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THE LIOUVILLE PROPERTY AND HILBERTIAN COMPRESSION

by Antoine GOURNAY

Abstract. — Lower bounds on the equivariant Hilbertian compression exponent α are obtained using random walks. More precisely, if the probability of return of the simple random walk is $\succeq \exp(-n^\gamma)$ in a Cayley graph then $\alpha \geqslant (1-\gamma)/(1+\gamma)$. This motivates the study of further relations between return probability, speed, entropy and volume growth. For example, if $|B_n| \preceq e^{n^\nu}$ then the speed exponent is $\leqslant 1/(2-\nu)$.

Under a strong assumption on the off-diagonal decay of the heat kernel, the lower bound on compression improves to $\alpha \geqslant 1-\gamma$. Using a result from Naor & Peres on compression and the speed of random walks, this yields very promising bounds on speed and implies the Liouville property if $\gamma < 1/2$.

Résumé. — Des bornes inférieures sur l'exposant de compression hilbertienne équivariante α sont données en utilisant les marches aléatoires. Plus précisément, si la probabilité de retour de la marche aléatoire est $\succeq \exp(-n^\gamma)$ pour un graphe de Cayley, alors $\alpha \geqslant (1-\gamma)/(1+\gamma)$. Ceci motive l'étude de relations supplémentaires entre la probabilité de retour, la vitesse, l'entropie et la croissance du volume. Par exemple, if $|B_n| \preceq e^{n^\nu}$, alors l'exposant de vitesse est $\leqslant 1/(2-\nu)$.

Avec une hypothèse plus forte sur le comportement du noyau de la chaleur hors de la diagonale, la borne inférieure sur la compression $\alpha \geqslant 1-\gamma$. Par un résultat de Naor et Peres sur la compression et la vitesse des marches aléatoires, ceci donne un estimé prometteur sur la vitesse et implique la propriété de Liouville si $\gamma < 1/2$.

1. Introduction

Throughout the text, G will be a finitely generated discrete group and it will be studied using its Cayley graph. The finite symmetric generating set S chosen to produce the Cayley graph will not be explicitly mentioned unless it is of importance; finiteness and symmetry will also always be assumed. P is the distribution of a lazy random walk. More precisely, it

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is obtained from a simple random walk distribution $P' = \mathbb{1}_S/|S|$ by $P = \frac{1}{2}(\delta_e + P')$. $P^{(n)}$ is the n^{th}-step distribution of the lazy random walk, i.e. the n^{th}-convolution of P with itself.

For further definitions, the reader should consult §2.

Theorem 1.1. — *If $P^{(n)}(e) \geqslant Le^{-Kn^\gamma}$ where $L, K > 0$ then the equivariant compression exponent of G, $\alpha(G)$, satisfies $\alpha(G) \geqslant (1-\gamma)/(1+\gamma)$.*

This improves a lower bound from Tessera [34, Proposition 15]: $\alpha(G) \geqslant (1-\gamma)/2$. The proof of Theorem 1.1 is contained in §3. Recall that return probability are stable under quasi-isometries between Cayley graphs (see Pittet & Saloff-Coste [30, Theorem 1.2]). There are many possible behaviours for γ, see Pittet & Saloff-Coste [31, Theorem 1.1].

The speed [or drift] of a random walk is defined as $\mathbb{E}|P^{(n)}| = \int |g| \mathsf{d}P^{(n)}(g)$ where $|g|$ is the word length of g (i.e. the graph distance in the Cayley graph between g and the identity element). The speed [or drift] exponent is $\beta = \sup\{c \in [0,1] \mid \text{there exists } K > 0 \text{ such that } \mathbb{E}|P^{(n)}| \geqslant Kn^c\}$. Surprisingly, there is little known on how much β depends on S.

Naor & Peres showed in [28, Theorem 1.1] that $\alpha(G) \leqslant 1/2\beta$. Since the map $n \mapsto \mathbb{E}|P^{(n)}|$ is sub-additive, the sequence $\mathbb{E}|P^{(n)}|/n$ always as a limit. Compression is a natural way to show that β is bounded away from 1 (for any generating set) and hence, that the afore-mentioned limit is 0. This is interesting because a group has the Liouville property if and only if $\mathbb{E}|P^{(n)}|$ is $\mathsf{o}(n)$.

However, the above result on compression only yields $\beta \leqslant (1+\gamma)/2(1-\gamma)$ which is non-trivial only if $\gamma < \frac{1}{3}$. Let B_n be the ball of radius n, i.e. $B_n = \{g \in G \mid |g| \leqslant n\}$. Recall that, if there are $K, L > 0$ so that

$$(1.1) \qquad \forall n, |B_n| \geqslant Ke^{Ln^v} \text{ then } \forall n, P^{(n)}(e) \leqslant K'e^{-L'n^c} \text{ with } c \leqslant \frac{v}{2+v},$$

for some $K', L' > 0$ (e.g. see [38, (14.5) Corollary]). Hence the bound on speed is not interesting from the point of view of the Liouville property: by (1.1), $\gamma < 1/3$ implies the group is of subexponential growth and so automatically Liouville. See §4 for more details. However, this bound motivates further investigations on possible relations between the various quantities in groups of intermediate growth.

Recall the entropy is defined by $H(P^{(n)}) := -\sum_{g \in G} P^{(n)}(g) \ln P^{(n)}(g)$.

Theorem 1.2. — *Assume G is so that $|B_n| \leqslant Le^{Kn^\nu}$ for some $K, L > 0$ and $\nu \in]0,1]$.*

(1) *Then $\mathbb{E}|P^{(n)}| \leqslant K'n^{1/(2-\nu)}$ (hence $\beta \leqslant 1/(2-\nu)$) and $H(P^{(n)}) \leqslant L'' + K''n^{\nu/(2-\nu)}$ for some $K', K'', L'' > 0$.*

(2) *$\alpha(G) \geqslant 1-\nu$.*
(3) *If $H(P^{(n)}) \geqslant K' + L'n^h$ for some $K', L' > 0$, then $\beta\nu \geqslant h$ and $2h\alpha \leqslant \nu$.*

For example, $\nu = \frac{k-1}{k}$ gives $\alpha \geqslant \frac{1}{k}$, $\mathbb{E}|P^{(n)}| \leqslant K'n^{k/(k+1)}$ and $H(P^{(n)}) \leqslant L'' + K''n^{(k-1)/(k+1)}$.

The bound on speed extends to measures with finite second moment and improves the $\frac{1+\nu}{2}$ bound from Erschler & Karlsson [21, Corollary 13]. The upper bound on entropy also holds for measures of finite second moment and implies a result of Coulhon, Grigor'yan & Pittet [16, Equation (7.5) in Corollary 7.4]. For finitely supported measures, these bounds are in Erschler [20, Lemma 5.1] (explicitly for the speed and implicitly in the proof for entropy).

The lower bound $\alpha(G) \geqslant 1-\nu$ is obtained as a corollary of Theorem 1.1 and of the estimate on return probabilities coming from volume growth: $P^{(n)}(e) \geqslant K''e^{L''n^{\nu/(2-\nu)}}$. This lower bound is already present in Tessera [34, Proposition 14] but comes here from a slightly different method.

For more discussions on the various exponents in groups, see §2.2 and §4.

