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NORMAL FORM OF HOLOMORPHIC VECTOR FIELDS WITH AN INVARIANT TORUS UNDER BRJUNO'S A CONDITION

by Claire CHAVAUDRET

Abstract. — This article proves the existence of an analytic normal form for some holomorphic differential systems in the neighborhood of a fixed point and of an invariant torus. Once a formal normal form is constructed, one shows that the initial system with quasilinear part S can be holomorphically conjugated to a normal form, i.e. a vector field which commutes with S, under two arithmetical conditions known as Brjuno's γ and ω conditions, and an algebraic condition known as Brjuno's A-condition, which requires the formal normal form to be proportional to S.

Résumé. — On prouve l'existence d'une forme normale analytique pour certains champs de vecteurs holomorphes au voisinage d'un point fixe et d'un tore invariant. Après avoir construit une forme normale formelle, on montre que le champ de vecteurs initial peut être analytiquement normalisé sous deux conditions arithmétiques et une condition algébrique, connues comme les conditions γ, ω et A de Brjuno.

1. Introduction

The present article considers the problem of normal forms for autonomous differential systems defined on a product of a torus with a disk, i.e. systems of the form

$$\dot{X} = f(X,Y);\ \dot{Y} = g(X,Y),\ X \in \mathbb{T}^d, Y \in \mathbb{C}^n \tag{1.1}$$

We also assume that $g(X,0) = 0$, $f(X,0) = \omega \in \mathbb{R}^d$ and $\partial_Y g(X,Y)_{|Y=0} = \Lambda$ where Λ is a diagonal matrix: thus, the system can be viewed as a perturbation of the integrable system

$$\dot{X} = \omega;\ \dot{Y} = \Lambda Y \tag{1.2}$$

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where $X \in \mathbb{T}^d, Y \in \mathbb{C}^n$. Such systems can appear when considering the restriction of a larger system on an invariant torus supporting a quasiperiodic motion. We require the system to be analytic in both variables: thus there exists a complex neighborhood $\mathcal{V}$ of $\mathbb{T}^d$ and a neighborhood $\mathcal{W}$ of the origin in $\mathbb{C}^n$ such that f and g are holomorphic on $\mathcal{V} \times \mathcal{W}$.

The invariant manifolds of the unperturbed system (1.2) can be easily computed and form a foliation of the phase space. What happens after perturbation, however, depends on possible resonances between ω and the spectrum $\sigma(\Lambda)$ of Λ. Assuming that ω and $\sigma(\Lambda)$ are jointly non resonant (in a sense that will be made explicit below), then linearization is possible: as shown in [1], an analytic perturbation of (1.2) is analytically linearizable if ω and $\sigma(\Lambda)$ are non resonant, and if they satisfy some arithmetical conditions known as Brjuno's γ and ω conditions.

The problem of holomorphically linearizing a system with non resonant linear part in the vicinity of a fixed point was extensively studied: Siegel [8] showed that a diophantine condition on the spectrum of the linear part implies the existence of an analytic linearization. Brjuno [2, 3] managed to relax the arithmetical condition on the spectrum. Giorgilli-Marmi [5] obtained a lower bound for the radius of convergence which is expressed in terms of the Brjuno function of the spectrum.

Systems with a fixed point and a resonant linear part have also been studied by Brjuno: in [2, 3], it is proved that for a vector field in $\mathbb{C}^n$ with a fixed point and a resonant linear part, a strong algebraic condition on the formal normal form, known as Brjuno's A condition and the well-known arithmetical "ω condition" are sufficient in order to have an analytic normalization (i.e. an analytic change of variables conjugating the initial vector field to a normal form). Also, the A condition and an arithmetical condition $\bar{\omega}$ which is weaker than ω are necessary for the analytic normalization.

In [9], other links are given between algebraic conditions on a formal normal form and the holomorphic normalizability of a vector field.

A refinement of the normal form theory near a fixed point can be found in Lombardi-Stolovitch [6]): if some eigenvalues of Λ are zero, then the normal form is defined with respect to a given unperturbed vector field which might be nonlinear. This makes it possible to distinguish the dynamics of distinct vector fields even if they have the same degenerate linear part.

In the case under consideration, periodicity with respect to the set of variables X implies a slightly different definition of the normal form: normal forms will be taken in the kernel of ad_S, where

$$S = \sum_{j=1}^{d} \omega_j \frac{\partial}{\partial X_j} + \sum_{j'=1}^{n} \lambda_{j'} Y_{j'} \frac{\partial}{\partial Y_{j'}} \tag{1.3}$$

i.e. S is the unperturbed vector field generating the system (1.2), where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix Λ and ω_j are the components of ω. More explicitly, a normal form for (1.1) is a formal vector field $NF(X,Y)$ which is a formal Fourier series in X and a formal series in Y and whose Fourier-Taylor development,

$$NF(X,Y) = \sum_{P\in\mathbb{Z}^d} \sum_{Q\in\mathbb{N}^n} NF_{P,Q} e^{i\langle P,X\rangle} Y^Q$$

only has non zero coefficients $NF_{P,Q}$ with indices P, Q satisfying

$$i\sum_{j=1}^{d} P_j\omega_j + \sum_{j=1}^{n} Q_j\lambda_j = 0$$

(such couples (P,Q) are the resonances between ω and $\sigma(\Lambda)$), and which is conjugate to (1.1) by a formal change of variables (the precise meaning of this formal conjugation will be given in Section 2).

The tools used when dealing with vector fields close to an invariant torus are similar to the ones that have been used before, in the study of vector fields with a fixed point. The main difference comes from the presence of a Fourier development which will make the small divisors more complicated to deal with. In [7], Meziani considers the real-valued non resonant case and obtains analytic normalization under a Siegel diophantine condition.

