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ON THE CANCELLATION PROBLEM FOR ALGEBRAIC TORI

by Adrien DUBOULOZ (*)

Dedicated to Wlodek Danielewski.

Abstract. — We address a variant of Zariski Cancellation Problem, asking whether two varieties which become isomorphic after taking their product with an algebraic torus are isomorphic themselves. Such cancellation property is easily checked for curves, is known to hold for smooth varieties of log-general type by virtue of a result of Iitaka-Fujita and more generally for non $\mathbb{A}^1_*$-uniruled varieties. We show in contrast that for smooth affine factorial $\mathbb{A}^1_*$-ruled varieties, cancellation fails in any dimension bigger than or equal to two.

Résumé. — Nous considérons une variante du Problème de Simplification de Zariski pour les tores algébriques: deux variétés algébriques dont les produits cartésiens avec un même tore algébrique sont isomorphes sont-elles isomorphes ? Un argument élémentaire montre que les courbes algébriques possèdent cette propriété de simplification. Un résultat très général de simplification du à Iitaka et Fujita implique qu'il en est de même pour les variétés de type log-général ou non $\mathbb{A}^1_*$-réglées. Dans cet article, nous construisons en toute dimension supérieure ou égale à deux des couples de variétés factorielles $\mathbb{A}^1_*$-réglées ne possèdant pas la propriété de simplification par des tores.

Since the late seventies, the Cancellation Problem is usually understood in its geometric form as the question whether two algebraic varieties X and Y with isomorphic cylinders $X\times\mathbb{A}^1$ and $Y\times\mathbb{A}^1$ are isomorphic themselves. This problem is intimately related to the geometry of rational curves on X and Y: in particular, if X or Y are smooth quasiprojective and not $\mathbb{A}^1$-uniruled, in the sense that they do not admit any dominant generically finite morphism from a variety of the form $Z\times\mathbb{A}^1$, then every isomorphism $\Phi : X\times\mathbb{A}^1 \xrightarrow{\sim} Y\times\mathbb{A}^1$ descends to an isomorphism between X and Y, a property which is sometimes called strong cancellation. Over an algebraically closed field of characteristic zero, the non $\mathbb{A}^1$-uniruledness of a

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smooth quasi-projective variety X is guaranteed in particular by the existence of nonzero pluri-forms with logarithmic poles at infinity on suitable projective completions of X. The existence of nonzero such pluri-forms of highest degree can in turn be read off from the non-negativity of a numerical invariant of X, called its (logarithmic) Kodaira dimension $\kappa(X)$, introduced by S. Iitaka [12] as the analogue of the usual notion of Kodaira dimension for complete varieties. In this setting, it was established by S. Iitaka et T. Fujita [13] that strong cancellation does hold for a large class of smooth varieties, namely whenever X or Y has non-negative Kodaira dimension. This general result implies in particular that cancellation holds for smooth affine curves, due to the fact that the affine line $\mathbb{A}^1$ is the only such curve with negative Kodaira dimension.

All these assumptions turned out to be essential, as shown by a famous unpublished example due to W. Danielewski [3] of a pair of non-isomorphic smooth complex $\mathbb{A}^1$-ruled affine surfaces with isomorphic cylinders. The techniques introduced by W. Danielewski have been sources of progress on the Cancellation Problem during the last decade but, except for the case of the affine plane $\mathbb{A}^2$ which was solved earlier affirmatively by M. Miyanishi and T. Sugie [17], the question whether cancellation holds for the complex affine space $\mathbb{A}^n$ remains one of the most challenging and widely open problem in affine algebraic geometry. In contrast, the same question in positive characteristic was recently settled in the negative by N. Gupta [9], who checked using algebraic methods developed by A. Crachiola and L. Makar-Limanov that a three-dimensional candidate constructed by T. Asanuma [1] is indeed a counter-example.

In this article, we consider another natural cancellation problem in which $\mathbb{A}^1$ is replaced by the punctured affine line $\mathbb{A}^1_* \simeq \mathrm{Spec}(\mathbb{C}[x^{\pm 1}])$ or, more generally, by an algebraic torus $\mathbb{T}^n = \mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$, $n \geqslant 1$. The question is thus whether two, say smooth quasi-projective, varieties X and Y such that $X \times \mathbb{T}^n$ is isomorphic to $Y \times \mathbb{T}^n$ are isomorphic themselves. When restricted to affine algebraic varieties, this cancellation problem is of course equivalent to the more algebraic question of the uniqueness of the coefficient ring in a Laurent polynomial ring. In contrast with the usual Cancellation Problem, this version seems to have received much less attention, one possible reason being that the analogue in this context of the Cancellation Problem for $\mathbb{A}^n$, namely the question whether an affine variety X such that $X \times \mathbb{T}^n$ is isomorphic to $\mathbb{T}^{n+m}$ is itself isomorphic to the torus $\mathbb{T}^n$, admits an elementary positive answer derived from the knowledge of the structure of the automorphism groups of algebraic tori: indeed, the action of the torus $\mathbb{T}^n = \mathrm{Spec}(\mathbb{C}[M'])$ by translations on the second factor of $X \times \mathbb{T}^n$ corresponds to a grading of the algebra of the torus $\mathbb{T}^{n+m} = \mathrm{Spec}(\mathbb{C}[M])$ by the lattice M' of characters of $\mathbb{T}^n$, induced by a surjective homomorphism $\sigma : M \to M'$ from the lattice of characters M of $\mathbb{T}^{n+m}$. The kernel of σ is a sub-lattice M'' of rank m of M for which we have isomorphisms of geometric quotients $X \simeq X \times \mathbb{T}^n/\mathbb{T}^n \simeq \mathbb{T}^{n+m}/\mathbb{T}^n \simeq \mathbb{T}^m = \mathrm{Spec}(\mathbb{C}[M''])$.

Without such precise information on automorphism groups, the question for general varieties X and Y is more complicated. Of course, appropriate conditions on the structure of invertible functions on X and Y can be imposed to guarantee that cancellation holds (see [6] for a detailed discussion of this point of view): this is the case for instance when either X or Y does not have non constant such functions. Indeed, given an isomorphism $\Phi : X \times \mathbb{A}^1_* \xrightarrow{\sim} Y \times \mathbb{A}^1_*$, the restriction to every closed fiber of the first projection $\mathrm{pr}_1 : X \times \mathbb{A}^1_* \to X$ of the composition of Φ with the second projection $\mathrm{pr}_2 : Y \times \mathbb{A}^1_* \to \mathbb{A}^1_*$ induces an invertible function on X, implying that Φ descends to an isomorphism between X and Y as soon as every such function on X is constant.

But from a geometric point of view, it seems that the cancellation property for $\mathbb{A}^1_*$ is again related to the nature of affine rational curves contained in the varieties X and Y, more specifically to the geometry of images of the punctured affine line $\mathbb{A}^1_*$ on them. It is natural to expect that strong cancellation holds for varieties which are not dominantly covered by images of $\mathbb{A}^1_*$, but this property is harder to characterize in terms of numerical invariants. In particular, in every dimension $\geqslant 2$, there exist smooth $\mathbb{A}^1_*$-uniruled affine varieties X of any Kodaira dimension $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X - 1\}$. In contrast, a smooth complex affine variety X of log-general type, i.e., of maximal Kodaira dimension $\kappa(X) = \dim X$, is not $\mathbb{A}^1_*$-uniruled, and another general result of I. Itaka and T. Fujita [13] does indeed confirm that strong cancellation holds for products of algebraic tori with smooth affine varieties of log-general type. Combined with the fact that $\mathbb{A}^1_*$ is the unique smooth affine curve of Kodaira dimension 0, this is enough for instance to conclude that cancellation holds for smooth affine curves.