The methods in the proof of Theorem 1.1 give a particularly interesting result if one makes a strong hypothesis on the off-diagonal behaviour of the heat kernel. The most natural estimate which is conjectural for most groups is: for some $M, N > 0$

$$P^{(n)}(g) \leqslant P^{(n)}(e)Ne^{-M|g|^2/n}. \tag{OD}$$

where $|g|$ is the word length of g (i.e. the distance between g and the identity in the Cayley graph). This estimate is true for groups of polynomial growth (and free groups) but there are no other groups where it is known to hold. Weaker forms are sufficient. Very recently, Brieussel & Zheng [12, Problem 9.3 and foregoing paragraphs] have used Theorem 1.3 below to give examples of groups where this estimate fails. See §2.3 for more on this topic.

Before stating the next result recall that $P^{(n)}(e) \geqslant Ke^{-Ln^\gamma}$ for $\gamma < 1$ implies the group is amenable (see Kesten [24]). Furthermore, a result known as "Gromov's trick" shows non-equivariant compression is equal to equivariant compression in amenable groups.

Theorem 1.3. — *Assume that $P^{(n)}(e) \geqslant Ke^{-Ln^\gamma}$ (with $\gamma < 1$) in some Cayley graph of G and* (OD) *holds in some [possibly different] Cayley graph of G. Then $\alpha(G') \geqslant 1-\gamma$ for any group G' with a Cayley graph quasi-isometric to that of G. Consequently, $\beta \leqslant \frac{1}{2(1-\gamma)}$ so that, if $\gamma < \frac{1}{2}$, the graph is Liouville.*

The bound obtained above is significantly more interesting; for example, if $\gamma = \frac{1}{3}$ it would yield $\beta \leqslant \frac{3}{4}$. Also, if $|B_n| \leqslant Ke^{Ln^\nu}$ it would yield, $\alpha(G) \geqslant \frac{1-\nu}{1-\nu/2}$. However the upper bound on the speed in Theorem 1.2 does not follow from Theorem 1.3 if the estimate on the probability of return is only given by volume growth.

Theorem 1.3, the discussion below and §5 motivated the author to make the following

CONJECTURE 1.4. — *If there are $K, L > 0$ so that $P^{(n)}(e) \geqslant Ke^{Ln^\gamma}$ in a Cayley graph of G then $\beta \leqslant 1/2(1-\gamma)$ in all Cayley graphs of G.*

This is now a theorem of Saloff-Coste & Zheng [33, Theorem 1.8] (their result is more precise than just an estimate on β and covers many measure P driving the random walk).

Sharpness of Theorems 1.1 and 1.3: Nothing indicates Theorem 1.1 is sharp. Sharpness of Theorem 1.3 (assuming the hypothesis is satisfied!) are discussed in detail in §5. In short, there are groups with $\gamma = 0, \frac{1}{3}, \frac{1}{2}$ or 1 for which, if (OD) were to hold, Theorem 1.3 is sharp (i.e. $\alpha = 1-\gamma$; also $\beta = 1/2(1-\gamma)$ if $\gamma \neq 1$). There are also groups with $\gamma = \frac{1}{3}, \frac{1}{2}$ or 1 where the conjectural bounds of Theorem 1.3 meet neither compression nor speed. Thus, it seems unlikely that there is a better estimate in terms of those quantities (see Question 4.5 for a possible improvement).

Bartholdi & Erschler [8, §1.2 and §7] showed that some groups of intermediate growth have arbitrarily bad compression, in particular $\alpha = 0$. Consequently, there are Liouville groups with arbitrarily quickly decaying return probability (hence return exponent $\gamma > 1/2$). Also, since growth is an invariant of quasi-isometry, the stability under quasi-isometry of the Liouville property is known for this class of groups.

On the other hand, recent work of M. Kotowski & Virág [26] shows there are groups with $-\ln P^{(n)}(e) \lesssim n^{1/2} + o(1)$ (the "error" being at most $\ln\ln n/\ln n$) which are not Liouville.

An interesting point of investigation would be to determine whether all groups with $P^{(n)}(e) \asymp e^{-n^{1/2}}$ are Liouville (or exhibit a counterexample).

Around Theorem 1.2: It is difficult to discuss the sharpness of Theorem 1.2 because the present construction of groups intermediate growth focus on controlling one parameter. These constructions often leave, in the meantime, the other parameters uncomputed (and hard to compute). It might, for this precise reason be even more interesting to have bounds between those quantities (see Amir [1] or Brieussel & Zheng [12] for recent developments). In fact, too good improvements of the bounds in Theorem 1.2 would lead to some forms of the gap conjecture on volume growth. This leads the author to believe that these are sharp.

Isoperimetry: How slowly must the Følner function of a group grow so that one can deduce that the group is Liouville? Theorem 1.3 hints at an answer using the link between the Følner function and return probability from Bendikov, Pittet & Sauer [10].

There are also descriptions in term of "adapted isoperimetry". For "Følner couples" the reader is referred to Coulhon, Grigor'yan & Pittet [16, Theorem 4.8]). For "controlled Følner sequences" (and its relation to compression) see Tessera [34, Corollary 13]. Of course, "adapted isoperimetry" mixes distances and isoperimetry, and are *a priori* not completely determined by the Følner function.

Amenability: The method presented in §3 is reminiscent of Bekka, Chérix & Valette [9]. To show amenable groups have the Haagerup property, they used $w_n = \mathbb{1}_{F_n}$ where F_n is a Følner sequence. See also Valette [36, Proposition 1 in §2].

Recently, M. Carette [13] showed that the Haagerup property is not an invariant of quasi-isometry; in [13, Appendix A], Arnt, Pillon & Valette use these same examples to show that the equivariant compression exponent is not an invariant of quasi-isometry.

Compression of Thompson's group F: It is straightforward to reread the paper of Naor & Peres [28] [and/or the current text] while keeping track of compression functions instead of taking only the exponent. Introduce $s^{-1}(k) = \inf\{k \in \mathbb{R} \mid \mathbb{E}|P^{(n)}| < k\}$. Under the (mild) assumption that ρ_- is sub-additive, then [ρ_- is up to multiplication by 2 concave and so] ρ_- is less (up to constants) than $k \mapsto (s^{-1}(k))^{1/2}$. Hence, a compression function strictly better than $n \mapsto Kn^{1/2}$ implies the Liouville property. As noted in [28] this improves a result of Guentner & Kaminker [22] (since the Liouville property implies amenability).

Here is an application of this remark. It seems known (see Kaimanovich in [23]) that Thompson's group F is not Liouville (this does not have any impact on its amenability). In the case of non-Liouville groups the sub-additivity [or concavity] hypothesis may be discarded (by using arguments from Austin, Naor & Peres [5]). This provides the answer to a question of Arzhantseva, Guba & Sapir [3, Question 1.4]: the best Hilbertian equivariant compression function for Thompson's group F is (up to constants) $\rho_-(x) \simeq x^{1/2}$.

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2. Definitions and preliminary results

Cayley graphs are defined by right-multiplication: x and y are neighbours if $\exists s \in S$ such that $xs = y$. Though common for the setting of random walks, this convention is slightly uncommon when one speaks of actions and convolutions.

The word length (for the implicit generating set S) of an element g will be noted $|g|$.

2.1. Compression

Definition 2.1. — *Let B be a Banach space and $\pi : G \to \mathrm{Isom} B$ be a representation of G in the isometries of B. An equivariant uniform embedding $f : G \to B$ is a map such that there exist an unbounded increasing function $\rho_- : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and a constant $C > 0$, satisfying $\forall x, y \in G$*

$$\rho_-(|y^{-1}x|) \leqslant \|f(x) - f(y)\| \leqslant C|y^{-1}x| + C,$$

and $f(\gamma x) = \pi(\gamma) f(x)$.
The function $\rho_- : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is called the compression function (associated to f). The [equivariant] compression exponent is $\alpha(f) = \sup\{c \in [0,1] \mid \exists K > 0$ such that $\rho_-(n) \geqslant K n^c\}$. The compression exponent of G, $\alpha(G)$, is the supremum over all $\alpha(f)$.