Here, we will combine techniques used by Brjuno, Stolovitch [9] and Aurouet [1] to prove a conjecture by Brjuno in [4], mentioned in [1], which is analytic normalization for systems of the form (1.1), where the linear part might be resonant, under Brjuno's γ, ω and A conditions:

Theorem 1.1. — *Consider the following autonomous differential system:*

$$\dot{X} = \omega + F(X,Y); \ \dot{Y} = \Lambda Y + G(X,Y) \tag{1.4}$$

where F and G are analytic on a neighborhood $\mathcal{V}$ of $\mathbb{T}^d \times \{0\}$, $F(X,0) = 0$, $G(X,0) = \partial_Y G(X,0) = 0, \Lambda$ is a diagonal $n \times n$ matrix and $\omega \in \mathbb{R}^d$. Let S be the quasilinear vector field given by (1.3)*. Assume that S satisfies Brjuno's "γ and ω conditions" (see Assumption 3.1 below) away from the resonances. If a formal normal form NF of* (1.4) *is such that there exists* *a formal series* $a(X,Y)$ *such that* $a(0,0)=1$ *and*

$$NF(X,Y)=a(X,Y)S(Y) \tag{1.5}$$

then there exists $\zeta_0>0$ *depending only on* Λ,ω,n,d *such that if the components of* F *and* G *are less than* ζ_0 *in the analytic norm, then* (1.4) *is holomorphically normalizable in a complex neighborhood* $\mathcal{W}$ *of* $\mathbb{T}^d\times\{0\}$ *and the holomorphic normal form has also the form* $b(X,Y)S(Y)$.

Remark. — The Assumption (1.5) is usually referred to as Brjuno's A condition. Note that $\mathcal{W}$ is strictly included in $\mathcal{V}$ in general: the loss of analyticity comes from the presence of small divisors.

A first step will be the construction of a normal form by a direct method; although the existence of a formal normal form is not a new result, one does need to give an explicit construction since the definition of a normal form in the present case might not be completely standard because of Fourier series. It is then proved that the algebraic Assumption (1.5) on one of the normal forms is preserved by the changes of variables that actually appear in the proof. Then, by a Newton method adapted from Stolovitch and Aurouet, one constructs a converging sequence of holomorphic diffeomorphisms conjugating the system to another one which is normalized up to arbitrary order, which gives the analytic normal form.

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2. Notations and general definitions

2.1. Topological setting

Let $r>0$; in the following, $\mathcal{V}_r\subset\mathbb{C}^d$ will denote a complex neighbourhood of the d-dimensional torus, of width r, and $\mathcal{W}_\delta\subset\mathbb{C}^n$ the ball centered at the origin in $\mathbb{C}^n$, of radius δ.

In the following, given a function f on $\mathbb{C}^d/\mathbb{Z}^d\times\mathbb{C}^n$, the notation $f_{P,Q}$ will refer to the complex coefficient appearing next to $e^{i\langle P,X\rangle}Y^Q$ in the Taylor-Fourier development of f, that is, the Taylor development near the origin w.r.t. the second variable together with the Fourier development with respect to the first variable (such a development might be purely formal). Also, for all indices $P=(P_1,\dots,P_d)\in\mathbb{Z}^d$ and $Q=(Q_1,\dots,Q_n)\in\mathbb{N}^n$, one will use the notation $|P|:=\sum_{j=1}^d|P_j|$ (where $|P_j|$ is the absolute value of P_j) and $|Q|:=\sum_{j'=1}^n Q_{j'}$.

The space of scalar-valued functions which are holomorphic on $\mathcal{V}_r \times \mathcal{W}_\delta$ is denoted by $C^\omega_{r,\delta}$. It is provided with the weighted norm

$$|f|_{r,\delta} = \sum_{P\in\mathbb{Z}^d} \sum_{Q\in\mathbb{N}^n} |f_{P,Q}| e^{r|P|} \delta^{|Q|}$$

where $f(X,Y) = \sum_{P\in\mathbb{Z}^d} \sum_{Q\in\mathbb{N}^n} f_{P,Q} e^{i\langle X,P\rangle} Y^Q$ is the Taylor-Fourier development of f.

Remark. — Such functions can be viewed as functions which are holomorphic in $\mathcal{W}_\delta$ with respect to the set of variables Y, holomorphic in a strip $\{X \in \mathbb{C}^d, \forall j, 1 \leqslant j \leqslant d, |ImX_j| \leqslant r\}$ with respect to the set of variables X, and 2π-periodic with respect to each variable X_j.

Also, notice that for all $f \in C^\omega_{r,\delta}$,

$$\sup_{(X,Y)\in\mathcal{V}_r\times\mathcal{W}_\delta} |f(X,Y)| \leqslant |f|_{r,\delta} \tag{2.1}$$

and a classical estimate on the Fourier coefficients and a Cauchy estimate imply that for all $P \in \mathbb{Z}^d, Q \in \mathbb{N}^n$,

$$|f_{P,Q}| \leqslant \sup_{(X,Y)\in\mathcal{V}_r\times\mathcal{W}_\delta} |f(X,Y)| e^{-|P|r} \delta^{-|Q|} \tag{2.2}$$

The space of vector fields defined on $\mathcal{V}_r \times \mathcal{W}_\delta$ with components in $C^\omega_{r,\delta}$ will be denoted by $VF_{r,\delta}$. For

$$F(X,Y) = \sum_{j=1}^{d} F_j(X,Y) \frac{\partial}{\partial X_j} + \sum_{j'=1}^{n} F'_{j'}(X,Y) \frac{\partial}{\partial Y_{j'}} \in VF_{r,\delta}$$

the norm of F is by definition

$$||F||_{r,\delta} = \max\{|F_j|_{r,\delta}, |F'_{j'}|_{r,\delta}, 1 \leqslant j \leqslant d, 1 \leqslant j' \leqslant n\}$$

For all $r, r', \delta, \delta' > 0$, when dealing with an operator sending $VF_{r,\delta}$ into $VF_{r',\delta'}$, we will denote its operator norm by $|||\cdot|||_{VF_{r,\delta}\to VF_{r',\delta'}}$.

2.2. Formal aspects

Definition 2.1.