Our main result, which can be summarized as follows, shows that similarly as in the case of the usual Cancellation Problem for $\mathbb{A}^1$, these assumptions are essential:

THEOREM. — *For every $d \geqslant 3$ and every $\kappa \in \{2, \ldots, d-1\}$, there exists non isomorphic smooth factorial affine $\mathbb{A}^1_*$-uniruled varieties X and Y of dimension d and Kodaira dimension κ with isomorphic $\mathbb{A}^1_*$-cylinders $X \times \mathbb{A}^1_*$*

and $Y \times \mathbb{A}^1_*$. *Furthermore, there exists non isomorphic smooth factorial affine surfaces of Kodaira dimension* 1 *with isomorphic* $\mathbb{A}^1_*$*-cylinders.*

In dimension $d \geqslant 3$ and Kodaira dimension $\kappa = d-1$, these families are obtained in the form of total spaces of suitable Zariski locally trivial $\mathbb{A}^1_*$-bundles over smooth affine varieties of log-general type. The construction guarantees the isomorphy between the corresponding $\mathbb{A}^1_*$-cylinders thanks to a fiber product argument reminiscent to the famous Danielewski fiber product trick in the case of the usual Cancellation Problem. The two-dimensional counter-examples are produced along the same lines, at the cost of replacing the base varieties of the $\mathbb{A}^1_*$-bundles involved in the construction by appropriate orbifold curves. The article is organized as follows: in the first section, we establish a variant of the Iitaka-Fujita strong cancellation Theorem for Zariski locally trivial $\mathbb{T}^n$-bundles over smooth affine varieties of log-general type. This criterion is applied in the second section to deduce the existence of families of Zariski locally trivial $\mathbb{A}^1_*$-bundles over smooth affine varieties of log-general type with non isomorphic total spaces but isomorphic $\mathbb{A}^1_*$-cylinders. The two-dimensional case is treated in a separate sub-section. The last section contains a generalization of some of these constructions to the cancellation problem for higher dimensional tori $\mathbb{T}^n$ over varieties of dimension at least three, leading in particular to the existence of counter-examples of arbitrary Kodaira dimension $\kappa \in \{2, \ldots, d-1\}$, and a complete discussion of the cancellation problem for $\mathbb{A}^1_*$ in the special case of smooth factorial affine surfaces.

1. A criterion for cancellation

1.1. Recollection on locally trivial $\mathbf{T}^n$-bundles

In what follows, we denote by $\mathbf{T}^n$ the spectrum of the Laurent polynomial algebra $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in n variables. We use the notation $\mathbb{T}^n$ to indicate that we consider $\mathbf{T}^n$ as an algebraic group: the direct product $\mathbb{G}_m^n$ of n-copies of the multiplicative group $\mathbb{G}_m$. The automorphism group $\mathrm{Aut}(\mathbf{T}^n)$ of $\mathbf{T}^n$ is isomorphic to the semi-direct product $\mathbb{T}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$, where $\mathbb{T}^n$ acts on $\mathbf{T}^n$ by translations and where $\mathrm{GL}_n(\mathbb{Z})$ acts by $(a_{ij})_{i,j=1,\ldots,n} \cdot (t_1, \ldots, t_n) = (\prod_{i=1}^n t_i^{a_{1i}}, \ldots, \prod_{i=1}^n t_i^{a_{ni}})$.

Definition 1.1. — *A* Zariski locally trivial $\mathbf{T}^n$-bundle *over a scheme* X*, is an* X*-scheme* $p : P \to X$ *for which every point of* X *has a Zariski open neighbourhood* $U \subset X$ *such that* $p^{-1}(U) \simeq U \times \mathbf{T}^n$ *as schemes over* U.

Isomorphy classes of Zariski locally trivial $\mathbf{T}^n$-bundles over X are in one-to-one correspondence with elements of the Čech cohomology group $\check{H}^1(X, \mathrm{Aut}(\mathbf{T}^n)_X)$, where $\mathrm{Aut}(\mathbf{T}^n)_X$ denotes the sheaf of groups on the category of schemes over X equipped with the Zariski topology which associates to every X-scheme S the group $\mathrm{Aut}(\mathbf{T}^n)(S) = \mathrm{Aut}_S(S \times \mathbf{T}^n)$ of S-automorphisms of $S \times \mathbf{T}^n$. Letting $\mathrm{GL}_n(\mathbb{Z})_X$ be the locally constant sheaf $\mathrm{GL}_n(\mathbb{Z})$ on X, we derive from the short exact sequence

$$0 \to \mathbb{T}^n_X = \mathbb{G}^n_{m,X} \to \mathrm{Aut}(\mathbf{T}^n)_X \to \mathrm{GL}_n(\mathbb{Z})_X \to 0$$

of sheaves of groups over X for the Zariski topology the following long exact sequence of pointed sets in non abelian Čech cohomology

$$\begin{aligned} 0 \to \check{H}^0(X, \mathbb{T}^n_X) &\longrightarrow \check{H}^0(X, \mathrm{Aut}(\mathbf{T}^n)_X) \longrightarrow \check{H}^0(X, \mathrm{GL}_n(\mathbb{Z})_X) \\ \to \check{H}^1(X, \mathbb{T}^n_X) &\longrightarrow \check{H}^1(X, \mathrm{Aut}(\mathbf{T}^n)_X) \longrightarrow \check{H}^1(X, \mathrm{GL}_n(\mathbb{Z})_X), \end{aligned}$$

(see e.g. [7, Proposition 3.4.1]). If X is irreducible, then $\check{H}^1(X, \mathrm{GL}_n(\mathbb{Z})_X) = 0$ and so, every Zariski locally trivial $\mathbf{T}^n$-bundle over X can be equipped with the additional structure of a principal homogeneous $\mathbb{T}^n$-bundle. Moreover, two principal homogeneous $\mathbb{T}^n$-bundles have isomorphic underlying $\mathbf{T}^n$-bundles if and only if their isomorphy classes in $\check{H}^1(X, \mathbb{T}^n_X) \simeq H^1(X, \mathbb{T}^n_X)$ belong to the same orbit of the natural action of $\check{H}^0(X, \mathrm{GL}_n(\mathbb{Z})_X) \simeq \mathrm{GL}_n(\mathbb{Z})$ which, for every $(a_{ij})_{i,j=1,\dots n} \in \mathrm{GL}_n(\mathbb{Z})$, sends the isomorphy class of the $\mathbb{T}^n$-bundle $p : P \to X$ with action $\mathbb{T}^n \times P \to P$, $((t_1, \cdots, t_n), p) \mapsto (t_1, \dots, t_n) \cdot p$ to the isomorphy class of $p : P \to X$ equipped with the action $((t_1, \dots t_n), p) \mapsto ((a_{ij})_{i,j=1,\dots,n} \cdot (t_1, \dots, t_n)) \cdot p$. In other words, for an irreducible X, isomorphy classes of Zariski locally trivial $\mathbf{T}^n$-bundles over X are in one-to-one correspondence with elements of $H^1(X, \mathbb{T}^n_X)/\mathrm{GL}_n(\mathbb{Z})$.

1.2. Cancellation for $\mathbf{T}^n$-bundles over varieties of log-general type

Recall that the *(logarithmic) Kodaira dimension* $\kappa(X)$ of a smooth complex algebraic variety X is the Iitaka dimension of the invertible sheaf $\omega_{\overline{X}/\mathbb{C}}(\log B) = (\det \Omega^1_{\overline{X}/\mathbb{C}}) \otimes \mathcal{O}_{\overline{X}}(B)$ on a smooth complete model $\overline{X}$ of X with reduced SNC boundary divisor $B = \overline{X} \setminus X$. So $\kappa(X)$ is equal to $\mathrm{tr.deg}_{\mathbb{C}}(\bigoplus_{m \geqslant 0} H^0(\overline{X}, \omega_{\overline{X}/\mathbb{C}}(\log B)^{\otimes m})) - 1$ if $H^0(\overline{X}, \omega_{\overline{X}/\mathbb{C}}(\log B)^{\otimes m}) \neq 0$ for sufficiently large m, and, by convention to $-\infty$ otherwise. The so-defined element of $\{-\infty\} \cup \{0, \dots, \dim_{\mathbb{C}} X\}$ is independent of the choice of a smooth complete model $(\overline{X}, B)$ [12] and coincides with the usual Kodaira dimension in the case where X is complete. A smooth variety X such that $\kappa(X) = \dim_{\mathbb{C}} X$ is said to be of log-general type.

The following Proposition is a variant for Zariski locally trivial $\mathbf{T}^n$-bundles of the Iitaka-Fujita's strong Cancellation Theorem [13, Theorem 3] for products of varieties of log-general type with affine varieties of Kodaira dimension equal to 0, such as algebraic tori $\mathbf{T}^n$.

Proposition 1.2. — *Let X and Y be smooth algebraic varieties and let $p : P \to X$ and $q : Q \to Y$ be Zariski locally trivial $\boldsymbol{T}^n$-bundles. If either X or Y is of log-general type then for every isomorphism of abstract algebraic varieties $\Phi : P \stackrel{\sim}{\to} Q$ between the total spaces of P and Q, there exists an isomorphism $\varphi : X \stackrel{\sim}{\to} Y$ such that the diagram*

$$\begin{array}{ccc} P & \stackrel{\Phi}{\to} & Q \\ p\downarrow & & \downarrow q \\ X & \stackrel{\varphi}{\to} & Y \end{array}$$

commutes.