It follows easily from the definition that changing the generating set does not change α.

An equivariant uniform embedding is, in fact, very constrained. Indeed, one may (by translating everything) always put $f(e) = 0 \in B$ for simplicity. Next, recall that an [surjective] isometry of a [real] Banach space is always affine (Mazur–Ulam theorem). Write $\pi(y)v = \lambda(y)v + b(y)$ where λ is a map from G into the linear isometries of B and b is a map from G to B. Note that $f(y) = \pi(y)f(e) = \pi(y)0 = b(y)$. Furthermore $\pi(xy)v = \pi(x)\pi(y)v$ (for all $v \in B$) implies that λ is a homomorphism and b satisfies the cocycle relation:

$$b(xy) = \lambda(x)b(y) + b(x).$$

The strategy that will be used here to make an interesting equivariant uniform embedding (i.e. a λ-cocycle) is to use a "virtual coboundary". A coboundary would be a cocycle defined by

$$f(y) = \lambda(y)v - v$$

for some $v \in B$. The idea is to define such a cocycle using a v which does belongs to B but to some bigger space $\widetilde{B}$ (to which the action λ extends). Note that if $f(s)$ belongs to B for any s in the generating set S, then this also holds for $f(g)$ for any $g \in G$ (thanks to the cocycle relation).

Finally, a quick calculation (using that λ is isometric and writing g as a word) shows that cocycles always satisfy the upper bound required by equivariant uniform embedding. Also, it suffices to check that $\|f(g)\| \geqslant \rho_-(|g|)$:

$$\|b(gh) - b(g)\| = \|\lambda(g)b(h)\| = \|b(h)\|$$

This explains why §3 only discusses this lower bound.

2.2. Probabilistic parameters for groups

The entropy of a probability measure Q is $H(Q) = -\sum_{g \in G} Q(g) \ln Q(g)$ (when convergent). The group G is Liouville (for the [finite symmetric] generating set S) if any of the following equivalent conditions hold:

(i) There are no non-constant bounded harmonic functions on the Cayley graph;

(ii) $H(P^{(n)})$ is $\mathsf{o}(n)$;

(iii) $\mathbb{E}|P^{(n)}|$ is $\mathsf{o}(n)$.

(iii) $\implies$ (ii) can be obtained as in Lemma 4.2; see also Erschler [19, Lemma 6]. The implication (ii) $\implies$ (i) may be found in Avez [6]. For a complete (and more modern) picture see Erschler & Karlsson [21] and references therein.

Recall that $\mathbb{E}|P^{(n+m)}| \leqslant \mathbb{E}|P^{(n)}|+\mathbb{E}|P^{(m)}|$ and $H(P^{(n+m)}) \leqslant H(P^{(n)})+H(P^{(m)})$. Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be increasing, unbounded, $f(n+m) \leqslant f(n)+f(m)$ and $f(0)=0$. Recall that $\lim_{n\to\infty} f(n)/n$ exists. One can also define two exponents:

$$\begin{aligned}
\overline{\phi} &= \inf\{c \in [0,1] \mid \exists K>0, L\in\mathbb{R} \text{ such that } f(n) \leqslant L+Kn^c \text{ for all } n\} \\
&= \sup\{c \in [0,1] \mid \exists K>0, L\in\mathbb{R} \text{ such that } f(n) \geqslant L+Kn^c \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for infinitely many } n\} \\
&= \limsup_{n\to\infty} \frac{\ln f(n)}{\ln n} \\
\underline{\phi} &= \sup\{c \in [0,1] \mid \exists K>0, L\in\mathbb{R} \text{ such that } f(n) \geqslant L+Kn^c \text{ for all } n\} \\
&= \inf\{c \in [0,1] \mid \exists K>0, L\in\mathbb{R} \text{ such that } f(n) \leqslant L+Kn^c \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for infinitely many } n\} \\
&= \liminf_{n\to\infty} \frac{\ln f(n)}{\ln n}
\end{aligned}$$

The constant L is unnecessary. The exponents are obviously related by $\underline{\phi} \leqslant \overline{\phi}$. Note that if $\underline{\phi} < 1$ then $f(n)$ is $\mathsf{o}(n)$ (since f is sub-additive).

Definition 2.2. — *Let B_n be the ball of radius n. Define*

$$\begin{array}{ll}
\gamma = \overline{\gamma} = \overline{\phi} \text{ for } f(n) = -\ln P^{(n)}(e) & \underline{\gamma} = \underline{\phi} \text{ for } f(n) = -\ln P^{(n)}(e) \\
\nu = \overline{\nu} = \overline{\phi} \text{ for } f(n) = \ln|B_n| & \underline{\nu} = \underline{\phi} \text{ for } f(n) = \ln|B_n| \\
\eta = \overline{\eta} = \overline{\phi} \text{ for } f(n) = H(P^{(n)}) & \underline{\eta} = \underline{\phi} \text{ for } f(n) = H(P^{(n)}) \\
\overline{\beta} = \overline{\phi} \text{ for } f(n) = \mathbb{E}|P^{(n)}| & \beta = \underline{\beta} = \underline{\phi} \text{ for } f(n) = \mathbb{E}|P^{(n)}|
\end{array}$$

Simple bounds between these quantities are explored in §4.

2.3. Off-diagonal decay

An estimate which goes back to Carne [14] and Varopoulos [37] on the "off-diagonal" behaviour of random walks is, for some $M, N > 0$,

$$P^{(n)}(g) \leqslant Ne^{-M|g|^2/n}. \tag{2.1}$$

Improvements of this theorem are known. For example, under a regularity hypothesis, there is a similar estimate due to Coulhon, Grigor'yan & Zucca, see [17, Theorem 5.2] but it concerns the ration $P^{(kn)}(g)/P^{(n)}(e)$ for some $k \geqslant 2$. When the group is of polynomial growth this actually implies (OD).

It seems challenging to produce groups which violate the off-diagonal estimate from (OD). As pointed out in Dungey [18, End of §1], an interpolation argument shows this estimate is close to be true in all groups. More precisely: there are constant $M, N > 0$ so that for any $\epsilon \in [0,1]$,

$$P^{(n)}(g) \leqslant P^{(n)}(e)^{1-\epsilon} N^{\epsilon} e^{-\epsilon M|g|^2/n}. \tag{2.2}$$

Note that the estimate (OD) is not the only estimate which would suffice for the proof of Theorem 1.3. The first obvious relaxation would be to have this estimate for $n < L|g|^{2-\epsilon}$ (for any $\epsilon > 0$ with $L = L(\epsilon)$). The following condition would be also sufficient for the proof: for any $\epsilon > 0$, there exists n_0, K, L such that for any $n > L|g|^{2+\epsilon}$ and $|g| > n_0$, one has $\frac{P^{(2n)}(g)}{P^{(2n)}(e)} \leqslant e^{-K|g|^2/n}$. Of course, any estimate with a fixed ϵ could also be of interest.

Recently, Brieussel & Zheng [12, Problem 9.3 and foregoing paragraphs] have shown that there are groups for which the conclusion of Theorem 1.3 cannot hold. They give a family of groups for which $\alpha < 1-\gamma$ (see Brieussel & Zheng [12, First line of Table 1]: $\alpha = 1/(1+\theta)$ while $\gamma = (1+\theta)/(3+\theta)$ and take $\theta > 1$). This implies that these groups violate (OD) (and its relaxations). Furthermore, as they also provide an estimate on the speed and entropy exponents (both equal $(1+\theta)/(2+\theta)$), so that these groups are Liouville. B. Virág pointed out to the author that the lamplighter on $\mathbb{Z}^3$ might also violate (OD) (by fine estimates on the return probability). Interestingly, all these groups have return exponent $> 1/2$.