- *A formal Fourier-Taylor series in* (X,Y) *is a series* f *of the form*

$$f(X,Y) = \sum_{P\in\mathbb{Z}^d, Q\in\mathbb{N}^n} f_{P,Q} e^{i\langle P,X\rangle} Y^Q \tag{2.3}$$

with coefficients $f_{P,Q} \in \mathbb{C}$ *and such that for all* $Q \in \mathbb{N}^n$,

$$\sum_{P\in\mathbb{Z}^d} |f_{P,Q}|^2 < \infty \tag{2.4}$$

i.e. for all $Y \in \mathbb{C}^n$, $f(\cdot, Y) \in L^2(\mathbb{T}^d)$ *(this restriction is motivated by the fact that we need products and finite sums of Fourier-Taylor series to be themselves Fourier-Taylor series).*

- *A formal vector field (on* $\mathbb{T}^d \times \mathbb{C}^n$ *) is a* $d+n$*-uple whose components are formal Fourier-Taylor series.*

Definition 2.2 (truncation of a formal series, of a vector field). — *Let* f *be a formal series. We will denote by* $T^k f$ *its truncation at order* $k \in \mathbb{N}$ *(or* k*-jet):*

$$T^k f(X,Y) = \sum_{P\in\mathbb{Z}^d, |Q|\leqslant k} f_{P,Q} e^{i\langle P,X\rangle} Y^Q$$

For a vector field (resp. diffeomorphism) F *with components* $(F_1, \dots, F_{d+n})$, *its truncation at order* k *is the vector field (resp. diffeomorphism)* $T^k F$ *with components*

$$(T^k F_1, \dots, T^k F_d, T^{k+1} F_{d+1}, \dots T^{k+1} F_{d+n})$$

Remark. — Thus for a vector field F, its truncation $T^k F$ has degree k (as a vector field).

Definition 2.3. — *A formal diffeomorphism (of* $\mathbb{C}^d/\mathbb{Z}^d \times \mathbb{C}^n$*) is a sequence of analytic diffeomorphisms* (Φ_k) *defined on a non-increasing sequence of domains* $\mathcal{V}_k \subset \mathbb{C}^d/\mathbb{Z}^d \times \mathbb{C}^n$ *(for the order induced by the inclusion), such that* Φ_k *has degree* k *and for all* $j \leqslant k$, $T^j \Phi_k = \Phi_{j|\mathcal{V}_k}$. *We will denote by* $T^k\Phi$ *the analytic diffeomorphism* Φ_k.

Remark. — An analytic diffeomorphism is a particular case of a formal diffeomorphism.

Definition 2.4 (order of a formal series). — *A formal series* $f(X,Y) = \sum_{P\in\mathbb{Z}^d, Q\in\mathbb{N}^n} f_{P,Q} e^{i\langle P,X\rangle} Y^Q$ *has order* $k \in \mathbb{N}$ *if* $|Q| \leqslant k-1 \Rightarrow f_{P,Q} = 0$.

Remark. — Only the order in Y is considered. Thus, the product of two formal series has order at least the sum of their orders. Notice that the order is not uniquely defined by maximality: if f has order k and $k' \leqslant k$, then f has order k'.

Definition 2.5 (order of a vector field). — *For*

$$F = \sum_{j=1}^{d} F_j(X,Y) \frac{\partial}{\partial X_j} + \sum_{j'=1}^{n} F'_{j'}(X,Y) \frac{\partial}{\partial Y_{j'}} \in VF_{r,\delta},$$

the vector field F has quasi-order $k \in \mathbb{N}$ if all the F_j have order k in Y and the $F'_{j'}$ have order $k+1$ in Y (as formal series).

Definition 2.6 (degree of a vector field). — *A vector field F has degree $k \in \mathbb{N}$ if its components F_j have degree k for $1 \leqslant j \leqslant d$ and have degree $k+1$ for $d+1 \leqslant j \leqslant d+n$.*

Remark. — In particular, if a vector field has degree k and order k, it is quasi-homogeneous of order k. For instance, the quasilinear vector field S defined in (1.3) is quasi-homogeneous of order 0.

Definition 2.7. — *For a vector field F, its quasilinear part is the vector field T^0F.*

Definition 2.8. — *A formal diffeomorphism $\Phi = (\Phi_1, \ldots, \Phi_{n+d})$ is tangent to identity if for all $1 \leqslant L \leqslant d$, $\Phi_L(X,Y) - X_L$ has order 1 and if for all $d+1 \leqslant L \leqslant d+n$, $\Phi_L(X,Y) - Y_{L-d}$ has order 2.*

2.3. Resonances

The quasilinear part S defines an equivalence relation $\sim$ on $\mathbb{Z}^d \times \mathbb{N}^n$ as follows: for all $(P,Q),(P',Q') \in \mathbb{Z}^d \times \mathbb{N}^n$,

$$(P,Q) \sim (P',Q') \iff i\langle P,\omega\rangle + \langle Q,\Lambda\rangle = i\langle P',\omega\rangle + \langle Q',\Lambda\rangle \tag{2.5}$$

Definition 2.9. — *The vectors ω and Λ are jointly non resonant if the equivalence classes of the relation $\sim$ are reduced to singletons.*

Remark. — It does not hold in general. Moreover, if an equivalence class is not a singleton, then it has infinitely many elements. In this article, ω and Λ are *not assumed to be jointly non resonant.*

For $c \in \mathbb{C}$, $\mathcal{C}_c$ denotes the equivalence class such that for all $(P,Q) \in \mathcal{C}_c$, $i\langle P,\omega\rangle + \langle Q,\Lambda\rangle = c$.

Let f be an analytic function. Then the Fourier-Taylor development of f has only indices in $\mathcal{C}_c$ if and only if $S(f) = cf$, where $S(f)$ is the Lie derivative of f along S:

$$S(f) = \sum_{j=1}^{d} \omega_j \frac{d}{dX_j} f + \sum_{j'=1}^{n} \lambda_{j'} Y_{j'} \frac{d}{dY_{j'}} f$$

Definition 2.10.

- *Monomials $f_{P,Q}e^{i\langle P,X\rangle}Y^Q$ with $i\langle P,\omega\rangle + \langle Q,\Lambda\rangle = 0$ are resonant, or invariant, monomials (indeed they are constant in the direction of the vector field S).*
- *For a formal series f, its resonant part is the sum of all its resonant monomials, and its non resonant part is the sum of all other monomials.*
- *A formal series is resonant if it is equal to its resonant part, and non resonant if it is equal to its non resonant part.*

Remark. — This means that a formal series is resonant if all its monomials are indexed by $(P,Q)\in\mathcal{C}_0$, i.e. if $S(f)=0$.

