ANNALES

DE

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Tome 66, nº 6 (2016), p. 2507-2523.

<http://aif.cedram.org/item?id=AIF_2016__66_6_2507_0>

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Ann. Inst. Fourier, Grenoble
66, 6 (2016) 2507-2523

LOCAL AND INFINITESIMAL RIGIDITY OF SIMPLY CONNECTED NEGATIVELY CURVED MANIFOLDS

by Kingshook BISWAS

Abstract. — Let (X, g_0) be a simply connected Riemannian manifold with sectional curvature $K \leqslant -1$. For a metric g on X which is equal to g_0 outside a compact the identity map of X induces a conformal map $\hat{id}_{g_0,g} : \partial_{g_0} X \to \partial_g X$ between the boundaries at infinity of X with respect to g_0 and g. We define a function $S(g)$ on the space of geodesics of (X, g_0), called the integrated Schwarzian of g, which measures the deviation of this conformal map from being Moebius. We use the integrated Schwarzian to prove local and infinitesimal rigidity results for such metric deformations.

Résumé. — Soit (X, g_0) un variété Riemannienne simplement connexe à courbure $K \leqslant -1$. Pour une métrique g qui est égale à g_0 en dehors d'un compact l'identité de X s'étend à une application conforme $\hat{id}_{g_0,g} : \partial_{g_0} X \to \partial_g X$ entre les bords à l'infini de X par rapport à g_0 et g. On définit une fonction $S(g)$ sur l'espace des geodésiques de (X, g_0), appelé le Schwarzian integré de g, qui quantifie la déviation de cette application d'être Moebius. On utilise le Schwarzian integré pour démontrer des théorèmes de rigidité locale et infinitesimale pour tels déformations métriques.

1. Introduction

The problems we consider in this article are motivated by rigidity results for negatively curved manifolds. Two cases have been intensively studied, namely closed negatively curved manifolds, and compact negatively curved manifolds with convex boundary. In both cases rigidity results have been obtained, to the effect that metric deformations preserving some form of a "length spectrum" are trivial, i.e. are isometric. For closed negatively curved manifolds, the role of length spectrum is played by the *marked length spectrum*, i.e. the function on free homotopy classes of closed curves

Keywords: Negatively curved manifolds, Moebius, cross-ratio.
Math. classification: 53C24.

which assigns to a homotopy class the length of the unique closed geodesic in that class, while for compact negatively curved manifolds with boundary, the role of length spectrum is played by the *boundary distance function*, i.e. the function which assigns to pairs of points on the boundary the geodesic distance between them.

To put the corresponding results in a general context, we may consider the moduli space $M(X)$ of negatively curved metrics on a manifold X, i.e. the quotient of the space of negatively curved metrics on X by the natural action of the group of diffeomorphisms of X. We are then given a "length spectrum map" $L : M(X) \to C^{\mathbb{R}^+}$ which assigns to a negatively curved metric g a length function $L(g) : C \to \mathbb{R}^+$, where C is a space parametrizing a certain collection of geodesics (in the two cases mentioned above, C would be the set of free homotopy classes of closed curves and the set of pairs of distinct points on the boundary respectively). There are then three rigidity questions one may pose: global rigidity (injectivity of the map L), local rigidity (local injectivity of L) and infinitesimal rigidity (injectivity of the differential of L).

We mention briefly some results obtained in the two cases previously mentioned. Guillemin and Kazhdan [7] proved an infinitesimal rigidity result for closed negatively curved surfaces. Otal proved that global rigidity holds for closed negatively curved surfaces [8] (giving an affirmative answer to the "marked length spectrum rigidity" problem of Burns–Katok [4] in dimension 2), and also for compact negatively curved surfaces with convex boundary [9]. Croke and Sharafutdinov have proven that infinitesimal rigidity holds for closed negatively curved n-manifolds [6], and Croke–Dairbekov–Sharafutdinov [5] have also proven a local rigidity result for compact negatively curved n-manifolds with convex boundary.

We prove local and infinitesimal rigidity results in a third case not previously considered, namely that of simply connected complete negatively curved manifolds. This is similar to the second case, only now the boundary is at infinity. In this case given a complete negatively curved metric g_0 on a simply connected manifold X, say with sectional curvatures bounded above by -1 so that X with the distance function induced by g_0 is a CAT(-1) space, while there is no natural notion of a length spectrum for the metric g_0, there is, given a compactly supported deformation g of g_0, a well-defined notion of a *relative length spectrum* for the pair (g_0, g_1). This is given by a function $S_{g_0}(g) : \partial^2_{g_0} X \to \mathbb{R}$ of pairs of distinct points on the boundary at infinity $\partial_{g_0} X$ of X with respect to g_0, called the *integrated Schwarzian* of g with respect to g_0, which is a renormalized version of the boundary distance function, measuring the difference between the g_0 and g distances between pairs of points on the boundary at infinity.

The integrated Schwarzian was first introduced in [1] in the study of Moebius and conformal maps between boundaries of CAT(-1) spaces. We recall that the boundary at infinity ∂X of a CAT(-1) space X carries a natural family of metrics called visual metrics. These metrics are Moebius equivalent to each other (i.e. the metric cross-ratios all coincide), and in particular conformal to each other, hence the notions of Moebius and conformal maps between boundaries of CAT(-1) spaces are well-defined, independent of choices of visual metrics (for basic properties of CAT(-1) spaces and their boundaries see [2], [3]). Given a conformal map between two boundaries $f : \partial X \to \partial Y$, the integrated Schwarzian of f is a function $S(f) : \partial^2 X \to \mathbb{R}$ which measures the deviation of the conformal map f from being Moebius. In the case of a pair of negatively curved metrics g_0, g_1 as above with $g - g_0$ compactly supported, the identity map $id : (X, g_0) \to (X, g)$ extends in fact to a conformal homeomorphism $\hat{id}_{g_0,g} : \partial_{g_0} X \to \partial_g X$. The integrated Schwarzian of g with respect to g_0 is then defined to be the integrated Schwarzian of the conformal map $\hat{id}_{g_0,g}$. It turns out that the integrated Schwarzian vanishes if the boundary map $\hat{id}_{g_0,g} : \partial_{g_0} X \to \partial_g X$ is Moebius, and conversely in the presence of a lower curvature bound for g_0, $\hat{id}_{g_0,g}$ is Moebius if $S_{g_0}(g)$ vanishes.