3. A lower bound using random walks

The idea will be to construct an equivariant uniform embedding of G into $\mathcal{H} := \oplus_{n\in\mathbb{N}} \ell^2 G$. The isometric action is simply the diagonal action of G on each factor by the right-regular representation. The idea is to define a cocycle using a virtual coboundary of the form $w = (a_n w_n)_{n\in\mathbb{N}} \in \oplus_{n\in\mathbb{N}} \ell^2 G$ where $w_n \in \ell^2 G$ and $a_n \in \mathbb{R}$. This yields a cocycle (in $\rho^{\mathbb{N}}_{\ell^2 G}$) if, for any $s \in S$,

$$\|w - \rho_s^{\mathbb{N}} w\|_2^2 = \sum_n a_n^2 \|w_n - \rho_s w_n\|_2^2 < +\infty.$$

Simply put $a_n^2 = \big(\max_{s\in S} \|w_n - \rho_s w_n\|_2\big)^{-2} n^{-1-\epsilon}$, where $\epsilon > 0$. The gradient of a function $f : G \to \mathbb{R}$ is defined by $\nabla f(x,y) = f(y) - f(x)$ for two adjacent vertices x, y in the Cayley graph. This operator is essentially build up by the various $f-\rho_s f$, and ∇f can be interpreted as a function $G\times S\to\mathbb{R}$. The gradient is a bounded operator (since S is finite) and its adjoint ∇^* can be used to form the Laplacian Δ. These are related to P by $\Delta=\nabla^*\nabla=|S|(I-P')=2|S|(I-P)$.

As mentioned in §2.1, it now remains to find a lower bound for the norm of $b(g)=\rho^{\mathbb{N}}(g)w-w$. Using that $\max_{s\in S}\|w_n-\rho_s w_n\|_2^2\leqslant\sum_{s\in S}\|w_n-\rho_s w_n\|^2=\|\nabla w_n\|_2^2$, one has

$$\|b(g)\|_2^2\geqslant\sum_{n\geqslant 1} n^{-1-\epsilon}\frac{\|w_n-\rho_g w_n\|_2^2}{\|\nabla w_n\|_2^2}$$

The idea will be to take $w_n=P^{(k_n)}$ for some $k_n\in[n,2n]$ (the k_n^{th}-step distribution of a lazy random walk starting at $e\in G$).

Lemma 3.1. — $\|P^{(n)}-\rho_{g^{-1}}P^{(n)}\|_2^2=2\big(P^{(2n)}(e)-P^{(2n)}(g)\big)$.

Proof. — Indeed,

$$\begin{aligned}\|P^{(n)}-\rho_g P^{(n)}\|_2^2&=\langle P^{(n)}-\rho_g P^{(n)}\mid P^{(n)}-\rho_g P^{(n)}\rangle\\&=2\|P^{(n)}\|^2-2\langle P^{(n)}\mid\rho_g P^{(n)}\rangle\end{aligned}$$

Since S is symmetric, note that $\langle f\mid P*g\rangle=\langle P*f\mid g\rangle$. Consequently,

$$\langle P^{(n)}\mid\rho_g P^{(n)}\rangle=\langle P^{(n)}\mid P^{(n)}*\delta_{g^{-1}}\rangle=\langle P^{(2n)}\mid\delta_{g^{-1}}\rangle=P^{(2n)}(g^{-1}).$$

To get the claimed equality, use that, similarly, $P^{(2n)}(e)=\|P^{(n)}\|_2^2$ and replace g by g^{-1}. □

Lemma 3.2. — $\|\nabla P^{(n)}\|_2^2=2|S|\big(P^{(2n)}(e)-P^{(2n+1)}(e)\big)$

Proof. — This is a simple calculation using the relation $\Delta=\nabla^*\nabla=2|S|(I-P)$:

$$\|\nabla P^{(n)}\|_2^2=\langle\Delta P^{(n)}\mid P^{(n)}\rangle=2|S|\big(P^{(2n)}(e)-P^{(2n+1)}(e)\big).\qquad\square$$

Putting Lemmas 3.1 and 3.2 together gives:

$$|S|\,\frac{\|P^{(n)}-\rho_g P^{(n)}\|^2}{\|\nabla P^{(n)}\|^2}=\frac{P^{(2n)}(e)-P^{(2n)}(g)}{P^{(2n)}(e)-P^{(2n+1)}(e)}=\frac{1-P^{(2n)}(g)/P^{(2n)}(e)}{1-P^{(2n+1)}(e)/P^{(2n)}(e)}$$

The next step is to find satisfying bounds for this quantity. There are reasonable estimates for the denominator, the following lemma is essentially from Tessera [34, Proof of Proposition 7.2]. For similar estimates on the entropy, see Erschler & Karlsson [21, Lemma 10] (see also Remark 3.5 below).

Lemma 3.3. — *If $P^{(n)}(e) \geqslant e^{-f_P(n)}$ for a positive sub-additive increasing function f_P. Then, for any n there is a $k \in [n, 2n]$*

$$\left(1 - \frac{P^{(2k+1)}(e)}{P^{(2k)}(e)}\right) \leqslant 8 f_P(n)/n.$$

Proof. — For readability, start with $F(n) = -\ln P^{(n)}(e)$. Let C_n be the largest real number such that, for any $q \in [n, 2n]$.

$$F(q+1) - F(q) \geqslant C_n f_P(n)/n.$$

This implies $F(2n) - F(n) \geqslant C_n f_P(n)$, and in particular $F(2n) \geqslant C_n f_P(n)$ (since $F(n) \geqslant 0$). By hypothesis, $F(2n) \leqslant f_P(2n) \leqslant 2 f_P(n)$ so that $C_n \leqslant 2$. Thus, for any n, there exists a $k \in [n, 2n]$ such that $F(k+1) - F(k) \leqslant 2 f_P(n)/n$. This implies

$$1 - \frac{P^{(k+1)}(e)}{P^{(k)}(e)} \leqslant 1 - e^{-2 f_P(n)/n} \leqslant \frac{2 f_P(n)}{n},$$

where the last inequality comes from $1 - e^{-x} \leqslant x$ for $x \geqslant 0$.

The actual statement is obtained by doing the same argument with $G(n) = F(2n)$ and noticing that an additional constant comes in since one then looks at the gradient defined for the generating set $S' = S^2$. □

Proof of Theorems 1.1 and 1.3. — By hypothesis, $P^{(n)}(e) \geqslant K e^{-Ln^{\gamma}}$ so that, in Lemma 3.3, $f_P(n) = -\ln K + Ln^{\gamma} \leqslant 2 \max(-\ln K, L) n^{\gamma}$. Using $w_n = P^{(k_n)}$ where $k_n \in [n, 2n]$ is given by Lemma 3.3 and the bound mentioned above for the numerator, one finds (using $1 \leqslant \frac{k_n}{n} \leqslant 2$) and $K'' = \big(16 \max(-\ln K, L)|S|\big)^{-1}$

$$\|b(g^{-1})\|_2^2 \geqslant \sum_{n \geqslant 1} K'' n^{-\gamma-\epsilon} (1 - P^{(2n)}(g)/P^{(2n)}(e))$$

Since $|g^{-1}| = |g|$, the question boils down to showing for which n one has, $\frac{P^{(2n)}(g)}{P^{(2n)}(e)} \leqslant 1/2$.

