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ON A MOTIVIC INVARIANT OF THE ARC-ANALYTIC EQUIVALENCE

by Jean-Baptiste CAMPESATO

Abstract. — To a Nash function germ, we associate a zeta function similar to the one introduced by J. Denef and F. Loeser. Our zeta function is a formal power series with coefficients in the Grothendieck ring $\mathcal{M}$ of $\mathcal{AS}$-sets up to $\mathbb{R}^*$-equivariant $\mathcal{AS}$-bijections over $\mathbb{R}^*$, an analog of the Grothendieck ring constructed by G. Guibert, F. Loeser and M. Merle. This zeta function generalizes the previous construction of G. Fichou but thanks to its richer structure it allows us to get a convolution formula and a Thom–Sebastiani type formula.

We show that our zeta function is an invariant of the arc-analytic equivalence, a version of the blow-Nash equivalence of G. Fichou. The convolution formula allows us to obtain a partial classification of Brieskorn polynomials up to arc-analytic equivalence by showing that the exponents are arc-analytic invariants.

Résumé. — À un germe Nash, nous associons une fonction zêta similaire à la fonction zêta motivique de J. Denef et F. Loeser. Il s'agit d'une série formelle à coefficients dans un anneau de Grothendieck $\mathcal{M}$ des ensembles $\mathcal{AS}$ au-dessus de $\mathbb{R}^*$ à $\mathcal{AS}$-bijection $\mathbb{R}^*$-équivariante près. Cet anneau de Grothendieck est analogue à celui construit par G. Guibert, F. Loeser et M. Merle. Cette fonction zêta généralise les précédentes constructions de G. Fichou. Sa richesse algébrique permet d'obtenir une formule de convolution ainsi qu'une formule de type Thom–Sebastiani.

On démontre que la fonction zêta considérée dans cet article est un invariant de l'équivalence arc-analytique, une caractérisation de l'équivalence blow-Nash de G. Fichou. La formule de convolution permet d'obtenir une classification partielle des polynômes de Brieskorn à équivalence arc-analytique près. Plus précisément, on montre que le type arc-analytique d'un tel polynôme détermine ses exposants.

1. Introduction

In the context of motivic integration, J. Denef and F. Loeser [11, 15] associate a formal power series to a function $f : X \to \mathbb{A}^1$ defined on a non-singular (scheme theoric) algebraic variety. This power series, called

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the motivic zeta function of f, comes in various forms by modifying the ring where its coefficients lie. For instance J. Denef and F. Loeser work with the classical Grothendieck ring of algebraic varieties in order to define the naive motivic zeta function or with an equivariant Grothendieck ring which encodes actions of the roots of unity in order to define the equivariant motivic zeta function. This equivariant structure allows them to get a convolution formula [13] which computes a modified equivariant zeta function of $f \oplus g(x, y) = f(x) + g(y)$ by applying coefficientwise a convolution product to the modified equivariant zeta functions of f and g.

The key lemma for the motivic change of variables formula [12] ensures that their zeta functions are rational. Thus they admit a limit at infinity whose multiplication by -1 is called motivic Milnor fibers since their known realizations coincide with the ones of the classical Milnor fiber. The convolution formula induces a Thom–Sebastiani type formula for these motivic Milnor fibers.

Similarly S. Koike and A. Parusiński [24] associate to a real analytic function germ a formal power series with coefficients in $\mathbb{Z}$ using the Euler characteristic with compact support. This way they define a naive zeta function and two zeta functions with sign (a positive one and a negative one) which play the role of the equivariant zeta function. Particularly these zeta functions with sign admit formulas similar to the ones of Denef–Loeser such as a convolution formula. Thanks to an adaptation to the real analytic case of the key lemma for the motivic change of variables formula, it turns out that Koike–Parusiński zeta functions are invariants of the blow-analytic equivalence of T.-C. Kuo [27] for real analytic germs.

G. Fichou [16] brings a richer structure by using the virtual Poincaré polynomial [16, 30, 31] for $\mathcal{AS}$-sets instead of the Euler characteristic with compact support. This way he defines a naive zeta function and two zeta functions with sign. In order to get a rationality formula, he has to restrict to Nash functions. For this reason he introduces a semialgebraic version of the blow-analytic equivalence for Nash germs, called the blow-Nash equivalence. As it is not known if this relation is an equivalence relation, he introduces a more general notion of blow-Nash equivalence [17] in terms of Nash modifications which is an equivalence relation but two Nash germs which are blow-Nash equivalent in this sense have to satisfy an additional condition to ensure they have the same zeta functions. Recently one uses a third definition of the blow-Nash equivalence, that is midway between the both previous ones [18, 19, 20] by adding the above cited additional condition, but it was not obvious originally whether it is an equivalence relation. We show it in Corollary 7.10 by using the notion of arc-analytic equivalence that we introduce in Section 7.

G. Guibert, F. Loeser and M. Merle [22] introduce an equivariant Grothendieck ring for actions of the multiplicative group on (scheme theoric) algebraic varieties over some fixed algebraic variety (this Grothendieck ring is equivalent to the one of Denef–Loeser for actions of the roots of unity). In this paper we first adapt this framework to $\mathcal{AS}$-sets up to $\mathbb{R}^*$-equivariant $\mathcal{AS}$-bijections over $\mathbb{R}^*$.

This will allow us to define a local zeta function similar to the one of Fichou but with additional structures. After having highlighted the links between our zeta function to the ones of Koike–Parusiński and Fichou, we show that our zeta function is also rational.

The main part of this work consists in proving that our additional structures permit to define a convolution formula allowing us to compute the zeta function of $f \oplus g$. More precisely, we construct a new convolution product on our Grothendieck ring which is compatible with a modified zeta function in the following sense. If we apply coefficientwise this convolution product to the modified zeta functions of f and g, we get the modified zeta function of $f \oplus g$.

We may notice that this modified zeta function has already appeared in [13] and [24] in their respective settings. In a way similar to [24] we prove that our modified zeta function contains the same information as our zeta function.

The next part of this paper is devoted to the study of the behavior of our zeta function under the blow-Nash equivalence. To this purpose we introduce a new relation, the arc-analytic equivalence, we show that it is an equivalence relation and that allows us to avoid using Nash modifications. Moreover, we show that the arc-analytic equivalence coincides with the blow-Nash equivalence in the sense of [18, 19, 20]. Our zeta function and the ones of Fichou are invariants of this relation.

Finally, the convolution formula allows us to prove that the exponents of Brieskorn polynomials are invariants of the arc-analytic equivalence.

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2. Geometric framework

K. Kurdyka [28] introduced semialgebraic arc-symmetric subsets of $\mathbb{R}^d$ which are semialgebraic subsets of $\mathbb{R}^d$ such that given a real analytic arc on $\mathbb{R}^d$ either this arc is entirely included in the subset or it meets it at isolated points only. We are going to work with $\mathcal{AS}$-sets, a slightly different notion introduced by A. Parusiński [34].

Definition 2.1 ([34]). — *We say that a semialgebraic subset $A \subset \mathbb{P}^n_{\mathbb{R}}$ is an $\mathcal{AS}$-set if for every real analytic arc $\gamma : (-1,1) \to \mathbb{P}^n_{\mathbb{R}}$ such that $\gamma((-1,0)) \subset A$ there exists $\varepsilon > 0$ such that $\gamma((0,\varepsilon)) \subset A$.*

Definition 2.2. — *By an $\mathcal{AS}$-map we mean a map between $\mathcal{AS}$-sets whose graph is $\mathcal{AS}$ and by an $\mathcal{AS}$-isomorphism we mean a bijection between $\mathcal{AS}$-sets whose graph is $\mathcal{AS}$.*

Remark 2.3 ([34, Remark 4.2]*).* — The family $\mathcal{AS}$ is the boolean algebra spanned by semialgebraic arc-symmetric subsets of $\mathbb{P}^n_{\mathbb{R}}$ (in the sense of [28, Définition 1.1]). Moreover closed (for the Euclidean topology) $\mathcal{AS}$-sets are exactly the semialgebraic arc-symmetric subsets of $\mathbb{P}^n_{\mathbb{R}}$.

The following proposition results from the proof of [34, Theorem 2.5]. It is an $\mathcal{AS}$ version of [28, Théorème 1.4].

Proposition 2.4. — *The closed $\mathcal{AS}$-subsets of $\mathbb{P}^n_{\mathbb{R}}$ are exactly the closed sets of a noetherian topology on $\mathbb{P}^n_{\mathbb{R}}$.*

Definition 2.5. — *Let Y, X, F be three $\mathcal{AS}$-sets and $p : Y \to X$ be an $\mathcal{AS}$-map. We say that p is a locally trivial $\mathcal{AS}$-fibration with fibre F if every $x \in X$ admits an open $\mathcal{AS}$ neighborhood U such that $\varphi : p^{-1}(U) \to U \times F$ is an $\mathcal{AS}$-isomorphism such that the following diagram commutes*

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi", "\simeq"'] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

The following proposition is a direct consequence of the noetherianity of the $\mathcal{AS}$ topology.

Proposition 2.6. — *A locally trivial $\mathcal{AS}$-fibration $p : Y \to X$ with fibre F is a piecewise trivial fibration with fiber F, i.e. we may break X into finitely many $\mathcal{AS}$-sets $X = \bigsqcup_{\alpha=1}^{k} X_\alpha$ such that $p^{-1}(X_\alpha)$ is $\mathcal{AS}$-isomorphic to $X_\alpha \times F$.*

Nash maps and Nash manifolds were introduced by J. Nash in [32] considering real analytic functions satisfying non-trivial polynomial equations. M. Artin and B. Mazur gave a new description of these objects in [3] in terms of maps which can be lifted to polynomial maps on non-singular irreducible real algebraic sets.

The benefits of Nash functions is that they share good algebraic properties with polynomial functions and good geometric properties with real analytic geometry.

Definition 2.7 (Nash functions and Nash maps [6, 32]). — *Let $U \subset \mathbb{R}^d$ be an open semialgebraic subset. A function $f : U \to \mathbb{R}$ is said to be Nash if it satisfies one of the two following equivalent conditions:*

(1) f is semialgebraic and C^∞.

(2) f is analytic and satisfies a nontrivial polynomial equation.

A map $f : U \to \mathbb{R}^n$ is Nash if its coordinate functions are Nash.

Remark 2.8. — Obviously, the zero locus of a Nash function is an $\mathcal{AS}$-set.

Definition 2.9. — *A Nash submanifold of dimension d is a semialgebraic subset M of $\mathbb{R}^n$ such that for every $x \in M$ there exist an open semialgebraic neighborhood U of x in $\mathbb{R}^n$, an open semialgebraic neighborhood V of 0 in $\mathbb{R}^n$ and a Nash isomorphism $\varphi : U \to V$ such that $\varphi(x) = 0$ and $\varphi(M \cap U) = (\mathbb{R}^d \times \{0\}) \cap V$.*

Example 2.10 ([6, Proposition 3.3.11]*).* — Let $V \subset \mathbb{R}^d$ be a non-singular[(1)] algebraic set. Then, by the Jacobian criterion [6, Proposition 3.3.8] and the Nash inverse theorem [6, Proposition 2.9.7], V is a Nash submanifold of $\mathbb{R}^d$. ⊡

Proposition 2.11. — *Let $f : V \to \mathbb{R}$ be a Nash function defined on an algebraic set V. There exists $\sigma : \tilde{V} \to V$ a finite sequence of (algebraic) blowings-up with non-singular centers such that $\tilde{V}$ is non-singular and $f \circ \sigma : \tilde{V} \to \mathbb{R}$ has only normal crossings*[(2)].

Proof. — Let $G(x,y) = \sum_{i=0}^{p} G_i(x) y^{p-i}$ be a non-trivial polynomial such that $G(x, f(x)) = 0$. We may assume that $G_p \neq 0$. Then f divides G_p seen as a Nash function.

There exists a finite sequence of algebraic blowings-up with non-singular centers such that $G_p \circ \sigma$ is monomial, thus $f \circ \sigma$ is monomial. □

3. A Grothendieck ring

In this section we adapt the Grothendieck ring introduced in [22] to our framework over $\mathbb{R}$. The classical Grothendieck ring of $\mathcal{AS}$-sets doesn't allow

(1) Every point of V are non-singular in the same dimension, see [6, Definition 3.3.4].

(2) i.e.every $x \in \tilde{V}$ admits an open $\mathcal{AS}$-neighborhood such that $f \circ \sigma$ equals a monomial times a nowhere vanishing function on this neighborhood.

one to get a convolution formula similar to the one of Section 6 and the lack of real roots of unity prevents us from using an equivariant Grothendieck ring similar to the one of [11] or [14] as in [13].

Definition 3.1. — *We denote by $K_0(\mathcal{AS})$ the free abelian group spanned by symbols $[X]$ where $X \in \mathcal{AS}$ with the relations*

(1) *If there is a bijection whose graph is $\mathcal{AS}$ between X and Y then $[X] = [Y]$*
(2) *If $Y \subset X$ is a closed $\mathcal{AS}$-subset then $[X \setminus Y] + [Y] = [X]$*

Moreover we have a ring structure induced by the cartesian product:

$$[X \times Y] = [X][Y]$$

The unit of the sum is $0 = [\varnothing]$ and the one of the product is $1 = [\mathrm{pt}]$.

Remark 3.2. — The group $K_0(\mathcal{AS})$ is well-defined since $\mathcal{AS}$ is a set.

Remark 3.3. — If $A, B \in \mathcal{AS}$ then $[A \sqcup B] = [A] + [B]$. We may prove this observation using that an $\mathcal{AS}$-set may be written as a finite disjoint union of locally closed $\mathcal{AS}$-sets.

We denote by $\mathbb{L}_{\mathcal{AS}} = [\mathbb{R}]$ the class of the affine line and by $\mathcal{M}_{\mathcal{AS}} = K_0(\mathcal{AS})\left[\mathbb{L}_{\mathcal{AS}}^{-1}\right]$ the localization by the class of the affine line.

We define the category $\mathrm{Var}^n_{\mathrm{mon}}$ whose objects are $\mathcal{AS}$-sets X endowed with an $\mathcal{AS}$ action of $\mathbb{R}^*$ (i.e. the graph of the map $\mathbb{R}^* \times X \to X$ defined by $(\lambda, x) \mapsto \lambda \cdot x$ is $\mathcal{AS}$) together with an $\mathcal{AS}$-map $\varphi_X : X \to \mathbb{R}^*$ such that $\varphi_X(\lambda \cdot x) = \lambda^n \varphi_X(x)$.

A morphism is an equivariant $\mathcal{AS}$ map $f : X \to Y$ over $\mathbb{R}^*$, i.e. $f(\lambda \cdot x) = \lambda \cdot f(x)$ and the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi_X"'] & & Y \arrow[dl, "\varphi_Y"] \\ & \mathbb{R}^* & \end{tikzcd}$$

Definition 3.4. — *We denote by K_0^n the free abelian group spanned by the symbols $[\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*]$ where $\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^* \in \mathrm{Var}^n_{\mathrm{mon}}$ with the relations*

(1) *If there is $f : X \to Y$ an equivariant bijection over $\mathbb{R}^*$ whose graph is $\mathcal{AS}$, i.e.*

$$\begin{tikzcd} X \arrow[rr, "f", "\simeq"'] \arrow[dr, "\varphi_X"'] & & Y \arrow[dl, "\varphi_Y"] \\ & \mathbb{R}^* & \end{tikzcd}$$

then $[\varphi_X : \mathbb{R}^* \circlearrowleft X \to \mathbb{R}^*] = [\varphi_Y : \mathbb{R}^* \circlearrowleft Y \to \mathbb{R}^*]$

(2) *If* Y *is a closed* $\mathcal{AS}$*-subset of* X *invariant by the action of* $\mathbb{R}^*$ *then*

$$[\varphi_X : \mathbb{R}^* \circlearrowleft X \to \mathbb{R}^*] = \left[\varphi_{X|X\setminus Y} : \mathbb{R}^* \circlearrowleft X \setminus Y \to \mathbb{R}^*\right] + \left[\varphi_{X|Y} : \mathbb{R}^* \circlearrowleft Y \to \mathbb{R}^*\right]$$

(3) *Let* $\varphi : \mathbb{R}^* \circlearrowleft_\tau Y \to \mathbb{R}^* \in \mathrm{Var}^n_{\mathrm{mon}}$ *and* $\psi = \varphi \circ \mathrm{pr}_Y : Y \times \mathbb{R}^m \to \mathbb{R}^*$. *Let* σ *and* σ' *be two actions on* $Y \times \mathbb{R}^m$ *that are two liftings of* τ *on* Y, *i.e.* $\mathrm{pr}_Y(\lambda \cdot_\sigma x) = \mathrm{pr}_Y(\lambda \cdot_{\sigma'} x) = \lambda \cdot_\tau \mathrm{pr}_Y(x)$. *Then* $\psi : \mathbb{R}^* \circlearrowleft_\sigma (Y \times \mathbb{R}^m) \to \mathbb{R}^*$ *and* $\psi : \mathbb{R}^* \circlearrowleft_{\sigma'} (Y \times \mathbb{R}^m) \to \mathbb{R}^*$ *are in* $\mathrm{Var}^n_{\mathrm{mon}}$ *and we add the relation*(3)

$$[\psi : \mathbb{R}^* \circlearrowleft_\sigma (Y \times \mathbb{R}^m) \to \mathbb{R}^*] = [\psi : \mathbb{R}^* \circlearrowleft_{\sigma'} (Y \times \mathbb{R}^m) \to \mathbb{R}^*]$$

Moreover, K_0^n *has a ring structure given by the fiber product over* $\mathbb{R}^*$ *where the action is diagonal. Furthermore the class* $\mathbb{1}_n = [\mathrm{id} : \mathbb{R}^* \circlearrowleft \mathbb{R}^* \to \mathbb{R}^*]$, *where* $\mathbb{R}^*$ *acts on* $\mathbb{R}^*$ *by* $\lambda \cdot a = \lambda^n a$, *is the unit of this product.*

Finally, the cartesian product induces a structure of $K_0(\mathcal{AS})$*-algebra by*

$$K_0(\mathcal{AS}) \to K_0^n,\ [A] \mapsto [A] \cdot \mathbb{1}_n = [A \times \mathbb{R}^* \to \mathbb{R}^*]$$

where the action of $\mathbb{R}^*$ *on* A *is trivial. Particularly we set* $\mathbb{L}_n = \mathbb{L}_{\mathcal{AS}} \mathbb{1}_n = [\mathrm{pr}_{\mathbb{R}^*} : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^*] \in K_0^n$ *and* $\mathcal{M}^n = K_0^n \left[\mathbb{L}_n^{-1}\right]$.

Notation 3.5. — We may simply denote $[\varphi_X : \mathbb{R}^* \circlearrowleft_\sigma X \to \mathbb{R}^*]$ by $[\varphi_X, \sigma]$.

We consider the directed partial order $\prec$ on $\mathbb{N} \setminus \{0\}$ defined by

$$n \prec m \Leftrightarrow \exists k \in \mathbb{N} \setminus \{0\}, n = km$$

For $n \prec m$, we define the morphism $\theta_{mn} : \mathrm{Var}^m_{\mathrm{mon}} \to \mathrm{Var}^n_{\mathrm{mon}}$ which keeps the same object, the same morphism but which replaces the action by $\lambda \cdot_n x = \lambda^k \cdot_m x$. Thus $(\mathrm{Var}^n_{\mathrm{mon}})_{n \geqslant 1}$ is an inductive system and we set $\mathrm{Var}_{\mathrm{mon}} = \varinjlim \mathrm{Var}^n_{\mathrm{mon}}$. We define $K_0 = \varinjlim K_0^n$ and $\mathcal{M} = \varinjlim \mathcal{M}^n$ in the same way.

Thereby K_0 has a natural structure of $K_0(\mathcal{AS})$-algebra and $\mathcal{M}$ has a natural structure of $\mathcal{M}_{\mathcal{AS}}$-algebra. The product unit of K_0 is $\mathbb{1} = \varinjlim \mathbb{1}_n \in K_0$. We notice that $\varinjlim \mathbb{L}_n = \mathbb{L}_{\mathcal{AS}} \mathbb{1} \in K_0$ (for the scalar multiplication) and we denote this class by $\mathbb{L}$. We also notice that $\mathcal{M} = K_0 \left[\mathbb{L}^{-1}\right]$.