We prove the following infinitesimal rigidity result:

Theorem 1.1. — *Let (X, g_0) be a complete simply connected Riemannian manifold with sectional curvatures bounded above by -1. Let $(g_t)_{0 \leqslant t \leqslant 1}$ be a 1-parameter family of Riemannian metrics on X such that:*

(1) *The symmetric $(0,2)$-tensors $g_t - g_0, 0 \leqslant t \leqslant 1$ are compactly supported with supports contained in a fixed compact $C \subset X$.*

(2) *The sectional curvatures of the metrics $g_t, 0 \leqslant t \leqslant 1$ are bounded above by -1.*

(3) *The metrics $g_t, 0 \leqslant t \leqslant 1$ depend smoothly on the parameter t, i.e. the map $[0,1] \times X \to T^*X^{\odot 2}, (t,x) \mapsto g_t(x)$ is smooth.*

(4) *All the boundary maps $\hat{id}_{g_0,g_t} : \partial_{g_0} X \to \partial_{g_t} X, 0 \leqslant t \leqslant 1$, are Moebius.*

Then there is a 1-parameter family of diffeomorphisms $f_t : X \to X$ such that $f_t^* g_t = g_0$. Moreover there is a compact $K \subset X$ such that $f_t = id_X$ on $X - K$ for $0 \leqslant t \leqslant 1$.

There are two main steps in the proof of the above theorem. The first is to prove a formula for the first variation of the integrated Schwarzian,

$$\frac{d}{dt} 2S_{g_0}(g_t) = I_{g_t}(\dot{g_t}) \circ \hat{id}_{g_0,g_t}$$

where I_{g_t} denotes the *ray transform* of the metric g_t and $\dot{g_t}$ the symmetric $(0,2)$-tensor $\frac{d}{dt}(g_t)$. Here by the ray transform of a metric g we mean the map which assigns to a compactly supported symmetric $(0,2)$-tensor u the function $I_g(u) : \partial^2 X \to \mathbb{R}$ on the space of bi-infinite geodesics of X obtained by integrating u along geodesics (when u is thought of as a function on the unit tangent bundle). This formula is an analogue of a formula for the first variation of the boundary distance function in the case of a compact manifold with boundary, and indeed is proved by passing to the limit in this formula.

The second step is to prove that the kernel of the ray transform consists of *exact* symmetric $(0,2)$-tensors, i.e. symmetric $(0,2)$-tensors arising as Lie derivatives of the metric with respect to vector fields. The analogous statement in the case of a compact manifold with boundary is known, and again the statement in the simply connected case is proved by passing to the limit. Combining these two steps, it follows that each $\dot{g_t}$ is exact, and integrating the corresponding vector fields gives the required family of diffeomorphisms $(f_t)_{0 \leqslant t \leqslant 1}$.

We also prove a local rigidity result:

Theorem 1.2. — *Let (X, g_0) be a complete simply connected Riemannian manifold with sectional curvature bounded above by -1. Given a compact $K \subset X$ and $0 < \alpha < 1$, there is an $\epsilon > 0$ such that the following holds:*

Let g be a metric with sectional curvatures bounded above by -1 such that the support of $g - g_0$ is contained in K, and such that the $C^{2,\alpha}$ norm of $g - g_0$ is less than ϵ. If the boundary map $\hat{id}_{g_0,g} : \partial_{g_0} X \to \partial_g X$ is Moebius and $Vol_{g_0}(K) = Vol_g(K)$ then g is isometric to g_0.

The proof of the above theorem follows along lines very similar to Croke–Dairbekov–Sharafutdinov's proof of local rigidity for compact manifolds with convex boundary, and is essentially an adaptation of their proof to the case at hand.

The paper is organized as follows. In Section 2 we recall background material on Moebius maps, conformal maps and the integrated Schwarzian, in the context of general CAT(-1) spaces. In Section 3 we consider the integrated Schwarzian in the case of a compactly supported deformation of a complete simply connected negatively curved Riemannian manifold, and derive the variational formula described above. In Section 4 we prove the assertion about the kernel of the ray transform. Finally in Section 5 we put these ingredients together to prove Theorems 1.1 and 1.2.

2. Moebius maps, conformal maps and the integrated Schwarzian

The material in this section is taken from [1].

Definition 2.1. — *A homeomorphism between metric spaces $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ with no isolated points is said to be* conformal *if for all $\xi \in Z_1$, the limit*

$$df_{\rho_1,\rho_2}(\xi) := \lim_{\eta \to \xi} \frac{\rho_2(f(\xi), f(\eta))}{\rho_1(\xi, \eta)}$$

exists and is positive. The positive function df_{ρ_1,ρ_2} is called the derivative of f with respect to ρ_1, ρ_2. We say f is C^1 conformal *if its derivative is continuous.*

Two metrics ρ_1, ρ_2 inducing the same topology on a set Z, such that Z has no isolated points, are said to be conformal (respectively C^1 conformal) if the map $id_Z : (Z, \rho_1) \to (Z, \rho_2)$ is conformal (respectively C^1 conformal). In this case we denote the derivative of the identity map by $\frac{d\rho_2}{d\rho_1}$.