Proof. — The proof is very similar to that of [13, Theorem 1]. We may assume without loss of generality that Y is of log-general type. It enough to show that $q \circ \Phi$ is constant on the fibers of p. Indeed, if so, then there exists a unique set-theoretic map $\varphi : X \to Y$ such that $\varphi \circ p = q \circ \Phi$, and the fact that p has local sections in the Zariski topology implies that φ is actually a morphism. Furthermore, since Φ is an isomorphism, φ will be bijective whence an isomorphism by virtue of Zariski Main Theorem [8, 8.12.6]. Now since $p : P \to X$ is Zariski locally trivial and $\kappa(\mathbf{T}^n) = 0$, it follows from [12] that for every prime Weil divisor D on X, the Kodaira dimension $\kappa(p^{-1}(D_{\mathrm{reg}}))$ of the inverse image of the regular part of D is at most equal to $\dim D$. This implies in turn that the restriction of $q \circ \Phi$ to $p^{-1}(D)$ cannot be dominant since otherwise we would have $\kappa(Y) \leqslant \kappa(p^{-1}(D_{\mathrm{reg}})) < \dim X = \dim Y$, in contradiction with the assumption that $\kappa(Y) = \dim Y$. So there exists a prime Weil divisor D' on Y such that the image of $p^{-1}(D)$ by Φ is contained in $q^{-1}(D')$, whence is equal to it since they are both irreducible of the same dimension. Now given any closed point $x \in X$, we can find a finite collection of prime Weil divisors $D_1, \ldots, D_n$ such that $D_1 \cap \cdots D_n = \{x\}$. Letting D'_i be a collection of prime Weil divisors on Y such that $\Phi(p^{-1}(D_i)) = q^{-1}(D'_i)$ for every $i = 1, \ldots, n$, we have

$$\begin{aligned} q^{-1}\Big(\bigcap_{i=1,\dots,n} D_i'\Big) = \bigcap_{i=1,\dots,n} q^{-1}(D_i') = \bigcap_{i=1,\dots,n} \Phi(p^{-1}(D_i)) \\ \simeq \Phi\Big(\bigcap_{i=1,\dots,n} p^{-1}(D_i)\Big) \\ \simeq \Phi(\{x\} \times \mathbf{T}^n) \simeq \mathbf{T}^n. \end{aligned}$$

So the intersection of the D_i', $i = 1, \dots, n$, consists of a unique closed point $y \in Y$ for which we have by construction $\Phi(p^{-1}(x)) = q^{-1}(y)$, as desired. □

Remark 1.3. — The proof above shows in fact that the conclusion of the proposition holds under the more geometric hypothesis that either X or Y is not $\mathbb{A}^1_*$-uniruled, i.e., does not admit any dominant generically finite morphism from a variety of the form $Z \times \mathbb{A}^1_*$. In particular strong cancellation holds for products of algebraic tori $\mathbf{T}^n$ with non $\mathbb{A}^1_*$-uniruled varieties.

COROLLARY 1.4. — *Two smooth curves C and C' admit Zariski locally trivial $\mathbf{T}^n$-bundles $p : P \to C$ and $p' : P' \to C'$ with isomorphic total spaces P and P' if and only if they are isomorphic.*

Proof. — If either C or C' is of (log-)general type, then the assertion follows from Proposition 1.2. Note further that C is affine if and only if so is P. Indeed, $p : P \to C$ is an affine morphism and conversely, if P is affine, then viewing P as a principal homogeneous $\mathbb{T}^n$ -bundle with geometric quotient $P//\mathbb{T}^n \simeq C$, the affineness of C follows from the fact that the algebraic quotient morphism $P \to P//\mathbb{T} = \mathrm{Spec}(\Gamma(P, \mathcal{O}_P)^{\mathbb{T}})$ is a categorical quotient in the category of algebraic varieties, so that $C \simeq \mathrm{Spec}(\Gamma(P, \mathcal{O}_P)^{\mathbb{T}})$. Thus C and C' are simultaneously affine or projective. In the first case, C and C' are isomorphic to either the affine line $\mathbb{A}^1$ or the punctured affine line $\mathbb{A}^1_*$ which both have a trivial Picard group. So P and P' are trivial $\mathbf{T}^n$-bundles and the isomorphy of C and C' follows by comparing invertible function on P and P'. In the second case, if either C or C' has genus 1, say C', then, being rational, the image of a fiber of $p : P \to C$ by an isomorphism $\Phi : P \stackrel{\sim}{\to} P'$ must be contained in a fiber of $p' : P' \to C'$, and we conclude similarly as in the proof of the previous proposition that Φ descends to an isomorphism between C and C'. □

The automorphism group $\mathrm{Aut}(X)$ of a scheme X acts on the set of isomorphy classes of principal homogeneous $\mathbb{T}^n$-bundles over X via the linear representation $\eta : \mathrm{Aut}\,(X) \to \mathrm{GL}(H^1(X, \mathbb{T}^n_X))$, where for an element

$\psi \in \mathrm{Aut}\,(X)$, $\eta(\psi)$ maps the isomorphy class of a principal homogeneous $\mathbb{T}^n$-bundle $p : P \to X$ to the one of the $\mathbb{T}^n$-bundle $\mathrm{pr}_2 : P \times_{p,X,\psi} X \to X$. This action commutes with action of $\mathrm{GL}_n(\mathbb{Z})$ introduced in page 2625 above, and Proposition 1.2 implies the following characterization:

Corollary 1.5. — *Over a smooth variety X of log-general type, the set $H^1(X, \mathbb{T}^n_X)/(\mathrm{Aut}(X) \times \mathrm{GL}_n(\mathbb{Z}))$ parametrizes isomorphy classes as abstract varieties of total spaces of Zariski locally trivial $\boldsymbol{T}^n$-bundles $p : P \to X$.*

2. Non-Cancellation for the 1-dimensional torus

Candidates for non-cancellation of the 1-dimensional torus $\mathbf{T} = \mathbb{A}^1_* = \mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$ can be constructed along the following lines: given say a smooth quasi-projective variety X and a pair of non-isomorphic principal homogeneous $\mathbb{G}_m$-bundles $p : P \to X$ and $q : Q \to X$ whose classes generate the same subgroup of $H^1(X, \mathbb{G}_m)$, the fiber product $W = P \times_X Q$ is a principal homogeneous $\mathbb{T}^2$-bundle over X, which inherits the structure of a principal $\mathbb{G}_m$-bundle over P and Q simultaneously, via the first and the second projection respectively. The pull-back $p^*P = P \times_X P \to P$ of P to itself having a section, it is a trivial $\mathbb{G}_m$-bundle, which implies in turn that the subgroup of $H^1(X, \mathbb{G}_m)$ generated by the class of P belongs to the kernel of $p^* : H^1(X, \mathbb{G}_m) \to H^1(P, \mathbb{G}_m)$. For the same reason, the subgroup of $H^1(X, \mathbb{G}_m)$ generated by the class of Q belongs to the kernel of $q^* : H^1(X, \mathbb{G}_m) \to H^1(Q, \mathbb{G}_m)$. The assumption that the classes of P and Q generate the same subgroup of $H^1(X, \mathbb{G}_m)$ guarantees that the classes of $\mathrm{pr}_1 : W \simeq p^*Q \to P$ and $\mathrm{pr}_2 : W \simeq q^*P \to Q$ in $H^1(P, \mathbb{G}_m)$ and $H^1(Q, \mathbb{G}_m)$ respectively are both trivial, and so, we obtain isomorphisms $P \times \mathbb{G}_m \simeq W \simeq Q \times \mathbb{G}_m$ of locally trivial $\mathbf{T}^2$-bundles over X.

Then we are left with finding appropriate choices of X and classes in $H^1(X, \mathbb{G}_m)$ which guarantee that the total spaces of the corresponding principal homogeneous $\mathbb{G}_m$-bundles $p : P \to X$ and $q : Q \to X$ are not isomorphic as abstract algebraic varieties.