Remark 3.6. — $\mathbb{L} = [\mathrm{pr}_{\mathbb{R}^*} : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^*]$

We have a forgetful morphism of $K_0(\mathcal{AS})$-modules

$$\overline{\cdot} : K_0^n \to K_0(\mathcal{AS})$$

(3) This relation will allow us to focus on the angular component, we can move out the other coefficients, e.g. in the rationality formula.

induced by

$$[\varphi_X : \mathbb{R}^* \circlearrowright X \to \mathbb{R}^*] \in K_0^n \mapsto [X] \in K_0(\mathcal{AS})$$

This morphism isn't compatible with the ring structures as shown in the next example. It extends to a morphism of $K_0(\mathcal{AS})$-modules

$$\overline{\cdot} : K_0 \to K_0(\mathcal{AS})$$

and to two morphisms of $\mathcal{M}_{\mathcal{AS}}$-modules

$$\overline{\cdot} : \mathcal{M}^n \to \mathcal{M}_{\mathcal{AS}} \text{ and } \overline{\cdot} : \mathcal{M} \to \mathcal{M}_{\mathcal{AS}}$$

Example 3.7.

$$\overline{\mathbb{1}} = [\mathbb{R}^*] = \mathbb{L}_{\mathcal{AS}} - 1 \neq 1 \in K_0(\mathcal{AS}) \qquad \boxdot$$

Remark 3.8. — Let $A \in K_0$ and $n \in \mathbb{N}$ so that $\frac{A}{\mathbb{L}^n} \in \mathcal{M}$ then

$$\overline{\left(\frac{A}{\mathbb{L}^n}\right)} = \frac{\overline{A}}{\mathbb{L}^n_{\mathcal{AS}}} \in \mathcal{M}_{\mathcal{AS}}$$

4. The motivic local zeta function

In [11] and [15], J. Denef and F. Loeser introduced and studied a motivic global zeta function and defined the motivic Milnor fiber as a limit of this zeta function. In their framework, the realizations of the motivic Milnor fiber and the classical Milnor fiber coincide for the known additive invariants.

In real geometry, a similar work was first initiated by S. Koike and A. Parusiński [24] using the Euler characteristic with compact support. They defined a naive motivic local zeta function for real analytic functions. They also introduced a positive and a negative zeta function in order to study the equivariant side. Next, G. Fichou [16] defined similar zeta functions of Nash functions using the virtual Poincaré polynomial. These constructions are used to classify real singularities respectively in terms of blow-analytic [26, 27] and blow-Nash equivalences.

Following J. Denef and F. Loeser, as well as G. Guibert, F. Loeser and M. Merle [22], we introduce a motivic local zeta function for Nash germs with coefficients in $\mathcal{M}$. This way, we obtain a richer zeta function which, in particular, encodes the equivariant aspects. By means of a resolution, we show that this zeta function is rational as all of the above-cited zeta functions. Finally, we shall exhibit the links with the zeta functions of S. Koike and A. Parusiński as well with those of G. Fichou.

4.1. Definition

Definition 4.1. — *For M a Nash manifold, we set*

$$\mathcal{L}(M) = \{\gamma : (\mathbb{R}, 0) \to M,\ \gamma \text{ real analytic}\}$$

and

$$\mathcal{L}_n(M) = \mathcal{L}(M)/\sim_n$$

where $\gamma_1 \sim_n \gamma_2 \Leftrightarrow \gamma_1 \equiv \gamma_2 \mod t^{n+1}$ in a local Nash coordinate system around $\gamma_1(0) = \gamma_2(0)$.

We have truncation maps

$$\pi_n : \mathcal{L}(M) \to \mathcal{L}_n(M) \text{ and } \pi_n^m : \mathcal{L}_m(M) \to \mathcal{L}_n(M)$$

where $m \geqslant n$. These maps are surjective.

J. Nash first studied truncation of arcs on algebraic varieties in order to study singularities in 1964 [33]. They were then studied by K. Kurdyka, M. Lejeune-Jalabert, A. Nobile, M. Hickel and many others. They are a centerpiece of motivic integration developped by M. Kontsevich and then by J. Denef and F. Loeser.

If $h : M \to N$ is Nash, then $h_* : \mathcal{L}(M) \to \mathcal{L}(N)$ and $h_{*n} : \mathcal{L}_n(M) \to \mathcal{L}_n(N)$ are well-defined and the following diagram commutes

$$\begin{tikzcd}
\mathcal{L}(M) \arrow[r, "h_*"] \arrow[d, "\pi_m"'] & \mathcal{L}(N) \arrow[d, "\pi_m"] \\
\mathcal{L}_m(M) \arrow[r, "h_{*m}"] \arrow[d, "\pi_n^m"'] & \mathcal{L}_m(N) \arrow[d, "\pi_n^m"] \\
\mathcal{L}_n(M) \arrow[r, "h_{*n}"] & \mathcal{L}_n(N)
\end{tikzcd}$$

We refer the reader to [7, §2.4] for the properties of $\mathcal{L}_n(M)$ and $\mathcal{L}(M)$.

Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash germ and let, for $n \geqslant 1$,

$$\mathfrak{X}_n(f) = \left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d),\ \gamma(0) = 0,\ f(\gamma(t)) \equiv ct^n \mod t^{n+1},\ c \neq 0\right\}$$

Then $\mathfrak{X}_n(f)$ is Zariski-constructible and $[\mathfrak{X}_n(f)]$ is well-defined in K_0^n by the morphism $\varphi : \mathfrak{X}_n(f) \to \mathbb{R}^*$ with $\varphi(\gamma) = \mathrm{ac}(f\gamma) = c$ and the action of $\mathbb{R}^*$ given by $\lambda \cdot \gamma(t) = \gamma(\lambda t)$.

Definition 4.2. — *The motivic local zeta function of f is defined by*

$$Z_f(T) = \sum_{n \geqslant 1} [\mathfrak{X}_n(f)]\, \mathbb{L}^{-nd} T^n \in \mathcal{M}[\![T]\!]$$

Example 4.3. — Let $f_k^\varepsilon = \varepsilon x^k$ where $\varepsilon \in \{\pm 1\}$. Then

$$Z_{f_k^\varepsilon}(T) = [f_k^\varepsilon : \mathbb{R}^* \to \mathbb{R}^*] \frac{\mathbb{L}^{-1}T^k}{\mathbb{1} - \mathbb{L}^{-1}T^k}$$

Indeed

$$[\mathfrak{X}_n(f_k^\varepsilon)]\,\mathbb{L}^{-n} = \begin{cases} [f_k^\varepsilon : \mathbb{R}^* \to \mathbb{R}^*]\mathbb{L}^{-q} & \text{if } n = kq \\ 0 & \text{otherwise} \end{cases}$$ ⊡

4.2. Link with previously defined motivic real zeta functions

4.2.1. Koike–Parusiński zeta functions

Definition 4.4. — *We denote by $K_0(SA)$ the free abelian group spanned by symbols $[X]$ where X is semialgebraic with the relations*

(1) If there is a semialgebraic homeomorphism $X \to Y$ then $[X] = [Y]$.
(2) If $Y \subset X$ is closed-semialgebraic then $[X \setminus Y] + [Y] = [X]$.

Moreover we have a ring structure induced by the cartesian product

(3) $[X \times Y] = [X][Y]$.

Remark 4.5 ([35]). — The Grothendieck ring of semialgebraic sets up to semialgebraic homeomorphisms is isomorphic to $\mathbb{Z}$ via the Euler characteristic with compact support, thus every additive invariant factorises through the Euler characteristic with compact support.

Notation 4.6. — We set $\mathbb{L}_{SA} = [\mathbb{R}] \in K_0(SA)$ and $\mathcal{M}_{SA} = K_0(SA)\left[\mathbb{L}_{SA}^{-1}\right]$.

Remark 4.7. — $K_0(SA) \simeq \mathbb{Z} \simeq \mathcal{M}_{SA}$

Remark 4.8. — The cartesian product induces a structure of $K_0(\mathcal{AS})$-module (resp. $\mathcal{M}_{\mathcal{AS}}$-module) on $K_0(SA)$ (resp. $\mathcal{M}_{SA}$).

Proposition 4.9. — *The maps*

$$F^> : \begin{array}{ccc} \mathrm{Var}^n_{\mathrm{mon}} & \longrightarrow & SA \\ (X, \sigma, \varphi : X \to \mathbb{R}^*) & \longmapsto & \varphi^{-1}(\mathbb{R}_{>0}) \end{array}$$

$$F^< : \begin{array}{ccc} \mathrm{Var}^n_{\mathrm{mon}} & \longrightarrow & SA \\ (X, \sigma, \varphi : X \to \mathbb{R}^*) & \longmapsto & \varphi^{-1}(\mathbb{R}_{<0}) \end{array}$$

induce morphisms of $K_0(\mathcal{AS})$-modules (resp. $\mathcal{M}_{\mathcal{AS}}$-modules)

$$F^> : K_0 \to K_0(SA) \qquad \text{(resp. } F^> : \mathcal{M} \to \mathcal{M}_{SA}\text{)}$$
$$F^< : K_0 \to K_0(SA) \qquad \text{(resp. } F^< : \mathcal{M} \to \mathcal{M}_{SA}\text{)}$$

Remark 4.10. — These morphisms are not compatible with the ring structures. Particularly, the following computation shows that the unit is not mapped to the unit

$$\chi_c(F^{>}(\mathbb{1})) = \chi_c(\mathbb{R}_{>0}) = -1 \neq 1 = \chi_c(\mathrm{pt})$$

Remark 4.11. — Given a rational fraction in $\mathcal{M}[\![T]\!]$, we can't directly apply the forgetful morphism or the morphisms $F^{>}, F^{<}$ to the coefficients in the numerator and the denominator. We first have to develop it in series.

For example, whereas $\sum_{n\geqslant 1} \mathbb{1}T^n = \frac{T}{\mathbb{1}-T} \in K_0[\![T]\!]$, we have

$$\sum_{n\geqslant 1} \overline{\mathbb{1}}T^n = (\mathbb{L}_{\mathcal{AS}} - 1)\frac{T}{1-T} \neq \frac{(\mathbb{L}_{\mathcal{AS}}-1)T}{(\mathbb{L}_{\mathcal{AS}}-1) - (\mathbb{L}_{\mathcal{AS}}-1)T} \in K_0(\mathcal{AS})[\![T]\!]$$

This phenomenon is due to the fact that these morphisms are not compatible with the ring structures.

Remark 4.12. — Let $A \in K_0$ then $F^{\varepsilon}\left(\frac{A}{\mathbb{L}^n}\right) = \frac{F^{\varepsilon}(A)}{\mathbb{L}^n_{SA}}$ where $\varepsilon \in \{<,>\}$.

PROPOSITION 4.13. — *Given $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ a Nash germ, we recover from $Z_f(T)$ the motivic zeta functions considered by S. Koike and A. Parusiński* [24] *applying the previous morphisms and the Euler characteristic with compact support at each coefficient:*

$$Z_f^{\chi_c}(T) = \sum_{n\geqslant 1} \chi_c\left(\overline{[\mathfrak{X}_n(f)]}\right)(-1)^{nd}T^n \in \mathbb{Z}[\![T]\!]$$

and

$$Z_f^{\chi_c,\varepsilon}(T) = \sum_{n\geqslant 1} \chi_c\left(F^{\varepsilon}\left([\mathfrak{X}_n(f)]\right)\right)(-1)^{nd}T^n \in \mathbb{Z}[\![T]\!]$$

where $\varepsilon \in \{>,<\}$.

4.2.2. Fichou zeta functions

C. McCrory and A. Parusiński [30] proved that there exists a unique additive invariant of real algebraic varieties which coincides with the Poincaré polynomial for compact non-singular varieties. This construction relies on the weak factorization theorem [1] in order to describe the Grothendieck ring of real algebraic varieties in terms of blowings-up. Then G. Fichou [16] extended this construction to $\mathcal{AS}$-sets up to Nash isomorphisms. Using an extension theorem of F. Guillén and V. Navarro Aznar [23], C. McCrory and A. Parusiński [31] proved the virtual Poincaré polynomial is in fact an invariant of $\mathcal{AS}$-sets up to bijections with $\mathcal{AS}$ graph.

Theorem 4.14 (The virtual Poincaré polynomial for $\mathcal{AS}$-sets [16, 30, 31]). — *There is a unique map $\beta : \mathcal{AS} \to \mathbb{Z}[u]$ which factorises through $K_0(\mathcal{AS})$ as a ring morphism $\beta : K_0(\mathcal{AS}) \to \mathbb{Z}[u]$,*

$$\begin{tikzcd} \mathcal{AS} \arrow[rr, "\beta"] \arrow[dr] & & \mathbb{Z}[u] \\ & K_0(\mathcal{AS}) \arrow[ur, "\beta"'] & \end{tikzcd}$$

such that

- *If $X \in \mathcal{AS}$ is non-empty, then* $\deg \beta(X) = \dim X$ *and the leading coefficient is positive.*
- *If $X \in \mathcal{AS}$ is compact and non-singular, $\beta(X) = \sum_i \dim H_i(X, \mathbb{Z}_2) u^i$.*

Remark 4.15. — We recall the argument of the proof of [31, Theorem 4.6] which explains why the virtual Poincaré polynomial is an invariant of $\mathcal{AS}$-sets up to $\mathcal{AS}$-isomorphism. Let $f : X \to Y$ be an $\mathcal{AS}$-isomorphism (i.e. a bijection whose graph is $\mathcal{AS}$). First we may break X into a finite decomposition of locally compact $\mathcal{AS}$-sets, $X = \sqcup X_i$. Since $f : X_i \to f(X_i)$ is semialgebraic we may break X_i into a finite decomposition of semialgebraic sets $X_i = \sqcup X_{ij}$, where $f : X_{ij} \to f(X_{ij})$ is continuous. As explained in the proof of [31, Theorem 4.6], we may assume that $X_{ij} \in \mathcal{AS}$ using the $\mathcal{AS}$-closure and the noetherianity of the $\mathcal{AS}$-topology. Now we repeat these arguments to $f^{-1} : f(X_{ij}) \to X_{ij}$ in order to get a finite decomposition $X = \sqcup X_{ijk}$ of X into locally compact $\mathcal{AS}$-sets such that $f : X_{ijk} \to f(X_{ijk})$ is a homeomorphism whose graph is $\mathcal{AS}$. By [31, Proposition 4.3], $\beta(X_{ijk}) = \beta(f(X_{ijk}))$. We conclude using the additivity of the virtual Poincaré polynomial.

Proposition 4.16. — *The maps*

$$F^+ : \begin{array}{ccc} \mathrm{Var}^n_{\mathrm{mon}} & \longrightarrow & \mathcal{AS} \\ (X, \sigma, \varphi : X \to \mathbb{R}^*) & \longmapsto & \varphi^{-1}(1) \end{array}$$

$$F^- : \begin{array}{ccc} \mathrm{Var}^n_{\mathrm{mon}} & \longrightarrow & \mathcal{AS} \\ (X, \sigma, \varphi : X \to \mathbb{R}^*) & \longmapsto & \varphi^{-1}(-1) \end{array}$$

induce morphisms of $K_0(\mathcal{AS})$-algebras (resp. $\mathcal{M}_{\mathcal{AS}}$-algebras)

$$F^+ : K_0 \to K_0(\mathcal{AS}) \qquad \textit{(resp. } F^+ : \mathcal{M} \to \mathcal{M}_{\mathcal{AS}}\textit{)}$$
$$F^- : K_0 \to K_0(\mathcal{AS}) \qquad \textit{(resp. } F^- : \mathcal{M} \to \mathcal{M}_{\mathcal{AS}}\textit{)}$$

Remark 4.17. — Let $A \in K_0$ then $F^\varepsilon\left(\frac{A}{\mathbb{L}^n}\right) = \frac{F^\varepsilon(A)}{\mathbb{L}^n_{\mathcal{AS}}}$ where $\varepsilon \in \{-, +\}$.

Proposition 4.18. — *Given $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ a Nash germ, we recover from $Z_f(T)$ the motivic zeta functions considered by G. Fichou in* [16][17] *applying the previous morphisms and the virtual Poincaré polynomial at each coefficient:*

$$Z_f^{\beta}(T) = \sum_{n \geqslant 1} \beta\left(\overline{[\mathfrak{X}_n(f)]}\right) u^{-nd} T^n \in \mathbb{Z}[u, u^{-1}][\![T]\!]$$

and

$$Z_f^{\beta,\varepsilon}(T) = \sum_{n \geqslant 1} \beta\left(F^{\varepsilon}\left([\mathfrak{X}_n(f)]\right)\right) u^{-nd} T^n \in \mathbb{Z}[u, u^{-1}][\![T]\!]$$

where $\varepsilon \in \{+, -\}$.

4.3. Rationality of the motivic local zeta function

4.3.1. Change of variables key lemma

The following lemma is a version of Denef–Loeser change of variables key lemma [12, Lemma 3.4] adapted for our framework. Denef–Loeser key lemma was introduced in order to generalize Kontsevich transformation rule.

Lemma 4.19 (Change of variables key lemma [7, Lemma 4.5]). — *Let $h : M \to \mathbb{R}^d$ be a proper generically one-to-one Nash map with M a non-singular Nash variety. For $e \in \mathbb{N}$, let*

$$\Delta_e = \{\gamma \in \mathcal{L}_n(M),\, \operatorname{ord}_t \operatorname{Jac}_h(\gamma) = e\}$$

Then for $n \geqslant 2e$, $h_{*n}(\pi_n \Delta_e)$ is an $\mathcal{AS}$-set and $h_{*n} : \pi_n \Delta_e \to h_{*n}(\pi_n \Delta_e)$ is a piecewise trivial fibration(4) *with fiber $\mathbb{R}^e$.*

4.3.2. Monomialization

Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash function. By Proposition 2.11 there exists $h : Y \to \mathbb{R}^d$ a finite sequence of algebraic blowings-up such that $f \circ h$ and the jacobian determinant Jac_h have simultaneously only normal crossings.

(4) We mean that we may break $h_{*n}(\pi_n \Delta_e)$ into disjoint $\mathcal{AS}$ parts B_i such that $h_{*n}^{-1}(B_i)$ is $\mathcal{AS}$ and $\mathcal{AS}$-isomorphic to $B_i \times \mathbb{R}^e$.

We denote by $X_0(f) = f^{-1}(0)$ the zero locus of f. We denote by $(E_i)_{i\in A}$ the irreducible $\mathcal{AS}$ components of $h^{-1}(X_0(f))$ and we set

$$N_i = \mathrm{ord}_{E_i} f \circ h$$

and

$$\nu_i - 1 = \mathrm{ord}_{E_i} \mathrm{Jac}\, h$$

We use the usual stratification of Y: for $I \subset A$, we set $E_I = \cap_{i\in I} E_i$ and $E_I^\bullet = E_I \setminus \cup_{j\in A\setminus I} E_j$. Thus $Y = \sqcup_{I\subset A} E_I^\bullet$ and $E_\varnothing = Y \setminus h^{-1}(X_0)$.

For $I \neq \varnothing$, we define U_I using the following classical construction [37, 3.2 Existence Theorem](5).

Let (U_k, x) (resp. (U_l, x')) be a coordinate system on Y around $E_I^\bullet$ with U_k (resp. U_l) an open $\mathcal{AS}$ set such that $(f\circ h)_{|U_k} = u(x)\prod_{i\in I} x_i^{N_i}$ with u a unit and $E_i : x_i = 0$ (resp. $(f \circ h)_{|U_l} = u(x')\prod_{i\in I} x_i'^{N_i}$ with u' a unit and $E_i : x_i' = 0$).

Assume that $U_k \cap U_l \neq \varnothing$. Then on $U_k \cap U_l$ we have $x_i' = \alpha_i^{kl}(x)x_i$ with α_i^{kl} a unit so that $f\circ\sigma(x') = u'(x')\prod_{i\in I} x_i'^{N_i} = \left(u'(x')\prod_{i\in I}\alpha_i^{kl}(x)^{N_i}\right)\prod_{i\in I} x_i^{N_i}$ hence $u(x) = u'(x')\prod_{i\in I}\alpha_i^{kl}(x)^{N_i}$. We index a family of such U_k covering $E_I^\bullet$ by $k \in K$. Then set

$$T = \left\{(x, (a_i), k) \in E_I^\bullet \times (\mathbb{R}^*)^{|I|} \times K,\, x \in U_k\right\}$$

and $U_I = T/\!\sim$ where

$$(x, (a_i), k) \sim (y, (b_i), l) \Leftrightarrow \begin{cases} x = y \\ b_i = \alpha_i^{kl}(x) a_i \end{cases}$$

so that $p_I : U_I \to E_I^\bullet$ is a locally trivial $\mathcal{AS}$-fibration with fiber $(\mathbb{R}^*)^{|I|}$.

For $x \in U_k$ we define $f_I^k : (E_I^\bullet \cap U_k) \times (\mathbb{R}^*)^{|I|} \to \mathbb{R}^*$ by

$$f_I^k(x, (a_i)) = u(x)\prod_{i\in I} a_i^{N_i}$$

This induces a map $f_I : U_I \to \mathbb{R}^*$.

Let $N_I = \gcd_{i\in I}(N_i)$ then there are $\alpha_i \in \mathbb{Z}$ such that $N_I = \sum_{i\in I}\alpha_i N_i$. We consider the action τ of $\mathbb{R}^*$ on U_I locally defined by

$$\lambda \cdot (x, (a_i)_i) = (x, (\lambda^{\alpha_i} a_i)_i)$$

Definition 4.20. — *The class $[f_I : U_I \to \mathbb{R}^*, \tau]$ is well-defined in $K_0^{N_I}$. We shall simply denote it by $[U_I]$.*

(5) Using the adjunction formula, we may prove that U_I is in fact the fiber product of the $U_{i|E_I^\bullet}$, $i \in I$, where U_i is the complement of the null section of the normal bundle of E_i in Y. Then $f_I : U_I \to \mathbb{R}^*$ is just the map induced by $f \circ h$.

