1987, p. 574-628.

[24] S. Zucker, "L_2 cohomology of warped products and arithmetic groups", *Invent. Math.* **70** (1982/83), no. 2, p. 169-218.

Manuscrit reçu le 20 décembre 2015,
révisé le 25 mars 2016,
accepté le 14 juin 2016.

Luca F. DI CERBO
Max-Planck-Institut für Mathematik
Vivatsgasse 7,
53111 Bonn, (Germany)
luca@mpim-bonn.mpg.de

Matthew STOVER
Department of Mathematics
Temple University
1805 N. Broad St.
Philadelphia, PA 19122 (USA)
mstover@temple.edu