2.1. Non-cancellation for smooth factorial affine varieties of dimension $\geqslant 3$

A direct application of the above strategy leads to families of smooth factorial affine varieties of any dimension $\geqslant 3$ for which cancellation fails:

Proposition 2.1. — *Let X be the complement of a smooth hypersurface D of degree ℓ in $\mathbb{P}^\kappa$, $\kappa \geqslant 2$, such that $\ell \geqslant \kappa + 2$ and $|(\mathbb{Z}/\ell\mathbb{Z})^*| \geqslant 3$, and let $p : P \to X$ and $q : Q \to X$ be the $\mathbb{G}_m$-bundles corresponding to the line bundles $\mathcal{O}_{\mathbb{P}^\kappa}(1)|_X$ and $\mathcal{O}_{\mathbb{P}^\kappa}(r)|_X$, for some $r \in \mathbb{Z}$ such that $\overline{r} \in (\mathbb{Z}/\ell\mathbb{Z})^* \setminus \{\overline{1}, \overline{\ell - 1}\}$ under the isomorphism $H^1(X, \mathbb{G}_m) \simeq \mathrm{Pic}(X) \simeq \mathbb{Z}/\ell\mathbb{Z}$. Then P and Q are not isomorphic as abstract algebraic varieties but $P \times \mathbb{A}^1_*$ and $Q \times \mathbb{A}^1_*$ are isomorphic as schemes over X.*

Proof. — The Picard group of X is isomorphic to the group $\mu_\ell \simeq \mathbb{Z}/\ell\mathbb{Z}$ of ℓ-th roots of unity, generated by the restriction of $\mathcal{O}_{\mathbb{P}^\kappa}(1)$ to X. Since r is relatively prime with ℓ, $\mathcal{O}_{\mathbb{P}^\kappa}(r)|_X$ is also a generator of $\mathrm{Pic}(X)$. This guarantees that $P \times \mathbb{A}^1_*$ is isomorphic to $Q \times \mathbb{A}^1_*$ by virtue of the previous discussion. Since $\ell \geqslant \kappa + 2$, the divisor $K_{\mathbb{P}^\kappa} + D$ is linearly equivalent to a positive multiple of a hyperplane section, and so X is of log-general type. We can therefore apply Proposition 1.2 to deduce that for every isomorphism of abstract algebraic varieties $\Phi : P \xrightarrow{\sim} Q$, there exists an automorphism φ of X such that P is isomorphic to $\varphi^* Q$ as a Zariski locally trivial $\mathbb{A}^1_*$-bundle over X. In view of Corollary 1.5, this means equivalently that as a $\mathbb{G}_m$-bundle over X, $\varphi^* Q$ is isomorphic to either P or its inverse P^{-1} in $H^1(X, \mathbb{G}_m)$. Since the choice of r guarantees that the $\mathbb{G}_m$-bundle Q is isomorphic neither to P nor to P^{-1}, the conclusion follows from the observation that the natural action of $\mathrm{Aut}(X)$ on $H^1(X, \mathbb{G}_m)$ is the trivial one. Indeed, through the open inclusion $X \hookrightarrow \mathbb{P}^\kappa$, we may consider an automorphism φ of X as a birational self-map of $\mathbb{P}^\kappa$ restricting to an isomorphism outside D. If φ is not biregular on the whole $\mathbb{P}^\kappa$, then D would be an exceptional divisor of φ^{-1}, in particular, D would be birationally ruled, in contradiction with the ampleness of its canonical divisor K_D guaranteed by the condition $\ell \geqslant \kappa + 2$. So every automorphism of X is the restriction of a linear automorphism of the ambient space $\mathbb{P}^\kappa$, and since $\mathrm{PGL}(\kappa + 1)$ acts trivially on $\mathrm{Pic}(\mathbb{P}^\kappa)$, it follows that $\mathrm{Aut}(X)$ acts trivially on $\mathrm{Pic}(X)$. □

Example 2.2. — In the setting of Proposition 2.1 above, an isomorphism $P \times \mathbb{A}^1_* \xrightarrow{\sim} Q \times \mathbb{A}^1_*$ can be constructed "explicitly" as follows. The complement $X \subset \mathbb{P}^\kappa = \mathrm{Proj}(\mathbb{C}[x_0, \ldots, x_\kappa])$ of a smooth hypersurface D defined by an equation $F(x_0, \ldots, x_\kappa) = 0$ for some homogeneous polynomial of degree ℓ can be identified with the quotient of the smooth factorial affine variety $\tilde{X} \subset \mathbb{A}^{\kappa+1}$ with equation $F(x_0, \ldots, x_r) = 1$ by the free action of the group $\mu_\ell = \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^\ell - 1))$ of ℓ-th roots of unity defined by $\varepsilon \cdot (x_0, \ldots, x_\kappa) = (\varepsilon x_0, \ldots, \varepsilon x_\kappa)$. The $\mathbb{G}_m$-bundles over X corresponding to the line bundles $\mathcal{O}_{\mathbb{P}^\kappa}(r)|_X$, $r \in \mathbb{Z}$, then coincide with the quotients of the trivial $\mathbb{A}^1_*$-bundles $\tilde{X} \times \mathbb{A}^1_* = \tilde{X} \times \mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$ by the respective μ_ℓ-actions

$\varepsilon \cdot (x_0, \ldots, x_\kappa, t) = (\varepsilon x_0, \ldots, \varepsilon x_\kappa, \varepsilon^r t)$, $r \in \mathbb{Z}$. Now let $q : Q \to X$ be the $\mathbb{G}_m$-bundle corresponding to $\mathcal{O}_{\mathbb{P}^\kappa}(r) \mid_X$ for some $r \in \{2, \ldots, \ell - 2\}$ relatively prime with ℓ, and let $a, b \in \mathbb{Z}$ be such that $ar - b\ell = 1$. Then the following isomorphism

$$\tilde{\Phi} : \tilde{X} \times \mathbf{T}^2 = \tilde{X} \times \operatorname{Spec}(\mathbb{C}[t_1^{\pm 1}, u_1^{\pm 1}]) \stackrel{\sim}{\to} \tilde{X} \times \mathbf{T}^2 = \tilde{X} \times \operatorname{Spec}(\mathbb{C}[t_2^{\pm 1}, u_2^{\pm 1}])$$
$$(t_1, u_1) \mapsto (t_2, u_2) = (t_1^r u_1^a, t_1^{b\ell} u_2)$$

of schemes over $\tilde{X}$ is equivariant for the actions, say $\mu_{\ell,1}$ and $\mu_{\ell,r}$, of μ_ℓ defined respectively by $\varepsilon \cdot (x_0, \ldots, x_\kappa, t_1, u_1) = (\varepsilon x_0, \ldots, \varepsilon x_\kappa, \varepsilon t_1, u_1)$ and $\varepsilon \cdot (x_0, \ldots, x_\kappa, t_2, u_2) = (\varepsilon x_0, \ldots, \varepsilon x_\kappa, \varepsilon^r t_2, u_2)$, hence descends to an isomorphism

$$\Phi : (\tilde{X} \times \mathbf{T}^2)/\mu_{\ell,1} \simeq P \times \mathbb{A}^1_* \stackrel{\sim}{\to} Q \times \mathbb{A}^1_* \simeq (\tilde{X} \times \mathbf{T}^2)/\mu_{\ell,r}$$

of schemes over $X \simeq \tilde{X}/\mu_\ell$.

2.2. Non-cancellation for smooth factorial affine surfaces

Since the Picard group of a smooth affine curve C of log-general type is either trivial if C is rational or of positive dimension otherwise, there is no direct way to adapt the previous construction using principal $\mathbb{G}_m$-bundles over algebraic curves to produce 2-dimensional candidate counter-examples for cancellation by $\mathbb{A}^1_*$. Instead, we will use locally trivial $\mathbb{A}^1_*$-bundles over certain orbifold curves $\tilde{C}$ which arise from suitably chosen $\mathbb{A}^1_*$-fibrations $\pi : S \to C$ on smooth affine surfaces S.