PROPOSITION 4.21. — $\overline{[U_I]} = (\mathbb{L}_{\mathcal{AS}} - 1)^{|I|}[E_I^\bullet] \in K_0(\mathcal{AS})$

Proof. — By Proposition 2.6, we have $E_I^\bullet = \bigsqcup_{\alpha=1}^k X_\alpha$ where

$$p_I^{-1}(X_\alpha) \simeq X_\alpha \times (\mathbb{R}^*)^{|I|}$$

Thus, in $K_0(\mathcal{AS})$, we have

$$\begin{aligned}\overline{[U_I]} = \sum_{\alpha=1}^{k}[p_I^{-1}(X_\alpha)] &= \sum_{\alpha=1}^{k}[X_\alpha](\mathbb{L}_{\mathcal{AS}} - 1)^{|I|} \\ &= [E_I^\bullet](\mathbb{L}_{\mathcal{AS}} - 1)^{|I|} \in K_0(\mathcal{AS})\end{aligned}$$ □

4.3.3. A rational expression of the motivic local zeta function

The following theorem is similar to the rationality results for the zeta functions of [15, 16, 22, 24, 29] in their respective frameworks.

THEOREM 4.22 (Rationality formula). — *Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash germ. Let $h : (Y, h^{-1}(0)) \to (\mathbb{R}^d, 0)$ be as in Section 4.3.2. Then*

$$Z_f(T) = \sum_{\varnothing \neq I \subset A} \left[U_I \cap (h \circ p_I)^{-1}(0)\right] \prod_{i\in I} \frac{\mathbb{L}^{-\nu_i}T^{N_i}}{\mathbb{1} - \mathbb{L}^{-\nu_i}T^{N_i}}$$

Proof. — The sum

$$[\mathfrak{X}_n(f)] = \sum_{e\geqslant 1}[\mathfrak{X}_n(f) \cap h_{*n}\Delta_e]$$

is finite since for $\gamma = h_{*n}\varphi \in \mathfrak{X}_n(f) \cap \Delta_e$ we have $\operatorname{ord}_t \operatorname{Jac}_h(\varphi) = \sum(\nu_i - 1)k_i$ and $\operatorname{ord}_t f\gamma = \sum N_i k_i$ where $k_i = \operatorname{ord}_{E_i}\varphi$.

By the change of variables key Lemma 4.19 we have

$$[\mathfrak{X}_n(f) \cap h_{*n}\Delta_e] = [h_{*n}^{-1}\mathfrak{X}_n(f) \cap \Delta_e]\mathbb{L}^{-e}$$

Next

$$[h_{*n}^{-1}\mathfrak{X}_n(f) \cap \Delta_e]\mathbb{L}^{-e} = \sum_{\varnothing \neq I \subset A}[h_{*n}^{-1}\mathfrak{X}_n(f) \cap \Delta_e \cap E_I^\bullet]\mathbb{L}^{-e}$$

and

$$\begin{aligned} &h_{*n}^{-1}\mathfrak{X}_n(f) \cap \Delta_e \cap E_I^\bullet \\ &\qquad = \left\{ \gamma \in \mathcal{L}_n(Y),\ \gamma(0) \in E_I^\bullet \cap h^{-1}(0),\ \begin{array}{l} \sum_{i\in I} k_i(\nu_i - 1) = e, \\ \sum_{i\in I} k_i N_i = n, \\ k_i = \operatorname{ord}_{E_i}\gamma \end{array} \right\}\end{aligned}$$

Let $\gamma \in \mathcal{L}_n(Y)$ with $\gamma(0) \in E_I^\bullet$ then $\operatorname{ac}_{f\circ h}(\gamma) = f_I\left(\gamma(0), (\operatorname{ac}\gamma_i)_{i\in I}\right)$ and $\operatorname{ord}_t(f \circ h)(\gamma(t)) = \sum_{i\in I} k_i N_i$ where $k_i = \operatorname{ord}_{E_i}\gamma$. The action $\lambda \cdot \gamma(t) =$

$\gamma(\lambda t)$ of $\mathbb{R}^*$ on $\mathcal{L}_n(Y)$ induces an action σ of $\mathbb{R}^*$ on U_I locally defined by $\lambda\cdot(x,(a_i)_i)=(x,(\lambda^{k_i}a_i)_i)$. We have $f_I(\lambda\cdot(x,(a_i)_i))=\lambda^n f_I(x,(a_i)_i)$ where $n=\sum_{i\in I}k_iN_i$. So the class $[f_I:U_I\to\mathbb{R}^*,\sigma]$ is well-defined in K_0^n. Let $\beta=\frac{n}{N_I}$, and consider the action σ' of $\mathbb{R}^*$ on U_I locally defined by $\lambda\cdot(x,(a_i)_i)=(x,(\lambda^{\beta\alpha_i}a_i)_i)$, then $[f_I,\sigma']$ is well-defined in K_0^n and $[f_I,\sigma']=[f_I,\tau]$ in K_0.

We locally define W_I and $g_I:W_I\to\mathbb{R}^*$ by

$$W_{I|U}=\{(x,r)\in(E_I^\bullet\cap U)\times\mathbb{R}^*,\,u(x)r^{N_I}\neq 0\}$$

and $g_{I|U}(x,r)=u(x)r^{N_I}$. We locally define an action of $\mathbb{R}^*$ on W_I by $\lambda\cdot(x,r)=(x,\lambda^{\frac{n}{N_I}}r)$ then $[g_I:W_I\to\mathbb{R}^*]$ is well-defined in K_0^n. Next $U_I\underset{\simeq}{\xrightarrow{\psi}}W_I\times\left\{a\in(\mathbb{R}^*)^I,\,\prod a_i^{\frac{N_i}{N_I}}=1\right\}$ where ψ and ψ^{-1} are locally defined by

$$\psi(x,a)=\left(x,r=\prod a_i^{\frac{N_i}{N_I}},(r^{-\alpha_i}a_i)_i\right)$$

and

$$\psi^{-1}(x,r,a)=(x,(r^{\alpha_i}a_i)_i)$$

Thus, by 3.4 (3), we have $[f_I,\sigma]=[f_I,\sigma']\in K_0^n$ since

$$\mathrm{pr}_{W_I}\psi(\lambda\cdot_\sigma y)=\lambda\cdot\mathrm{pr}_{W_I}\psi(y)=\mathrm{pr}_{W_I}\psi(\lambda\cdot_{\sigma'}y)$$

So $[f_I,\sigma]=[f_I,\tau]$ in K_0 (the last one being the class $[U_I]$ of Definition 4.20). Then, we have

$$\begin{aligned}Z_f(T)&=\sum_{n\geqslant1}\sum_{e\geqslant1}\sum_{\varnothing\neq I\subset A}\sum_{\substack{k\in\mathbb{N}^{|I|}\\ \sum k_iN_i=n\\ \sum k_i(\nu_i-1)=e}}\left[U_I\cap(h\circ p_I)^{-1}(0)\right]\mathbb{L}^{-\sum k_i-e}T^n\\ &=\sum_{\varnothing\neq I\subset A}\left[U_I\cap(h\circ p_I)^{-1}(0)\right]\sum_{k\in(\mathbb{N}\setminus\{0\})^{|I|}}\mathbb{L}^{-\sum k_i\nu_i}T^{\sum k_iN_i}\\ &=\sum_{\varnothing\neq I\subset A}\left[U_I\cap(h\circ p_I)^{-1}(0)\right]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i}T^{N_i}}{\mathbb{1}-\mathbb{L}^{-\nu_i}T^{N_i}}\end{aligned}\qquad\square$$

Definition 4.23. — *Denote by $\mathcal{M}[\![T]\!]_{\mathrm{sr}}$ the $\mathcal{M}$-submodule of $\mathcal{M}[\![T]\!]$ spanned by $\mathbb{1}$ and finite products of terms of the forms $\frac{\mathbb{L}^\nu T^N}{\mathbb{1}-\mathbb{L}^\nu T^N}$ and $\frac{\mathbb{1}}{\mathbb{1}-\mathbb{L}^\nu T^N}$ where $N\in\mathbb{N}_{>0}$ and $\nu\in\mathbb{Z}$.*

Remark 4.24. — There exists a unique morphism of $\mathcal{M}$-modules

$$\lim_{T\infty}:\mathcal{M}[\![T]\!]_{\mathrm{sr}}\to\mathcal{M}$$

such that, for $(\nu_i, N_i)_{i\in I} \in (\mathbb{Z}\times\mathbb{N}_{>0})^I$ and $(\nu_j, N_j)_{j\in J} \in (\mathbb{Z}\times\mathbb{N}_{>0})^J$ where I and J are two finite sets, we have

$$\lim_{T\infty}\left(\prod_{i\in I}\frac{\mathbb{L}^{\nu_i}T^{N_i}}{\mathbb{1}-\mathbb{L}^{\nu_i}T^{N_i}}\prod_{j\in J}\frac{\mathbb{1}}{\mathbb{1}-\mathbb{L}^{\nu_j}T^{N_j}}\right)=\begin{cases}(-1)^{|I|} & \text{if } J=\varnothing\\ 0 & \text{otherwise}\end{cases}$$

Definition 4.25. — *Let $f:(\mathbb{R}^d,0)\to(\mathbb{R},0)$ be a Nash germ. By Theorem 4.22,*

$$\mathscr{S}_f = -\lim_{T\infty} Z_f(T) \in \mathcal{M}$$

is well-defined. It is called the motivic Milnor fiber of f.

Remark 4.26. — Given h as in Section 4.3.2, we have the following explicit formula

$$\mathscr{S}_f = \sum_{\varnothing\neq I\subset A} (-1)^{|I|+1}\left[U_I \cap (h\circ p_I)^{-1}(0)\right]$$

Example 4.27. — Let $f(x,y) = x^3 - y^3$, let h be the blowing-up along the origin.

In the y-chart, h is given by $h(X,Y) = (XY, Y)$ then

$$fh(X,Y) = Y^3(X^3-1)$$

where $E_1 : Y = 0$ is the exceptional divisor and $E_2 : X^3 - 1 = 0$ is the strict transform. We also have

$$\mathrm{Jac}_h(X,Y) = \begin{vmatrix} Y & X \\ 0 & 1 \end{vmatrix} = Y$$

And after the change of variables $\tilde{Y} = Y$ and $\tilde{X} = X - 1$ we get

$$fh(\tilde{X},\tilde{Y}) = \tilde{Y}^3\tilde{X}(\tilde{X}^2 + 3\tilde{X} + 3)$$

where $E_1 : \tilde{Y} = 0$ and $E_2 : \tilde{X} = 0$.

In the x-chart, h is given by $h(X',Y') = (X', X'Y')$ then

$$fh(X',Y') = X'^3(1-Y'^3)$$

where $E_1 : X' = 0$ and $E_2 : 1 - Y'^3 = 0$.

We also have

$$\mathrm{Jac}_h(X',Y') = \begin{vmatrix} 1 & 0 \\ Y' & X' \end{vmatrix} = X'$$

And after the change of variables $\tilde{X}' = X'$ and $\tilde{Y}' = Y' - 1$ we get

$$fh(\tilde{X}',\tilde{Y}') = \tilde{X}'^3\tilde{Y}'(-\tilde{Y}'^2 - 3\tilde{Y}' - 3)$$

where $E_1 : \tilde{X}' = 0$ and $E_2 : \tilde{Y}' = 0$.

Thus $N_1 = 3$ and $N_2 = 1$, i.e. $f \circ h^{-1}(0) = 3E_1 + 1E_2$. Also $\nu_1 = 2$ and $\nu_2 = 1$.

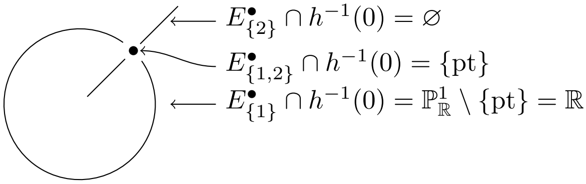
Notice that $(X', Y') = \left(XY, \frac{1}{X}\right)$ and thus that

$$(\tilde{X}', \tilde{Y}') = \left(\tilde{Y}(\tilde{X}+1), \tilde{X}\frac{-1}{\tilde{X}+1}\right)$$

and

$$(\tilde{X}, \tilde{Y}) = \left(\tilde{Y}'\frac{-1}{\tilde{Y}'+1}, \tilde{X}'(\tilde{Y}'+1)\right).$$

Now we can construct $f_{\{1\}} : U_{\{1\}} \to \mathbb{R}^*$, $f_{\{2\}} : U_{\{2\}} \to \mathbb{R}^*$ and $f_{\{1,2\}} : U_{\{1,2\}} \to \mathbb{R}^*$ following the construction before Definition 4.20.



Thus

- $U_{\{1\}} \cap (h \circ p_I)^{-1}(0) = (\mathbb{P}^1 \setminus [1:1]) \times \mathbb{R}^*$ and $f_{\{1\}}([r:s], a) = a^3(r^3 - s^3)$.
- $U_{\{2\}} \cap (h \circ p_I)^{-1}(0) = \varnothing$.
- $U_{\{1,2\}} \cap (h \circ p_I)^{-1}(0) = (\mathbb{R}^*)^2$ and $f_{\{1,2\}}(a,b) = a^3 b$.

Finally

$$\begin{aligned} Z_f(T) &= \left[f_{\{1\}} : (\mathbb{P}^1 \setminus [1:1]) \times \mathbb{R}^* \to \mathbb{R}^*\right] \frac{\mathbb{L}^{-2}T^3}{\mathbb{1} - \mathbb{L}^{-2}T^3} \\ &\quad + \left[f_{\{1,2\}} : (\mathbb{R}^*)^2 \to \mathbb{R}^*\right] \frac{\mathbb{L}^{-2}T^3}{\mathbb{1} - \mathbb{L}^{-2}T^3} \frac{\mathbb{L}^{-1}T}{\mathbb{1} - \mathbb{L}^{-1}T} \\ &= \mathbb{L}\frac{\mathbb{L}^{-2}T^3}{\mathbb{1} - \mathbb{L}^{-2}T^3} + (\mathbb{L} - \mathbb{1})\frac{\mathbb{L}^{-2}T^3}{\mathbb{1} - \mathbb{L}^{-2}T^3} \frac{\mathbb{L}^{-1}T}{\mathbb{1} - \mathbb{L}^{-1}T} \end{aligned}$$ ⊡

We recover the rationality of Koike–Parusiński or Fichou zeta functions.

Proposition 4.28 ([16, Proposition 3.2 & Proposition 3.5]). — *For $\varepsilon \in \{+,-\}$, we have*

$$Z_f^{\beta}(T) = \sum_{\varnothing \neq I \subset A} (u-1)^{|I|} \beta\left(E_I^{\bullet} \cap h^{-1}(0)\right) \prod_{i \in I} \frac{u^{-\nu_i} T^{N_i}}{1-u^{-\nu_i} T^{N_i}} \in \mathbb{Z}[u,u^{-1}][\![T]\!]$$

$$Z_f^{\beta,\varepsilon}(T) = \sum_{\varnothing \neq I \subset A} \beta\left(U_I \cap (h \circ p_I)^{-1}(0) \cap f_I^{-1}(\varepsilon 1)\right) \prod_{i \in I} \frac{u^{-\nu_i} T^{N_i}}{1-u^{-\nu_i} T^{N_i}} \in \mathbb{Z}[u,u^{-1}][\![T]\!]$$

Proof. — We apply the forgetful morphism (resp. $F^{\pm}$) to the coefficients of

$$Z_f(T) = \sum_{\varnothing \neq I \subset A} \left[U_I \cap (h \circ p_I)^{-1}(0)\right] \sum_{k \in (\mathbb{N}\setminus\{0\})^{|I|}} \mathbb{L}^{-\sum k_i \nu_i} T^{\sum k_i N_i} \qquad \square$$

Example 4.29. — Let $f(x,y) = x^3 - y^3$. We deduce from Example 4.27 that

$$Z_f^{\beta}(T) = u(u-1)\frac{u^{-2}T^3}{1-u^{-2}T^3} + (u-1)^2 \frac{u^{-2}T^3}{1-u^{-2}T^3}\frac{u^{-1}T}{1-u^{-1}T}$$

$$Z_f^{\beta,+}(T) = Z_f^{\beta,-}(T) = u\frac{u^{-2}T^3}{1-u^{-2}T^3} + (u-1)\frac{u^{-2}T^3}{1-u^{-2}T^3}\frac{u^{-1}T}{1-u^{-1}T} \qquad \boxdot$$

Proposition 4.30 ([24, (1.1)&(1.2)]). — *For $\varepsilon \in \{<,>\}$, we have*

$$Z_f^{\chi_c}(T) = \sum_{\varnothing \neq I \subset A} (-2)^{|I|} \chi_c\left(E_I^{\bullet} \cap h^{-1}(0)\right) \prod_{i \in I} \frac{(-1)^{\nu_i} T^{N_i}}{1-(-1)^{\nu_i} T^{N_i}} \in \mathbb{Z}[\![T]\!]$$

$$Z_f^{\chi_c,\varepsilon}(T) = \sum_{\varnothing \neq I \subset A} \chi_c\left(U_I \cap (h \circ p_I)^{-1}(0) \cap f_I^{-1}(\mathbb{R}_{\varepsilon 0})\right) \prod_{i \in I} \frac{(-1)^{\nu_i} T^{N_i}}{1-(-1)^{\nu_i} T^{N_i}} \in \mathbb{Z}[\![T]\!]$$

Example 4.31. — Let $f(x,y) = x^3 - y^3$. We deduce from Example 4.27 that

$$Z_f^{\chi_c}(T) = 2\frac{T^3}{1-T^3} + 4\frac{T^3}{1-T^3}\frac{T}{1+T}$$

$$Z_f^{\chi_c,>}(T) = Z_f^{\chi_c,<}(T) = \frac{T^3}{1-T^3} + 2\frac{T^3}{1-T^3}\frac{T}{1+T} \qquad \boxdot$$

5. Example: non-degenerate polynomials

In this section we follow G. Guibert [21, §2.1] to compute the motivic local zeta function of a non-degenerate polynomial. We may find similar construction for the topological case [10, §5] and for the p-adic case [9]. Some ideas of these constructions go back to [25] and [38].

We may find the first adaptation in the real non-equivariant case using the virtual Poincaré polynomial in [20].

5.1. The Newton polyhedron of a polynomial

We first recall some definitions related to the Newton polyhedron of a polynomial. We refer the reader to [2, §8] for more details.

Definition 5.1. — *The Newton polyhedron Γ_f of*

$$f(x) = \sum_{\nu \in \mathbb{N}^d} c_\nu x^\nu \in \mathbb{R}[x_1, \dots, x_d]$$

is the convex hull of

$$\bigcup_{\nu \in \mathbb{N}^d,\, c_\nu \neq 0} \nu + (\mathbb{R}_+)^d$$

in $(\mathbb{R}_+)^d$*.*

Definition 5.2. — *Given a face $\tau \in \Gamma_f$, we set $f_\tau(x) = \sum_{\nu \in \tau} c_\nu x^\nu$.*

Definition 5.3. — *A polynomial f is said to be non-degenerate if for every compact face*[(6)] *τ of Γ_f, the polynomials*

$$f_\tau,\ \frac{\partial f_\tau}{\partial x_i},\ 1 \leqslant i \leqslant d,$$

have no common zero in $(\mathbb{R}^*)^d$*.*

Definition 5.4. — *For $k \in (\mathbb{R}_+)^d$, we define the supporting function*

$$m(k) = \inf_{x \in \Gamma_f} \{k \cdot x\}$$

Remark 5.5. — Actually $m(k) = \min_{x \in \Gamma_f} \{k \cdot x\}$ since we can take the infimum in the compact set $C = \mathrm{Conv}(\{\nu,\, c_\nu \neq 0\})$ using that

$$\Gamma_f = C + (\mathbb{R}_+)^d$$

(6) We mean the proper faces (not only the facets) and Γ_f itself.

Definition 5.6. — *For $k \in (\mathbb{R}_+)^d$, we define the trace of k by*

$$\tau(k) = \{x \in \Gamma_f, k \cdot x = m(k)\}$$

Proposition 5.7.

- $\tau(0) = \Gamma_f$
- *For $k \neq 0$, $\tau(k)$ is a proper face of Γ_f*
- *$\tau(k)$ is a compact face if and only if $k \in (\mathbb{R}_+ \setminus \{0\})^d$*

Notation 5.8. — For τ a face of Γ_f, we define the cone of τ by

$$\sigma(\tau) = \left\{k \in (\mathbb{R}_+)^d, \tau(k) = \tau\right\}$$

and we set $\tilde{\sigma}(\tau) = \sigma(\tau) \cap \mathbb{N}^d$.

Notation 5.9. — Let τ be a facet (i.e. a face of codimension 1), then τ is supported by a hyperplane which contains at least one point with integer coefficients. So this hyperplane has a unique equation

$$\sum_{i=1}^{d} a_i x_i = N$$

with $a_i, N \in \mathbb{N}$ and $\gcd\{a_i\} = 1$. Thus $e^\tau = (a_1, \dots, a_d)$ is the unique primitive vector in $\mathbb{N}^d \setminus \{0\}$ which is perpendicular to τ.