Definition 2.2. — *Let Z be a set with at least four points. For a metric ρ on Z we define the metric cross-ratio with respect to ρ of a quadruple of distinct points (ξ, ξ', η, η') of Z by*

$$[\xi\xi'\eta\eta']_\rho := \frac{\rho(\xi, \eta)\rho(\xi', \eta')}{\rho(\xi, \eta')\rho(\xi', \eta)}$$

Definition 2.3. — *A map between metric spaces $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ is said to be Moebius if it preserves metric cross-ratios. A map f is locally Moebius if every $\xi \in Z_1$ has a neighbourhood U such that $f_{|U}$ is Moebius. Two metrics ρ_1, ρ_2 on a set Z are Moebius equivalent if the identity map $id : (Z, \rho_1) \to (Z, \rho_2)$ is Moebius.*

We recall the following facts:

Proposition 2.4. — *We have the following:*

1. *A locally Moebius map between metric spaces with no isolated points is C^1 conformal.*
2. *A Moebius map $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ between metric spaces with no isolated points satisfies the "geometric mean-value theorem":*

$$\rho_2(f(\xi), f(\eta))^2 = df_{\rho_1,\rho_2}(\xi) df_{\rho_1,\rho_2}(\eta) \rho_1(\xi, \eta)^2$$

Let X be a CAT(-1) space. We recall that the boundary at infinity of X, denoted by ∂X, comes equipped with a family of metrics $(\rho_x)_{x\in X}$ called visual metrics, defined by $\rho_x(\xi,\eta) = \exp(-(\xi|\eta)_x)$, where $(.|.)_x$ denotes the Gromov inner product (extended to the boundary) with respect to the basepoint x. These metrics are all Moebius equivalent, so the notions of Moebius and conformal maps between boundaries of CAT(-1) spaces $f : \partial X \to \partial Y$ are well-defined independent of the choices of visual metrics on ∂X and ∂Y. For $(\xi,\eta) \in \partial^2 X$, we denote the bi-infinite geodesic in X with endpoints ξ,η also by (ξ,η).

Definition 2.5. — *Let $f : \partial X \to \partial Y$ be a conformal map between boundaries of CAT(-1) spaces equipped with visual metrics. The integrated Schwarzian of f is the function $S(f) : \partial^2 X \to \mathbb{R}$ defined by*

$$S(f)(\xi,\eta) := \log(df_{\rho_x,\rho_y}(\xi)df_{\rho_x,\rho_y}(\eta)) \quad (\xi,\eta) \in \partial^2 X$$

where x, y are any two points $x \in (\xi,\eta), y \in (f(\xi), f(\eta))$ (it is easy to see that the quantity defined above is independent of the choices of x and y).

It is easy to see (using the geometric mean-value theorem) that the integrated Schwarzian of a Moebius map is identically zero. We recall that a CAT(-1) space is said to be *geodesically complete* if every geodesic segment can be extended to a bi-infinite geodesic. The choice of the name *integrated Schwarzian* is perhaps explained by the following theorem from [1], which asserts that just as the classical Schwarzian derivative determines when a conformal map of $\mathbb{C}$ is Moebius, for conformal maps between boundaries of certain CAT(-1) spaces the integrated Schwarzian plays a similar role:

Theorem 2.6. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures satisfying $-b^2 \leqslant K \leqslant -1$ for some $b \geqslant 1$, and let Y be a proper geodesically complete CAT(-1) space. A C^1 conformal map $f : U \subset \partial X \to V \subset \partial Y$ between open subsets of $\partial X, \partial Y$ is Moebius on U (i.e. preserves cross-ratios) if and only if its integrated Schwarzian $S(f)$ vanishes on $\partial^2 U$. Two C^1 conformal maps $f, g : U \subset \partial X \to V \subset \partial Y$ differ by post-composition with a Moebius map $h : V \to V$ if and only if their integrated Schwarzians coincide, $S(f) = S(g)$.*

We recall that the Busemann function of a CAT(-1) space X is the function $B : \partial X \times X \times X \to \mathbb{R}$ defined by

$$B(\xi,x,y) := \lim_{a\to\xi} (d_X(x,a) - d_X(y,a))$$

where $a \in X$ converges radially towards ξ.

We have the following formula for the derivatives of visual metrics on ∂X:

$$\frac{d\rho_y}{d\rho_x}(\xi) = \exp(B(\xi,x,y))\ \xi \in \partial X, x, y \in X.$$

3. The integrated Schwarzian for compactly supported deformations

Let (X, g_0) be a complete simply connected Riemannian manifold with sectional curvatures bounded above by -1. Let g be a Riemannian metric on X with sectional curvatures bounded above by -1 such that the symmetric $(0,2)$-tensor $g - g_0$ is compactly supported. Note that (X, g_0) and (X, g) are both CAT(-1) spaces. We denote the corresponding distance functions on X by d_{g_0}, d_g, the boundaries by $\partial_{g_0} X, \partial_g X$ and the visual metrics by $\rho_{x,g_0}, \rho_{x,g}, x \in X$.

Lemma 3.1. — *The identity map $id : (X, g_0) \to (X, g)$ extends to a locally Moebius homeomorphism $\hat{id}_{g_0,g} : \partial_{g_0} X \to \partial_g X$ (in particular the boundary map is conformal).*

Proof. — Clearly $id : (X, g_0) \to (X, g)$ is bi-Lipschitz, hence extends to a homeomorphism $\hat{id}_{g_0,g} : \partial_{g_0} X \to \partial_g X$. Fix a basepoint $x_0 \in X$ and let $B \subset X$ be a large g_0-ball around x_0 containing the support of $g - g_0$. Given a point $\xi_0 \in \partial_{g_0} X$, we may choose a neighbourhood U of ξ_0 small enough such that for all $\xi, \eta \in U, \xi \neq \eta$, the bi-infinite g_0-geodesic with endpoints ξ, η lies outside B. We may also choose x_1 lying along the g_0-geodesic ray joining x_0 to ξ_0 far enough from x_0 such that for all $\xi \in U$, the g_0-geodesic ray joining x_1 to ξ lies outside B. Since g_0-geodesics lying outside B are also g_1-geodesics, it follows that $\hat{id}_{g_0,g} : (U, \rho_{x_1,g_0}) \to (V = \hat{id}_{g_0,g}(U), \rho_{x_1,g})$ is an isometry. Since the metrics $\rho_{x_0,g_0}, \rho_{x_1,g_0}$ are Moebius equivalent, as are the metrics $\rho_{x_0,g}, \rho_{x_1,g}$, it follows that $\hat{id}_{g_0,g} : (U, \rho_{x_0,g_0}) \to (V, \rho_{x_0,g})$ is Moebius. □