Let S be a smooth affine surface equipped with a flat fibration $\pi : S \to C$ over a smooth affine rational curve C whose fibers, closed or not, are all isomorphic to $\mathbb{A}^1_*$ over the corresponding residue fields when equipped with their reduced structure [(1)]. It follows from the description of degenerate fibers of $\mathbb{A}^1_*$-fibrations given in [16, Theorem 1.7.3] that S admits a relative completion into a $\mathbb{P}^1$-fibered surface $\overline{\pi} : \overline{S} \to C$ obtained from a trivial $\mathbb{P}^1$-bundle $\mathrm{pr}_1 : C \times \mathbb{P}^1 \to C$ with a fixed pair of disjoint sections H_0 and H_∞ by performing finitely many sequences of blow-ups of the following type: the first step consists of the blow-up of a closed point $c_i \in H_0$, $i = 1, \ldots, s$, with exceptional divisor $E_{1,i}$ followed by the blow-up of the intersection point of $E_{1,i}$ with the proper transform of the fiber $F_i = \mathrm{pr}_1^{-1}(\mathrm{pr}_1(c_i))$, the next steps consist of the blow-up of an intersection point of the last exceptional divisor produced with the proper transform of the union of F_i

(1) In particular, π is an untwisted $\mathbb{A}^1_*$-fibration in the sense of [16, §1.7 p. 201].

and the previous ones, in such a way that the total transform of F_i is a chain of proper rational curves with the last exceptional divisors produced, say E_{i,n_i}, as the unique irreducible component with self-intersection -1. The projection $\mathrm{pr}_1 : C \times \mathbb{P}^1 \to C$ lifts on the so-constructed surface $\overline{S}$ to a $\mathbb{P}^1$-fibration $\overline{\pi} : \overline{S} \to C$ and S isomorphic to the complement of the union of the proper transforms of H_0 and H_∞ and of the divisors $F_i \cup E_{i,1} \cup \cdots \cup E_{i,n_i-1}$, $i = 1, \ldots, s$. The restriction of $\overline{\pi}$ to S is indeed an $\mathbb{A}^1_*$-fibration $\pi : S \to C$ with s degenerate fibers $\pi^{-1}(\mathrm{pr}_1(c_i))$ isomorphic to $E_{i,n_i} \cap S \simeq \mathbb{A}^1_*$ when equipped with their reduced structure and whose respective multiplicities depend on the sequences of blow-ups performed.

It follows in particular from this construction that S admits a proper action of the multiplicative group $\mathbb{G}_m$ which lifts the one on $(C \times \mathbb{P}^1) \setminus (H_0 \cup H_\infty) \simeq C \times \mathbb{A}^1_*$ by translations on the second factor. The local descriptions given in [5] can then be re-interpreted for our purpose as the fact that the $\mathbb{A}^1_*$-fibration $\pi : S \to C$ factors through an étale locally trivial $\mathbb{A}^1_*$-bundle $\tilde{\pi} : S \to \tilde{C}$ over an orbifold curve $\delta : \tilde{C} \to C$ obtained from C by replacing the finitely many points $c_1, \ldots, c_s$ over the which the fiber $\pi^{-1}(c_i)$ is multiple, say of multiplicity $m_i > 1$, by suitable orbifold points $\tilde{c}_i$ depending only on the multiplicity m_i. More precisely, $\tilde{C}$ is a smooth separated Deligne-Mumford stack of dimension 1, of finite type over $\mathbb{C}$ and with trivial generic stabilizer, which, Zariski locally around $\delta^{-1}(c_i)$ looks like the quotient stack $[\tilde{U}_{c_i}/\mathbb{Z}_{m_i}]$, where $\tilde{U}_{c_i} \to U_{c_i}$ is a Galois cover of order m_i of a Zariski open neighborhood U_{c_i} of c_i, totally ramified over c_i and étale elsewhere [2].

Example 2.3. — Let $\mathbb{G}_m$ act on $\mathbb{A}^2_* = \mathrm{Spec}(\mathbb{C}[x,y]) \setminus \{(0,0)\}$ by $t \cdot (x,y) = (t^2 x, t^5 y)$. The quotient $\mathbb{P}(2,5) = \mathbb{A}^2_*/\mathbb{G}_m$ is isomorphic to $\mathbb{P}^1$ and the quotient morphism $q : \mathbb{A}^2_* \to \mathbb{P}^1 = \mathbb{A}^2_*/\mathbb{G}_m$ is an $\mathbb{A}^1_*$-fibration with two degenerate fibers $q^{-1}([0:1])$ and $q^{-1}([1:0])$ of multiplicities 5 and 2 respectively, corresponding to the orbits of the points $(0,1)$ and $(1,0)$. In contrast, the quotient stack $[\mathbb{A}^2_*/\mathbb{G}_m]$ is the Deligne-Mumford curve $\mathbb{P}[2,5]$ obtained from $\mathbb{P}^1$ by replacing the points $[0:1]$ and $[1:0]$ by "stacky points" with respective Zariski open neighborhoods isomorphic to the quotients $[\mathbb{A}^1/\mathbb{Z}_5]$ and $[\mathbb{A}^1/\mathbb{Z}_2]$ for the actions of $\mathbb{Z}_5$ and $\mathbb{Z}_2$ on $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[z])$ given by $z \mapsto \exp(2i\pi/5)z$ and $z \mapsto -z$. The quotient morphism $q : \mathbb{A}^2_* \to \mathbb{P}^1$ factors through the canonical morphism $\tilde{q} : \mathbb{A}^2_* \to \mathbb{P}[2,5]$ which is an étale local trivial $\mathbb{A}^1_*$-bundle, and the induced morphism $\delta : \mathbb{P}[2,5] \to \mathbb{P}^1 = \mathbb{A}^2_*/\mathbb{G}_m$ is an isomorphism over the complement of the points $[0:1]$ and $[1:0]$.

In the next paragraphs, we construct two smooth affine surfaces S_1 and S_2 with an $\mathbb{A}^1_*$-fibration $\pi_i : S_i \to \mathbb{A}^1$, $i = 1, 2$, factoring through a locally trivial $\mathbb{A}^1_*$-bundle $\tilde{\pi} : S_i \to \mathbb{A}^1[2,5]$ over the affine Deligne-Mumford curve $\mathbb{A}^1[2,5]$ obtained from the one $\mathbb{P}[2,5]$ of Example 2.3 above by removing a general scheme-like point.

The first one S_1 is equal to the complement in the projective plane $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[x,y,z])$ of the union of the cuspidal curve $D_1 = \left\{x^5 - y^2z^3 = 0\right\}$ and the line $L_z = \{z = 0\}$. Equivalently, S_1 is the complement in $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x,y]) = \mathbb{P}^2 \setminus L_z$ of the curve $D_1 \cap \mathbb{A}^2 = \{x^5 - y^2 = 0\}$. This curve being an orbit with trivial isotropy of the $\mathbb{G}_m$-action $t\cdot(x,y) = (t^2x, t^5y)$ on $\mathbb{A}^2$, the composition of the inclusion $S_1 \hookrightarrow \mathbb{A}^2_*$ with the canonical morphism $q : \mathbb{A}^2_* \to \mathbb{P}[2,5] = [\mathbb{A}^2_*/\mathbb{G}_m]$ defines a locally trivial $\mathbb{A}^1_*$-bundle $\tilde{\pi}_1 : S_1 \to \mathbb{P}[2,5] \setminus q(D_1 \cap \mathbb{A}^2_*) \simeq \mathbb{A}^1[2,5]$. The rational pencil $\xi_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ induced by $\pi_1 = \delta \circ \tilde{\pi}_1 : S_1 \to \mathbb{A}^1$ coincide with that generated by the pairwise linearly equivalent divisors D_1, $5L_x$ and $3L_z + 2L_y$, where L_x, L_y and L_z denote the lines $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$ in $\mathbb{P}^2$ respectively. A minimal resolution $\tilde{\xi}_1 : \tilde{S}_1 \to \mathbb{P}^1$ of ξ_1 is depicted in Figure 2.1. A relatively minimal SNC completion $(\overline{S}_1, B_1)$ of S_1, with boundary $B_1 = D_1 \cup H_{\infty,1} \cup H_{0,1} \cup E_{\infty,3} \cup E_{\infty,1} \cup E_{0,1} \cup E_{0,2} \cup E_{\infty,2} \cup E_{0,3}$, on which π_1 extends to a $\mathbb{P}^1$-fibration $\overline{\pi}_1 : \overline{S}_1 \to \mathbb{P}^1$ is then obtained from $\tilde{S}_1$ by contracting the proper transform of L_z.