Lemma 5.10. — *Let τ be a proper face of Γ_f, denote by $\tau_1, \dots, \tau_l$ the facets containing τ. Then*

$$\sigma(\tau) = \left\{\sum_{i=1}^{l} \alpha_i e^{\tau_i},\ \alpha_i \in \mathbb{R}_+^*\right\}$$

5.2. The motivic zeta function and Milnor fiber of a non-degenerate polynomial

We may easily adapt the proof of [36, Proposition 3.13] in order to get the following lemma.

Lemma 5.11. — *For τ a compact face of Γ_f and $k \in \tilde{\sigma}(\tau)$ we define the class*

$$\left[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0), \sigma_k\right] \in K_0^{m(k)}$$

where the morphism is f_τ and the action is given by $\lambda \cdot_{\sigma_k} x = (\lambda^{k_i} x_i)_i$. Then for $k, k' \in \sigma(\tau)$,

$$\left[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0), \sigma_k\right] = \left[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0), \sigma_{k'}\right] \in K_0$$

We simply denote it by $\left[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0)\right]$.

Lemma 5.12. — *Let $f:\mathbb{R}^d\to\mathbb{R}$ be a weighted homogeneous polynomial of weight $(k_1,\dots,k_d;m)$ with $k_i\in\mathbb{N}_{>0}$ such that $f,\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_d}$ have no common zero in $(\mathbb{R}^*)^d$. For $l\geqslant 1$, we consider*

$$A_l=\left\{\gamma\in\mathcal{L}_{m+l}(\mathbb{R}^d),\ \operatorname{ord}_t\gamma=(k_1,\dots,k_d),\ \operatorname{ord}_t f\gamma=m+l\right\}$$

with the morphism $\varphi:A_l\to\mathbb{R}^*$ which associates to γ the angular component of $f\gamma$ and with the action $\lambda\cdot\gamma(t)=\gamma(\lambda t)$. Thus $[A_l]\in K_0^{m+l}$ is well-defined. Then

$$[A_l]=\left[f^{-1}(0)\cap(\mathbb{R}^*)^d\right]\mathbb{L}^{l(d-1)+md-\sum_{r=1}^d k_r}\in K_0^{m+l}$$

where $\left[f^{-1}(0)\cap(\mathbb{R}^*)^d\right]\in K_0(\mathcal{AS})$.

Proof. — We set $\gamma(t)=(\gamma_1(t),\dots,\gamma_d(t))$ where $\gamma_r(t)=t^{k_r}\left(\sum_{i=0}^{m-k_r}a_{ri}t^i\right)$ with $a_{r0}\neq 0$. The coefficient of t^m in $f\gamma(t)$ is $f(a_{10},\dots,a_{d0})$ and, for $i=1,\dots,l$, the one of t^{m+i} is of the form

$$g_i(a_{1i},\dots,a_{di})-P_i$$

where g_i is a linear form in $(a_{1i},\dots,a_{di})$ which is non-zero since $\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_d}$ have no common zero in $(\mathbb{R}^*)^d$ and where P_i is a polynomial in $a_{1j},\dots,a_{dj}$ for $j<i$.

Thus

$$A_l=\left\{(a_{ri})_{\substack{r=1,\dots,d\\ i=0,\dots,m-k_r}}\ \begin{array}{lcll} f(a_{10},\dots,a_{d0}) & = & 0 & \\ g_i(a_{1i},\dots,a_{di})-P_i & = & 0 & \text{for i=1,\dots,l-1}\\ g_l(a_{1l},\dots,a_{dl})-P_l & \neq & 0 & \end{array}\right\}$$

Up to reordering the coordinates we may assume that the coefficient of a_{dl} in g_l is non-zero. Then we set

$$B_l=\left\{(\tilde a_{ri})_{\substack{r=1,\dots,d\\ i=0,\dots,m-k_r}}\ \begin{array}{lcll} f(\tilde a_{10},\dots,\tilde a_{d0}) & = & 0 & \\ g_i(\tilde a_{1i},\dots,\tilde a_{di})-P_i & = & 0 & \text{for i=1,\dots,l-1}\\ \tilde a_{dl} & \neq & 0 & \end{array}\right\}$$

and we define the following $\mathcal{AS}$-bijection over $\mathbb{R}^*$

$$\begin{tikzcd} A_l \arrow[rr] \arrow[dr, "x^{m+l}\circ(g_l-P_l)"'] & & B_l \arrow[dl, "x^{m+l}\circ\mathrm{pr}_{\tilde a_{dl}}"] \\ & \mathbb{R}^* & \end{tikzcd}$$

by $\tilde a_{ri}=a_{ri}$ if $(r,i)\neq(d,l)$ and $\tilde a_{dl}=g_l(a_{1l},\dots,a_{dl})-P_l$

Therefore

$$\begin{aligned}[A_l] &= [B_l] \\ &= \left[f^{-1}(0) \cap (\mathbb{R}^*)^d\right] \left(\mathbb{L}^{d-1}\right)^{l-1} \mathbb{L}^{d-1} \mathbb{1} \mathbb{L}^{\sum_{r=1}^{d}(m-k_r)} \\ &= \left[f^{-1}(0) \cap (\mathbb{R}^*)^d\right] \mathbb{L}^{l(d-1)-md-\sum_{r=1}^{d} k_r} \end{aligned}$$

□

Definition 5.13. — *For $m \in \mathbb{Z}$ we define $\mathcal{F}^m\mathcal{M}$ as the subgroup of $\mathcal{M}$ spanned by elements $[S]\mathbb{L}^{-i}$ with $i - \dim S \geqslant m$. We denote by $\widehat{\mathcal{M}}$ the completion of $\mathcal{M}$ with respect to the filtration $\mathcal{F}^{\cdot}\mathcal{M}$.*

Remark 5.14. — The ring $\widehat{\mathcal{M}}$ allows us to handle terms of the form $\sum_i \mathbb{L}^{-ki}$ that may appear in the following formula.

Theorem 5.15. — *Let f be non-degenerate, then the following equality holds in $\widehat{\mathcal{M}}[\![T]\!]$*

$$Z_f(T) = \sum_{\tau \text{ compact face}} \left(\left[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0)\right] + \left[f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d\right] \frac{\mathbb{L}^{-1}T}{\mathbb{1} - \mathbb{L}^{-1}T} \right) \cdot \sum_{k \in \tilde{\sigma}(\tau)} \mathbb{L}^{-\sum k_i} T^{m(k)}$$

where $\left[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0)\right]$ is defined in Lemma 5.11 and $\left[f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d\right] \in K_0(\mathcal{AS})$.

Proof. — We first notice that

$$(\mathbb{N} \setminus \{0\})^d = \bigsqcup_{\tau \text{ compact face of } \Gamma_f} \tilde{\sigma}(\tau)$$

Thus[7]

$$\begin{aligned} &Z_f(T) \\ &= \sum_{n \geqslant 1} [\mathfrak{X}_n(f)]\, \mathbb{L}^{-nd} T^n \\ &= \sum_{\tau \text{ compact face}} \sum_{k \in \tilde{\sigma}(\tau)} \sum_{n \geqslant m(k)} \left[\gamma \in \mathcal{L}_n(\mathbb{R}^d),\, \operatorname{ord}_t \gamma = k,\, \operatorname{ord}_t f\gamma = n\right] \mathbb{L}^{-nd} T^n \end{aligned}$$

(7) The condition $\gamma(0) = 0$ is satisfied since $k \in (\mathbb{N} \setminus \{0\})^d$.

$$\begin{aligned}
&= \sum_{\tau} \Bigg(\sum_{k\in\tilde\sigma(\tau)} \big[\gamma\in\mathcal{L}_{m(k)}(\mathbb{R}^d),\, \mathrm{ord}_t\,\gamma = k,\, \mathrm{ord}_t\, f\gamma = m(k)\big]\, \mathbb{L}^{-m(k)d} T^{m(k)} \\
&\qquad + \sum_{k\in\tilde\sigma(\tau)} \sum_{l\geqslant 1} \big[\gamma\in\mathcal{L}_{m(k)+l}(\mathbb{R}^d),\, \mathrm{ord}_t\,\gamma = k,\, \mathrm{ord}_t\, f\gamma = m(k)+l\big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{L}^{-(m(k)+l)d} T^{m(k)+l} \Bigg) \\
&= \sum_{\tau} \big(Z_\tau^{=}(T) + Z_\tau^{>}(T)\big)
\end{aligned}$$

Fix τ a compact face of Γ_f and $k \in (\mathbb{N}\setminus\{0\})^d$ such that $\tau(k) = \tau$. Let $\gamma \in \mathcal{L}_{m(k)}(\mathbb{R}^d)$ satisfying $\mathrm{ord}_t\,\gamma = k$ and $\mathrm{ord}_t\, f\gamma = m(k)$. Then $\gamma(t) = (t^{k_i}a_i(t))_i$ with $a_i(0) \neq 0$ and

$$f\gamma(t) = \sum_\nu c_\nu a(t)^\nu t^{k\cdot\nu} = f_\tau(a(0))t^{m(k)} + t^{m(k)+1}R(t)$$

Thus $\mathrm{ord}_t\, f\gamma = \mathrm{ord}_t\, f_\tau\gamma$ and $\mathrm{ac}\, f\gamma = \mathrm{ac}\, f_\tau\gamma$ and

$$Z_\tau^{=}(T) = \sum_{k\in\tilde\sigma(\tau)} \big[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0), \sigma_k\big]\, \mathbb{L}^{-\sum k_i} T^{m(k)}$$

where the morphism is f_τ and the action σ_k is the one induced by the action on $\mathcal{L}_{m(k)}(\mathbb{R}^d)$, i.e. $\lambda \cdot_{\sigma_k} (x_1,\dots,x_d) = (\lambda^{k_1}x_1,\dots,\lambda^{k_d}x_d)$. Thus $\big[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0), \sigma_k\big]$ is well-defined in $K_0^{m(k)}$. By Lemma 5.11, we get

$$Z_\tau^{=}(T) = \big[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0)\big] \sum_{k\in\tilde\sigma(\tau)} \mathbb{L}^{-\sum k_i} T^{m(k)}$$

Now let $l \geqslant 1$ and $\gamma \in \mathcal{L}_{m(k)+l}(\mathbb{R}^d)$ satisfying $\mathrm{ord}_t\,\gamma = k$ and $\mathrm{ord}_t\, f\gamma = m(k)+l$. We set $\gamma(t) = (t^{k_1}a_1(t),\dots,t^{k_d}a_d(t))$ with $a_i(0) \neq 0$. By Lemma 5.12 we get

$$\begin{aligned}
Z_\tau^{>}(T) &= \sum_{k\in\tilde\sigma(\tau)} \sum_{l\geqslant 1} \big[f_\tau^{-1}(0)\cap(\mathbb{R}^*)^d\big]\, \mathbb{L}^{-\sum k_i - l} T^{m(k)+l} \\
&= \big[f_\tau^{-1}(0)\cap(\mathbb{R}^*)^d\big] \frac{\mathbb{L}^{-1}T}{\mathbb{1} - \mathbb{L}^{-1}T} \sum_{k\in\tilde\sigma(\tau)} \mathbb{L}^{-\sum k_i} T^{m(k)} \\
&= \big[f_\tau^{-1}(0)\cap(\mathbb{R}^*)^d\big] \frac{\mathbb{L}^{-1}T}{\mathbb{1} - \mathbb{L}^{-1}T} \sum_{k\in\tilde\sigma(\tau)} \mathbb{L}^{-\sum k_i} T^{m(k)} \qquad \square
\end{aligned}$$

Corollary 5.16. — *If f is non-degenerate then*

$$S_f = -\sum_{\tau \text{ compact face}} (-1)^{d-\dim\tau}\left(\left[(\mathbb{R}^*)^d \setminus f_\tau^{-1}(0)\right] - \left[f_\tau^{-1}(0)\cap(\mathbb{R}^*)^d\right]\cdot \mathbb{1}\right) \in \widehat{\mathcal{M}}$$

Proof. — It's a direct application of [8, p. 1006–1007] [21, Lemme 2.1.5], noticing that $\dim\tau + \dim\sigma(\tau) = d$. ☐

Example 5.17. — Let $f(x,y) = x^3 - y^3$. The Newton polyhedron of f has 3 compact faces:

(1) $\tau_1 = \{\lambda(0,3) + (1-\lambda)(3,0),\ \lambda \in [0,1]\}$ with $\tilde\sigma(\tau_1) = \mathbb{N}_{>0}(1,1)$.
(2) $\tau_2 = \{(0,3)\}$ with $\tilde\sigma(\tau_2) = (2,1) + \mathbb{N}(1,1) + \mathbb{N}(0,1)$.
(3) $\tau_3 = \{(3,0)\}$ with $\tilde\sigma(\tau_3) = (1,2) + \mathbb{N}(1,1) + \mathbb{N}(0,1)$.

Thus

$$\begin{aligned}
Z_f(T) &= \Bigg([(x,y)\in(\mathbb{R}^*)^2, x^3-y^3\neq 0] \\
&\qquad + \overline{[(x,y)\in(\mathbb{R}^*)^2, x^3-y^3=0]}\frac{\mathbb{L}^{-1}T}{\mathbb{1}-\mathbb{L}^{-1}T}\Bigg)\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3} \\
&\quad + \left[(x,y)\in(\mathbb{R}^*)^2 \mapsto x^3\right]\frac{\mathbb{1}}{\mathbb{L}-\mathbb{1}}\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3} \\
&\quad + \left[(x,y)\in(\mathbb{R}^*)^2 \mapsto -y^3\right]\frac{\mathbb{1}}{\mathbb{L}-\mathbb{1}}\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3} \\
&= \Bigg(\left[(x,y)\in(\mathbb{R}^*)^2, x^3-y^3\neq 0\right] + \left[x\in\mathbb{R}^*\mapsto x^3\right] + \left[y\in\mathbb{R}^*\mapsto -y^3\right] \\
&\qquad + (\mathbb{L}-\mathbb{1})\frac{\mathbb{L}^{-1}T}{\mathbb{1}-\mathbb{L}^{-1}T}\Bigg)\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3} \\
&= \left((\mathbb{L}-\mathbb{1}-\mathbb{1}) + \mathbb{1} + \mathbb{1} + (\mathbb{L}-\mathbb{1})\frac{\mathbb{L}^{-1}T}{\mathbb{1}-\mathbb{L}^{-1}T}\right)\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3} \\
&= \left(\mathbb{L} + (\mathbb{L}-\mathbb{1})\frac{\mathbb{L}^{-1}T}{\mathbb{1}-\mathbb{L}^{-1}T}\right)\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3}
\end{aligned}$$

The third equality comes from the fact that the following diagram commutes

$$\begin{tikzcd}
\left\{(x,y)\in(\mathbb{R}^*)^2, x^3-y^3\neq 0\right\} \arrow[rr, "\psi"] \arrow[dr, "f"'] & & \mathbb{R}^*\times\mathbb{R}\setminus\{0,1\} \arrow[dl, "\tilde f"] \\
& \mathbb{R}^* &
\end{tikzcd}$$

where $f(x,y)=x^3-y^3$, $\tilde{f}(a,b)=a$, $\psi(x,y)=\left(x^3-y^3,\frac{y}{x}\right)$ and $\psi^{-1}(a,b)=\left(\left(\frac{a}{1-b^3}\right)^{\frac{1}{3}},\left(\frac{a}{1-b^3}\right)^{\frac{1}{3}}b\right)$. ⊡

6. A convolution formula for the motivic local zeta function

The goal of this section is to express $Z_{f_1\oplus f_2}(T)$ in terms of $Z_{f_1}(T)$ and $Z_{f_2}(T)$ where the $f_i:(\mathbb{R}^{d_i},0)\to(\mathbb{R},0)$ are two Nash germs and

$$f_1\oplus f_2(x_1,x_2)=f_1(x_1)+f_2(x_2)$$

The idea of the proof is the following, given two arcs $\gamma_i\in\mathcal{L}(\mathbb{R}^d,0)$, we have two cases to distinguish: either $\operatorname{ord}_t f_1\gamma_1\neq\operatorname{ord}_t f_2\gamma_2$, let say $\operatorname{ord}_t f_1\gamma_1<\operatorname{ord}_t f_2\gamma_2$, and then $\operatorname{ord}_t(f_1\oplus f_2)(\gamma_1,\gamma_2)=\operatorname{ord}_t f_1\gamma_1$ and $\operatorname{ac}(f_1\oplus f_2)(\gamma_1,\gamma_2)=\operatorname{ac} f_1\gamma_1$ or $\operatorname{ord}_t f_1\gamma_1=\operatorname{ord}_t f_2\gamma_2$ and then two phenomena may appear. In this case, either $\operatorname{ac} f_1\gamma_1+\operatorname{ac} f_2\gamma_2\neq 0$ and then $\operatorname{ord}_t(f_1\oplus f_2)(\gamma_1,\gamma_2)=\operatorname{ord}_t f_1\gamma_1=\operatorname{ord}_t f_2\gamma_2$ and $\operatorname{ac}(f_1\oplus f_2)(\gamma_1,\gamma_2)=\operatorname{ac} f_1\gamma_1+\operatorname{ac} f_2\gamma_2$ or $\operatorname{ac} f_1\gamma_1+\operatorname{ac} f_2\gamma_2=0$ and then $\operatorname{ord}_t(f_1\oplus f_2)(\gamma_1,\gamma_2)>\operatorname{ord}_t f_1\gamma_1=\operatorname{ord}_t f_2\gamma_2$.

In [13] or [29], the authors work with an equivariant Grothendieck ring over $\mathbb{C}$ with actions of the roots of unity (with the additional hypothesis that $\operatorname{ac} f\gamma=1$ in $\mathfrak{X}_n(f)$). Then they consider the motives of

$$\{x^n+y^n=\varepsilon\}\times(\mathfrak{X}_n(f_1)\times\mathfrak{X}_n(f_2))\Big/(\lambda\cdot(x,y),(\gamma_1,\gamma_2))\sim((x,y),\lambda\cdot(\gamma_1,\gamma_2))$$

where $\lambda\in\mu_n$ and $\varepsilon\in\{0,1\}$, to handle the case $\operatorname{ord}_t f_1\gamma_1=\operatorname{ord}_t f_2\gamma_2$.

The lack of real roots of unity and the quotient don't allow us to adapt these constructions. However our ring K_0, adapted from the one of [22], remembers the angular component morphisms $\operatorname{ac}_f:\gamma\mapsto\operatorname{ac} f\gamma$. Thus, following [22, §5.1], we may define a convolution product in order to get a convolution formula(8). The definition of this convolution product is motivated by the previous discussion.

Notation 6.1. — We denote by $*:K_0^m\times K_0^n\to K_0^{nm}$ the unique $K_0(\mathcal{AS})$-bilinear map satisfying

$$[X,\sigma,\varphi]*[Y,\tau,\psi]=-[Z_1,\mu_1,f_1]+[Z_2,\mu_2,f_2]$$

(8) We may also notice that the convolution [22, §5.1] is compatible with the one of [29] by [22, (5.1.8)].

where

$$\begin{cases} Z_1 = X \times Y \setminus (\varphi+\psi)^{-1}(0) \\ f_1 = \varphi + \psi \\ \lambda \cdot_{\mu_1} (x,y) = (\lambda^n \cdot_\sigma x, \lambda^m \cdot_\tau y) \end{cases}$$

and

$$\begin{cases} Z_2 = (\varphi+\psi)^{-1}(0) \times \mathbb{R}^* \\ f_2 = \mathrm{pr}_{\mathbb{R}^*} \\ \lambda \cdot_{\mu_2} (x,y,r) = (\lambda^n \cdot_\sigma x, \lambda^m \cdot_\tau y, \lambda^{mn} r) \end{cases}$$

Remark 6.2. — The map $* : K_0^m \times K_0^n \to K_0^{nm}$ induces a $K_0(\mathcal{AS})$-bilinear map $* : K_0 \times K_0 \to K_0$ and a $\mathcal{M}_{\mathcal{AS}}$-bilinear map $* : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$.