For example, assuming the estimate (OD) holds, one sees this is true for $n \leqslant M|g|^2/\ln(2N)$ (since, necessarily, $N \geqslant 1$). Hence, restricting the sum to those values of n:

$$\|b(g)\|_2^2 \geqslant \sum_{n \leqslant M'|g|^2/\ln(2N)} \tfrac{K''}{2} n^{-\gamma-\epsilon} \geqslant \widetilde{K} |g|^{2(1-\gamma-\epsilon)}.$$

where the second inequality can be obtained from the Euler–Maclaurin approximation method and $\widetilde{K} = K''(M/\ln(2N))^{1-\gamma-\epsilon}/4(1-\gamma-\epsilon)$. Letting $\epsilon \to 0$ proves Theorem 1.3 (even though the constant gets worse as $\epsilon \to 0$).

Using (2.1) instead of (OD), one must restrict the sum to $n < K'|g|^{2/(1+\gamma)}$ where $K' = \frac{1}{4L}\big(\sqrt{(\ln(2N/K))^2 + 4LM} - \ln(2N/K)\big)$. This yields a weaker lower bound of $\alpha \geqslant \frac{1-\gamma}{1+\gamma}$ (but is true in any group) and proves Theorem 1.1. □

If the reader is interested in compression functions (rather than exponents), then it is fairly easy to check that, given f_P as in Lemma 3.3, $\rho_-(k) \geqslant k^{1/(1+\gamma)}/f_P(k^{2/(1+\gamma)})^{1/2}$ and, if (OD) holds, $\geqslant k/f_P(k^2)^{1/2}$.

Remark 3.4. — It would be interesting to generalise this proof by picking v_n elements which are in V_{λ_n} with $\lambda_n \to 0$, where V_λ is the image of the spectral projection (of the Laplacian) to eigenvalues $\leqslant \lambda$. This would ensure a good bound for the denominator. For the numerator, one needs to elucidate how to relate bound on the von Neumann dimension of V_λ to upper estimates on $\langle v \mid \rho_g v\rangle$ for $v \in V_\lambda$.

More precisely, if $\lambda_n = 1/n$ and $v_n \in V_{1/n}$ then one would require

- either, for some $K > 0$, $\langle v_n \mid \rho_g v_n\rangle \leqslant 1/2$ when $|g|^{2-2\gamma} > Kn$;
- or, for some $K, K' > 0$ and $\epsilon > 0$, $\langle v_n \mid \rho_g v_n\rangle \leqslant \exp(-K|g|^{2-2\gamma}/n)$ when $n > K'|g|^{2-2\gamma+\epsilon}$.

Using the results of Bendikov,Pittet & Sauer [10], note that $P^{(n)}(e) \succcurlyeq \exp(-n^\gamma)$ (near infinity) corresponds to the fact that the von Neumann dimension of $V_\lambda \succcurlyeq \exp(-\lambda^{\gamma/(1-\gamma)}$ (near zero).

Remark 3.5. — There is an alternative proof of Lemma 3.3 along the lines of Erschler & Karlsson [21, Lemma 10]. Let $F(n) = -\ln P^{(2n)}(e)$. Then it is well-known that $F(n+1) - F(n)$ is decreasing, see Woess' book [38, (10.1) Lemma].

4. Some relations between the exponents

The aim of this section is to relate the return, speed, entropy and growth exponents. An elementary computation (see Avez [7, Theorem 3]) shows, using concavity of ln, that

$$H(P^{(n)}) \geqslant -\ln\Big(\sum_{g\in G} P^{(n)}(g)^2\Big) = -\ln \|P^{(n)}\|_2^2 = -\ln P^{(2n)}(e). \tag{4.1}$$

Hence, $\underline{\gamma} \leqslant \underline{\eta}$ and $\overline{\gamma} \leqslant \overline{\eta}$. (With Kesten's criterion [24], this shows Liouville $\Longrightarrow$ amenable.)

(2.1) gives $P^{(n)}(g) \leqslant Ne^{-M|g|^2/n}$. This, together with convexity of $x \mapsto x^2$, gives another useful bound, found in either Amir & Virág [2, Proposition 8] or Erschler [19, Lemma 7.(i)]:

$$H(P^{(n)}) \geqslant \ln N + M \sum_{g\in G} P^{(n)}(g)\tfrac{|g|^2}{n} \geqslant \ln N + \tfrac{M}{n}(\mathbb{E}|P^{(n)}|)^2. \tag{4.2}$$

Thanks to Erschler & Karlsson [21, Corollary 9.ii], this inequality is also true for measures with finite second moment. This implies that $\underline{\beta} \leqslant \frac{1+\underline{\eta}}{2}$ and $\overline{\beta} \leqslant \frac{1+\overline{\eta}}{2}$ and constitutes a proof of (ii) $\Longrightarrow$ (iii) in the equivalences of the Liouville property described in §2.2.

There is also "classical" bound obtained by Varopoulos' method (see e.g. Woess' book [38, (14.5) Corollary]) relating growth and return exponent: $\underline{\gamma} \geqslant \frac{\underline{\nu}}{2+\underline{\nu}}$.

The following lemma (see e.g. [11, §1.2]) will be useful.

Lemma 4.1. — *Let $f : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be a sub-additive, non-decreasing function with $f(0) = 0$. If g is the concave hull of f then $f(x) \leqslant g(x) \leqslant 2f(x)$.*

The upcoming lemma is an improvement of a standard inequality (see e.g. Erschler [19, Lemma 6]) and of the simple inequality $H(P^{(n)}) \leqslant \ln |B_n|$ (see Erschler & Karlsson [21, Lemma 1]). For finitely supported measures and volume growth $|B_n| \leqslant K \exp(Ln^{\nu})$, it is implicit in Erschler [20, Lemma 5.1].

Since it might be of larger use, it will be stated in full generality, namely P will be some measure and S^* some finite (symmetric) generating set.

Lemma 4.2. — *Let $|g|_*$ be the word length for S^*. Assume P has finite first moment (i.e. $\sum_{g \in G} P(g)|g|_* < +\infty$), and $B_n = \{g \in G \mid |g|_* \leqslant n\}$. Let $|B_n| = e^{f_V(n)}$ and assume $|B_n|$ is at least quadratic in n. Then*

$$H(P^{(n)}) \leqslant L + 4 f_V(\mathbb{E}|P^{(n)}|_*).$$

In particular, $\underline{\beta}\overline{\nu} \geqslant \underline{\eta}$, $\overline{\beta}\underline{\nu} \geqslant \underline{\eta}$ and $\overline{\beta}\overline{\nu} \geqslant \overline{\eta}$.

Proof. — The idea is to compare a measure m to a measure m' which is uniform on spheres. First,

$$H(m) - \sum_{g \in G} m(g) \ln(\frac{1}{m'(g)}) = \sum_{g \in G} m(g) \Big(-\ln \frac{m(g)}{m'(g)} \Big) \leqslant 0$$

using $-\ln t \leqslant \frac{1}{t} - 1$. Now let $a_i = |\delta B_i|$ where $\delta B_i = B_i \setminus B_{i-1}$ and $B_{-1} = \emptyset$ and $m'(g) = \phi(|g|_*)/a_{|g|_*}$ where $\phi(k) = L_1 |B_k|^{-1}$ and L_1 chosen so that $\sum_{k \geqslant 0} \phi(k) = 1$. Then,

$$H(m) \leqslant \sum_{g \in G} m(g) \big(\ln a_{|g|_*} - \ln \phi(|g|_*) \big).$$

Then, one has (with $L' = \ln(L_1)$)

$$H(m) \leqslant L' + 2 \sum_{g \in G} m(g) f_V(|g|_*) \leqslant L' + 4 f_V \Big(\sum_{g \in G} m(g) |g|_* \Big)$$

by passing to the concave hull of f_V and using Lemma 4.1 to bound this by $2f_V$. This shows $H(P^{(n)}) \leqslant L' + 4f_V(\mathbb{E}|P^{(n)}|_*)$, as desired.