The second surface S_2 is obtained as follows. In the Hirzebruch surface $\rho : \mathbb{F}_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) \to \mathbb{P}^1$ with exceptional section C_0 of self-intersection -3, we choose a section C of ρ in the linear system $|C_0 + 4F|$, where F denotes a general fiber of ρ, and a section C_1 in the linear system $|C_0 + 3F|$ intersecting C with multiplicity 4 in a unique point p_0. The fact that such pairs of sections exists follows for instance from [4, Lemma 3.2]. Let $\xi_2 : \mathbb{F}_3 \dashrightarrow \mathbb{P}^1$ be the pencil generated by the linearly equivalent divisors C_0+5C and $6C_1+2F_0$ where $F_0 = \overline{\rho}^{-1}(\rho(p_0))$. Let D_2 be a general member of ξ and let $S_2 \subset \mathbb{F}_3$ be the complement of $C_0 \cup C_1 \cup D_2$. A minimal resolution $\tilde{\xi}_2 : \tilde{S}_2 \to \mathbb{P}^1$ of $\xi_2 : \mathbb{F}_3 \dashrightarrow \mathbb{P}^1$ is depicted in Figure 2.2. A relatively minimal SNC completion $(\overline{S}_2, B_2)$ of S_2 with boundary $B_2 = D_2 \cup H_{\infty,2} \cup H_{0,2} \cup E_{\infty,1} \cup E_{0,5} \cup C_0 \cup \bigcup_{i=6}^{9} E_{0,i}$, on which π_2 extends to a morphism $\overline{\pi}_2 : \overline{S}_2 \to \mathbb{P}^1$ is then obtained from $\tilde{S}_2$ by contracting successively the proper transforms of C_1, $E_{0,4}$, $E_{0,3}$, $E_{0,2}$ $E_{0,1}$. The proper transform of C in $\overline{S}_2$ is a unique (-1)-curve in the fiber $\overline{\pi}_2^{-1}(\overline{\pi}_2(C))$, and the image of $C_0 \subset \overline{\pi}_2^{-1}(\overline{\pi}_2(C))$ by the successive contractions of C,$E_{0,6}$, $E_{0,7}$ and $E_{0,9}$ is a smooth rational curve of self-intersection 0. Similarly, F_0 is a unique (-1)-curve in $\overline{\pi}_2^{-1}(\overline{\pi}_2(C))$ and the image of $E_{\infty,1}$ by the

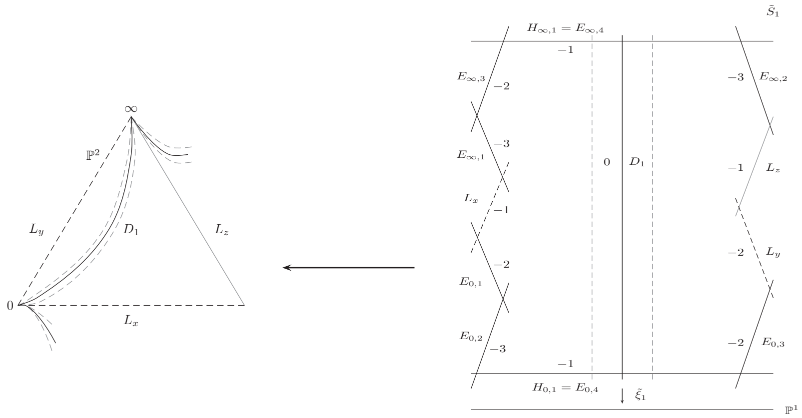

Figure 2.1. Minimal resolution of the pencil $\xi_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. The exceptional divisors $E_{0,i}$ and $E_{\infty,i}$, $i = 1, \ldots, 4$, over the respective proper base points $0 = [0 : 0 : 1]$ and $\infty = [0 : 1 : 0]$ of ξ_1 are numbered according to the order of their extraction.

contractions of F_0 and $E_{0,5}$ is a smooth rational curve of self-intersection 0. Counting the number of points blown-up from $\mathbb{F}_3$ and of curves contracted, we conclude that the smooth projective surface obtained from $\overline{S}_2$ by making all these contractions has Picard rank 2, hence is a Hirzebruch surface, in which the images of C_0 and $E_{\infty,1}$ are fibers of a $\mathbb{P}^1$-bundle structure having the images of $H_{\infty,2}$ and $H_{0,2}$ as disjoint sections. It follows that $\overline{\pi}_2 : \overline{S}_2 \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration which has $\overline{\pi}_2^{-1}(\overline{\pi}_2(C))$ and $\overline{\pi}_2^{-1}(\overline{\pi}_2(F_0))$ has unique reducible fibers. By construction, $\overline{\pi}_2$ restricts on S_2 to an $\mathbb{A}^1_*$-fibration $\pi_2 : S_2 \to \mathbb{A}^1 = \mathbb{P}^1 \setminus \xi(D_2)$ with two degenerate fibers: one of multiplicity 5 supported by $C \cap S_2 \simeq \mathbb{A}^1_*$, and one of multiplicity 2 supported by $F_0 \cap S_2 \simeq \mathbb{A}^1_*$. So by virtue of page 2631, π_2 factors through a locally trivial $\mathbb{A}^1_*$-bundle $\tilde{\pi}_2 : S_2 \to \mathbb{A}^1[2, 5]$.

Proposition 2.4. — *The surfaces S_1 and S_2 are smooth affine rational and factorial. They are non isomorphic but $S_1 \times \mathbb{A}^1_*$ is isomorphic to $S_2 \times \mathbb{A}^1_*$.*

Proof. — Since S_1 is a principal open subset of $\mathbb{A}^2$, it is smooth affine rational and factorial. The smoothness and the rationality of S_2 are also clear. Since D_2 belongs to the linear system $6C_0 + 20F$, it is ample by virtue of [11, Theorem 2.17]. This implies in turn that $C_0 + C_1 + D_2$ is the support of an ample divisor, whence that S_2 is affine. Since the divisor class group of $\mathbb{F}_3$ is generated by C_0 and F, the identity $F \sim 7C_1 - C_0 - D_2$ in the divisor

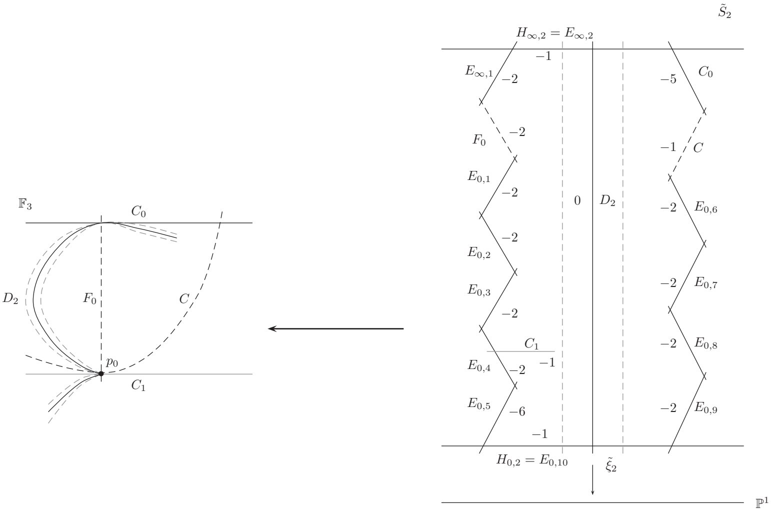

Figure 2.2. Minimal resolution of the pencil $\xi_2 : \mathbb{F}_3 \dashrightarrow \mathbb{P}^1$. The exceptional divisors $E_{\infty,1}$, $E_{\infty,2}$ and $E_{0,i}$, $i = 1, \ldots, 10$, over the respective proper base points $\infty = F_0 \cap C_0$ and $0 = p_0$ of ξ_2 are numbered according to the order of their extraction.

class group of $\mathbb{F}_3$ implies that every Weil divisor on S_2 is linearly equivalent to one supported on the boundary $\mathbb{F}_3 \setminus S_2 = C_0 \cup C_1 \cup D_2$. So S_2 is factorial. By construction, S_1 and S_2 both have the structure of locally trivial $\mathbb{A}^1_*$-bundles $\tilde{\pi}_i : S_i \to \mathbb{A}^1[2,5]$. The fiber product $W = S_1 \times_{\mathbb{A}^1[2,5]} S_2$ thus inherits via the first and the second projection respectively the structure of an étale locally trivial $\mathbb{A}^1_*$-bundle over S_1 and S_2. Since $H^1_{\text{ét}}(S_i, \mathbb{G}_m) \simeq H^1(S_i, \mathbb{G}_m)$ by virtue of Hilbert's Theorem 90 and S_i is factorial, it follows that W is simultaneously isomorphic to the trivial $\mathbb{A}^1_*$-bundles $S_1 \times \mathbb{A}^1_*$ and $S_2 \times \mathbb{A}^1_*$. It remains to check that S_1 and S_2 are not isomorphic. Suppose on the contrary that there exists an isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ and consider its natural extension as a birational map $\varphi : \overline{S}_1 \dashrightarrow \overline{S}_2$ between the smooth SNC completions $(\overline{S}_1, B_1)$ and $(\overline{S}_2, B_2)$ of S_1 and S_2 constructed above. Then φ must be a biregular isomorphism. Indeed, if either φ or φ^{-1}, say φ, is not regular then we can consider a minimal resolution $\overline{S}_1 \xleftarrow{\sigma} X \xrightarrow{\sigma'} \overline{S}_2$ of it. By definition of the minimal resolution, there is no (-1)-curve in the union B of the total transforms of B_1 and B_2 by σ and σ'respectively which is exceptional for σ and σ' simultaneously, and σ' consists of the contraction of a sequence of successive (-1)-curves supported on B. The only possible (-1)-curves in B which are not exceptional for σ are the proper transforms of D_1 and of the two sections $H_{0,1}$ and $H_{\infty,1}$ of the $\mathbb{P}^1$-fibration $\overline{\pi}_1 : \overline{S}_1 \to \mathbb{P}^1$, but the contraction of any of these would lead to a boundary which would no longer be SNC, which is excluded by the fact that B_2 is SNC. It follows that every isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ is the restriction of an isomorphism of pairs $(\overline{S}_1, B_1) \xrightarrow{\sim} (\overline{S}_2, B_2)$. But no such isomorphism can exist due the fact that the intersection forms of the boundaries B_1 and B_2 are different. Thus S_1 and S_2 are not isomorphic, which completes the proof. □