Proposition 6.3. — *The convolution product $*$ in K_0 (resp. $\mathcal{M}$) is commutative and associative. The class $\mathbb{1}$ is the unit of this product.*

Proof. — We easily adapt the proof of [22, Proposition 5.2]. □

Lemma 6.4. — *Let $g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash germ. Then, in $K_0(\mathcal{AS})$, we have the relation*

$$\left[\left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\, \mathrm{ord}_t\, g\gamma > n\right\}\right] = \mathbb{L}_{\mathcal{AS}}^{nd} \left(1 - \sum_{i=1}^{n} [\mathfrak{X}_i(g)] \mathbb{L}_{\mathcal{AS}}^{-id}\right)$$

Proof. — $\mathcal{L}_n(\mathbb{R}^d, 0) = \bigsqcup_{i=1}^n (\pi_i^n)^{-1}(\mathfrak{X}_i(g)) \sqcup \left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\, \mathrm{ord}_t\, g\gamma > n\right\}$. So $\mathbb{L}_{\mathcal{AS}}^{nd} = \left(\sum_{i=1}^n [\mathfrak{X}_i(g)] \mathbb{L}_{\mathcal{AS}}^{(n-i)d}\right) + \left[\left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\, \mathrm{ord}_t\, g\gamma > n\right\}\right]$. Finally, $\left[\left\{\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\, \mathrm{ord}_t\, g\gamma > n\right\}\right] = \mathbb{L}_{\mathcal{AS}}^{nd} - \sum_{i=1}^n [\mathfrak{X}_i(g)] \mathbb{L}_{\mathcal{AS}}^{(n-i)d}$ □

Definition 6.5. — *We define the motivic naive local zeta function by*

$$Z_f^{\mathrm{naive}}(T) = \sum_{n \geqslant 1} \overline{[\mathfrak{X}_n(f)]} \mathbb{L}^{-nd} T^n \in \mathcal{M}[\![T]\!]$$

Definition 6.6. — *Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash germ. We define the modified zeta function by*

$$\tilde{Z}_f(T) = \sum_{n \geqslant 1} [\mathfrak{Y}_n(f)]\, \mathbb{L}^{-nd} T^n \in \mathcal{M}[\![T]\!]$$

where

$$[\mathfrak{Y}_n(f)] = [\mathfrak{X}_n(f)] - \overline{[\gamma \in \mathcal{L}_n(\mathbb{R}^d, 0),\, \mathrm{ord}_t\, f\gamma > n]} \cdot \mathbb{1}$$

Proposition 6.7.

$$\tilde{Z}_f(T) = Z_f(T) - \frac{\mathbb{1} - Z_f^{\mathrm{naive}}(T)}{\mathbb{1} - T} + \mathbb{1}$$

Proof. — Lemma 6.4 allows to rewrite $[\mathfrak{Y}_n(f)]$ as follows

$$[\mathfrak{Y}_n(f)] = [\mathfrak{X}_n(f)] - \mathbb{L}^{nd}\left(\mathbb{1} - \sum_{i=1}^{n} \overline{[\mathfrak{X}_i(f)]}\mathbb{L}^{-id}\right)$$

Then

$$\begin{aligned}
\tilde{Z}_f(T) &= \sum_{n\geqslant 1} [\mathfrak{Y}_n(f)]\,\mathbb{L}^{-nd}T^n \\
&= \sum_{n\geqslant 1} [\mathfrak{X}_n(f)]\,\mathbb{L}^{-nd}T^n - \sum_{n\geqslant 1} T^n + \sum_{n\geqslant 1}\sum_{i=1}^{n} \overline{[\mathfrak{X}_i(f)]}\mathbb{L}^{-id}T^n \\
&= Z_f(T) - \frac{T}{\mathbb{1} - T} + \sum_{i\geqslant 1}\sum_{n\geqslant i} \overline{[\mathfrak{X}_i(f)]}\mathbb{L}^{-id}T^n \\
&= Z_f(T) - \frac{T - \mathbb{1} + \mathbb{1}}{\mathbb{1} - T} + \frac{\mathbb{1}}{\mathbb{1} - T}\sum_{i\geqslant 1} \overline{[\mathfrak{X}_i(f)]}\mathbb{L}^{-id}T^i \\
&= Z_f(T) - \frac{\mathbb{1} - Z_f^{\text{naive}}(T)}{\mathbb{1} - T} + \mathbb{1}
\end{aligned}$$ □

Corollary 6.8.

$$-\lim_{T\infty} \tilde{Z}_f(T) = \mathscr{S}_f - \mathbb{1}$$

Remark 6.9. — Applying the forgetful morphism or the morphisms $F^{>}, F^{<}$ and the Euler characteristic with compact support to the coefficients of $\tilde{Z}_f(T)$ we recover the modified zeta functions of S. Koike and A. Parusiński [24].

$$\begin{aligned}
\tilde{Z}_f^{\chi_c,>}(T) &= \sum_{n\geqslant 1} \chi_c\left(\gamma\in\mathcal{L}_n(\mathbb{R}^d,0),\ f\gamma(t) = ct^n \mod t^{n+1},\ c\geqslant 0\right)(-1)^{nd}T^n \\
&= \frac{1 - Z_f^{\chi_c}(T)}{1 - T} - 1 + Z_f^{\chi_c,>}(T) \in \mathbb{Z}[\![T]\!] \\
\tilde{Z}_f^{\chi_c,<}(T) &= \sum_{n\geqslant 1} \chi_c\left(\gamma\in\mathcal{L}_n(\mathbb{R}^d,0),\ f\gamma(t) = ct^n \mod t^{n+1},\ c\leqslant 0\right)(-1)^{nd}T^n \\
&= \frac{1 - Z_f^{\chi_c}(T)}{1 - T} - 1 + Z_f^{\chi_c,<}(T) \in \mathbb{Z}[\![T]\!] \\
\tilde{Z}_f^{\chi_c}(T) &= \sum_{n\geqslant 1} \Big(\chi_c\left(\gamma \in \mathcal{L}_n(\mathbb{R}^d,0),\ \operatorname{ord}_t f\gamma(t) = n\right) \\
&\qquad + 2\chi_c\left(\gamma \in \mathcal{L}_n(\mathbb{R}^d,0),\ \operatorname{ord}_t f\gamma > n\right)\Big)(-1)^{nd}T^n \\
&= \sum_{n\geqslant 1} \Big(\chi_c\left(\gamma \in \mathcal{L}_n(\mathbb{R}^d,0),\ \operatorname{ord}_t f\gamma(t) \geqslant n\right) \\
&\qquad + \chi_c\left(\gamma \in \mathcal{L}_n(\mathbb{R}^d,0),\ \operatorname{ord}_t f\gamma > n\right)\Big)(-1)^{nd}T^n
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n\geqslant 1} \Big(\chi_c\left(\gamma \in \mathcal{L}_n(\mathbb{R}^d,0),\ f\gamma(t) = ct^n \mod t^{n+1},\ c\geqslant 0\right) \\
&\qquad + \chi_c\left(\gamma \in \mathcal{L}_n(\mathbb{R}^d,0),\ f\gamma(t) = ct^n \mod t^{n+1},\ c\leqslant 0\right) \Big) \\
&\qquad\qquad \cdot (-1)^{nd}T^n \\
&= \tilde{Z}_f^{\chi_c,>}(T) + \tilde{Z}_f^{\chi_c,<}(T) \\
&= 2\frac{1-Z_f^{\chi_c}(T)}{1-T} - 2 + Z_f^{\chi_c}(T) \in \mathbb{Z}[\![T]\!]
\end{aligned}$$

Example 6.10. — Let $f_k^\varepsilon(x) = \varepsilon x^k$ where $\varepsilon \in \{\pm 1\}$, then

$$\begin{aligned}
\tilde{Z}_{f_k^\varepsilon}(T) &= -\sum_{r=1}^{k-1} T^r + \sum_{q\geqslant 1}\left(([f_k^\varepsilon : \mathbb{R}^* \to \mathbb{R}^*] - \mathbb{1})\mathbb{L}^{-q}T^{kq} - \sum_{r=1}^{k-1}\mathbb{L}^{-q}T^{kq+r}\right) \\
&= -\sum_{q\geqslant 0}\sum_{r=1}^{k-1}\mathbb{L}^{-q}T^{kq+r} - \sum_{q\geqslant 1}(\mathbb{1} - [f_k^\varepsilon : \mathbb{R}^* \to \mathbb{R}^*])\mathbb{L}^{-q}T^{kq} \\
&= -T - \cdots - T^{k-1} \\
&\quad - (\mathbb{1} - [f_k^\varepsilon])\,\mathbb{L}^{-1}T^k - \mathbb{L}^{-1}T^{k+1} - \cdots - \mathbb{L}^{-1}T^{2k-1} \\
&\quad - (\mathbb{1} - [f_k^\varepsilon])\,\mathbb{L}^{-2}T^{2k} - \mathbb{L}^{-2}T^{2k+1} - \cdots
\end{aligned}$$

Indeed

$$[\mathfrak{X}_{kq+r}(f_k^\varepsilon)]\,\mathbb{L}^{-(kq+r)} = \begin{cases} [f_k^\varepsilon : \mathbb{R}^* \to \mathbb{R}^*]\mathbb{L}^{-q} & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{[\mathfrak{X}_{kq+r}(f_k^\varepsilon)]}\mathbb{L}^{-(kq+r)} = \begin{cases} (\mathbb{L} - \mathbb{1})\mathbb{L}^{-q} & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\begin{aligned}
&[\mathfrak{Y}_{kq+r}(f_k^\varepsilon)]\,\mathbb{L}^{-(kq+r)} \\
&\qquad = \begin{cases} [f_k^\varepsilon : \mathbb{R}^* \to \mathbb{R}^*]\mathbb{L}^{-q} + \displaystyle\sum_{i=1}^{q}(\mathbb{L} - \mathbb{1})\mathbb{L}^{-i} - \mathbb{1} & \text{if } r = 0 \\ \displaystyle\sum_{i=1}^{q}(\mathbb{L} - \mathbb{1})\mathbb{L}^{-i} - \mathbb{1} & \text{otherwise} \end{cases} \\
&\qquad = \begin{cases} ([f_k^\varepsilon : \mathbb{R}^* \to \mathbb{R}^*] - \mathbb{1})\mathbb{L}^{-q} & \text{if } r = 0 \\ -\mathbb{L}^{-q} & \text{otherwise} \end{cases}
\end{aligned}$$ ⊡

Definition 6.11. — *We define the motivic naive modified zeta function by*

$$\tilde{Z}_f^{\mathrm{naive}}(T) = \sum_{n\geqslant 1} \overline{[\mathfrak{Y}_n(f)]}\mathbb{L}^{-nd}T^n \in \mathcal{M}[\![T]\!]$$

Proposition 6.12.

$$\frac{\mathbb{L} - \tilde{Z}_f^{\mathrm{naive}}(T)}{\mathbb{L} - T} = \frac{\mathbb{1} - Z_f^{\mathrm{naive}}(T)}{\mathbb{1} - T}$$

Proof. — From

$$\tilde{Z}_f(T) = Z_f(T) - \frac{\mathbb{1} - Z_f^{\mathrm{naive}}(T)}{\mathbb{1} - T} + \mathbb{1}$$

we deduce

$$\tilde{Z}_f^{\mathrm{naive}}(T) = Z_f^{\mathrm{naive}}(T) - (\mathbb{L} - \mathbb{1})\frac{\mathbb{1} - Z_f^{\mathrm{naive}}(T)}{\mathbb{1} - T} + \mathbb{L} - \mathbb{1}$$

$$\tilde{Z}_f^{\mathrm{naive}}(T) - \mathbb{L} = (Z_f^{\mathrm{naive}}(T) - \mathbb{1}) + (Z_f^{\mathrm{naive}}(T) - \mathbb{1})\frac{\mathbb{L} - \mathbb{1}}{\mathbb{1} - T}$$

$$\tilde{Z}_f^{\mathrm{naive}}(T) - \mathbb{L} = (Z_f^{\mathrm{naive}}(T) - \mathbb{1})\frac{\mathbb{L} - T}{\mathbb{1} - T}$$

□

Remark 6.13. — We recover the following formula of [24, p. 2070]

$$\frac{1 - Z_f^{\chi_c}(T)}{1 - T} = \frac{1 + \tilde{Z}_f^{\chi_c}(T)}{1 + T}$$

Corollary 6.14. — *We may compute $\tilde{Z}_f(T)$ from $Z_f(T)$ and $Z_f(T)$ from $\tilde{Z}_f(T)$. Thus they encode the same information. More precisely, we have*

$$\tilde{Z}_f(T) = Z_f(T) - \frac{\mathbb{1} - Z_f^{\mathrm{naive}}(T)}{\mathbb{1} - T} + \mathbb{1}$$

and

$$Z_f(T) = \tilde{Z}_f(T) + \frac{\mathbb{L} - \tilde{Z}_f^{\mathrm{naive}}(T)}{\mathbb{L} - T} - \mathbb{1}$$

Theorem 6.15. — *Let $f_1 : (\mathbb{R}^{d_1}, 0) \to (\mathbb{R}, 0)$ and $f_2 : (\mathbb{R}^{d_2}, 0) \to (\mathbb{R}, 0)$ be two Nash germs. Then*

$$\tilde{Z}_{f_1 \oplus f_2}(T) = -\tilde{Z}_{f_1}(T) \circledast \tilde{Z}_{f_2}(T)$$

where the product $\circledast$ is the Hadamard product which consists in applying the convolution product coefficientwise.

Remark 6.16. — Similar formulas are known when the angular component is fixed to be 1 with an action of the roots of unity [13, Main Theorem 4.2.4] or for the Euler characteristic with compact support [24, Theorem 2.3].

Remark 6.17. — In the definition of the modified zeta function, it could have been multiplied by a factor $(-1)^d$ in order to avoid the sign in Theorem 6.15.

Example 6.18. — Let $f(x,y) = x^3 - y^3$. We deduce from Example 6.10 and Theorem 6.15 that

$$\tilde{Z}_f(T) = -\sum_{q\geqslant 0}\sum_{r=1}^{2}\mathbb{L}^{-2q}T^{3q+r} = -\frac{T+T^2}{\mathbb{1}-\mathbb{L}^{-2}T^3}$$

We recover that

$$\begin{aligned}
Z_f(T) &= \tilde{Z}_f(T) + \frac{\mathbb{L} - \tilde{Z}_f^{\text{naive}}(T)}{\mathbb{L}-T} - \mathbb{1} \\
&= -\frac{T+T^2}{\mathbb{1}-\mathbb{L}^{-2}T^3} + \frac{\mathbb{L} + (\mathbb{L}-\mathbb{1})\frac{T+T^2}{\mathbb{1}-\mathbb{L}^{-2}T^3}}{\mathbb{L}-T} - \mathbb{1} \\
&= \frac{T^3 - \mathbb{L}^{-2}T^4}{(\mathbb{L}-T)(\mathbb{1}-\mathbb{L}^{-2}T^3)} \\
&= \frac{\mathbb{L}^{-1}T^3 - \mathbb{L}^{-3}T^4}{(\mathbb{1}-\mathbb{L}^{-1}T)(\mathbb{1}-\mathbb{L}^{-2}T^3)} \\
&= \mathbb{L}\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3} + (\mathbb{L}-\mathbb{1})\frac{\mathbb{L}^{-1}T}{\mathbb{1}-\mathbb{L}^{-1}T}\frac{\mathbb{L}^{-2}T^3}{\mathbb{1}-\mathbb{L}^{-2}T^3}
\end{aligned}$$ $\boxdot$

Lemma 6.19 ([29, Lemma 7.6][13, Proposition 5.1.2]). — *Let* $Z_1(T)$, $Z_2(T) \in \mathcal{M}[\![T]\!]_{\mathrm{sr}}$ *then*

$$\lim_{T\infty} Z_1(T) \circledast Z_2(T) = -\left(\lim_{T\infty} Z_1(T)\right) * \left(\lim_{T\infty} Z_2(T)\right)$$

Corollary 6.20 (Motivic Thom–Sebastiani formula).

$$\mathscr{S}_{f_1\oplus f_2} = -\mathscr{S}_{f_1} * \mathscr{S}_{f_2} + \mathscr{S}_{f_1} + \mathscr{S}_{f_2}$$

Proof.

$$\begin{aligned}
\mathscr{S}_{f_1\oplus f_2} - \mathbb{1} &= -\lim_{T\infty}\tilde{Z}_{f_1\oplus f_2}(T) && \text{by Corollary 6.8} \\
&= \lim_{T\infty}\left(\tilde{Z}_{f_1}(T)\circledast\tilde{Z}_{f_2}(T)\right) && \text{by Theorem 6.15} \\
&= -\left(\lim_{T\infty}\tilde{Z}_{f_1}(T)\right) * \left(\lim_{T\infty}\tilde{Z}_{f_2}(T)\right) && \text{by Lemma 6.19} \\
&= -(-\mathscr{S}_{f_1}+\mathbb{1}) * (-\mathscr{S}_{f_2}+\mathbb{1}) && \text{by Corollary 6.8} \\
&= -\mathscr{S}_{f_1} * \mathscr{S}_{f_2} + \mathscr{S}_{f_1} + \mathscr{S}_{f_2} - \mathbb{1}
\end{aligned}$$ $\square$

Proof of Theorem 6.15. — In order to shorten the formulas, we set $\mathcal{L}(M,A)=\{\gamma\in\mathcal{L}(M),\,\gamma(0)\in A\}$.

Our goal is to compute, for $n\in\mathbb{N}\setminus\{0\}$, $\mathfrak{X}_n(f_1\oplus f_2)$, so let $(\gamma_1,\gamma_2)\in\mathfrak{X}_n(f_1\oplus f_2)$, i.e. $\gamma_i\in\mathcal{L}_n(\mathbb{R}^{d_i},0)$ such that $\operatorname{ord}_t(f_1\gamma_1+f_2\gamma_2)=n$.

We first restrict to the case $\operatorname{ord}_t f_1\gamma_1\neq\operatorname{ord}_t f_2\gamma_2$. Then either

$$n=\operatorname{ord}_t f_1\gamma_1<\operatorname{ord}_t f_2\gamma_2 \text{ with } \operatorname{ac}\left((f_1\oplus f_2)(\gamma_1,\gamma_2)\right)=\operatorname{ac}(f_1\gamma_1)$$

or

$$\operatorname{ord}_t f_1\gamma_1>\operatorname{ord}_t f_2\gamma_2=n \text{ with } \operatorname{ac}\left((f_1\oplus f_2)(\gamma_1,\gamma_2)\right)=\operatorname{ac}(f_2\gamma_2)$$

Therefore, if we set

$$\mathfrak{X}_n^{\neq}(f_1\oplus f_2)=\big\{(\gamma_1,\gamma_2)\in\mathcal{L}_n(\mathbb{R}^{d_1+d_2},0),\,\operatorname{ord}_t f_1\gamma_1\neq\operatorname{ord}_t f_2\gamma_2,\\ \operatorname{ord}_t(f_1\gamma_1+f_2\gamma_2)=n\big\}$$

with the natural action on jets and the natural morphism induced by the angular component, we get

$$\begin{aligned}\left[\mathfrak{X}_n^{\neq}(f_1\oplus f_2)\right]&=\overline{[\gamma_2\in\mathcal{L}_n(\mathbb{R}^{d_2},0),\,\operatorname{ord}_t f_2\gamma_2>n]}\,[\mathfrak{X}_n(f_1)]\\&\quad+\overline{[\gamma_1\in\mathcal{L}_n(\mathbb{R}^{d_1},0),\,\operatorname{ord}_t f_1\gamma_1>n]}\,[\mathfrak{X}_n(f_2)]\\&=\left(1-\sum_{\omega=1}^{n}\overline{[\mathfrak{X}_\omega(f_2)]}\mathbb{L}^{-\omega d_2}\right)[\mathfrak{X}_n(f_1)]\mathbb{L}^{nd_2}\\&\quad+\left(1-\sum_{\omega=1}^{n}\overline{[\mathfrak{X}_\omega(f_1)]}\mathbb{L}^{-\omega d_1}\right)[\mathfrak{X}_n(f_2)]\mathbb{L}^{nd_1}\end{aligned}$$

where the overline means that the class is in $K_0(\mathcal{AS})$ and that we use the scalar multiplication. The second equality comes from Lemma 6.4.

It remains to manage the case $\operatorname{ord}_t f_1\gamma_1=\operatorname{ord}_t f_2\gamma_2$. Set

$$\mathfrak{X}_n^{=}(f_1\oplus f_2)=\left\{(\gamma_1,\gamma_2)\in\mathcal{L}_n(\mathbb{R}^{d_1+d_2},0),\ \begin{matrix}\operatorname{ord}_t f_1\gamma_1=\operatorname{ord}_t f_2\gamma_2,\\ \operatorname{ord}_t(f_1\gamma_1+f_2\gamma_2)=n\end{matrix}\right\}$$

with the natural action on jets and the natural morphism induced by the angular component.

Let $(\gamma_1,\gamma_2)\in\mathfrak{X}_n^{=}(f_1\oplus f_2)$. Either $\operatorname{ord}_t(f_1\gamma_1)=\operatorname{ord}_t(f_2\gamma_2)=n$ with $\operatorname{ac}(f_1\gamma_1)+\operatorname{ac}(f_2\gamma_2)\neq0$ or $\operatorname{ord}_t(f_1\gamma_1)=\operatorname{ord}_t(f_2\gamma_2)<n$ with $\operatorname{ac}(f_1\gamma_1)+\operatorname{ac}(f_2\gamma_2)=0$. We now focus on this second case.