Definition 2.11. — *Let $F=\sum_{j=1}^d F_j\frac{\partial}{\partial X_j}+\sum_{j'=1}^n F_{d+j'}\frac{\partial}{\partial Y_{j'}}$ be a vector field on a domain in $\mathbb{C}^d/\mathbb{Z}^d\times\mathbb{C}^n$.*

- *F is non resonant if $F_1,\dots,F_d$ are non resonant formal series and if for all $1\leqslant j\leqslant n$, $F_{d+j}(X,Y)Y_j^{-1}$ is a non resonant formal series.*
- *F is resonant if $F_1,\dots,F_d$ are resonant formal series and if for all $1\leqslant j\leqslant n$, $F_{d+j}(X,Y)Y_j^{-1}$ is a resonant formal series.*
- *There is a unique decomposition $F=F_{res}+F_{nr}$ where F_{res} is a resonant vector field and F_{nr} is a non resonant vector field; then F_{res} is called the resonant part of F and F_{nr} its non resonant part.*

Remark. — Let j with $1\leqslant j\leqslant n$ and write the decomposition

$$F_{d+j}(X,Y)=\sum_{P,Q}F_{d+j,P,Q}e^{i\langle P,X\rangle}Y^Q$$

Then $Y_j^{-1}F_{d+j}(X,Y)$ is resonant if

$$F_{d+j,P,Q}\neq 0\Rightarrow i\langle P,\omega\rangle+\langle Q,\Lambda\rangle-\lambda_j=0$$

This coincides with Aurouet's definition of the invariants.

Definition 2.12. — *Let Φ be a formal diffeomorphism. Suppose that for all $1\leqslant j\leqslant d$, $\Phi_j(X,Y)=X_j+\tilde\Phi_j(X,Y)$ where $\tilde\Phi_j(X,Y)$ has order 1, and that for all $1\leqslant j'\leqslant n$, $\Phi_{d+j'}(X,Y)=Y_{j'}\tilde\Phi_{d+j'}(X,Y)$ where $\tilde\Phi_{d+j'}(X,Y)$ has order 1.*

- *Φ is non resonant up to order k if for all $1\leqslant j\leqslant n+d$, $T^k\tilde\Phi_j$ is non resonant;*
- *Φ is resonant up to order k if for all $1\leqslant j\leqslant n+d$, $T^k\tilde\Phi_j$ is resonant;*
- *Φ is non resonant (resp. resonant) if for all $k\in\mathbb{N}$ it is non resonant up to order k (resp. resonant up to order k).*

Remark. — This definition applies in particular if Φ is an analytic diffeomorphism.

Definition 2.13. — *Let F be a formal vector field and G be an analytic vector field; G is formally conjugate to F if there exists a formal diffeomorphism Φ such that for all $k \in \mathbb{N}$,*

$$T^k(DT^k\Phi \cdot F) = T^k(G \circ T^k\Phi)$$

Remark. — If an analytic vector field G is formally conjugate to a formal vector field F and if G is also analytically conjugate to an analytic vector field H, then H is formally conjugate to F. Indeed, assume that for all $k \in \mathbb{N}$

$$T^k(DT^k\Phi \cdot F) = T^k(G \circ T^k\Phi)$$

and that there exists an analytic diffeomorphism $\tilde{\Phi}$ such that

$$D\tilde{\Phi} \cdot G = H \circ \tilde{\Phi}.$$

Let $\Psi_k = T^k(\tilde{\Phi} \circ T^k\Phi)$; then $D\Psi_k = T^k(D\tilde{\Phi} \circ T^k\Phi \cdot DT^k\Phi)$; thus for all k,

$$\begin{aligned} T^k(D\Psi_k \cdot F) &= T^k[D\tilde{\Phi} \circ T^k\Phi \cdot DT^k\Phi \cdot F] \\ &= T^k[D\tilde{\Phi} \circ T^k\Phi \cdot G \circ T^k\Phi] \\ &= T^k[H \circ \tilde{\Phi} \circ T^k\Phi] \\ &= T^k[H \circ \Psi_k] \end{aligned} \tag{2.6}$$

therefore the sequence (Ψ_k) defines a formal diffeomorphism conjugating H to F.

Other notations. — For $I = (i_1, \ldots, i_{d+n}) \in \mathbb{N}^{d+n}$, the symbol ∂_I will be a compact notation for the partial derivative $\partial_{X_1}^{i_1} \ldots \partial_{X_d}^{i_d} \partial_{Y_1}^{i_{d+1}} \ldots \partial_{Y_n}^{i_{d+n}}$. Sometimes the remainder of order k in Y in a formal series or a formal vector field will be denoted by $O(Y^k)$.

If $F \in VF$ and g is a function, the Lie derivative of g with respect to F will be denoted by $F(g)$.

3. Setting and arithmetical conditions

Let $r_0 > 0, \delta_0 > 0$. Consider the following system on the neighborhood $\mathcal{V}_{r_0} \times \mathcal{W}_{\delta_0} \subset \mathbb{C}^d/\mathbb{Z}^d \times \mathbb{C}^n$ of $\mathbb{T}^d \times \{0\}$:

$$\left\{\begin{array}{l} \dot{X}_1 = \omega_1 + R_1(X,Y) \\ \vdots \\ \dot{X}_d = \omega_d + R_d(X,Y) \\ \dot{Y}_1 = \lambda_1 Y_1 + R_{d+1}(X,Y) \\ \vdots \\ \dot{Y}_n = \lambda_n Y_n + R_{d+n}(X,Y) \end{array}\right. \tag{3.1}$$

where $(X_1, \dots, X_d) \in \mathcal{V}_{r_0}$, $(Y_1, \dots, Y_n) \in \mathcal{W}_{\delta_0}$, $R_1, \dots, R_d$ have order 1 and $R_{d+1}, \dots, R_{d+n}$ have order 2. Suppose that $R_1, \dots, R_{d+n}$ are all in $C^\omega_{r_0,\delta_0}$.