3. Complements and open questions

3.1. Non-cancellation for higher dimensional tori

Continuing the same idea as in section 2 above, it is possible to construct more generally pairs of principal homogeneous $\mathbb{T}^n$-bundles over a given smooth variety X whose total spaces become isomorphic after taking their products with $\mathbf{T}^n$ but not with any other lower dimensional tori. For instance, one can start with two collections $\{[p_1], \ldots, [p_n]\}$ and $\{[q_1], \ldots, [q_n]\}$ of classes in $H^1(X, \mathbb{G}_m)$ which generate the same sub-group G of $H^1(X, \mathbb{G}_m)$ and consider a pair of principal homogeneous $\mathbb{T}^n$-bundles $p : P \to X$ and $q : Q \to X$ representing the classes $([p_1], \ldots, [p_n])$ and $([q_1], \ldots, [q_n])$ in $H^1(X, \mathbb{T}_X) \simeq H^1(X, \mathbb{G}_m)^{\oplus n}$. Since G is contained in the kernels of the natural homomorphism $p^* : H^1(X, \mathbb{G}_m) \to H^1(P, \mathbb{G}_m)$ and $q^* : H^1(X, \mathbb{G}_m) \to H^1(Q, \mathbb{G}_m)$ (see Lemma 3.2), it follows that as a locally trivial $\mathbf{T}^n \times \mathbf{T}^n$-bundle over X, $P \times_X Q$ is simultaneously isomorphic to $P \times \mathbf{T}^n$ and $Q \times \mathbf{T}^n$. Then again, it remains to make appropriate choices for X, P and Q which guarantee that for every $n' = 0, \ldots, n-1$, $P \times \mathbf{T}^{n'}$ and $Q \times \mathbf{T}^{n'}$ are not isomorphic as abstract algebraic varieties.

Theorem 3.1. — *Let $\kappa \geqslant 2$, let $\ell \geqslant \kappa + 2$ be a product $n \geqslant 2$ distinct prime numbers $5 \leqslant \ell_1 < \cdots < \ell_n$ and let $X \subset \mathbb{P}^\kappa$ be the complement of a smooth hypersurface of degree ℓ. Let $[p_i], [q_i] \in H^1(X, \mathbb{G}_m) \simeq \mu_\ell$, $i = 1, \ldots, n$, be the classes corresponding via the isomorphism $\mu_\ell \simeq \prod_{i=1}^n \mu_{\ell_i}$ to the elements $(1, \ldots, \exp(2\pi/\ell_i), \ldots, 1)$ and $(1, \ldots, \exp(2\pi r_i/\ell_i), \ldots, 1)$ for some $r_i \in \{2, \ldots, \ell_i - 2\}$, and let $p : P \to X$ and $q : Q \to X$ be principal homogeneous $\mathbb{T}^n$-bundles representing respectively the classes $([p_1], \ldots, [p_n])$ and $([q_1], \ldots, [q_n])$ in $H^1(X, \mathbb{T}^n_X)$.*

Then for every $n' = 0, \ldots, n-1$, the varieties $P \times \mathbf{T}^{n'}$ and $Q \times \mathbf{T}^{n'}$ are not isomorphic while $P \times \mathbf{T}^n$ and $Q \times \mathbf{T}^n$ are isomorphic as schemes *over X. In particular, $P\times \mathbf{T}^{n-1}$ and $Q\times \mathbf{T}^{n-1}$ are non isomorphic varieties of dimension $d=\kappa+n-1$ and Kodaira dimension κ with isomorphic $\mathbb{A}^1_*$-cylinders.*

Proof. — Our choices guarantee that for every $0\leqslant n'<n$, the classes $([p_1],\dots,[p_n],[1],\dots[1])$ and $([q_1],\dots,[q_n],[1],\dots[1])$ in $H^1(X,\mathbb{T}^n_X\times\mathbb{T}^{n'}_X)$ belong to distinct $\mathrm{GL}_{n+n'}(\mathbb{Z})$-orbits. Since $\ell\geqslant\kappa+2$, X is of general type and $\mathrm{Aut}(X)$ acts trivially on $H^1(X,\mathbb{G}_m)$ (see the proof of Theorem 2.1). So the fact that $P\times\mathbf{T}^{n'}$ and $Q\times\mathbf{T}^{n'}$ are not isomorphic as abstract algebraic varieties follows again from Corollary 1.5. On the other hand, since the classes $[p_1],\dots,[p_n]$ and $[q_1],\dots,[q_n]$ both generate $H^1(X,\mathbb{G}_m)$, $P\times\mathbf{T}^n$ and $Q\times\mathbf{T}^n$ are isomorphic X-schemes by virtue of the previous discussion. Alternatively, one can observe that choosing $a_i,b_i\in\mathbb{Z}$ such that $a_ir_i+b_i\ell_i=1$ for every $i=1,\dots,n$, the following matrices A and B in $\mathrm{GL}_{2n}(\mathbb{Z})$

$$A=\begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & \ddots & 0 & & 0_n & \\ 0 & 0 & 1 & & & \\ r_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & r_n & 0 & 0 & 1 \end{pmatrix}\qquad B=\begin{pmatrix} a_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & a_n & 0 & 0 & 1 \\ 1 & 0 & 0 & & & \\ 0 & \ddots & 0 & & 0_n & \\ 0 & 0 & 1 & & & \end{pmatrix}$$

map respectively the classes $([p_1],\dots,[p_n],[1],\dots[1])$ and $([q_1],\dots[q_n],[1],\dots,[1])$ onto the one $([p_1],\dots,[p_n],[q_1],\dots,[q_n])$, providing isomorphisms $P\times\mathbf{T}^n\simeq P\times_X Q$ and $Q\times\mathbf{T}^n\simeq P\times_X Q$ of Zariski locally trivial $\mathbf{T}^{2n}$-bundles over X. □

The following lemma relating the Picard group of the total space of principal homogeneous $\mathbb{T}^n$-bundle with the Picard group of its base is certainly well known. We include it here because of the lack of appropriate reference.

Lemma 3.2. — *Let X be a normal variety, let $[p_1],\dots,[p_n]$ be a collection of classes in $H^1(X,\mathbb{G}_m)$, and let $p:P\to X$ be the principal homogeneous $\mathbb{T}^n$-bundle with class $([p_1],\dots,[p_n])\in H^1(X,\mathbb{T}^n_X)=H^1(X,\mathbb{G}_m)^{\oplus n}$. Then $H^1(P,\mathbb{G}_m)\simeq H^1(X,\mathbb{G}_m)/G$ where $G=\langle[p_1],\dots,[p_n]\rangle$ is the subgroup generated by $[p_1],\dots,[p_n]$.*

Proof. — The Picard sequence [15] for the fibration $p : P \to X$ reads

$$0 \to H^0(X,\mathcal{U}_X) \longrightarrow H^0(P,\mathcal{U}_E) \longrightarrow H^0(\mathbb{G}_m^n,\mathcal{U}_{\mathbb{G}_m^n}) \xrightarrow{\delta} H^1(X,\mathbb{G}_m) \longrightarrow H^1(P,\mathbb{G}_m) \longrightarrow H^1(\mathbb{G}_m^n,\mathbb{G}_m) = 0$$

where for a variety Y, $\mathcal{U}_Y$ denotes the sheaf cokernel of the homomorphism $\mathbb{C}^*_Y \to \mathbb{G}_{m,Y}$ from the constant sheaf $\mathbb{C}^*$ on Y to the sheaf $\mathbb{G}_{m,Y}$ of germs of invertible functions on Y. We may choose a basis $(e_1,\dots,e_n)$ of $H^0(\mathbb{G}_m^n,\mathcal{U}_{\mathbb{G}_m^n}) \simeq \mathbb{Z}^n$ in such a way that the connecting homomorphism δ maps e_i to $[g_i]$ for every $i = 1,\dots,n$. The assertion follows. $\square$