Let $h_i:M_i\to\mathbb{R}^{d_i}$ be as in Section 4.3.2. We lift the jets in $\mathfrak{X}_n^{=}(f_1\oplus f_2)$ to jets in $\mathcal{L}_m(\mathbb{R}^{d_1+d_2})$ (second equality below) with $m\geqslant n$ big enough to apply the change of variables key lemma 4.19 (fourth equality) in order to work locally with jets in $M_1 \times M_2$ with origin in $E_I^\bullet \times E_J^\bullet$ (fifth equality) where $f_i h_i$ is a monomial times a unit.

$$\begin{aligned}
(*) &= \left[(\gamma_1,\gamma_2)\in\mathcal{L}_n(\mathbb{R}^{d_1+d_2},0),\ \begin{array}{l}\mathrm{ord}_t\, f_1\gamma_1=\mathrm{ord}_t\, f_2\gamma_2<n,\\ \mathrm{ord}_t(f_1\gamma_1+f_2\gamma_2)=n\end{array}\right]\\
&= \left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(\mathbb{R}^{d_1+d_2},0),\ \begin{array}{l}\mathrm{ord}_t\, f_1\gamma_1=\mathrm{ord}_t\, f_2\gamma_2<n,\\ \mathrm{ord}_t(f_1\gamma_1+f_2\gamma_2)=n\end{array}\right]\mathbb{L}^{(d_1+d_2)(n-m)}\\
&= \sum_{\substack{e\geqslant 1\\ e'\geqslant 1}}\left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(\mathbb{R}^{d_1+d_2},0),\ \begin{array}{l}\mathrm{ord}_t\, f_1\gamma_1=\mathrm{ord}_t\, f_2\gamma_2<n,\\ \mathrm{ord}_t(f_1\gamma_1+f_2\gamma_2)=n,\\ \gamma_1\in h_{1*m}\pi_m\Delta_e,\\ \gamma_2\in h_{2*m}\pi_m\Delta_{e'}\end{array}\right]\\
&\qquad\qquad\times\mathbb{L}^{(d_1+d_2)(n-m)}\\
&= \sum_{\substack{e\geqslant 1\\ e'\geqslant 1}}\left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(M_1\times M_2),\ \begin{array}{l}\gamma_1(0)\in h_1^{-1}(0),\ \gamma_2(0)\in h_2^{-1}(0)\\ \mathrm{ord}_t\, f_1h_1\gamma_1=\mathrm{ord}_t\, f_2h_2\gamma_2<n,\\ \mathrm{ord}_t(f_1h_1\gamma_1+f_2h_2\gamma_2)=n,\\ \gamma_1\in\pi_m\Delta_e,\ \gamma_2\in\pi_m\Delta_{e'}\end{array}\right]\\
&\qquad\qquad\times\mathbb{L}^{(d_1+d_2)(n-m)-e-e'}\\
&= \sum_{\substack{e\geqslant 1\\ e'\geqslant 1\\ I\neq\varnothing\\ J\neq\varnothing}}\left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(M_1\times M_2),\ \begin{array}{l}\gamma_1(0)\in h_1^{-1}(0)\cap E_I^\bullet,\\ \gamma_2(0)\in h_2^{-1}(0)\cap F_J^\bullet,\\ \mathrm{ord}_t\, f_1h_1\gamma_1=\mathrm{ord}_t\, f_2h_2\gamma_2<n,\\ \mathrm{ord}_t(f_1h_1\gamma_1+f_2h_2\gamma_2)=n,\\ \gamma_1\in\pi_m\Delta_e,\ \gamma_2\in\pi_m\Delta_{e'}\end{array}\right]\\
&\qquad\qquad\times\begin{array}{l}\mathbb{L}^{(d_1+d_2)(n-m)}\\ \mathbb{L}^{-e-e'}\end{array}
\end{aligned}$$

In a neighborhood of $E_I^\bullet$ we have

$$f_1h_1(x) = u(x)\prod_{i\in I} x_i^{N_i(f_1)}$$

and in a neighborhood of $E_J^\bullet$ we have

$$f_2h_2(y) = v(y)\prod_{j\in J} y_j^{N_j(f_2)}$$

where u, v are units and $E_i : x_i = 0, F_j : y_j = 0$. Let $(\gamma_1,\gamma_2)\in\mathcal{L}(M_1\times M_2)$ satisfying the conditions of the last equality. Set $\gamma_{1i}(t) = t^{k_i}a_i(t)$ with $a_i(0)\neq 0$ and $\gamma_{2j}(t) = t^{l_j}b_j(t)$ with $b_j(0)\neq 0$.

Denote by $\omega = ord_t(f_1\gamma_1) = \mathrm{ord}_t(f_2\gamma_2)$. Then the coefficient of t^ω in $f_1h_1(\gamma_1(t)) + f_2h_2(\gamma_2(t))$ is

$$u(\gamma_1(0))\prod_{i\in I} a_{i0}^{N_i(f_1)} + v(\gamma_2(0))\prod_{j\in J} a_{j0}^{N_j(f_2)}$$

which is just $\mathrm{ac}(f_1h_1\gamma_1) + \mathrm{ac}(f_2h_2\gamma_2)$.

The coefficient of $t^{\omega+l}$ for $l = 1, \dots, n-\omega$ is of the form

$$g_l(a_{il}, b_{jl}) + P_l$$

where g_l is a non-zero linear form in $a_{il}, b_{jl}, i \in I, j \in J$ and where P_l is a polynomial in $a_{ik}, b_{jk}, i \in I, j \in J, k < l$.

Since the coefficient of t^ω is zero, it brings the equation $\mathrm{ac}(f_1h_1\gamma_1) + \mathrm{ac}(f_2h_2\gamma_2) = 0$. The coefficients of $t^{\omega+l}$ must be zero for $l = 1, \dots, n-\omega-1$, that brings the factor $\left(\mathbb{L}_{\mathcal{AS}}^{|I|+|J|-1}\right)^{n-\omega-1} = \mathbb{L}_{\mathcal{AS}}^{(|I|+|J|-1)(n-\omega-1)}$. The coefficient of t^n must be non-zero and is the one which contributes to the angular component, hence it brings the factor $\mathbb{L}_{\mathcal{AS}}^{|I|+|J|-1}\cdot\mathbb{1}$. We have no condition for the other coefficients of $\gamma_{1i}, i \in I$ and $\gamma_{2j}, j \in J$, that brings the factor $\prod_{i\in I}\mathbb{L}_{\mathcal{AS}}^{m-k_i-n+\omega} = \mathbb{L}_{\mathcal{AS}}^{(m-n+\omega)|I|-\sum_I k_i}$ and similarly $\mathbb{L}_{\mathcal{AS}}^{(m-n+\omega)|J|-\sum_J l_j}$. We have no condition on the components of γ_1 (resp. γ_2) not indexed by I (resp. J). This brings the factor $\mathbb{L}_{\mathcal{AS}}^{(d_1-|I|)m}$ and $\mathbb{L}_{\mathcal{AS}}^{(d_2-|J|)m}$.

Next

$$\begin{aligned}
&\left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(M_1\times M_2),\ \begin{array}{l}\gamma_1(0)\in h_1^{-1}(0)\cap E_I^\bullet,\\ \gamma_2(0)\in h_2^{-1}(0)\cap F_J^\bullet,\\ \mathrm{ord}_t\, f_1h_1\gamma_1 = \mathrm{ord}_t\, f_2h_2\gamma_2 < n,\\ \mathrm{ord}_t(f_1h_1\gamma_1 + f_2h_2\gamma_2) = n,\\ \gamma_1\in\pi_m\Delta_e,\ \gamma_2\in\pi_m\Delta_{e'}\end{array}\right]\\
&\quad = \overline{[h_1^{-1}(0)\cap E_I^\bullet][h_2^{-1}(0)\cap F_J^\bullet]}\,\overline{[f_1h_1+f_2h_2\neq 0]}\,\mathbb{L}^{\omega-n+(d_1+d_2)m}\\
&\qquad\times\sum_{\omega=1}^{n-1}\sum_{\substack{k_i\in\mathbb{N},l_j\in\mathbb{N}\\ \sum k_i(\nu_i(f_1)-1)=e\\ \sum l_j(\nu_j(f_2)-1)=e'\\ \sum k_iN_i(f_1)=\sum l_jN_j(f_2)=\omega}}\mathbb{L}^{-\sum k_i-\sum l_j}
\end{aligned}$$

and we may similarly check that the RHS is also equal to

$$\overline{\sum_{\omega=1}^{n-1}\left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(M_1\times M_2),\ \begin{array}{l}\gamma_1(0)\in h_1^{-1}(0)\cap E_I^\bullet,\\ \gamma_2(0)\in h_2^{-1}(0)\cap F_J^\bullet,\\ \mathrm{ord}_t\, f_1h_1\gamma_1 = \mathrm{ord}_t\, f_2h_2\gamma_2 = \omega,\\ \mathrm{ac}(f_1h_1\gamma_1)+\mathrm{ac}(f_2h_2\gamma_2) = 0,\\ \gamma_1\in\pi_m\Delta_e,\ \gamma_2\in\pi_m\Delta_{e'}\end{array}\right]}\mathbb{L}^{\omega-n}$$

We now come back to arcs on $\mathbb{R}^{d_1+d_2}$.

$$(*) = \sum_{\substack{e\geqslant 1\\ e'\geqslant 1\\ I\neq\varnothing\\ J\neq\varnothing\\ \omega=1,\dots,n-1}} \overline{\left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(M_1\times M_2),\ \begin{array}{l}\gamma_1(0)\in h_1^{-1}(0)\cap E_I^\bullet,\\ \gamma_2(0)\in h_2^{-1}(0)\cap F_J^\bullet,\\ \mathrm{ord}_t\, f_1h_1\gamma_1=\mathrm{ord}_t\, f_2h_2\gamma_2=\omega,\\ \mathrm{ac}(f_1h_1\gamma_1)+\mathrm{ac}(f_2h_2\gamma_2)=0,\\ \gamma_1\in\pi_m\Delta_e,\ \gamma_2\in\pi_m\Delta_{e'}\end{array}\right]} \times \mathbb{L}^{\omega-n+(d_1+d_2)(n-m)-e-e'}$$

$$= \sum_{\omega=1}^{n-1} \overline{\left[(\gamma_1,\gamma_2)\in\mathcal{L}_m(\mathbb{R}^{d_1+d_2},0),\ \begin{array}{l}\mathrm{ord}_t\, f_1\gamma_1=\mathrm{ord}_t\, f_2\gamma_2=\omega\\ \mathrm{ac}(f_1\gamma_1)+\mathrm{ac}(f_2\gamma_2)=0\end{array}\right]} \times \mathbb{L}^{\omega-n+(d_1+d_2)(n-m)}$$

$$= \sum_{\omega=1}^{n-1} \overline{\left[(\gamma_1,\gamma_2)\in\mathcal{L}_\omega(\mathbb{R}^{d_1+d_2},0),\ \begin{array}{l}\mathrm{ord}_t\, f_1\gamma_1=\mathrm{ord}_t\, f_2\gamma_2=\omega\\ \mathrm{ac}(f_1\gamma_1)+\mathrm{ac}(f_2\gamma_2)=0\end{array}\right]} \times \mathbb{L}^{\omega-n+(d_1+d_2)(n-m)-(d_1+d_2)(\omega-m)}$$

$$= \sum_{\omega=1}^{n-1} \overline{[(\gamma_1,\gamma_2)\in\mathfrak{X}_\omega(f_1)\times\mathfrak{X}_\omega(f_2),\ \mathrm{ac}\, f_1\gamma_1+\mathrm{ac}\, f_2\gamma_2=0]} \times \mathbb{L}^{(d_1+d_2-1)(n-\omega)}$$

Finally we get

$$\begin{aligned}
&[\mathfrak{X}_n^{=}(f_1\oplus f_2)]\\
&= [(\gamma_1,\gamma_2)\in\mathfrak{X}_n(f_1)\times\mathfrak{X}_n(f_2),\ \mathrm{ac}\, f_1\gamma_1+\mathrm{ac}\, f_2\gamma_2\neq 0]\\
&\quad + \sum_{\omega=1}^{n-1} \overline{[(\gamma_1,\gamma_2)\in\mathfrak{X}_\omega(f_1)\times\mathfrak{X}_\omega(f_2),\ \mathrm{ac}\, f_1\gamma_1+\mathrm{ac}\, f_2\gamma_2=0]} \times \mathbb{L}^{(d_1+d_2-1)(n-\omega)}\\
&= [(\gamma_1,\gamma_2)\in\mathfrak{X}_n(f_1)\times\mathfrak{X}_n(f_2),\ \mathrm{ac}\, f_1\gamma_1+\mathrm{ac}\, f_2\gamma_2\neq 0]\\
&\quad - [(\gamma_1,\gamma_2)\in\mathfrak{X}_n(f_1)\times\mathfrak{X}_n(f_2),\ \mathrm{ac}\, f_1\gamma_1+\mathrm{ac}\, f_2\gamma_2=0]\\
&\quad + \sum_{\omega=1}^{n} \overline{[(\gamma_1,\gamma_2)\in\mathfrak{X}_\omega(f_1)\times\mathfrak{X}_\omega(f_2),\ \mathrm{ac}\, f_1\gamma_1+\mathrm{ac}\, f_2\gamma_2=0]} \times \mathbb{L}^{(d_1+d_2-1)(n-\omega)}\\
&= -[\mathfrak{X}_n(f_1)]*[\mathfrak{X}_n(f_2)]\\
&\quad + \sum_{\omega=1}^{n} \overline{[(\gamma_1,\gamma_2)\in\mathfrak{X}_\omega(f_1)\times\mathfrak{X}_\omega(f_2),\ \mathrm{ac}\, f_1\gamma_1+\mathrm{ac}\, f_2\gamma_2=0]} \times \mathbb{L}^{(d_1+d_2-1)(n-\omega)}
\end{aligned}$$

This ends the second case.

Therefore the computations of the beginning of the proof give

$$\begin{aligned}
&[\mathfrak{X}_n(f_1\oplus f_2)]\,\mathbb{L}^{-n(d_1+d_2)}\\
&\quad=\left([\mathfrak{X}_n^{\neq}(f_1\oplus f_2)]+[\mathfrak{X}_n^{=}(f_1\oplus f_2)]\right)\mathbb{L}^{-n(d_1+d_2)}\\
&\quad=\left(1-\sum_{\omega=1}^{n}\overline{[\mathfrak{X}_\omega(f_2)]}\mathbb{L}^{-\omega d_2}\right)[\mathfrak{X}_n(f_1)]\mathbb{L}^{-nd_1}\\
&\qquad\quad+\left(1-\sum_{\omega=1}^{n}\overline{[\mathfrak{X}_\omega(f_1)]}\mathbb{L}^{-\omega d_1}\right)[\mathfrak{X}_n(f_2)]\mathbb{L}^{-nd_2}\\
&\qquad\quad-[\mathfrak{X}_n(f_1)]\,\mathbb{L}^{-nd_1}*[\mathfrak{X}_n(f_2)]\,\mathbb{L}^{-nd_2}\\
&\qquad\quad+\sum_{\omega=1}^{n}\overline{[(\gamma_1,\gamma_2)\in\mathfrak{X}_\omega(f_1)\times\mathfrak{X}_\omega(f_2),\ \mathrm{ac}(f_1\gamma_1)+\mathrm{ac}(f_2\gamma_2)=0]}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\mathbb{L}^{-\omega(d_1+d_2)}\mathbb{L}^{\omega-n}
\end{aligned}$$

Using again a monomialization, we get that

$$\begin{aligned}
&\overline{[(\gamma_1,\gamma_2)\in\mathcal{L}_n(\mathbb{R}^{d_1+d_2}),\ \mathrm{ord}_t(f_1\gamma_1+f_2\gamma_2)>n]}\cdot\mathbb{1}\\
&\quad=\overline{[\mathrm{ord}_t\,f_1>n,\mathrm{ord}_t\,f_2>n]}\cdot\mathbb{1}\\
&\qquad\quad+\sum_{\omega=1}^{n}\overline{[\mathrm{ord}_t\,f_1=\mathrm{ord}_t\,f_2=\omega,\ \mathrm{ord}_t(f_1+f_2)>n]}\cdot\mathbb{1}\\
&\quad=\overline{[\mathrm{ord}_t\,f_1>n,\mathrm{ord}_t\,f_2>n]}\cdot\mathbb{1}\\
&\qquad\quad+\sum_{\omega=1}^{n}\overline{[(\gamma_1,\gamma_2)\in\mathfrak{X}_\omega(f_1)\times\mathfrak{X}_\omega(f_2),\ \mathrm{ac}(f_1\gamma_1)+\mathrm{ac}(f_2\gamma_2)=0]}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\mathbb{L}^{(n-\omega)(d_1+d_2-1)}
\end{aligned}$$

This allows us to conclude as follows.

$$\begin{aligned}
&[\mathfrak{Y}_n(f_1\oplus f_2)]\,\mathbb{L}^{-n(d_1+d_2)}\\
&\qquad=[\mathfrak{X}_n(f_1\oplus f_2)]\,\mathbb{L}^{-n(d_1+d_2)}\\
&\qquad\qquad-\overline{[(\gamma_1,\gamma_2)\in\mathcal{L}_n(\mathbb{R}^{d_1+d_2}),\ \mathrm{ord}_t(f_1\gamma_1+f_2\gamma_2)>n]}\mathbb{L}^{-n(d_1+d_2)}\\
&\qquad=\left(1-\sum_{\omega=1}^{n}\overline{[\mathfrak{X}_\omega(f_2)]}\mathbb{L}^{-\omega d_2}\right)[\mathfrak{X}_n(f_1)]\mathbb{L}^{-nd_1}\\
&\qquad\qquad+\left(1-\sum_{\omega=1}^{n}\overline{[\mathfrak{X}_\omega(f_1)]}\mathbb{L}^{-\omega d_1}\right)[\mathfrak{X}_n(f_2)]\mathbb{L}^{-nd_2}\\
&\qquad\qquad-[\mathfrak{X}_n(f_1)]\,\mathbb{L}^{-nd_1}*[\mathfrak{X}_n(f_2)]\,\mathbb{L}^{-nd_2}\\
&\qquad\qquad-\overline{[\mathrm{ord}_t\,f_1>n]}\mathbb{L}^{-nd_1}\overline{[\mathrm{ord}_t\,f_2>n]}\mathbb{L}^{-nd_2}
\end{aligned}$$

$$= -\left([\mathfrak{Y}_n(f_1)]\,\mathbb{L}^{-nd_1}\right) * \left([\mathfrak{Y}_n(f_2)]\,\mathbb{L}^{-nd_2}\right) \qquad \square$$

We recover the convolution [24, Theorem 2.3] thanks to the following lemma.

Lemma 6.21. — *Let $\varepsilon \in \{>, <\}$ then*

$$\chi_c(F^\varepsilon(x * y)) = -\chi_c(F^\varepsilon x)\chi_c(F^\varepsilon y)$$

We need the following lemma in order to prove the previous one.

Lemma 6.22. — *Let X be an $\mathcal{AS}$-set endowed with an $\mathcal{AS}$ action of $\mathbb{R}^*$ and $\varphi : X \to \mathbb{R}^*$ be an $\mathcal{AS}$-map such that $\varphi(\lambda \cdot x) = \lambda^n \varphi(x)$. Then φ is a trivial semialgebraic fibration over $\mathbb{R}_{>0}$ and over $\mathbb{R}_{<0}$ (or over $\mathbb{R}^*$ if n is odd).*

Proof. — Indeed, the following diagram commutes (for the case > 0)

$$\begin{tikzcd}
\varphi^{-1}(\mathbb{R}_{>0}) \arrow[rr, "\Psi"] \arrow[dr, "\varphi"'] & & \varphi^{-1}(1) \times \mathbb{R}_{>0} \arrow[dl, "\mathrm{pr}_{\mathbb{R}_{>0}}"] \\
& \mathbb{R}_{>0} &
\end{tikzcd}$$

where $\Psi(x) = (\varphi(x)^{-\frac{1}{n}} \cdot x, \varphi(x))$ and $\Psi^{-1}(x, \lambda) = \lambda^{\frac{1}{n}} \cdot x$ $\square$

Proof of Lemma 6.21. — Assume that $\varepsilon = >$. Then

$$\begin{aligned}
\chi_c\left(F^{>}([\varphi_1 : X_1 \to \mathbb{R}^*] * [\varphi_2 : X_2 \to \mathbb{R}^*])\right) \\
&= -\chi_c\left((\varphi_1 + \varphi_2) > 0\right) + \chi_c\left((\varphi_1 + \varphi_2 = 0) \times \mathbb{R}_{>0}\right) \\
&= -\chi_c\left((\varphi_1 + \varphi_2) > 0\right) - \chi_c\left((\varphi_1 + \varphi_2 = 0)\right)
\end{aligned}$$

Where the last equality comes from the fact that $\chi_c(\mathbb{R}_{>0}) = -1$. Thus

$$\begin{aligned}
\chi_c\left(F^{>}([\varphi_1 : X_1 \to \mathbb{R}^*] * [\varphi_2 : X_2 \to \mathbb{R}^*])\right) \\
&= -\chi_c\left((\varphi_1 + \varphi_2) \geqslant 0\right) \\
&= -\chi_c\left(\varphi_1 \geqslant 0, \varphi_2 \geqslant 0\right) + \chi_c\left((\varphi_1 + \varphi_2) \geqslant 0, \varphi_1 < 0\right) \\
&\quad + \chi_c\left((\varphi_1 + \varphi_2) \geqslant 0, \varphi_2 < 0\right)
\end{aligned}$$

Since φ_i is trivial over $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{<0}$) and $\chi_c(a + b \geqslant 0, a < 0) = 0$, we get

$$\begin{aligned}
\chi_c\left(F^{>}([\varphi_1 : X_1 \to \mathbb{R}^*] * [\varphi_2 : X_2 \to \mathbb{R}^*])\right) \\
&= -\chi_c\left(\varphi_1 \geqslant 0, \varphi_2 \geqslant 0\right) \\
&= -\chi_c\left(\varphi_1 > 0, \varphi_2 > 0\right) - \chi_c\left(\varphi_1 \geqslant 0, \varphi_2 = 0\right) - \chi_c\left(\varphi_1 = 0, \varphi_2 \geqslant 0\right)
\end{aligned}$$

Since φ_i is trivial over $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{<0}$) and $\chi_c(a \geqslant 0, b = 0) = 0$, we finally have

$$\begin{aligned}\chi_c\left(F^{>}([\varphi_1 : X_1 \to \mathbb{R}^*] * [\varphi_2 : X_2 \to \mathbb{R}^*])\right) \\ &= -\chi_c\left(\varphi_1 > 0, \varphi_2 > 0\right) \\ &= -\chi_c\left(F^{>}[\varphi_1 : X_1 \to \mathbb{R}^*]\right)\chi_c\left(F^{>}[\varphi_2 : X_2 \to \mathbb{R}^*]\right) \quad \square\end{aligned}$$

Corollary 6.23 ([24, Theorem 2.3]). — *Let $\varepsilon \in \{>, <\}$ then*

$$\tilde{Z}^{\chi_c,\varepsilon}_{f\oplus g}(T) = \tilde{Z}^{\chi_c,\varepsilon}_{f}(T) \odot \tilde{Z}^{\chi_c,\varepsilon}_{g}(T)$$

where the product $\odot$ is the Hadamard product which consists in applying the classical product of $\mathbb{Z}$ coefficientwise.