Definition 3.2. — *The integrated Schwarzian of g with respect to g_0 is defined to be the integrated Schwarzian of the above conformal boundary map, $S_{g_0}(g) := S(\hat{id}_{g_0,g}) : \partial^2_{g_0} X \to \mathbb{R}$.*

The following lemma says that the integrated Schwarzian may be viewed as a renormalized limit of boundary distance functions:

Lemma 3.3. — *For $(\xi, \eta) \in \partial^2_{g_0} X$, the following limit exists and equals the integrated Schwarzian:*

$$\lim_{p,q \in (\xi,\eta), p \to \xi, q \to \eta} (d_g(p,q) - d_{g_0}(p,q)) = S_{g_0}(g)(\xi, \eta)$$

Proof. — Let $f = \hat{id}_{g_0,g}$. Let $\gamma_0 : \mathbb{R} \to X$ be the unique bi-infinite g_0-geodesic with endpoints $\gamma_0(-\infty) = \eta, \gamma_0(+\infty) = \xi$ in $\partial_{g_0} X$, and let $\gamma : \mathbb{R} \to X$ be the unique bi-infinite g-geodesic with endpoints $\gamma(-\infty) = \eta' := f(\eta), \gamma(+\infty) = \xi' := f(\xi)$ in $\partial_g X$. By definition, $d_{g_0}(\gamma_0(t), \gamma(t))$ is bounded as $t \to -\infty$ and as $t \to +\infty$. Since $g - g_0$ is compactly supported, for $R > 0$ large enough, $\gamma^+_R := \gamma_{|[R,+\infty)}$ and $\gamma^-_R := \gamma_{|(-\infty,-R]}$ are g_0-geodesic rays with endpoints ξ, η respectively in $\partial_{g_0} X$.

Let $x_t = \gamma_0(t), y_t = \gamma_0(-t)$ for $t > 0$. For $t > 0$ large enough, let $x'_t \in \gamma^+_R, y'_t \in \gamma^-_R$ be such that $B(\xi, x_t, x'_t) = 0, B(\eta, y_t, y'_t) = 0$, where $B : \partial_{g_0} X \times X \times X \to \mathbb{R}$ is the Busemann function of (X, g_0). Note that $d_{g_0}(x_t, x'_t) \to 0, d_{g_0}(y_t, y'_t) \to 0$ as $t \to +\infty$ by exponential convergence of asymptotic geodesic rays in the CAT(-1) space (X, g_0).

From the formula for the derivatives of visual metrics, we have

$$\frac{d\rho_{x_t,g_0}}{d\rho_{x'_t,g_0}} = \frac{d\rho_{y_t,g_0}}{d\rho_{y'_t,g_0}} = 1$$

Since in the previous Lemma we saw that the restrictions of f to small neighbourhoods U, U' of ξ, η respectively were isometries $f : (U, \rho_{x'_t,g_0}) \to (f(U), \rho_{x'_t,g}), f : (U, \rho_{y'_t,g_0}) \to (f(U), \rho_{y'_t,g})$ for t large enough, it follows that

$$df_{\rho_{x_t,g_0},\rho_{x'_t,g}}(\xi) = df_{\rho_{y_t,g_0},\rho_{y'_t,g}}(\eta) = 1$$

Hence by definition of the integrated Schwarzian and the chain rule we have

$$\begin{aligned}
S_{g_0}(g)(\xi,\eta) &= \log df_{\rho_{x_t,g_0},\rho_{x'_t,g}}(\xi) + \log df_{\rho_{x_t,g_0},\rho_{x'_t,g}}(\eta) \\
&= \log df_{\rho_{x_t,g_0},\rho_{x'_t,g}}(\eta) \\
&= \log \frac{d\rho_{y'_t,g}}{d\rho_{x'_t,g}}(\eta) + \log \frac{d\rho_{x_t,g_0}}{d\rho_{y_t,g_0}}(\eta) \\
&= d_g(x'_t, y'_t) - d_{g_0}(x_t, y_t) \\
&= d_g(x_t, y_t) - d_{g_0}(x_t, y_t) + o(1)
\end{aligned}$$

as $t \to +\infty$, since for t large, we have

$$d_g(x'_t, x_t) = d_{g_0}(x'_t, x_t) \to 0, d_g(y'_t, y_t) = d_{g_0}(y'_t, y_t) \to 0.$$

The result follows. □

We also have the following elementary lemma:

Lemma 3.4. — *There is a constant $C > 0$ such that for any $p, q \in X$ we have*

$$|d_g(p,q) - d_{g_0}(p,q)| \leqslant C$$

Proof. — Let B be an open g_0-ball containing the support of $g - g_0$. Since $\overline{B}$ is compact we may assume that p, q are not both contained in $\overline{B}$. We give the proof only in the case when both p and q lie outside B, the argument being similar in the other case when one of the two points lies outside B and the other inside B.