Denote by S the quasilinear part of the vector field generating the system (3.1):

$$S(Y) = \sum_{j=1}^d \omega_j \frac{\partial}{\partial X_j} + \sum_{j'=1}^n \lambda_{j'} Y_{j'} \frac{\partial}{\partial Y_{j'}}$$

Thus for all $r, \delta > 0$, $||S||_{r,\delta} = \max\{\omega_j, \lambda_{j'}\delta, 1 \leqslant j \leqslant d, 1 \leqslant j' \leqslant n\}$ and if $\delta \leqslant \min_{j'} \max_j |\omega_j/\lambda_{j'}|$, then $||S||_{r,\delta} \leqslant \max_j |\omega_j| := C_\omega$. For short, denote $F = S + R$ where R is the vector field with components $R_1, \dots, R_{d+n}$ defined in (3.1).

The following arithmetical condition, which will be assumed throughout the article, is a reformulation of Brjuno's γ and ω conditions (also used in [1]). It will be used in the theorem of analytic normalization.

Assumption 3.1. — *There exists a positive increasing unbounded function g on $[1, +\infty[$, an increasing sequence of integers $(m_k)_{k\geqslant 0}$ and two sequences of positive real numbers $(\epsilon_k)_{k\geqslant 0}, (r_k)_{k\geqslant 0}$ such that:*

(1) $m_0 = 1; m_{k+1} \leqslant 2m_k + 1;$
(2) $B := \sum_{k\geqslant 1} \frac{\ln g(m_k)}{m_k} < +\infty,$
(3) $\forall k \geqslant 0, (n+2)g(m_k)\ln g(m_k) \geqslant m_k,$
(4) $\forall P \in \mathbb{Z}^d, \forall Q \in \mathbb{N}^n,$
$|P| > m_k, |Q| \leqslant m_k, i\langle P,\omega\rangle + \langle Q,\Lambda\rangle \neq 0 \Rightarrow |i\langle P,\omega\rangle + \langle Q,\Lambda\rangle|^{-1} \leqslant e^{\epsilon_k |P|},$
(5) $\forall P \in \mathbb{Z}^d, \forall Q \in \mathbb{N}^n,$
$|P| \leqslant m_k, |Q| \leqslant m_k, i\langle P,\omega\rangle + \langle Q,\Lambda\rangle \neq 0 \Rightarrow |i\langle P,\omega\rangle + \langle Q,\Lambda\rangle|^{-1} \leqslant g(m_k),$
(6) $\forall k, 4^{4d+6}(r_k - r_{k+1} - \epsilon_k)^{-4d-6} \leqslant g(m_k),$
(7) $\forall k \geqslant 0, r_k > \frac{1}{2}.$

Remark. — The case when $m_k = 2^k$ corresponds to Brjuno's standard ω condition. However, the Assumption 3.1.1 seems to prevent (m_k) from increasing faster than 2^k.

Note that the set of elements $g, (m_k), (\epsilon_k), (r_k)$ satisfying Assumption 3.1 is non-empty. In [1], the conditions γ and ω correspond to the case $m_k = 2^k, r_k = r_0 \prod_{j=1}^k \frac{1}{g(m_j)^{\frac{1}{2^j}} 2^{\frac{2j}{2^j}}}, \epsilon_k > \frac{1}{4}(1 - \frac{1}{2^{\frac{k}{2^k}}})$, and $g(m_k) = \max\{(i\langle P, \omega\rangle + \langle Q, \Lambda\rangle)^{-1}/|P| \leqslant m_k, |Q| \leqslant m_k\}$, with r_0 big enough in order to have $r_k > \frac{1}{2}$.

The case where $g(m_k) = Cm_k^\tau$ for $m_k = 2^k$, for a positive constant C and an exponent $\tau \geqslant d + n + 1$, corresponds to the situation where the vector (ω, Λ) satisfies a Siegel diophantine condition; therefore the set of $(\omega, \Lambda) \in \mathbb{C}^{d+n}$ satisfying the Assumption 3.1 has a large Lebesgue measure.

Also, Assumption 3.1 does not contain any upper bound on $g(m_0)$: the only upper bound on g comes from 2 and it is logarithmic; therefore multiplying $g(m_k)$ by a constant does not change the condition. In Section 5.2, we shall assume that for all $k \geqslant 0$, $g(m_k)$ is larger than a fixed constant.

4. Formal normal form

This section contains an explicit construction of a formal normal form for the system (1.1). A direct method will be used, i.e. the formal conjugation will be written explicitly, by a Taylor development, as a recurrence relation between two consecutive orders of truncation of the change of variables. The following lemma explains why the recurrence can be solved.

LEMMA 4.1. — *Let $\bar{\Phi}_1, \dots, \bar{\Phi}_d$ be analytic Fourier-Taylor series in $(X, Y) \in \mathbb{T}^d \times \mathbb{C}^n$ which are of order 1 in Y and $\bar{\Phi}_{d+1}, \dots, \bar{\Phi}_{d+n}$ be analytic Fourier-Taylor series in $(X, Y) \in \mathbb{T}^d \times \mathbb{C}^n$ which are of order 2 in Y. Let $I = (i_1, \dots, i_{n+d}) \in \mathbb{N}^{d+n}$. For any analytic vector field F, let*

$$D_I F(X, Y) = c_I \partial_I F(X, Y) \cdot \bar{\Phi}_1(X, Y)^{i_1} \dots \bar{\Phi}_{d+n}(X, Y)^{i_{d+n}} \tag{4.1}$$

where $c_I \in \mathbb{C}^*$ and ∂_I is a compact notation for $\partial_{X_1}^{i_1} \dots \partial_{X_d}^{i_d} \partial_{Y_1}^{i_{d+1}} \dots \partial_{Y_n}^{i_{d+n}}$. If F has order k in Y, then $D_I F$ has order $k + i_1 + \dots + i_{d+n}$.

Proof. — It is enough to prove the statement for elementary $(i_1, \dots, i_{d+n})$. For $1 \leqslant j \leqslant d$, the derivative ∂_{X_j} does not change the order in Y and multiplying by $\bar{\Phi}_j$ adds a degree in Y. Thus the order of $D_{e_j} F$ is $k + 1$ if F has order k.