We showed in the proof of 6.22 that $\chi_c(F^{>}(a)) = -\chi_c(F^{+}(a))$. In the same way we may prove that $\chi_c(F^{<}(a)) = -\chi_c(F^{-}(a))$. From these facts we derive the following lemma.

Lemma 6.24. — *Let $\varepsilon \in \{+, -\}$ then*

$$\chi_c(F^{\varepsilon}(x * y)) = \chi_c(F^{\varepsilon}x)\chi_c(F^{\varepsilon}y)$$

Corollary 6.25. — *Let $\varepsilon \in \{+, -\}$ then*

$$\tilde{Z}^{\chi_c,\varepsilon}_{f\oplus g}(T) = -\tilde{Z}^{\chi_c,\varepsilon}_{f}(T) \odot \tilde{Z}^{\chi_c,\varepsilon}_{g}(T)$$

where the product $\odot$ is the Hadamard product which consists in applying the classical product of $\mathbb{Z}$ coefficientwise.

7. Arc-analytic equivalence

T.-C Kuo [27] defined the notion of blow-analytic equivalence for real analytic function germs: two germs are blow-analytically equivalent if we can get one from the other by composing with a homeomorphism which is blow-analytic and such that the inverse is also blow-analytic. In order to prove that this is an equivalence relation, he gave a characterization in terms of real modifications [27, Proposition 2]. Similarly, G. Fichou [16, Definition 4.1] defined the notion of blow-Nash equivalence for Nash function germs, which is an algebraic version of the blow-analytic equivalence of T.-C. Kuo. G. Fichou proved that his zeta functions are invariants of this relation. However it is not clear that this notion is an equivalence relation.

The definition of blow-Nash equivalence evolved, and we now use the following one [18, 19, 20]: two Nash function germs are blow-Nash equivalent if they are equivalent via a blow-Nash isomorphism in the sense of [17, Definition 1.1]. The assumption "via a blow-Nash isomorphism" is needed to ensure that the zeta functions of G. Fichou are invariants of this relation, but with this assumption, it is still not clear whether it is an equivalence relation.

In this section, we introduce a new relation on Nash function germs, the arc-analytic equivalence which is an equivalence relation. Moreover it coincides with the current version of the blow-Nash equivalence. To introduce it, we do not need Nash modifications. Finally, our zeta function is an invariant of this relation.

Definition 7.1. — *A semialgebraic function $f : V \to \mathbb{R}$ defined on an algebraic set V is said to be blow-Nash if there exists $\sigma : \tilde{V} \to V$ a finite sequence of (algebraic) blowings-up with non-singular centers such that $\tilde{V}$ is non-singular and $f \circ \sigma : \tilde{V} \to \mathbb{R}$ is Nash.*

Definition 7.2. — *A real analytic arc on $\mathbb{R}^d$ is a real analytic map $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^d$.*

We recall the following useful result of Bierstone–Milman.

Theorem 7.3 ([4, Theorem 1.1]). — *A semialgebraic function $f : U \to \mathbb{R}$ defined on a non-singular real algebraic set is blow-Nash if and only if it sends real analytic arcs to real analytic arcs by composition.*

Definition 7.4 ([28]). — *A map that sends real analytic arcs to real analytic arcs by composition is called* arc-analytic.

Definition 7.5. — *Two germs $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ are said to be arc-analytic equivalent if there exists a semialgebraic homeomorphism $h : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ satisfying $f = g \circ h$ such that h is arc-analytic and such that there exists $c > 0$ with $|\det \mathrm{d}h| > c$ where $\mathrm{d}h$ is defined.*

Remark 7.6. — Let $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ and $h : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$ be as in Definition 7.5. Then there exists $\varphi : M \to \mathbb{R}^d$ a finite sequence of blowings-up with non-singular centers such that $\tilde{\varphi} = h \circ \varphi$ is Nash:

$$\begin{tikzcd}
& M \arrow[dl, "\varphi"'] \arrow[dr, "\tilde{\varphi}"] & \\
\mathbb{R}^d \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathbb{R}^d \arrow[dl, "g"] \\
& \mathbb{R} &
\end{tikzcd}$$

Notice that then $\tilde{\varphi}$ is proper generically one-to-one Nash. Moreover, by [7, Corollary 4.17], $\tilde{\varphi}$ induces a bijection between arcs on M and arcs on $\mathbb{R}^d$ not entirely included in some nowhere dense subset of $\mathbb{R}^d$.

Proposition 7.7. — *Arc-analytic equivalence is an equivalence relation.*

Proof. — The reflexivity is obvious, the symmetry comes from [7, Theorem 3.5]. Thus it suffices to prove the transitivity. We have the following diagram

$$\begin{tikzcd}
\mathbb{R}^d \arrow[r, "h_1"] \arrow[dr, bend right, "f_1"'] & \mathbb{R}^d \arrow[r, "h_2"] \arrow[d, "f_2"'] & \mathbb{R}^d \arrow[dl, bend left, "f_3"] \\
& \mathbb{R} &
\end{tikzcd}$$

where the f_i are Nash germs and the h_i are as in Definition 7.5. Obviously there exists $c > 0$ such that $|\det \mathrm{d}(h_2 \circ h_1)| > c$. The composition $h_2 \circ h_1$ is obviously semialgebraic and arc-analytic (as the composition of such maps). $\square$

Proposition 7.8. — *Two Nash germs which are blow-Nash equivalent in the sense of* [16, Definition 4.1] *are arc-analytic equivalent.*

Proof. — Assume that f_1 and f_2 are blow-Nash equivalent in the sense of [16]. Then we have

$$\begin{tikzcd}
M_1 \arrow[rr, "\Phi"] \arrow[d, "\nu_1"'] & & M_2 \arrow[d, "\nu_2"] \\
\mathbb{R}^d \arrow[rr, "\phi"] \arrow[dr, "f_1"'] & & \mathbb{R}^d \arrow[dl, "f_2"] \\
& \mathbb{R} &
\end{tikzcd}$$

with ϕ a semialgebraic homeomorphism, ν_i two proper birational algebraic maps and Φ a Nash isomorphism which preserves the multiplicities of the jacobian determinants of ν_1 and ν_2.

Since we may lift analytic arcs by ν_1, ϕ is arc-analytic. By the chain rule, since Φ preserves the multiplicities of the jacobian determinants of ν_1 and ν_2, we deduce that $|\det \mathrm{d}\phi| > c$ for some $c > 0$. $\square$

Proposition 7.9. — *Two Nash germs are arc-analytic equivalent if and only if they are blow-Nash equivalent via a blow-Nash isomorphism in the sense of* [17, Definition 1.1].

Proof. — Assume that f_1 and f_2 are blow-Nash equivalent via a blow-Nash isomorphism in the sense of [17]. Then we have

$$\begin{tikzcd}
M_1 \arrow[rr, "\Phi"] \arrow[d, "\nu_1"'] & & M_2 \arrow[d, "\nu_2"] \\
\mathbb{R}^d \arrow[rr, "\phi"] \arrow[dr, "f_1"'] & & \mathbb{R}^d \arrow[dl, "f_2"] \\
& \mathbb{R} &
\end{tikzcd}$$

with ϕ a semialgebraic homeomorphism, ν_i two Nash modifications[(9)] and Φ a Nash isomorphism which preserves the multiplicities of the jacobian determinants of ν_1 and ν_2.

Since we may lift analytic arcs by ν_1, ϕ is arc-analytic. By the chain rule, since Φ preserves the multiplicities of the jacobian determinants of ν_1 and ν_2, we deduce that $|\det \mathrm{d}\phi| > c$ for some $c > 0$.

Assume that f_1 and f_2 are arc-analytic equivalent via h: $f_1 = f_2 \circ h$ with h semialgebraic and arc-analytic satisfying $|\det \mathrm{d}h| > c > 0$. Since h and h^{-1} are blow-Nash there exist $\nu_1 : M_1 \to \mathbb{R}^d$ and $\nu_2 : M_2 \to \mathbb{R}^d$ two finite sequences of blowings-up with non-singular centers such that $\alpha_1 = h \circ \nu_1$ and $\alpha_2 = h^{-1} \circ \nu_2$ are Nash. Let N_1 (resp. N_2) be the fiber product of α_1 and ν_2 (resp. α_2 and ν_1). Then $N_1 = N_2 = N$ in $M_1 \times M_2$. Notice that π_i and N_i are Nash. Let $\sigma : \tilde{N} \to N$ be a resolution of singularities (for Nash spaces, see [5, p. 234]). Then for $i = 1, 2$, $\nu_i \pi_i \sigma$ is a Nash modification.

$$\begin{tikzcd}
& \tilde{N} \arrow[d, "\sigma"'] & \\
N_1 \arrow[r, equal] \arrow[d, "\pi_1"'] \arrow[drr, "\rho_1"] & N \arrow[r, equal] & N_2 \arrow[d, "\pi_2"] \arrow[dll, "\rho_2"'] \\
M_1 \arrow[d, "\nu_1"'] \arrow[drr, "\alpha_1"] & & M_2 \arrow[d, "\nu_2"] \arrow[dll, "\alpha_2"'] \\
\mathbb{R}^d \arrow[rr, shift left, "h"] \arrow[dr, "f_1"'] & & \mathbb{R}^d \arrow[ll, shift left, "h^{-1}"] \arrow[dl, "f_2"] \\
& \mathbb{R} &
\end{tikzcd}$$

□

(9) A Nash modification is a proper surjective Nash map whose complexification is proper and bimeromorphic.

Corollary 7.10. — *Blow-Nash equivalence via a blow-Nash isomorphism in the sense of* [17, Definition 1.1] *is an equivalence relation.*

Theorem 7.11. — *If two Nash germs $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are arc-analytic equivalent then $Z_f(T) = Z_g(T)$.*

Proof. — We have the following diagram

$$\begin{tikzcd} & M \arrow[dl, "\varphi"'] \arrow[dr, "\tilde{\varphi}"] & \\ \mathbb{R}^d \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathbb{R}^d \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$

where h is a semialgebraic homeomorphism, φ a finite sequence of blowings-up with non-singular centers and $\tilde{\varphi}$ a Nash map.

Notice that in the statement of Theorem 4.22 we only need that h is proper generically 1-to-1 Nash and not merely birational. Therefore we may also apply this theorem to $\tilde{\varphi}$ in order to compute $Z_g(T)$.

Up to adding more blowings-up we may assume that $f \circ \varphi$, $g \circ \tilde{\varphi} = g \circ h \circ \varphi$, $\operatorname{Jac} \varphi$ and $\operatorname{Jac} \tilde{\varphi}$ are simultaneously normal crossings. Denote by $(E_i)_{i \in A}$ the irreducible components of the zero set of $f \circ \varphi = g \circ \tilde{\varphi}$. Set $f \circ \varphi = \sum_{i \in A} N_i E_i$, $g \circ \tilde{\varphi} = \sum_{i \in A} \tilde{N}_i E_i$, $\operatorname{Jac} \varphi = \sum_{i \in A} (\nu_i - 1) E_i$ and $\operatorname{Jac} \tilde{\varphi} = \sum_{i \in A} (\tilde{\nu}_i - 1) E_i$.

By [7, Lemma 4.15], $\forall i \in A$, $\nu_i = \tilde{\nu}_i$. And since $f \circ \varphi = g \circ \tilde{\varphi}$, we have $\forall i \in A$, $N_i = \tilde{N}_i$ and that U_I is well-defined and doesn't depend on φ or $\tilde{\varphi}$.

Thus by Theorem 4.22

$$Z_f(T) = Z_g(T) = \sum_{\varnothing \neq I \subset A} \left[U_I \cap (h \circ p_I)^{-1}(0) \right] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}} \qquad \square$$

Remark 7.12. — Particularly, by Proposition 7.9, we recover [17, Proposition 2.6].

By Proposition 7.8, [16, Theorem 4.3] works as it is in our settings (see also [17, Theorem 1.5] and [27, Theorem 1]).

Theorem 7.13. — *Let $F : (\mathbb{R}^d, 0) \times (0,1)^k \to (\mathbb{R}, 0)$ be a Nash function such that $\forall t \in (0,1)^k$, $F(\cdot, t) : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ has an isolated singularity at 0 and there exists an algebraic proper birational map $\sigma : M \to \mathbb{R}^d \times (0,1)^k$ such that $F \circ \sigma$ has only normal crossings. Then the elements of the family $F_t(\cdot) = F(\cdot, t)$ represent a finite number of arc-analytic equivalence classes.*

In the same way, we recover a version of [16, Proposition 4.17].

Proposition 7.14. — *Let $F : (\mathbb{R}^d, 0) \times (0,1)^k \to (\mathbb{R}, 0)$ be a Nash function such that $\forall t \in (0,1)^k$, $F(\cdot, t) : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ has an isolated singularity at 0 and there exists an algebraic proper birational map $\sigma : M \to \mathbb{R}^d$ such that $F \circ (\sigma, \mathrm{id}_{(0,1)^k})$ has only normal crossings. Then the elements of the family $F_t(\cdot) = F(\cdot, t)$ represent a unique arc-analytic equivalence class.*

Again, the following corollary is just a version of [16, Corollary 4.5].

Corollary 7.15. — *Let $F : (\mathbb{R}^d, 0) \times (0,1) \to (\mathbb{R}, 0)$ be a Nash function such that $F(\cdot, t) : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ are weighted homogeneous polynomials with the same weights and have an isolated singularity at 0. Then the elements of the family $F_t(\cdot) = F(\cdot, t)$ represent a unique arc-analytic equivalence class.*

Example 7.16. — The Whitney family [39, Example 13.1]

$$f_t(x, y) = xy(y - x)(y - tx), \; t \in (0, 1)$$

has only one arc-analytic equivalence class. ⊡

8. Classification of Brieskorn polynomials

In [24], S. Koike and A. Parusiński gave a complete classification of the Brieskorn polynomials in two variables up to blow-analytic equivalence using their zeta functions and the Fukui invariants. In [16], G. Fichou classified the Brieskorn polynomials in three variables up to blow-Nash equivalence (and thus up to arc-analytic equivalence) thanks to his zeta functions. In this section we use the convolution formula to prove that two arc-analytic equivalent Brieskorn polynomials share the same exponents.

Definition 8.1. — *A Brieskorn polynomial is a polynomial of the following form*

$$f(x) = \sum_{i=1}^{d} \varepsilon_i x^{k_i}$$

where $\varepsilon_i \in \{\pm 1\}$ and $k_i \geqslant 1$.

Remark 8.2. — We will assume that $k_i \geqslant 2$ since otherwise the polynomial is non-singular. Up to reordering the variables, we will always assume that $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$.

Proposition 8.3. — *Let*

$$f(x)=\sum_{i=1}^{d}\varepsilon_i x^{k_i}$$

be a Brieskorn polynomial. We may recover $(k_1,\dots,k_d)$ *from* d *and* $Z_f(T)$*.*

Corollary 8.4. — *Let*

$$f(x)=\sum_{i=1}^{d}\varepsilon_i x^{k_i}\qquad\text{and}\qquad g(x)=\sum_{i=1}^{d}\eta_i x^{l_i}$$

be two Brieskorn polynomials. If f *and* g *are blow-Nash equivalent then* $\forall i=1,\dots,d$ *we have* $k_i=l_i$*.*

Remember from Example 6.10 that

$$\begin{aligned}\tilde{Z}_{\varepsilon_i x_i^{k_i}}(T)=&-T-\cdots-T^{k_i-1}\\&-\left(\mathbb{1}-[\varepsilon_i x_i^{k_i}]\right)\mathbb{L}^{-1}T^{k_i}-\mathbb{L}^{-1}T^{k_i+1}-\cdots-\mathbb{L}^{-1}T^{2k_i-1}\\&-\left(\mathbb{1}-[\varepsilon_i x_i^{k_i}]\right)\mathbb{L}^{-2}T^{2k_i}-\mathbb{L}^{-2}T^{2k_i+1}-\cdots\end{aligned}$$

Notice that the coefficients of terms whose degrees are not multiples of k_i are of the form $-\mathbb{L}^{-\alpha}$ where α is the integral part of the degree divided by k_i.

Denote by $a_n\in\mathcal{M}$ the coefficients of the modified zeta function of f so that

$$\tilde{Z}_f(T)=\sum_{n\geqslant 1}a_nT^n\in\mathcal{M}[\![T]\!]$$

Using the convolution formula and that $\mathbb{L}^{-\alpha}*\mathbb{L}^{-\beta}=\mathbb{L}^{-(\alpha+\beta)}$ ($\mathcal{M}_{\mathcal{AS}}$-bilinearity of the convolution product) we deduce from the previous remark that if k_i doesn't divide n for all i then

$$a_n=-\mathbb{L}^{-\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor}$$

Moreover, we may recover the degree $\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor$ using the forgetful morphism $\overline{\cdot}:\mathcal{M}\to\mathcal{M}_{\mathcal{AS}}$ and the natural extension of the virtual Poincaré polynomial to $\mathcal{M}_{\mathcal{AS}}$, $\beta:\mathcal{M}_{\mathcal{AS}}\to\mathbb{Z}[u,u^{-1}]$. Indeed,

$$\begin{aligned}\beta\left(\overline{a_n}\right)=\beta\left(\overline{-\mathbb{L}^{-\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor}}\right)&=\beta\left(-\mathbb{L}_{\mathcal{AS}}^{-\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor}(\mathbb{L}_{\mathcal{AS}}-1)\right)\\&=-u^{-\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor}(u-1)\end{aligned}$$

and thus

$$\sum_{i=1}^{d} \left\lfloor \frac{n}{k_i} \right\rfloor = -\deg \beta(\overline{a_n}) + 1$$

The idea of the proof of Proposition 8.3 is to use the previous fact to reduce to a combinatorial problem.

LEMMA 8.5. — *Fix $m \in \mathbb{N}_{>0}$. Let $x = \sum_{i=1}^{m} \frac{1}{l_i}$ with $l_i \in \mathbb{N}_{>0}$. Then there exists a finite number of m-tuples $(l'_i)_{i=1,\dots,m}$ such that $x = \sum_{i=1}^{m} \frac{1}{l'_i}$.*

Proof. — Assume that the statement is true for some $m \in \mathbb{N}_{>0}$ and let $x = \sum_{i=1}^{m+1} \frac{1}{l_i}$. We may assume that $l_1 \leqslant \cdots \leqslant l_{m+1}$. Then $l_1 \leqslant \frac{m+1}{x}$. Then there is a finite number of choices for l_1 and for every choice $x - \frac{1}{l_1} = \sum_{i=2}^{m+1} \frac{1}{l_i}$ admits a finite number of expressions of this form. □

Proof of Proposition 8.3. — Let

$$\tilde{Z}_f(T) = \sum_{n \geqslant 1} a_n T^n \in \mathcal{M}[\![T]\!]$$

Denote by $\mathcal{P}$ the set of primes. For $p \in \mathcal{P}$ big enough, p is not a multiple of a k_i. Thus

$$\lim_{p \in \mathcal{P}} \frac{-\deg\left(\beta(\overline{a_p})\right) + 1}{p} = \lim_{p \in \mathcal{P}} \frac{\sum_{i=1}^{d} \left\lfloor \frac{p}{k_i} \right\rfloor}{p} = \sum_{i=1}^{d} \frac{1}{k_i}$$

We deduce from the previous computation and Lemma 8.5 that we may derive from $\tilde{Z}_f(T)$ (or equivalently from $Z_f(T)$) an integer K such that $k_1 \leqslant \cdots \leqslant k_d \leqslant K$.