Let γ_0, γ be g_0 and g geodesics respectively joining p, q. If either one of the curves γ_0, γ does not intersect B then neither does the other (since a g_0-geodesic lying entirely outside B is also a g-geodesic and vice-versa, and g_0, g-geodesics joining points are unique) and $d_g(p,q) = d_{g_0}(p,q)$. Otherwise choose points a, b on $\gamma_0 \cap \partial B$ and points a', b' on $\gamma \cap \partial B$ such that the g_0-geodesic segments $[p,a]$ and $[b,q]$ and the g-geodesic segments $[p,a'], [b',q]$ are all disjoint from B, then

$$|d_g(p,a') - d_{g_0}(p,a)| = |d_{g_0}(p,a') - d_{g_0}(p,a)| \leqslant d_{g_0}(a',a),$$
$$|d_g(b',q) - d_{g_0}(b,q)| = |d_{g_0}(b',q) - d_{g_0}(b,q)| \leqslant d_{g_0}(b',b),$$

while $|d_g(a',b') - d_{g_0}(a,b)|$ is bounded by $diam_g(B) + diam_{g_0}(B)$, so that $|d_g(p,q) - d_{g_0}(p,q)| \leqslant diam_g(B) + 3diam_{g_0}(B)$. □

We now consider a 1-parameter family of Riemannian metrics $(g_t)_{0 \leqslant t \leqslant 1}$ on X satisfying the following hypotheses:

(1) There is a fixed compact K containing the supports of the symmetric $(0,2)$-tensors $g_t - g_0, 0 \leqslant t \leqslant 1$.
(2) The sectional curvatures of the metrics $g_t, 0 \leqslant t \leqslant 1$, are bounded above by -1.
(3) The metrics $g_t, 0 \leqslant t \leqslant 1$ depend smoothly on the parameter t, i.e. the map $[0,1] \times X \to T^*X^{\odot 2}, (t,x) \mapsto g_t(x)$ is smooth.

Lemma 3.5. — *Fix $p, q \in X, p \neq q$. Let $a = d_{g_0}(p,q)$, and for $0 \leqslant t \leqslant 1$, let $\gamma_t : [0,a] \to X$ be the unique g_t-geodesic segment with endpoints p, q. Then the map $[0,1] \times [0,a] \to X, (t,s) \mapsto \gamma_t(s)$ is smooth.*

Proof. — Let $\exp_t : T_pX \to X$ denote the exponential mapping of the metric g_t based at p. By smooth dependence of solutions to ODE's on initial conditions and on coefficients, the map $\Phi : [0,1] \times T_pX \to X, (t,v) \mapsto \exp_t(v)$ is smooth. Since the metrics g_t are nonpositively curved, for each t there is a unique $v_t \in T_pX$ such that $\Phi(t, v_t) = q$. Moreover each map $\exp_t$ is a diffeomorphism, hence applying the Implicit Function Theorem to Φ it follows that the map $t \mapsto v_t$ is smooth. Since $\gamma_t(s) = \Phi(t, (s/a)v_t)$, the lemma follows. □

For notational convenience, given a symmetric $(0,m)$-tensor field u on X and a tangent vector $\xi \in TX$, we denote $u(\xi, \ldots, \xi)$ by simply $u(\xi)$.

Lemma 3.6. — *With the same notation as above, we have:*

$$\frac{d}{dt}\left(\frac{d_{g_t}(p,q)^2 - d_{g_0}(p,q)^2}{d_{g_0}(p,q)}\right) = \int_0^a \dot{g}_t(\dot{\gamma}_t(s))\ ds$$

where $\dot{g}_t$ is the symmetric $(0,2)$-tensor $\frac{d}{dt}(g_t)$.

Proof. — This formula may be found in [10]. We reproduce the proof for the benefit of the reader. It suffices to prove the formula for $t = 0$. Let $\gamma_t(s) = (\gamma_t^i(s)), g_t = (g_{t,ij}), \dot{g}_0 = (f_{ij})$ in local coordinates. We have (using Einstein summation convention)

$$\frac{d_{g_t}(p,q)^2}{d_{g_0}(p,q)} = \int_0^a g_{t,ij}(\gamma_t(s))\dot{\gamma}_t^i(s)\dot{\gamma}_t^j(s)\ ds$$

Differentiating the above equality with respect to t and putting $t = 0$ gives

$$\frac{d}{dt}_{|t=0}\left(\frac{d_{g_t}(p,q)^2 - d_{g_0}(p,q)^2}{d_{g_0}(p,q)}\right) = \int_0^a f_{ij}(\gamma_0(s))\dot{\gamma}_0^i(s)\dot{\gamma}_0^j(s)\ ds + A$$

where A is the integral

$$A = \int_0^a \left(\frac{\partial}{\partial x^k} g_{0,ij}(\gamma_0(s))\dot{\gamma}_0^i(s)\dot{\gamma}_0^j(s)\frac{\partial}{\partial t}_{|t=0}\gamma_t^k(s)\right.$$
$$\left. + 2g_{0,ij}(\gamma_0(s))\dot{\gamma}_0^i(s)\frac{\partial}{\partial t}_{|t=0}\dot{\gamma}_t^j(s)\right) ds$$

Now the integral A is equal to zero, since $\frac{\partial}{\partial t}\gamma_t(0) = \frac{\partial}{\partial t}\gamma_t(a) = 0$, and the curve γ_0 is an extremal of the energy functional

$$E_0(\gamma) = \int_0^a g_{0,ij}(\gamma(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s)\ ds$$

The lemma follows. □

Denote by $C_c^\infty(T^*X^{\odot 2})$ the space of smooth compactly supported symmetric $(0,2)$-tensors on X.

Definition 3.7. — *Let g be a complete Riemannian metric on X with sectional curvatures bounded above by -1. The ray transform of g is the linear map*

$$I_g : C_c^\infty(T^*X^{\odot 2}) \to C_c(\partial_g^2 X)$$
$$f \mapsto \left(I_g(f) : (\xi,\eta) \mapsto \int_{-\infty}^{\infty} f(\dot{\gamma}_{(\xi,\eta)}(s))\ ds\right)$$

where $\gamma_{(\xi,\eta)}$ is the unique (up to translation) bi-infinite g-geodesic with unit g-speed and endpoints $\gamma_{(\xi,\eta)}(-\infty) = \eta, \gamma_{(\xi,\eta)}(+\infty) = \xi$. The domain of I_g is the space of smooth symmetric $(0,2)$-tensors on X with compact *support, and the range the space of continuous functions on $\partial_g^2 X$ with compact support.*

We can now prove the first variation formula for the integrated Schwarzian:

THEOREM 3.8. — *With the same notation as above, for $(\xi,\eta) \in \partial_{g_0}^2 X$ we have:*

$$\frac{d}{dt} 2S_{g_0}(g_t)(\xi,\eta) = \left(I_{g_t}(\dot{g}_t) \circ \hat{id}_{g_0,g_t} \right)(\xi,\eta)$$

Proof. — For $R > 0$, let $p_R = \gamma_{(\xi,\eta)}(R), q_R = \gamma_{(\xi,\eta)}(-R)$ where $\gamma_{(\xi,\eta)}$ is a bi-infinite unit speed g_0-geodesic with endpoints ξ, η. Let $\gamma_{t,R} : [-R,R] \to X$ be the unique g_t-geodesic segment with endpoints p_R, q_R. Define real-valued functions on $[0,1]$ by

$$h_R : t \mapsto \frac{d_{g_t}(p_R,q_R)^2 - d_{g_0}(p_R,q_R)^2}{d_{g_0}(p_R,q_R)}$$

Then it is easy to see using Lemma 3.3 and Lemma 3.4 that the pointwise limit as $R \to +\infty$ of the functions h_R is the function $h : t \mapsto 2S_{g_0}(g_t)(\xi,\eta)$. Moreover from Lemma 3.6 it follows that each h_R is differentiable with derivative

$$h'_R : t \mapsto \int_{-R}^{R} \dot{g}_t(\dot{\gamma}_{t,R}(s)) \; ds$$

By Lemma 3.6, $\gamma_{t,R}$ depends smoothly on t, hence h'_R is continuous and h_R is in fact C^1. Moreover as $R \to +\infty$, for each fixed t it follows from a standard argument for CAT(-1) spaces that the g_t-geodesic segments $\gamma_{t,R}$ converge uniformly on compacts to a bi-infinite g_t-geodesic $\gamma_t : \mathbb{R} \to X$ with endpoints $\hat{id}_{g_0,g_t}(\xi), \hat{id}_{g_0,g_t}(\eta)$, which is unit speed (because $d_{g_t}(p_R,q_R)/d_{g_0}(p_R,q_R) \to 1$ as $R \to +\infty$). The same argument for CAT(-1) spaces also gives that the convergence of $\gamma_{t,R}$ to γ_t is uniform in t (the upper bound on the distance between $\gamma_{t,R}$ and γ_t only depends on the upper sectional curvature bound for g_t, which is -1 independent of t, and on the visual distance $\rho_{x,g_t}(\hat{id}_{g_0,g_t}(\xi), \hat{id}_{g_0,g_t}(\eta))$, which is bounded below by a positive constant independent of t for a fixed basepoint $x \in X$). Again due to negative curvature, C^0-convergence of $\gamma_{t,R}$ to γ_t actually implies C^1-convergence of $\gamma_{t,R}$ to γ_t. It follows that as $R \to +\infty$, the functions h'_R converge uniformly to the function

$$f : t \mapsto \int_{-\infty}^{\infty} \dot{g}_t(\dot{\gamma}_t(s)) \; ds = \left(I_{g_t}(\dot{g}_t) \circ \hat{id}_{g_0,g_t} \right)(\xi,\eta)$$

It follows that h is C^1 with derivative equal to f. $\square$

We will need the following lemma in the proof of the local rigidity result:

Lemma 3.9. — *Let g_1 be a negatively curved metric on X with $g_1 - g_0$ compactly supported, then for any $(\xi,\eta) \in \partial^2_{g_0} X$ we have:*

$$I_{g_0}(g_1 - g_0)(\xi,\eta) \geqslant 2S_{g_0}(g_1)(\xi,\eta)$$

Proof. — For $p,q \in X$ and $i = 0,1$ let $\gamma^i_{p,q} : [0, d_{g_0}(p,q)] \to X$ be the unique g_i-geodesic joining p to q. Letting p,q tend to ξ,η radially along g_0 geodesics, we have:

$$\begin{aligned}
&I_{g_0}(g_1 - g_0)(\xi,\eta)\\
&\qquad = \lim_{p\to\xi,q\to\eta}\left(\int_0^{d_{g_0}(p,q)} g_1((\gamma^0)'(t))dt - \int_0^{d_{g_0}(p,q)} g_0((\gamma^0)'(t))dt\right)\\
&\qquad \geqslant \lim_{p\to\xi,q\to\eta}\left(\int_0^{d_{g_0}(p,q)} g_1((\gamma^1)'(t))dt - \int_0^{d_{g_0}(p,q)} g_0((\gamma^0)'(t))dt\right)\\
&\qquad = \lim_{p\to\xi,q\to\eta}\left(\frac{d^2_{g_1}(p,q)}{d_{g_0}(p,q)} - d_{g_0}(p,q)\right)\\
&\qquad = \lim_{p\to\xi,q\to\eta}(d_{g_1}(p,q) - d_{g_0}(p,q))\left(\frac{d_{g_1}(p,q) + d_{g_0}(p,q)}{d_{g_0}(p,q)}\right)\\
&\qquad = 2S_{g_0}(g_1)(\xi,\eta)
\end{aligned}$$

where in the last step we have used Lemmas 3.3 and 3.4. □

4. The kernel of the ray transform

We keep the notation of the previous section. For $m \geqslant 1$ we denote by $\sigma : T^*X^{\otimes m} \to T^*X^{\odot m}$ the symmetrization operator on $(0,m)$ tensors. Covariant differentiation with respect to the Levi–Civita connection of g_0 defines an operator $\nabla^{g_0} : \Gamma(T^*X^{\otimes m}) \to \Gamma(T^*X^{\otimes m+1})$ (where for E a vector bundle over X, $\Gamma(E)$ denotes as usual the space of smooth sections of E). Restricting to symmetric tensors and composing with the symmetrization operator, we obtain a differential operator acting on symmetric tensors,

$$d^{g_0} := \sigma \circ \nabla^{g_0} : \Gamma(T^*X^{\odot m}) \to \Gamma(T^*X^{\odot m+1})$$