For $1 \leqslant j \leqslant n$ and $1 \leqslant l \leqslant d$, the derivative $\partial_{Y_j} F_l$ has order $k - 1$ and multiplying by $\bar{\Phi}_{d+j}$ adds two degrees in Y, thus $D_{e_{d+j}} F$ has order $k + 1$;

for $d+1 \leqslant l \leqslant d+n$, F_l has order $k+1$, therefore the derivative $\partial_{Y_j} F_l$ has order k, and multiplying by $\bar{\Phi}_{d+j}$ adds two degrees in Y.

Finally, $D_{e_{d+j}} F$ has order $k+1$ (as a vector field). □

The following Proposition gives a construction of a formal normal form:

Proposition 4.2. — *The system* (3.1) *can be formally normalized, i.e. there exist a formal diffeomorphism* Φ *and a formal resonant vector field* NF *such that for all* k,

$$T^k[D(T^k\Phi) \cdot NF] = T^k(F \circ T^k\Phi) = T^k((S+R) \circ T^k\Phi)$$

Proof. — One looks for a formal diffeomorphism Φ tangent to identity, i.e. such that

$$T^0(\Phi - Id) = 0$$

and for a resonant vector field $NF = S + N$ where N has order 1, such that

$$T^k(DT^k\Phi \cdot NF) = S \circ T^k\Phi + T^k(R \circ T^k\Phi) \tag{4.2}$$

The formal diffeomorphism Φ and the normal form NF will be constructed gradually, every quasi-homogeneous part of degree k being given formally by the data F and by the truncations $T^{k-1}NF$ and $T^{k-1}\Phi$.

Developing the composition $R \circ T^k\Phi$ in Taylor series (recall that F is analytic), letting $\bar{\Phi} = \Phi - Id$ and developing in $\bar{\Phi}$, one obtains

$$R \circ (Id + T^k\bar{\Phi}) = R + \sum_{\substack{I=(i_1,\dots,i_{d+n}) \\ \neq(0,\dots,0)}} c_I \partial_I R \cdot T^k\bar{\Phi}_1^{i_1} \dots T^k\bar{\Phi}_{d+n}^{i_{d+n}} \tag{4.3}$$

where $c_I = \frac{1}{i_1! \dots i_{d+n}!}$ and ∂_I is the partial derivative $\partial_{X_1}^{i_1} \dots \partial_{Y_n}^{i_{d+n}}$, that is to say,

$$R \circ (Id + T^k\bar{\Phi})(X,Y) = R(X,Y) + \sum_{\substack{(i_1,\dots,i_{d+n}) \\ \neq(0,\dots,0)}} D_{i_1,\dots,i_{n+d}} R(X,Y) \tag{4.4}$$

where $D_{i_1,\dots,i_{n+d}}$ is as in Lemma 4.1 as long as the $\bar{\Phi}_j$ have order 1 for $j \leqslant d$ and order 2 for $j \geqslant d+1$, i.e. as long as Φ is tangent to identity; the operator $D_{i_1,\dots,i_{n+d}}$ therefore increases the order by $i_1 + \dots + i_{n+d}$. Although the sum in (4.4) does not necessarily define a formal series since it has an infinite number of terms, its truncation has a finite number of terms, therefore it defines a formal series.

Thus if Φ is tangent to identity, the conjugation (4.2) can be rewritten

$$\begin{aligned}(4.5)\quad & T^k NF + T^k(DT^k\bar\Phi \cdot NF) \\ & = S\circ T^k\Phi + T^kR + \sum_{\substack{(i_1,\dots,i_{d+n})\\ \neq(0,\dots,0)}} T^k(D_{i_1,\dots,i_{n+d}}T^{k-i_1\cdots-i_{n+d}}R)\end{aligned}$$

Let $k \geqslant 1$; suppose that $T^{k-1}NF$ and $T^{k-1}\Phi$ are known and analytic, and that $T^0\bar\Phi = 0$. In Equation (4.5), the sum in the right-hand side is given because R has order 1 and therefore for all I,

$$D_IT^{k-|I|}R = c_I\partial_I T^{k-|I|}R\cdot T^{k-1}\bar\Phi^I + O(Y^k)$$

On the left-hand side, since the N_j have order 1 for $1\leqslant j\leqslant d$ and order 2 for $d+1\leqslant j\leqslant d+n$, then

$$T^k(DT^k\bar\Phi\cdot N) = T^k\left(\sum_j \partial_{X_j}T^{k-1}\bar\Phi N_j + \sum_{j'}\partial_{Y_{j'}}T^{k-1}\bar\Phi N_{j'+d}\right)$$

and since $T^0\bar\Phi = 0$, then

$$\begin{aligned}(4.6)\quad & T^k(DT^k\bar\Phi\cdot N) \\ & = T^k\left(\sum_j \partial_{X_j}T^{k-1}\bar\Phi T^{k-1}N_j + \sum_{j'}\partial_{Y_{j'}}T^{k-1}\bar\Phi T^kN_{j'+d}\right)\end{aligned}$$

Note that $T^{k-1}N_j$ and $T^kN_{j'+d}$ are exactly the components of $T^{k-1}N$, which is assumed to be a known analytic vector field. Since $T^{k-1}\bar\Phi$ is also known and analytic, then the whole quantity (4.6) is given and analytic. Thus, in (4.5), only $T^kN - T^{k-1}N + T^k(D\bar\Phi\cdot S) - T^{k-1}(D\bar\Phi\cdot S) - (S\circ T^k\Phi - S\circ T^{k-1}\Phi)$ is unknown. One therefore has to solve

$$T^kN - T^{k-1}N + T^k(D\bar\Phi\cdot S) - T^{k-1}(D\bar\Phi\cdot S) - (S\circ T^k\Phi - S\circ T^{k-1}\Phi) = G$$

where G is a given analytic vector field of degree k, namely

$$\begin{aligned}G = \sum_I T^k(D_IT^{k-|I|}R) - S - T^{k-1}N \\ - T^k(DT^k\bar\Phi\cdot N) - T^{k-1}D\bar\Phi\cdot S + S\circ T^{k-1}\bar\Phi\end{aligned}$$

which is homogeneous in $Y\in\mathcal{W}_{\delta_0}$, of degree k, by construction of $T^{k-1}N$ and $T^{k-1}\Phi$. The Fourier-Taylor coefficient $G_{P,Q}$ is well defined since the sum runs over a finite number of indices.