The bound $\overline{\eta} \leqslant \overline{\beta}\overline{\nu}$ follows directly while the others follow by applying the inequality for infinitely many n. □

If one assumes $|B_n| \leqslant Le^{Kn^\nu}$, one can also obtain the statement $H(P^{(n)}) \leqslant L' + K'(\mathbb{E}|P^{(n)}|_*)^\nu$ with K' as close as desired to K, as in Erschler & Karlsson [21, Lemma 1].

Corollary 4.3. — *Assume $|B_n| \leqslant Le^{Kn^\nu}$ and $|B_n|$ is more than quadratic. For any measure of finite second moment (i.e. $\sum_{g\in G} P(g)|g|^2 < +\infty$), one has:*

- $\mathbb{E}|P^{(n)}| \leqslant K'n^{1/(2-\nu)}$,
- $H(P^{(n)}) \leqslant L'' + K''n^{\nu/(2-\nu)}$,
- *and* $P^{(2n)}(e) \geqslant \exp\big(-H(P^{(n)})\big) \geqslant L''\exp(-K''n^{\nu/2-\nu})$.

In particular,

$$\underline{\beta} \leqslant \overline{\beta} \leqslant \tfrac{1}{2-\overline{\nu}} \quad \text{and} \quad \overline{\gamma} \leqslant \overline{\eta} \leqslant \overline{\beta}\overline{\nu} \leqslant \frac{\overline{\nu}}{2-\overline{\nu}}.$$

Proof. — Using first (4.2) (which extends to measures of finite second moment by Erschler & Karlsson [21, Corollary 9.ii]) then Lemma 4.2, one has $(\mathbb{E}|P^{(n)}|)^2 \leqslant n\big(\tilde{L} + 4(\ln K)(\mathbb{E}|P^{(n)}|)^\nu\big)$. Putting $K' = \big(4\ln K + \tilde{L}/\mathbb{E}|P^{(1)}|^\nu\big)^{1/(2-\nu)}$, this implies the first claim. The second claim is obtained by concatenating Lemma 4.2 and the bound on speed just obtained. The relation (4.1) is also used in the sequence of inequalities in term of exponents. □

Lemma 4.2 and Corollary 4.3 finish the proof of Theorem 1.2.

Let us mention an additional inequality. This inequality is already present in Coulhon & Grigoryan [15, §6] in a sharper form but with extra hypothesis. The proof presented here is elementary if one knows (2.2) and could be improved in the case of polynomial growth (though it does not meet [15]).

Lemma 4.4. — *Assume $|B_n| = e^{f_V(n)}$ is at least cubic. Let f be the concave hull of f_V, and F the inverse function of [the strictly increasing function] $k \mapsto k^2/f(k)$. Then $P^{(n)}(e) \geqslant K''|B_{F(L''n)}|^{-2}F(L''n)^{-1}$ for some $K'', L'' > 0$.*

Proof. — Write $|B_n| = e^{f_V(n)}$ as before.Then, using the bound (2.2) one has, for any $\epsilon \in]0,1[$,

$$1 = \sum_{g \in G} P^{(n)}(g) \leqslant \sum_{k=0}^{n} |B_k| P^{(n)}(e)^{1-\epsilon} N^\epsilon e^{-M\epsilon k^2/n} \leqslant P^{(n)}(e)^{1-\epsilon} \sum_{k=0}^{n} N^\epsilon e^{f(k)-M\epsilon k^2/n},$$

where f is the concave hull of f_V. Let $n_0 = \inf\{k \mid k^2/f(k) \geqslant n/M\epsilon\}$. Note that $k \mapsto k^2/f(k)$ is strictly increasing. Indeed, since f is concave and $f(0) = 0$ one has $f(n) = \sum_{i=1}^n f(i) - f(i-1) \geqslant n\big(f(n) - f(n-1)\big)$. That $(k+1)^2/f(k+1) > k^2/f(k)$ then follows from:

$$k^2\big(f(k+1) - f(k)\big) \leqslant \tfrac{k^2 f(k+1)}{k+1} < k\big(f(k) + f(1)\big) \leqslant 2kf(k) < (2k+1)f(k).$$

Hence, the exponent of the exponential is negative if $k \geqslant n_0$. Since $P^{(n)}(e)^{1-\epsilon} n \to 0$ for some $\epsilon \in]0,1[$ (because $|B_n| \geqslant Kn^3$ implies $P^{(n)}(e) \leqslant K' n^{-3/2}$), one may write (with $\delta_n \to 0$ as $n \to \infty$)

$$1 - \delta_n \leqslant P^{(n)}(e)^{1-\epsilon} \sum_{k=0}^{n_0} N^\epsilon e^{f(k)-M\epsilon k^2/n} \leqslant P^{(n)}(e)^{1-\epsilon} \sum_{k=0}^{n_0} N^\epsilon e^{f(k)} \leqslant n_0 P^{(n)}(e)^{1-\epsilon} e^{f(n_0)}.$$

This implies that $P^{(n)}(e) \geqslant K'' e^{-f(n_0)} n_0^{-1}$. To conclude apply Lemma 4.1: $f(x) \leqslant 2f_V(x)$. □

The preceding lemma implies $\overline{\gamma} \leqslant \overline{\nu}/(2-\overline{\nu})$ and $\underline{\gamma} \leqslant \underline{\nu}/(2-\overline{\nu})$, but these inequalities already follows for a larger class of measures from (4.1) and Corollary 4.3. One cannot deduce $\underline{\gamma} \leqslant \underline{\nu}/(2-\underline{\nu})$ from Lemma 4.4.

Lastly, the estimate $\overline{\gamma} \geqslant \frac{\overline{\nu}}{2+\overline{\nu}}$ can be deduced from Coulhon, Grigoryan & Pittet [16, Corollary 7.2]. The estimates cited or proved in this paper can also be summed up by:

$$\beta \overset{ii}{\leqslant} \frac{1+\eta}{2}, \quad \frac{\underline{\nu}}{2+\underline{\nu}} \leqslant \underline{\gamma} \overset{i}{\leqslant} \underline{\eta} \overset{i}{\leqslant} \min(\underline{\beta}\overline{\nu}, \overline{\beta}\underline{\nu}) \overset{ii}{\leqslant} \frac{\underline{\nu}}{2-\overline{\nu}}$$

$$\text{and} \quad \frac{\overline{\nu}}{2+\overline{\nu}} \leqslant \overline{\gamma} \overset{i}{\leqslant} \overline{\eta} \overset{i}{\leqslant} \overline{\beta}\overline{\nu} \overset{ii}{\leqslant} \frac{\overline{\nu}}{2-\overline{\nu}}$$

where i (resp. ii) denotes inequality which hold for measures with finite first (resp. second) moment, the remaining inequalities hold only for finitely supported measures and the absence of bars [above or below] the exponent mean it holds if bars are put on both sides at the same place.