For $m = 1$, if $u \in \Gamma(T^*X)$ is a smooth 1-form, then the symmetric $(0,2)$-tensor $d^{g_0}u$ coincides with the Lie derivative $\mathcal{L}_v g_0$ where $v \in \Gamma(TX)$ is the vector field dual to u with respect to the metric g_0. For $m \geqslant 1$,

$u \in \Gamma(T^*X^{\odot m})$, and any g_0-geodesic $\gamma : (a,b) \to X$, the following equality is valid:

$$\frac{d}{dt}(u(\dot\gamma(t))) = (d^{g_0}u)(\dot\gamma(t))$$

It follows immediately that for a smooth compactly supported 1-form v, we have $I_{g_0}(d^{g_0}v) = 0$. Conversely we have the following:

Theorem 4.1. — *Let $f \in \Gamma(T^*X^{\odot 2})$ be a smooth symmetric $(0,2)$-tensor field with compact support. If $I_{g_0}(f) = 0$ then $f = d^{g_0}v$ for some $v \in \Gamma(T^*X)$ with compact support. Moreover the support of v only depends on the support of f.*

The proof of the above theorem will follow easily from the characterization of the kernel of the ray transform on a completely dissipative Riemannian manifold given by Sharafutdinov in [10]. In order to state his result, we first recall the relevant notions from [10].

Definition 4.2. — *A compact Riemannian manifold with boundary (M, g_0) is called a completely dissipative Riemannian manifold (or CDRM for short) if the following conditions are satisfied:*

1. *The boundary ∂M is strictly convex, i.e. the second fundamental form of the boundary is positive-definite.*
2. *For every $x \in M$ and every $\xi \in T_xM - \{0\}$ the maximal geodesic γ with initial conditions $\gamma(0) = x, \gamma'(0) = \xi$ is defined on a finite interval.*

It is not hard to see in our situation that any closed g_0-metric ball $M \subset X$ is a CDRM. We define for M a CDRM the following:

$$\partial T^1_\pm M := \{(x,\xi) \in T^1M | x \in \partial M, \pm < \xi, \nu(x) >\geqslant 0\}$$

where $\nu(x)$ denotes the outward normal to the boundary.

Definition 4.3. — *Let (M, g_0) be a CDRM. The ray transform of M is the linear operator I_M defined by*

$$\begin{aligned} I_M : \Gamma(T^*M^{\odot 2}) &\to C^\infty(\partial T^1_+ M) \\ u &\mapsto \left(I_M(u) : (x,\xi) \mapsto \int_{\tau_-(x,\xi)}^0 u(\dot\gamma_{(x,\xi)}(s))\, ds \right) \end{aligned}$$

where $\gamma_{(x,\xi)} : [\tau_-(x,\xi), 0] \to M$ is the maximal geodesic with initial conditions $\gamma_{(x,\xi)}(0) = x, \gamma'_{(x,\xi)}(0) = \xi$.

We may now state a version of Sharafutdinov's result suited to our purposes:

Theorem 4.4. — *Let (M, g_0) be a CDRM of nonpositive sectional curvature and let $f \in \Gamma(T^*M^{\odot 2})$. If $I_M(f) = 0$ then $f = d^{g_0} v$ for some $v \in \Gamma(T^*M)$ such that $v_{|\partial M} = 0$.*

Proof of Theorem 4.1. — Given $f \in \Gamma(T^*X^{\odot 2})$ with compact support such that $I_{g_0}(f) = 0$, choose a closed g_0-metric ball M containing the support of f in its interior. Then M is a CDRM, and moreover, since the support of f is contained in M, the integral of f over any maximal geodesic of M coincides with the integral of f over the extension of the geodesic segment to a bi-infinite geodesic of X. Hence the equality $I_{g_0}(f) = 0$ implies $I_M(f_{|M}) = 0$. Applying the previous theorem, there exists $v \in \Gamma(T^*M)$ such that $f_{|M} = d^{g_0} v$ and $v_{|\partial M} = 0$. Extending v to be zero outside M so that v has compact support the conclusion of the theorem follows. □

5. Proofs of theorems

5.1. Infinitesimal rigidity

Proof of Theorem 1.1. — Since $S_{g_0}(g_t) = 0$ for $0 \leqslant t \leqslant 1$, it follows from Theorem 3.8 that $I_{g_t}(\dot{g}_t) = 0$ for $0 \leqslant t \leqslant 1$. By Theorem 4.1, this implies existence of vector fields v_t for $0 \leqslant t \leqslant 1$ such that $\dot{g}_t = \mathcal{L}_{v_t} g_t$. Moreover by the hypothesis on the supports of $g_t - g_0$, it follows that the supports of the vector fields v_t are contained in a fixed compact. Hence we may integrate to obtain a 1-parameter family of diffeomorphisms $f_t : X \to X$ which are equal to the identity outside a fixed compact, such that $f_t^* g_t = g_0$ for $0 \leqslant t \leqslant 1$. □

5.2. Local rigidity

We first recall some lemmas and notation from [5].

Let (M, g_0) be a CDRM. We denote by δ^{g_0} the *divergence operator* of the metric g_0 acting on symmetric tensors, which is a first-order differential operator formally adjoint to the operator d^{g_0} (we refer to [10] for the precise definition). Symmetric tensors f such that $\delta^{g_0}(f) = 0$ are called "solenoidal".

For $k \geqslant 1$ an integer and $0 < \alpha < 1$ a real number, we denote by $C^{k,\alpha}(T^*M^{\odot 2})$ the space of $C^{k,\alpha}$-smooth symmetric $(0, 2)$ tensor fields on M. Endowing $C^{k,\alpha}(T^*M^{\odot 2})$ with the natural $C^{k,\alpha}$ topology turns it into a topological Banach space, i.e. a topological vector space whose topology can be defined by some norm making it a Banach space.

We denote by $\mathrm{Diff}_0^{k,\alpha}(M)$ the set of all $C^{k,\alpha}$-smooth diffeomorphisms of M that are the identity on the boundary, and endow $\mathrm{Diff}_0^{k,\alpha}(M)$ with the natural $C^{k,\alpha}$ topology (defined using some finite atlas; the resulting topology is independent of the choice of atlas).