Letting

$$T^k N - T^{k-1} N = G_{res}$$

and

$$\bar{\Phi}_{L,P,Q} = (i\langle P, \omega\rangle + \langle Q, \Lambda\rangle)^{-1} G_{L,P,Q}$$

for non resonant P, Q such that $|Q| = k$ and $L \leqslant d$, and

$$\bar{\Phi}_{L,P,Q} = (i\langle P, \omega\rangle + \langle Q, \Lambda\rangle - \lambda_{L-d})^{-1} G_{L,P,Q}$$

for all P, Q such that $i\langle P, \omega\rangle + \langle Q, \Lambda\rangle - \lambda_{L-d} \neq 0$, with $|Q| = k$ and $L \geqslant d+1$, the formal diffeomorphism Φ and of the normal form NF are defined up to order k. □

5. Analytic normal form

This section deals with the problem of the existence of an analytic normalization.

5.1. Preliminary observations

The following is a lemma stating that being resonant is preserved by composition with a resonant diffeomorphism.

LEMMA 5.1. — *Let Φ be a resonant holomorphic diffeomorphism from $\mathcal{V} \times \mathcal{W} \subset \mathbb{C}^d/\mathbb{Z}^d \times \mathbb{C}^n$ to $\mathcal{V}' \times \mathcal{W}' \subset \mathbb{C}^d/\mathbb{Z}^d \times \mathbb{C}^n$, which is tangent to identity. If R is an analytic resonant vector field on $\mathcal{V}' \times \mathcal{W}'$, then $R \circ \Phi$ is also resonant.*

Proof. — Developing in Taylor series, for all $(X, Y) \in \mathcal{V} \times \mathcal{W}$,

$$R \circ \Phi(X, Y) = \sum_{I \in \mathbb{N}^{n+d}} c_I \partial_I R(X, Y)[(\Phi - Id)(X, Y)]^I$$

with $c_I = \frac{1}{I_1! \dots I_{d+n}!}$. Now Φ is resonant, which implies (see Definition 2.12) that

$$(\Phi - Id)(X, Y)^I Y_1^{-i_{d+1}} \dots Y_n^{-i_{d+n}}$$

is a resonant series.

Thus, for all I in the sum, $\partial_I R(X, Y)[(\Phi - Id)(X, Y)]^I$ is a resonant vector field, and this property passes to the sum. □

LEMMA 5.2. — *Let R be a formal resonant vector field and Φ a resonant analytic diffeomorphism which is tangent to identity. Then $D\Phi \cdot R$ is a formal resonant vector field.*

Proof. — Let $\tilde{\Phi}_1, \dots, \tilde{\Phi}_{d+n}$ be such that

$$\Phi = (X_1 + \tilde{\Phi}_1, \dots, X_d + \tilde{\Phi}_d, Y_1\tilde{\Phi}_{d+1}, \dots, Y_n\tilde{\Phi}_{d+n})$$

then

$$(5.1) \quad D\Phi \cdot R = \begin{pmatrix} R_1 + \partial_{X_1}\tilde{\Phi}_1 R_1 + \dots + \partial_{X_d}\tilde{\Phi}_1 R_d + \partial_{Y_1}\tilde{\Phi}_1 R_{d+1} + \dots + \partial_{Y_n}\tilde{\Phi}_1 R_{d+n} \\ \vdots \\ R_d + \partial_{X_1}\tilde{\Phi}_d R_1 + \dots + \partial_{X_d}\tilde{\Phi}_d R_d + \partial_{Y_1}\tilde{\Phi}_d R_{d+1} + \dots + \partial_{Y_n}\tilde{\Phi}_d R_{d+n} \\ Y_1 \sum_{j=1}^d \partial_{X_j}\tilde{\Phi}_{d+1} R_j + \tilde{\Phi}_{d+1} R_{d+1} + Y_1 \sum_{j'=1}^n \partial_{Y_{j'}}\tilde{\Phi}_{d+1} R_{d+j'} \\ \vdots \\ Y_n \sum_{j=1}^d \partial_{X_j}\tilde{\Phi}_{d+n} R_j + \tilde{\Phi}_{d+n} R_{d+n} + Y_n \sum_{j'=1}^n \partial_{Y_{j'}}\tilde{\Phi}_{d+n} R_{d+j'} \end{pmatrix}$$

Since all the $\tilde{\Phi}_j$ are resonant by assumption, and since R is a resonant vector field, then $D\Phi \cdot R$ is a resonant vector field. □

The following proposition states that if a system has been normalized up to some order k, then the formal normalizing diffeomorphism of this partially normalized system has to be resonant up to order k.

PROPOSITION 5.3. — *Let Φ be a formal diffeomorphism. Suppose that Φ formally conjugates an analytic vector field $S + N_k + R_k$ where N_k is resonant of order 1 and R_k has order $k+1$, to a formal resonant vector field $NF_1 = S + R$, where R has order 1 : for all $j \in \mathbb{N}$,*

$$T^j(DT^j\Phi \cdot NF_1) = T^j[(S + N_k + R_k) \circ T^j\Phi]$$

Then $T^k\Phi$ is a resonant diffeomorphism.

Proof. — By assumption and since R_k has order $k+1$, for all $l \leqslant k$,

$$T^l(DT^l\Phi \cdot NF_1) = S \circ T^l\Phi + T^l(N_k \circ T^l\Phi)$$

and since N_k has order 1,

$$T^l(DT^l\Phi \cdot NF_1) = S \circ T^l\Phi + T^l(N_k \circ T^{l-1}\Phi)$$

Now $NF_1 = S + R$ where R has order 1, thus

$$T^l(DT^l\Phi \cdot S) + T^l(DT^{l-1}\Phi \cdot R) = S \circ T^l\Phi + T^l(N_k \circ T^{l-1}\Phi)\cdot$$

By induction, let $l \leqslant k-1$ and assume that $T^{l-1}\Phi$ is a resonant diffeomorphism. By Lemma 5.1, this implies that $T^l(N_k \circ T^{l-1}\Phi)$ is a resonant vector field. Moreover, $T^l(DT^{l-1}\Phi \cdot R)$ is resonant by Lemma 5.2. Thus, $T^l(DT^l\Phi \cdot S) - S \circ T^l\Phi$ is a resonant vector field.