3.2. Non Cancellation for smooth factorial affine varieties of low Kodaira dimension?

Recall that by [13, Theorem 3], cancellation for $\mathbf{T}^n$ holds over smooth affine varieties of log-general type. On the other hand, since they arise as Zariski locally trivial $\mathbb{A}^1_*$-bundles over varieties of log-general type, it follows from Iitaka [13] and Kawamata [14] addition theorems that all the counter-examples X constructed in subsection 2.1 have Kodaira dimension $\dim X - 1 \geqslant 2$. Similarly, the examples constructed in Theorem 3.1 as well as their products by low dimensional tori have Kodaira dimension at least 2. One can also check directly that the two surfaces constructed in subsection 2.2 have Kodaira dimension equal to 1. This raises the question whether cancellation holds for smooth factorial affine varieties of small Kodaira dimension. The following proposition implies in particular that a potential counter-example to cancellation for varieties of Kodaira dimension 0 or $-\infty$ has to be of dimension at least 3:

PROPOSITION 3.3. — *Let S and S' be smooth factorial affine surfaces. If $S \times \mathbb{A}^1_*$ and $S' \times \mathbb{A}^1_*$ are isomorphic and $\kappa(S)$ (or, equivalently, $\kappa(S')$) is not equal to* 1, *then S and S' are isomorphic.*

Proof. — In view of the Iitaka-Fujita strong cancellation Theorem [13], we only have to consider the cases where $\kappa(S) = \kappa(S') = -\infty$ or 0. In the first case, S and S' are isomorphic to products of punctured smooth affine rational curves with $\mathbb{A}^1$ (see e.g. [10]) and so, the assertion follows from a combination of the existing positive results for cancellation by $\mathbb{A}^1$ and $\mathbb{A}^1_*$. Namely, let $S = C \times \mathbb{A}^1$ and $S' = C' \times \mathbb{A}^1$, where C and C' are punctured affine lines. If either C or C', say C, is not isomorphic to $\mathbb{A}^1$, then $\kappa(C\times\mathbb{A}^1_*)=\kappa(C)\geqslant 0$ and so, by the Iitaka-Fujita strong cancellation Theorem for $\mathbb{A}^1$, every isomorphism between $S\times\mathbb{A}^1_*$ and $S'\times\mathbb{A}^1_*$ descends to an isomorphism between $C\times\mathbb{A}^1_*$ and $C'\times\mathbb{A}^1_*$. Since cancellation by $\mathbb{A}^1_*$ holds for smooth affine curves, we deduce in turn that C and C' are isomorphic, whence that S and S' are isomorphic.

In the case where $\kappa(S)=\kappa(S')=0$, we already observed that cancellation holds if every invertible function on S or S' is constant. Therefore, we may assume that S and S' both have non constant units whence, by virtue of [10, §5], belong up to isomorphism to the following list of surfaces: $V_0=\mathbb{A}^1_*\times\mathbb{A}^1_*$, and the complements V_k in the Hirzebruch surfaces $\rho_k:\mathbb{F}_k\to\mathbb{P}^1$, $k\geqslant 1$, of a pair of sections $H_{0,k}$ and $H_{\infty,k}$ of ρ_k with self-intersection k intersecting each others in a unique point p_k, and a fiber F of ρ_k not passing through p_k. All the surfaces V_k, $k\geqslant 1$, admit an $\mathbb{A}^1_*$-fibration $\pi_k:V_k\to\mathbb{A}^1_*=\operatorname{Spec}(\mathbb{C}[x^{\pm 1}])$ induced by the restriction of the pencil on $\mathbb{F}_k$ generated by $H_{0,k}$ and $H_{\infty,k}$. The unique degenerate fiber of π_k, say $\pi_k^{-1}(1)$ up to a linear change of coordinate on $\mathbb{A}^1_*$, is reducible, consisting of the union of the intersection with V_k of the exceptional section $C_{0,k}$ of ρ_k, and of the fiber F_k of ρ_k passing through p_k, counted with multiplicity k. Note that V_0 does not contain any curve isomorphic to $\mathbb{A}^1$ whereas each surface V_k, $k\geqslant 1$, contains exactly two such curves: the intersections $F_k\cap V_k$ and $C_{0,k}\cap V_k$. It follows in particular that $V_0\times\mathbb{A}^1_*$ cannot be isomorphic to any $V_k\times\mathbb{A}^1_*$, $k\geqslant 1$. Now suppose that there exists an isomorphism $\Phi:V_k\times\mathbb{A}^1_*\xrightarrow{\sim}V_{k'}\times\mathbb{A}^1_*$ for some $k,k'\geqslant 1$. Since $\kappa((F_k\cap V_k)\times\mathbb{A}^1_*)=\kappa((C_{0,k}\cap V_k)\times\mathbb{A}^1_*)=-\infty$, their respective images by Φ cannot be mapped dominantly on $V_{k'}$ by the first projection and since $F_{k'}\cap V_{k'}$ and $C_{0,k'}\cap V_{k'}$ are the unique curves isomorphic to $\mathbb{A}^1$ on $V_{k'}$, we conclude similarly as in the proof of Proposition 1.2 that Φ map $(\pi_k^{-1}(1))_{\mathrm{red}}\times\mathbb{A}^1_*$ isomorphically onto $(\pi_{k'}^{-1}(1))_{\mathrm{red}}\times\mathbb{A}^1_*$. This implies in turn that Φ restricts to an isomorphism between the open subsets $U_k=\pi_k^{-1}(\mathbb{A}^1_*\setminus\{1\})\times\mathbb{A}^1_*$ and $U_{k'}=\pi_{k'}^{-1}(\mathbb{A}^1_*\setminus\{1\})\times\mathbb{A}^1_*$ of $V_k\times\mathbb{A}^1_*$ and $V_{k'}\times\mathbb{A}^1_*$ respectively. Now $\mathbb{A}^1_*\setminus\{1\}$ is of log-general type and since the restrictions of π_k and $\pi_{k'}$ to U_k and $U_{k'}$ are trivial $\mathbb{A}^1_*$-bundles, we deduce from the Iitaka-Fujita strong cancellation Theorem [13] that the restriction of Φ to U_k descends to an isomorphism $\varphi:\mathbb{A}^1_*\setminus\{1\}\xrightarrow{\sim}\mathbb{A}^1_*\setminus\{1\}$ for which the following diagram commutes

$$\begin{array}{ccc} U_k\times\mathbb{A}^1_*\simeq(\mathbb{A}^1_*\setminus\{1\})\times\mathbb{A}^1_*\times\mathbb{A}^1_* & \xrightarrow{\Phi} & (\mathbb{A}^1_*\setminus\{1\})\times\mathbb{A}^1_*\times\mathbb{A}^1_*\simeq U_{k'}\times\mathbb{A}^1_* \\ \downarrow{\scriptstyle \pi_k\circ\mathrm{pr}_1} & & \downarrow{\scriptstyle \pi_{k'}\circ\mathrm{pr}_1} \\ \mathbb{A}^1_*\setminus\{1\} & \xrightarrow{\varphi} & \mathbb{A}^1_*\setminus\{1\}. \end{array}$$

Summing up, if it exists, an isomorphism $\Phi : V_k \times \mathbb{A}^1_* \xrightarrow{\sim} V_{k'} \times \mathbb{A}^1_*$ must be compatible with the $\mathbf{T}^2$-fibrations $\pi_k \circ \mathrm{pr}_1 : V_k \times \mathbb{A}^1_* \to \mathbb{A}^1_*$ and $\pi_{k'} \circ \mathrm{pr}_1 : V_{k'} \times \mathbb{A}^1_* \to \mathbb{A}^1_*$. But this impossible since the multiplicity of the irreducible component $F_k \cap V_k$ of $\pi_k^{-1}(1)$ is different for every $k \geqslant 1$. In conclusion, the surfaces V_k, $k \geqslant 0$, are pairwise non isomorphic, with pairwise non isomorphic $\mathbb{A}^1_*$-cylinders, which completes the proof. □

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