The lower bound $\underline{\beta} \geqslant \underline{\nu}/\overline{\nu}(2+\underline{\nu})$ is not optimal (B. Virág gave a [sharp] lower bound of $\frac{1}{2}$; see Lee & Peres [27]).

Other inequalities which could be interesting to explore are: $\underline{\eta} \leqslant \frac{\nu}{2-\underline{\nu}}$? $\underline{\gamma} \leqslant \frac{\nu}{2-\underline{\nu}}$? A more interesting one (since a positive answer combined with (4.2) gives a proof of Conjecture 1.4) is

Question 4.5. — *Does the inequality $\underline{\eta} \leqslant \frac{\overline{\gamma}}{1-\overline{\gamma}}$ hold? Could it even hold for all measures with finite second moment?*

This has been answered in the positive by Saloff-Coste & Zheng [33, Theorem 1.8].

Let us conclude with this possibly well-known lemma.

Lemma 4.6. — *Assume $\psi : G \twoheadrightarrow H$ is a surjective homomorphism. Let $S = \mathrm{Supp} P$ be generating for G (hence $\psi(S)$ generates H). Let $P' = \psi^* P$, i.e. $P'(A) = P\big(\psi^{-1}A\big)$. Then $\mathbb{E}|P^{(n)}_{e_G}| \geqslant \mathbb{E}|P'^n_{e_H}|$ (where the word lengths $|\cdot|$ are for S and $\psi(S)$ respectively).*

Proof. — Let d_H be the distance of the Cayley graph with respect to $S_H =$ support of P'. Define the function $d' : G \to \mathbb{N}$ by $d'(\gamma) = d_H\big(\psi(\gamma), e_H\big)$. Note that $d'(\gamma) \leqslant d_G(\gamma, e)$: indeed $d_H(h_1, h_2) = d_G\big(\psi^{-1}(h_1), \psi^{-1}(h_2)\big)$, so that $d'(\gamma) = d_G(\gamma N, N)$ where $N = \ker \psi$. Let W^G_n be the random walker on G and W^H_n be the random walker on H (which moves according to P' as in the statement). Note that $\mathbb{P}\big(d_H(W^H_n, e_H) = i\big) = \mathbb{P}\big(d'(W^G_n) = i\big)$. This implies

$$\mathbb{E}|P'^n| = \mathbb{E}\big(d_H(W^H_n, e_H)\big) = \mathbb{E}\big(d'(W^G_n)\big) \leqslant \mathbb{E}\big(d_G(W^G_n, e_G)\big) = \mathbb{E}|P^{(n)}| \quad \square$$

In particular, this proves that $\mathbb{E}|P^{(n)}| \geqslant K_P n^{1/2}$ for any G with a non-trivial homomorphism to $\mathbb{Z}$ (this is true for any group, due to Virág, see [27]).

The statement of Lemma 4.6 may be generalised to coverings of graphs and more general maps. Here is a classical example. Define "levels" in the k-regular tree by looking at points which are at the same distance to some [fixed] point at infinity. The "level maps" gives a morphism from the tree to the line $\mathbb{Z}$. The arguments of the above Lemma apply to this map, but yield a biased random walk on $\mathbb{Z}$. This gives a rather precise estimate of the speed.

5. Some known values

Below is a table containing cases where $\alpha, \underline{\beta}$ and $\overline{\gamma}$ are known. The convention for wreath products $L \wr H$ is that L is the "lamp state" group, e.g. $\mathbb{Z}_2 \wr \mathbb{Z}$ is the usual lamplighter on the line. One could complete the table for many other wreath products using Naor & Peres [28, Theorem 6.1], Naor & Peres [29, Theorem 3.1], Pittet & Saloff-Coste [31, Theorem 3.11 and Remark (ii) after Theorem 3.15] and Revelle [32, Theorem 1].

The lower bound of Theorem 1.3 [assuming (OD) holds] meets compression in (A), (C), (D) if $d = 2$, (E) if H has polynomial growth, (H) and (I). It also meets speed, except in the last two cases. The lower bound meets neither speed nor compression in (B), (E) if H is polycyclic [since $\overline{\gamma} = \frac{3}{5}$] and (F) if $k \geqslant 3$. All the groups mentioned that have $\overline{\gamma} = \frac{1}{2}$ are Liouville.

Except in (H) and (I), the upper bound $\alpha \leqslant 1/2\underline{\beta}$ of Naor & Peres [28] meets compression. "Incompressible" (i.e. of compression exponent 0) amenable groups were first constructed by Austin (a solvable group, see [4]) and, more recently, Bartholdi & Erschler [8, §1.2 and §7]. It seems reasonable to believe there is an amenable group where the compression meets neither the upper bound of [28] nor the lower bound of Theorem 1.3 [assuming (OD) holds].

Group	$\underline{\beta}$	$\overline{\gamma}$	$1-\overline{\gamma}$	α	$1/2\underline{\beta}$
A:Polynomial growth	$\frac{1}{2}^{(1)}$	$0^{(7)}$	1	$1^{(2)}$	1
B:Polycyclic of exponential growth or $F\wr\mathbb{Z}$ with F finite	$\frac{1}{2}^{(2)}$	$\frac{1}{3}^{(8)}$	$\frac{2}{3}$	$1^{(2)}$	1
C:$\mathbb{Z}\wr\mathbb{Z}$	$\frac{3}{4}^{(3)}$	$\frac{1}{3}^{(9)}$	$\frac{2}{3}$	$\frac{2}{3}^{(11)}$	$\frac{2}{3}$
D:$F\wr H$ with F finite or $\mathbb{Z}$ and H polynomial growth of degree $d\geqslant 2$	$1^{(4)}$	$\frac{d}{d+2}^{(9)}$	$\frac{2}{d+2}$	$\frac{1}{2}^{(12)}$	$\frac{1}{2}$
E:$H\wr\mathbb{Z}^2$ with H amenable and $\alpha(H)\geqslant\frac{1}{2}$	$1^{(4)}$	$\geqslant\frac{1}{2}^{(9)}$	$\leqslant\frac{1}{2}$	$\frac{1}{2}^{(13)}$	$\frac{1}{2}$
F:$(\dots((\mathbb{Z}\wr\mathbb{Z})\wr\mathbb{Z})\dots)\wr\mathbb{Z}$ iterated wreath product with k "$\mathbb{Z}$", $k\geqslant 1$	$1-\frac{1}{2^k}^{(3)}$	$\frac{k-1}{k+1}^{(9)}$	$\frac{2}{k+1}$	$\frac{1}{2-2^{1-k}}^{(11)}$	$\frac{1}{2-2^{1-k}}$
G:Intermediate growth $e^{n^v}\preceq\lvert S^n\rvert\preceq e^{n^\nu}$	$[\frac{1}{2},\frac{1}{2-\nu}]^{(5)}$	$[\frac{v}{2+v},\frac{\nu}{2-\nu}]^{(10)}$	$[\frac{1-\nu}{1-\nu/2},\frac{1}{1+v/2}]$	?	$[1-\frac{\nu}{2},1]$
H:"Incompressible" amenable groups	?	1	0	$0^{(14)}$	$\geqslant\frac{1}{2}$
I:Property (T) groups	$1^{(6)}$	$1^{(6)}$	0	$0^{(6)}$	$\frac{1}{2}$