Denote by $\mathcal{P}'$ the set of primes lower or equal to K. For $p \in \mathcal{P}'$, we denote by γ_p the greatest exponent such that $p^{\gamma_p} \leqslant K$.

Set

$$Q = \left\{ \prod_{p \in \mathcal{P}'} p^{\alpha_p},\ 0 \leqslant \alpha_p \leqslant \gamma_p \right\}$$

so that $\{k_1, \dots, k_d\} \subset Q$. Up to adding elements in Q, we may assume that $2, 3, 5, 7 \in \mathcal{P}'$ and that $\gamma_2 \geqslant 3, \gamma_3 \geqslant 2, \gamma_5 \geqslant 1, \gamma_7 \geqslant 1$.

For $q \in Q$, set $\operatorname{mult}(q) = \#\{k_i,\ k_i = q\}$. Thus our goal is to compute $\operatorname{mult} q$, $q \in Q$, from $Z_f(T)$.

The main idea of the proof consists, for a number $q \in Q$, to use the Chinese remainder theorem to find n such that $n-1$ and $n+1$ are not multiple of a term in $Q \setminus \{1\}$ and such that the only factors of n that are in Q are the factors of q. The first condition ensures that no exponent divides $n-1$ and $n+1$ so that we can recover the degrees of a_{n-1} and a_{n+1}

as previously. The second condition ensures that the difference of these degrees is exactly

$$\sum_{k|q} \operatorname{mult} k$$

This allows us to compute $\operatorname{mult} q$ recursively by increasing the number of factors in q. This is exactly the first item below.

Unfortunately this method won't work when $2 \nmid q$ or $3 \nmid q$ since then 3 or 2 may divide $n-1$ or $n+1$ (whereas 2 and 3 may appear as exponents in the polynomial). Thus we have to manage these cases separately in a similar, but more sophisticated, way.

We first notice that $\operatorname{mult}(1) = 0$.

Equations involving $\operatorname{mult} q$ *for* $q \in Q$ *satisfying* $6|q$. — Let α_2, α_3 be such that $1 \leqslant \alpha_2 \leqslant \gamma_2$ and $1 \leqslant \alpha_3 \leqslant \gamma_3$ and for each $p \in \mathcal{P}' \setminus \{2,3\}$, let α_p be such that $0 \leqslant \alpha_p \leqslant \gamma_p$.

The Chinese remainder theorem ensures the existence of n such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0 \\ n \equiv 2 \mod p & \text{otherwise} \end{cases}$$

Thus, for $p \in \mathcal{P}'$, p doesn't divide $n-1$ and $n+1$. This ensures that, for $q \in Q$, $n-1$ and $n+1$ are not multiple of q and particularly of an exponent k_i. So $a_{n-1} = -\mathbb{L}^{-\sum_{i=1}^{d} \left\lfloor \frac{n-1}{k_i} \right\rfloor}$ and $a_{n+1} = -\mathbb{L}^{-\sum_{i=1}^{d} \left\lfloor \frac{n+1}{k_i} \right\rfloor}$.

Moreover the elements of Q which divide n are exactly those of the form $\prod_{p\in\mathcal{P}'} p^{\beta_p}$ with $0 \leqslant \beta_p \leqslant \alpha_p$.

Therefore

$$-\deg \beta(\overline{a_{n+1}}) + \deg \beta(\overline{a_{n-1}}) = \sum_{0\leqslant\beta_p\leqslant\alpha_p} \operatorname{mult} \left(\prod_{p\in\mathcal{P}'} p^{\beta_p} \right)$$

Computation of $\operatorname{mult} 2$. — Assume that $\alpha_2 = 2$, $\alpha_3 = \alpha_5 = 1$ and $\alpha_p = 0$ for $p \in \mathcal{P}' \setminus \{2,3,5\}$. Let n be such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0 \\ n \equiv 3 \mod p & \text{if } \alpha_p = 0 \end{cases}$$

Then $n = 60n'$ where no term in $\mathcal{P}'$ divides n'. No term in Q divide $n-1$ and $n+1$. And 2 but no other term in Q divides $n-2$ and $n+2$. Thus

$$a_{n-2} = -\alpha\mathbb{L}^{-\sum_{i=1}^{d} \left\lfloor \frac{n-2}{k_i} \right\rfloor} \quad \text{and} \quad a_{n+2} = -\alpha\mathbb{L}^{-\sum_{i=1}^{d} \left\lfloor \frac{n+2}{k_i} \right\rfloor}$$

where α is of the form $*_{i=1}^{\operatorname{mult} 2} \left(\mathbb{1} - [\varepsilon_i x^2 : \mathbb{R}^* \to \mathbb{R}^*] \right)$. Thus

$$\beta(F^+(a_{n-2})) = \beta\left(-\mathbb{L}_{\mathcal{AS}}^{-\sum_{i=1}^{d} \left\lfloor \frac{n-2}{k_i} \right\rfloor} F^+(\alpha) \right) = -u^{-\sum_{i=1}^{d} \left\lfloor \frac{n-2}{k_i} \right\rfloor} \beta\left(F^+(\alpha)\right)$$

and similarly

$$\beta(F^+(a_{n+2})) = \beta\left(-\mathbb{L}_{\mathcal{AS}}^{-\sum_{i=1}^d \left\lfloor \frac{n+2}{k_i} \right\rfloor} F^+(\alpha)\right) = -u^{-\sum_{i=1}^d \left\lfloor \frac{n+2}{k_i} \right\rfloor} \beta\left(F^+(\alpha)\right)$$

We deduce from Lemma 6.24 and $\chi_c\left(F^+\left(\mathbb{1} - [\pm x^2]\right)\right) = \mp 1$ that

$$\beta\left(F^+(\alpha)\right)(u=-1) = \chi_c(F^+(\alpha)) = \prod_{i=1}^{\operatorname{mult} 2} \chi_c(F^+(\mathbb{1} - [\varepsilon_i x^2])) = \pm 1$$

Thus $\beta\left(F^+(\alpha)\right) \neq 0$ and

$$-\deg \beta(F^+(a_{n+2})) + \deg \beta(F^+(a_{n-2})) = \operatorname{mult} 2 + \sum_{q \in Q, q | 60} \operatorname{mult}(q)$$

Notice that in the first case we got an equation of the form

$$cst = \sum_{q \in Q, q|60} \operatorname{mult}(q)$$

Therefore $\operatorname{mult} 2$ may be derived from $Z_f(T)$.

Computation of $\operatorname{mult} q$ *with* $2 \nmid q$ *and* $3 \nmid q$. — Assume that $\alpha_2 = \alpha_3 = 0$ and that $0 \leqslant \alpha_p \leqslant \gamma_p$ for each $p \in \mathcal{P}' \setminus \{2,3\}$. Let n be such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0 \\ n \equiv 1 \mod 8 & \\ n \equiv 1 \mod p & \text{if } p \neq 2,\ \alpha_p = 0 \end{cases}$$

Then the only elements of Q which divide n are those of the form $\prod_{p \in \mathcal{P}'} p^{\beta_p}$ with $0 \leqslant \beta_p \leqslant \alpha_p$, the only element in Q which divides $n+1$ and $n-3$ is 2, no element in Q divides $n-2$ and 6 divides $n-1$. Thus

$$\begin{aligned} -\deg \beta(\overline{a_{n+1}}) &+ \deg \beta(\overline{a_{n-3}}) \\ &= \sum_{q \in Q, q|n-1} \operatorname{mult}(q) + \sum_{0 \leqslant \beta_p \leqslant \alpha_p} \operatorname{mult}\left(\prod_{p \in \mathcal{P}'} p^{\beta_p}\right) + \operatorname{mult}(2) \end{aligned}$$

Notice that since in the first case we got an equation of the form

$$cst = \sum_{q \in Q, q|n-1} \operatorname{mult}(q)$$

and that we already know $\operatorname{mult} 2$, thus we get an equation of the form

$$cst = \sum_{0 \leqslant \beta_p \leqslant \alpha_p} \operatorname{mult}\left(\prod_{p \in \mathcal{P}'} p^{\beta_p}\right)$$

Remark: either $5|n$ (if $5|q$) or $5|n-1$.

Therefore we may recursively compute $\operatorname{mult} q$ for each $q \in Q$ such that $2 \nmid q$ and $3 \nmid q$ by varying α_p for each $p \in \mathcal{P}' \setminus \{2,3\}$.

Computation of $\operatorname{mult} 3$ *and* $\operatorname{mult} 4$. — Let n be such that

$$\begin{cases} n \equiv 4 \mod 8 \\ n \equiv 4 \mod 9 \\ n \equiv 4 \mod 25 \\ n \equiv 5 \mod 7 \\ n \equiv 4 \mod p \qquad \text{if } p \in \mathcal{P}' \setminus \{2,3,5,7\} \end{cases}$$

Then $2, 4$ are the only element of Q dividing n, 3 is the only element of Q dividing $n-1$, 5 is the only element of Q dividing $n+1$, 2 is the only element of Q dividing $n-2$, 6 divides $n+2$ and no element of Q divides $n-3, n+3$. Thus

$$-\deg \beta(\overline{a_{n+3}}) + \deg \beta(\overline{a_{n-3}}) = \sum_{q \in Q, q \mid n+2} \operatorname{mult}(q) + 2 \operatorname{mult} 2 + \operatorname{mult} 3 + \operatorname{mult} 4 + \operatorname{mult} 5$$

Notice that we already know $\operatorname{mult} 2$, $\operatorname{mult} 5$ and that in the previous case we got an equation of the form

$$c = \sum_{q \in Q, q \mid n+2} \operatorname{mult}(q)$$

So we have an equation of the form

$$\operatorname{mult} 3 + \operatorname{mult} 4 = cst \tag{8.1}$$

Now let $\alpha_2 = 3, \alpha_3 = 2, \alpha_5 = 1, \alpha_7 = 1$. Let n be such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0 \\ n \equiv 5 \mod p & \text{if } \alpha_p = 0 \end{cases}$$

Then $2^3 \cdot 3^2 \cdot 5 \cdot 7$ divides n, no term of Q divides $n-1, n+1$, only 2 divides $n-2, n+2$, only 3 divides $n-3, n+3$ and only $2, 4$ divide $n-4, n+4$.

Since $\chi_c\left(F^+\left(\mathbb{1} - [\pm x^2]\right)\right) = \pm 1$ and $\chi_c\left(F^+\left(\mathbb{1} - [\pm x^4]\right)\right) = \pm 1$, we have

$$-\deg \beta(F^+(a_{n+4})) + \deg \beta(F^+(a_{n-4})) = 3 \operatorname{mult} 2 + 2 \operatorname{mult} 3 + \operatorname{mult} 4 + \sum_{q \in Q, q \mid n} \operatorname{mult}(q)$$

but in the first case we got an equation of the form $\sum_{q \in Q, q \mid n} \operatorname{mult}(q) = cst$ and we already know $\operatorname{mult} 2$, thus we get an equation of the form

$$2 \operatorname{mult} 3 + \operatorname{mult} 4 = cst \tag{8.2}$$

We compute mult 3 and mult 4 from the system given by (8.1) and (8.2).

Computation of mult q *with* $2|q$, $4\nmid q$, $3\nmid q$ *and* $5\nmid q$. — Let $\alpha_2 = 1$, $\alpha_3 = \alpha_5 = 0$ and $0 \leqslant \alpha_p \leqslant \gamma_p$ otherwise. Let n be such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0, p \neq 2 \\ n \equiv 2 \mod 8 & \\ n \equiv 2 \mod 9 & \\ n \equiv 6 \mod 25 & \\ n \equiv 2 \mod 7 & \text{if } \alpha_7 = 0 \\ n \equiv 4 \mod p & \text{if } \alpha_p = 0 \text{ and } p \neq 3,5,7 \end{cases}$$

Then the only terms of Q dividing n are the divisors of $\prod p^{\alpha_p}$, the only term of Q dividing $n-1$ is 5, the only term of Q dividing $n+1$ is 3, the only terms of Q dividing $n+2$ are $2, 4$, no term of Q divides $n-3, n+3$ and 6 divides $n-2$. Thus

$$\begin{aligned} -\deg\beta(\overline{a_{n+3}}) + \deg\beta(\overline{a_{n-3}}) \\ = \operatorname{mult} 2 + \operatorname{mult} 3 + \operatorname{mult} 4 + \operatorname{mult} 5 \\ + \sum_{q\in Q, q|n-2} \operatorname{mult}(q) + \sum_{0\leqslant\beta_p\leqslant\alpha_p} \operatorname{mult}\left(\prod_{p\in\mathcal{P}'} p^{\beta_p}\right) \end{aligned}$$

from which we derive an equation of the form

$$\sum_{\substack{\beta_2=1 \\ 0\leqslant\beta_p\leqslant\alpha_p}} \operatorname{mult}\left(\prod_{p\in\mathcal{P}'} p^{\beta_p}\right) = cst$$

since we already know mult q for $q \in Q$ with $2 \nmid q$ and $3 \nmid q$ and since we got in the first case an equation of the form

$$\sum_{q\in Q, q|n-2} \operatorname{mult}(q) = cst$$

Therefore we may recursively compute mult q for each $q \in Q$ such that $2|q$, $4\nmid q$, $3\nmid q$ and $5\nmid q$ by varying the α_p.

Computation of $\operatorname{mult} q$ *with* $4|q$, $3 \nmid q$ *and* $5 \nmid q$. — Let $2 \leqslant \alpha_2 \leqslant \gamma_2$, $\alpha_3 = \alpha_5 = 0$ and $0 \leqslant \alpha_p \leqslant \gamma_p$ otherwise. Let n be such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0 \\ n \equiv 2 \mod 9 & \\ n \equiv 6 \mod 25 & \\ n \equiv 2 \mod 7 & \text{if } \alpha_7 = 0 \\ n \equiv 4 \mod p & \text{if } \alpha_p = 0, p \neq 3,5,7 \end{cases}$$

Then the only terms of Q dividing n are the divisors of $\prod p^{\alpha_p}$, the only term of Q dividing $n-1$ is 5, the only term of Q dividing $n+1$ is 3, the only term of Q dividing $n+2$ is 2, 6 divides $n-2$ and no term of Q divides $n-3, n+3$. Thus

$$\begin{aligned} -\deg \beta(\overline{a_{n+3}}) + \deg \beta(\overline{a_{n-3}}) \\ &= \operatorname{mult} 2 + \operatorname{mult} 3 + \operatorname{mult} 5 \\ &\quad + \sum_{q \in Q, q|n-2} \operatorname{mult}(q) + \sum_{0 \leqslant \beta_p \leqslant \alpha_p} \operatorname{mult}\left(\prod_{p \in \mathcal{P}'} p^{\beta_p}\right) \end{aligned}$$

from which we derive an equation of the form

$$\sum_{\substack{2 \leqslant \beta_2 \leqslant \alpha_2 \\ 0 \leqslant \beta_p \leqslant \alpha_p}} \operatorname{mult}\left(\prod_{p \in \mathcal{P}'} p^{\beta_p}\right) = cst$$

Therefore we may recursively compute $\operatorname{mult} q$ for each $q \in Q$ such that $4|q$, $3 \nmid q$ and $5 \nmid q$ by varying the α_p.

Computation of $\operatorname{mult} q$ *with* $2|q$, $4 \nmid q$, $3 \nmid q$, $5|q$. — Let $\alpha_2 = 1$, $\alpha_3 = 0$, $1 \leqslant \alpha_5 \leqslant \gamma_5$ and $0 \leqslant \alpha_p \leqslant \gamma_p$ otherwise. Let n be such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0 \text{ and } p \neq 2 \\ n \equiv 2 \mod 8 & \\ n \equiv 2 \mod 9 & \\ n \equiv 3 \mod p & \text{if } \alpha_p = 0 \text{ and } p \neq 3 \end{cases}$$

Then the only terms of Q dividing n are the divisors of $\prod p^{\alpha_p}$, the only term in Q dividing $n+1$ is 3 and the only terms in Q dividing $n+2$ are $2, 4$.

No term of Q divides $n-1, n+3$. Thus

$$-\deg\beta(\overline{a_{n+3}})+\deg\beta(\overline{a_{n-1}}) = \operatorname{mult}3+\operatorname{mult}2+\operatorname{mult}4+\sum_{0\leqslant\beta_p\leqslant\alpha_p}\operatorname{mult}\left(\prod_{p\in\mathcal{P}'}p^{\beta_p}\right)$$

We conclude as in the previous cases.

Computation of $\operatorname{mult} q$ *with* $4|q$, $3\nmid q$, $5|q$. — Let $2\leqslant\alpha_2\leqslant\gamma_2$, $\alpha_3=0$, $1\leqslant\alpha_5\leqslant\gamma_5$ and $0\leqslant\alpha_p\leqslant\gamma_p$ otherwise. Let n be such that

$$\begin{cases} n\equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p>0\\ n\equiv 2 \mod 9 & \\ n\equiv 3 \mod p & \text{if } \alpha_p=0 \text{ and } p\neq 3\end{cases}$$

Then the only terms of Q dividing n are the divisors of $\prod p^{\alpha_p}$, the only term in Q dividing $n+1$ is 3 and the only terms in Q dividing $n+2$ are 2. No term of Q divides $n-1, n+3$. Thus

$$-\deg\beta(\overline{a_{n+3}})+\deg\beta(\overline{a_{n-1}}) = \operatorname{mult}3+\operatorname{mult}2+\sum_{0\leqslant\beta_p\leqslant\alpha_p}\operatorname{mult}\left(\prod_{p\in\mathcal{P}'}p^{\beta_p}\right)$$

We conclude as in the previous cases.

Computation of $\operatorname{mult} q$ *with* $3|q$, $2\nmid q$ *and* $5\nmid q$. — Let $1\leqslant\alpha_3\leqslant\gamma_3$, $\alpha_2,\alpha_5=0$ and $0\leqslant\alpha_p\leqslant\gamma_p$ otherwise. Let n be such that

$$\begin{cases} n\equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p>0\\ n\equiv 3 \mod 8 & \\ n\equiv 4 \mod 25 & \\ n\equiv 3 \mod p & \text{if } \alpha_p=0,\ p\neq 2,\ p\neq 5\end{cases}$$

Then the only terms of Q dividing n are the divisors of $\prod p^{\alpha_p}$, 2 is the only term of Q dividing $n-1$, $2,4,5,10,20$ are the only terms of Q dividing $n+1$ and no term in Q divides $n-2, n+2$. Thus

$$-\deg\beta(\overline{a_{n+2}})+\deg\beta(\overline{a_{n-2}}) = \operatorname{mult}2+\sum_{q\in Q, q|20}\operatorname{mult}(q)+\sum_{0\leqslant\beta_p\leqslant\alpha_p}\operatorname{mult}\left(\prod_{p\in\mathcal{P}'}p^{\beta_p}\right)$$

Notice that in the previous case we got an equation of the form

$$c = \sum_{q \in Q, q|20} \operatorname{mult}(q)$$

We conclude as in the previous cases.

Computation of $\operatorname{mult} q$ *with* $3|q$, $2 \nmid q$ *and* $5|q$. — Let $1 \leqslant \alpha_3 \leqslant \gamma_3$, $\alpha_2 = 0$, $1 \leqslant \alpha_5 \leqslant \gamma_5$ and $0 \leqslant \alpha_p \leqslant \gamma_p$ otherwise. Let n be such that

$$\begin{cases} n \equiv p^{\alpha_p} \mod p^{\alpha_p+1} & \text{if } \alpha_p > 0 \\ n \equiv 3 \mod 8 & \\ n \equiv 3 \mod p & \text{if } \alpha_p = 0,\, p \neq 2 \end{cases}$$

Then the only terms of Q dividing n are the divisors of $\prod p^{\alpha_p}$, 2 is the only term of Q dividing $n-1$, $2, 4$ are the only terms of Q dividing $n+1$ and no term in Q divides $n-2, n+2$. Thus

$$-\deg \beta(\overline{a_{n+2}}) + \deg \beta(\overline{a_{n-2}}) = 2 \operatorname{mult} 2 + \operatorname{mult} 4 + \sum_{0 \leqslant \beta_p \leqslant \alpha_p} \operatorname{mult}\left(\prod_{p \in \mathcal{P}'} p^{\beta_p}\right)$$

We conclude as in the previous cases.

Computation of $\operatorname{mult} q$ *for* $q \in Q$ *satisfying* $6|q$. — Let α_2, α_3 be such that $1 \leqslant \alpha_2 \leqslant \gamma_2$ and $1 \leqslant \alpha_3 \leqslant \gamma_3$ and for each $p \in \mathcal{P}' \setminus \{2, 3\}$, let α_p be such that $0 \leqslant \alpha_p \leqslant \gamma_p$.

Now that we know $\operatorname{mult} q$ for $q \in Q$ satisfying $2 \nmid q$ or $3 \nmid q$ we may deduce from the equation of the first case an equation of the form

$$\sum_{\substack{1 \leqslant \beta_2 \leqslant \alpha_2 \\ 1 \leqslant \beta_3 \leqslant \alpha_3 \\ 0 \leqslant \beta_p \leqslant \alpha_p}} \operatorname{mult}\left(\prod_{p \in \mathcal{P}'} p^{\beta_p}\right) = cst$$

And we conclude as in the previous cases by varying the α_p. □

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