Theorem 2.1 of [5] states:

Theorem 5.1 ([5]). — *For every neighborhood $U \subset \mathrm{Diff}_0^{k,\alpha}(M)$ of the identity there is a neighborhood $W \subset C^{k,\alpha}(T^*M^{\odot 2})$ of the metric tensor g_0 such that for every metric $g \in W$ there exists a diffeomorphism $\phi \in U$ for which the tensor field $\phi^* g$ is solenoidal, i.e., $\delta^{g_0}(\phi^* g) = 0$.*

The metric g_0 defines in the usual way inner products on all spaces of tensor fields on M, in particular on $\Gamma(T^*M^{\odot 2})$; we denote the inner product on this space by $(.,.)_{L^2(T^*M^{\odot 2})}$. Then Proposition 4.1 of [5] states:

Proposition 5.2 ([5]). — *There is an $\epsilon > 0$ such that if $f \in C^0(T^*M^{\odot 2})$ satisfies $||f||_{C^0(T^*M^{\odot 2})} < \epsilon$ and $Vol_{g_0+f}(M) \leqslant Vol_{g_0}(M)$ then*

$$(g_0, f)_{L^2(T^*M^{\odot 2})} \leqslant \frac{2}{3}||f||^2_{L^2(T^*M^{\odot 2})}$$

The following lemma follows from the arguments in §6 of [5]:

Lemma 5.3 ([5]). — *Suppose M is nonpositively curved. There is a constant $C > 0$ and a quadratic first-order differential operator L on functions on $\partial T^1_+ M$ such that for any $f \in C^{2,\alpha}(T^*M^{\odot 2})$ such that $\delta^{g_0}(f) = 0$ and $f_{|\partial M} = 0$,*

$$||f||^2_{L^2(T^*M^{\odot 2})} \leqslant C \int_{\partial T^1_+ M} L(I_M(f)) d\Sigma^{2n-2}$$

where $d\Sigma^{2n-2}$ denotes the natural Liouville volume form on the manifold $\partial T^1_+ M$ induced by the metric g_0.

The arguments used to prove Lemma 6.1 of [5] yield the following:

Lemma 5.4 ([5]). — *There is a constant $C > 0$ such that the following holds:*

If $f \in C^{2,\alpha}(T^*M^{\odot 2})$ is such that $I_M(f) \geqslant 0$ on all of $\partial_+ T^1 M$, then

$$\int_{\partial T^1_+ M} L(I_M(f)) d\Sigma^{2n-2} \leqslant C||f||_{C^2}(g_0, f)_{L^2(T^*M^{\odot 2})}$$

We now have all the ingredients required to prove Theorem 1.2:

Proof of Theorem 1.2. — Let M be a closed g_0-ball containing the given compact K in its interior. Let g be a metric on X with sectional curvatures bounded above by -1 such that the support of $g - g_0$ is contained in K, $\hat{id}_{g_0,g}$ is Moebius and $Vol_g(K) = Vol_{g_0}(K)$. By Theorem 5.1, we may choose an $\epsilon > 0$ such that if $||g - g_0||_{C^{2,\alpha}(T^*M^{\odot 2})} < \epsilon$ then there is a diffeomorphism $\phi \in \mathrm{Diff}_0^{2,\alpha}(M)$ such that $\delta^{g_0}(\phi^* g) = 0$ on M. Extending ϕ to be the identity outside M, the same identity $\delta^{g_0}(\phi^* g) = 0$ holds on X. Let g_1 be the metric $\phi^* g$ on X, then $\hat{id}_{g_0,g_1}$ is also Moebius (since $\phi = id$ outside M), $Vol_{g_1}(M) = Vol_{g_0}(M)$, and the tensor $f := g_1 - g_0$ has support contained in M and vanishes identically on ∂M. It also follows from Theorem 5.1 that given $\epsilon' > 0$ we can always choose $\epsilon > 0$ small enough such that f satisfies $||f||_{C^{2,\alpha}(T^*M^{\odot 2})} < \epsilon'$. Thus choosing $\epsilon > 0$ small enough we may ensure that Proposition 5.2 applies to f, so that the hypothesis $Vol_{g_1}(M) = Vol_{g_0}(M)$ implies that

$$(g_0, f)_{L^2(T^*M^{\odot 2})} \leqslant \frac{2}{3} ||f||^2_{L^2(T^*M^{\odot 2})}$$

Since the support of f is contained in M, we have that for any $v \in \partial T^1_+ M$, if $\xi, \eta \in \partial_{g_0} X$ denote the endpoints of the bi-infinite g_0-geodesic with initial velocity v, then

$$I_M(f)(v) = I_{g_0}(f)(\xi, \eta) \geqslant 2S_{g_0}(g_1)(\xi, \eta) = 0$$

(using Lemma 3.9 and the fact that $\hat{id}_{g_0,g_1}$ is Moebius).

It now follows from Lemmas 5.3 and 5.4 that there are constants $C_1, C_2 > 0$ such that

$$\begin{aligned} ||f||^2_{L^2(T^*M^{\odot 2})} &\leqslant C_1 \int_{\partial T^1_+ M} L(I_M(f)) d\Sigma^{2n-2} \\ &\leqslant C_1 C_2 ||f||_{C^2(T^*M^{\odot 2})} (g_0, f)_{L^2(T^*M^{\odot 2})} \\ &\leqslant C_1 C_2 \epsilon \frac{2}{3} ||f||^2_{L^2(T^*M^{\odot 2})} \end{aligned}$$

thus choosing ϵ small enough so that $\epsilon \cdot C_1 C_2 \frac{2}{3} < 1$ implies that $||f||^2_{L^2(T^*M^{\odot 2})} = 0$, so $f = 0$ on M and hence on all of X, so $g_1 = g_0$, and g is isometric to g_0. □

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Manuscrit reçu le 11 janvier 2016,
accepté le 24 mars 2016.

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