Table's references

(1) The upper bound is classical; see §4. The (general) matching lower bound is due to Virág (see Lee & Peres [27]); this particular instance could be obtained by arguments of §4.
(2) The value of compression (from Tessera [34, Theorems 9 and 10]) imply the value of speed. For finer estimates on speed see Thompson [35, Theorem 1].
(3) This may be found either in Erschler [19, Thm. 1] or Revelle [32, Thm. 1].
(4) See Erschler [19, Theorem 1] or Naor & Peres [28, Theorem 6.1].
(5) The upper bound is easy; see §4. The lower bound is the general one due to Virág, see the introduction of Lee & Peres [27].
(6) Kesten's criterion for amenability [24] shows $\gamma = 1$, use Kesten [25, Theorem 5] or Lemma 4.2 to get $\beta = 1$. Property (T) groups do not have the Haagerup property. In particular, they have no proper affine action on a Hilbert space; hence $\alpha = 0$.
(7) 0 should be interpreted as arbitrarily small. This is the classical estimate of Varopoulos, see Woess' book [38, (14.5) Corollary].
(8) Due to Varopoulos; see [31, §1.1] for a list of possible references.
(9) See Pittet & Saloff-Coste [31, Theorems 3.11 and 3.15]
(10) For the lower bound see Woess' book [38, (14.5) Corollary]. The upper bound is Coulhon, Grigor'yan & Pittet [16, Corollary 7.4]; see also §4 of the present text.
(11) See Naor & Peres [28, Corollary 1.3].
(12) See Naor & Peres [29, Theorem 3.1].
(13) See Naor & Peres [28, Remark 3.4].
(14) See Austin [4] or Bartholdi & Erschler [8, §1.2 and §7]. $\alpha=0$ implies $\gamma=1$.

BIBLIOGRAPHY

[1] G. Amir, "On the joint behaviour of speed and entropy of random walks on groups", http://arxiv.org/abs/1509.00256.

[2] G. Amir & B. Virág, "Speed exponents for random walks on groups", http://arxiv.org/abs/1203.6226.

[3] G. N. Arzhantseva, V. S. Guba & M. V. Sapir, "Metrics on diagram groups and uniform embeddings in a Hilbert space", *Comment. Math. Helv.* **81** (2006), no. 4, p. 911-929.

[4] T. Austin, "Amenable groups with very poor compression into Lebesgue spaces", *Duke Math. J.* **159** (2011), no. 2, p. 187-222.

[5] T. Austin, A. Naor & Y. Peres, "The wreath product of $\mathbb{Z}$ with $\mathbb{Z}$ has Hilbert compression exponent 2/3", *Proc. Amer. Math. Soc.* **137** (2009), no. 1, p. 85-90.

[6] A. Avez, "Entropie des groupes de type fini", *C. R. Acad. Sci. Paris Sér. A-B* **275** (1972), p. A1363-A1366.

[7] ———, "Théorème de Choquet-Deny pour les groupes à croissance non exponentielle", *C. R. Acad. Sci. Paris Sér. A* **279** (1974), p. 25-28.

[8] L. Bartholdi & A. Erschler, "Imbeddings into groups of intermediate growth", http://arxiv.org/abs/1403.5584.

[9] B. Bekka, A. Chérix & A. Valette, "Proper affine isometric actions of amenable groups", in *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, London Math. Soc. Lecture Note Ser., vol. 227, Cambridge Univ. Press, Cambridge, 1995, p. 1-4.

[10] A. Bendikov, C. Pittet & R. Sauer, "Spectral distribution and L^2-isoperimetric profile of Laplace operators on groups", *Math. Ann.* **354** (2012), p. 43-72.

[11] Y. Benyamini & J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, providence, 2000.

[12] J. Brieussel & T. Zheng, "Speed of random walks, isoperimetry and compression of finitely generated groups", https://arxiv.org/abs/1510.08040.

[13] M. Carette, S. Arnt, P. Thibault & A. Valette, "The Haagerup property is not invariant under quasi-isometry", https://arxiv.org/abs/1403.5446.

[14] T. K. Carne, "A transmutation formula for Markov chains", *Bull. Sci. Math.* **109** (1985), no. 4, p. 399-405.

[15] T. Coulhon & A. Grigor'yan, "On-diagonal lower bounds for heat kernels and Markov chains", *Duke Math. J.* **89** (1997), no. 1, p. 133-199.

[16] T. Coulhon, A. Grigor'yan & C. Pittet, "A geometric approach to on-diagonal heat kernels lower bounds on groups", *Ann. Inst. Fourier* **51** (2001), no. 6, p. 1763-1827.

[17] T. Coulhon, A. Grigor'yan & F. Zucca, "The discrete integral maximum principle and its applications", *Tohoku Math. J.* **57** (2005), no. 4, p. 447-621.

[18] N. Dungey, "Properties of random walks on discrete groups: Time regularity and off-diagonal estimates", *Bull. Sci. math.* **132** (2008), p. 359-381.

[19] A. Erschler, "On drift and entropy growth for random walks on groups", *Ann. Probab.* **31** (2003), no. 3, p. 1193-1204.

[20] ———, "Critical constants for recurrence of random walks on G-spaces", *Ann. Inst. Fourier* **55** (2005), no. 2, p. 493-509.

[21] A. Erschler & A. Karlsson, "Homomorphisms to $\mathbb{R}$ constructed from random walks", *Ann. Inst. Fourier* **60** (2010), no. 6, p. 2095-2113.

[22] E. Guentner & J. Kaminker, "Exactness and uniform embeddability of discrete groups", *J. London Math. Soc.* **70** (2004), no. 3, p. 703-718.

[23] V. Kaimanovich, "Boundary behaviour of Thompson's group", in preparation.

[24] H. Kesten, "Full Banach mean values on countable groups", *Math. Scand.* **7** (1959), p. 146-156.

[25] ———, "Symmetric random walks on groups", *Trans. Amer. Math. Soc.* **92** (1959), p. 336-354.

[26] M. Kotowski & B. Virág, "Non-Liouville groups with return probability exponent at most 1/2", http://arxiv.org/abs/1408.6895.

[27] J. Lee & Y. Peres, "Harmonic maps on amenable groups and a diffusive lower bound for random walks", *Ann. Probab.* **41** (2013), no. 5, p. 3392-3419.

[28] A. Naor & Y. Peres, "Embeddings of discrete groups and the speed of random walks", *Int. Math. Res. Not.* (2008), p. 34.

[29] ———, "L_p-compression, traveling salesmen, and stable walks", *Duke Math. J.* **157** (2011), no. 1, p. 53-108.

[30] C. Pittet & L. Saloff-Coste, "On the stability of the behavior of random walks on groups", *J. Geom. Anal.* **10** (2000), no. 4, p. 713-737.

[31] ———, "On random walks in wreath products", *Ann. Probab.* **30** (2002), no. 2, p. 948-977.

[32] D. Revelle, "Rate of escape of random walks on wreath products and related groups", *Ann. Probab.* **31** (2003), no. 4, p. 1917-1934.

[33] L. Saloff-Coste & T. Sheng, "Random walks and isoperimetric profiles under moment conditions", https://arxiv.org/abs/1501.05929.

[34] R. Tessera, "Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces", *Comment. Math. Helv.* **86** (2011), no. 3, p. 499-535.

[35] R. Thompson, "The rate of escape of random walks on polycyclic and metabelian groups", *Ann. Inst. Henri Poincaré Probab. Stat.* **49** (2013), no. 1, p. 270-287.

[36] A. Valette, *Nouvelles approches de la propriété (T) de Kazhdan*, Astérisque, vol. 294, Société Mathématique de France, 2004, vii+97-124 pages.

[37] N. T. Varopoulos, "Long range estimates for Markov chains", *Bull. Sci. Math.* **139** (1985), no. 3, p. 225-252.

[38] W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge tracts in mathematics, vol. 138, Cambridge University Press, 200, xi+344 pages.

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