Let $j \leqslant d$. Since

$$T^l(DT^l\Phi_j \cdot S) - \omega_j = \sum_{P,|Q|\leqslant l} T^l\Phi_{j,P,Q}(i\langle P,\omega\rangle + \langle Q,\Lambda\rangle)e^{i\langle P,X\rangle}Y^Q - \omega_j$$

and since this quantity is a resonant function as shown above, then for all non-resonant (P,Q) with $|Q| \leqslant l$, $\Phi_{j,P,Q} = 0$ and therefore Φ_j is resonant up to order l. Similarly, if $j \geqslant d+1$,

$$T^l(DT^l\Phi_j \cdot S) - \lambda_{j-d}T^l\Phi_j = \sum_{P,|Q|\leqslant l} \Phi_{j,P,Q}(i\langle P,\omega\rangle + \langle Q,\Lambda\rangle - \lambda_{j-d})e^{i\langle P,X\rangle}Y^Q$$

and since $Y_{j-d}^{-1}(T^l(DT^l\Phi_j \cdot S) - \lambda_{j-d}T^l\Phi_j)$ is resonant, then $\Phi_{j,P,Q} = 0$ if $i\langle P,\omega\rangle + \langle Q,\Lambda\rangle - \lambda_{j-d} \neq 0$ for all $|Q| \leqslant l$, which implies that $T^l\Phi_j(X,Y)Y_{j-d}^{-1}$ is a resonant function.

Therefore, $T^l\Phi$ is a resonant diffeomorphism. By immediate induction, the formal diffeomorphism Φ is resonant up to order k. □

The next proposition states that the algebraic property (1.5) is preserved by partial normalization tangent to identity; in other terms, if a vector field has been normalized analytically up to order k, i.e. if it is analytically conjugate to a vector field $N_k + R_k$ where N_k is resonant and R_k has order $k+1$, and if the same vector field is formally conjugate to a resonant vector field of the form (1.5), then the partial normal form N_k also satisfies (1.5).

Proposition 5.4. — *Let F_2 be an analytic vector field on $\mathcal{V} \times \mathcal{W}$ with quasilinear part S. Assume that F_2 is formally normalizable, i.e. there exists a formal diffeomorphism Φ which is tangent to identity and a formal vector field N_1 such that Φ conjugates F_2 to N_1: for all $k \in \mathbb{N}$,*

$$T^k(DT^k\Phi \cdot N_1) = T^k(F_2 \circ T^k\Phi) \tag{5.2}$$

For all $k \in \mathbb{N}$, if T^kF_2 is a resonant analytic vector field and if $T^kN_1 = a_kS$ where a_k is L^2 in the first set of variables X and has degree k in Y, with constant part 1, then $T^kF_2 = bS$, for a resonant analytic function b which has constant part 1.

Proof. — By (5.2) and by assumption on N_1, one has

$$T^k(DT^k\Phi \cdot a_kS) = T^k(F_2 \circ T^k\Phi) \tag{5.3}$$

Now let $\bar{\Phi} = \Phi - Id$. If $1 \leqslant l \leqslant d$,

$$
\begin{aligned}
(5.4)\quad & DT^k\Phi_l(X,Y)\cdot a_k(X,Y)S \\
&= a_k(X,Y)DT^k\Phi_l(X,Y)\cdot S \\
&= a_k(X,Y)[\sum_{j=1}^{d}\partial_{X_j}T^k\Phi_l(X,Y)\cdot\omega_j + \sum_{j'=1}^{n}\partial_{Y_{j'}}T^k\Phi_l(X,Y)\cdot\lambda_{j'}Y_{j'}] \\
&= a_k(X,Y)[\sum_{j=1}^{d}\delta_{j,l}\omega_j + \partial_{X_j}T^k\bar{\Phi}_l(X,Y)\cdot\omega_j + \sum_{j'=1}^{n}\partial_{Y_{j'}}T^k\bar{\Phi}_l(X,Y)\cdot\lambda_{j'}Y_{j'}] \\
&= a_k(X,Y)[\omega_l + \sum_{P,|Q|\leqslant k}\bar{\Phi}_{l,P,Q}e^{i\langle P,X\rangle}Y^Q(i\langle P,\omega\rangle + \langle Q,\Lambda\rangle)]
\end{aligned}
$$

and the resonant part is merely $a_k(X,Y)\omega_l$. If $d+1 \leqslant l \leqslant d+n$,

$$
\begin{aligned}
(5.5)\quad & DT^k\Phi_l(X,Y)\cdot a_k(X,Y)S \\
&= a_k(X,Y)[\sum_{j=1}^{d}\partial_{X_j}T^k\Phi_l(X,Y)\cdot\omega_j + \sum_{j'=1}^{n}\partial_{Y_{j'}}T^k\Phi_l(X,Y)\cdot\lambda_{j'}Y_{j'}] \\
&= a_k(X,Y)[\sum_{j=1}^{d}\partial_{X_j}T^k\bar{\Phi}_l(X,Y)\cdot\omega_j + \sum_{j'=1}^{n}(\partial_{Y_{j'}}T^k\bar{\Phi}_l(X,Y) + \delta_{j',l})\lambda_{j'}Y_{j'}] \\
&= a_k(X,Y)[\sum_{P,|Q|\leqslant k}\bar{\Phi}_{P,Q}e^{i\langle P,X\rangle}Y^Q(i\langle P,\omega\rangle + \langle Q,\Lambda\rangle) + \lambda_l Y_l]
\end{aligned}
$$

therefore the resonant part of $Y_l^{-1}DT^k\Phi_l(X,Y)\cdot a_k(X,Y)S$ is $\lambda_l a_k(X,Y)$. On the other side, if one assumes that T^kF_2 is resonant, then

$$T^k(F_2\circ T^k\Phi) = T^k(N_k\circ T^k\Phi + R_k\circ T^k\Phi)$$

where R_k has order $k+1$ and N_k is resonant. Since the composition increases the order,

$$T^k(F_2\circ T^k\Phi) = T^k(N_k\circ T^k\Phi)$$

By Proposition 5.3, $T^k\Phi$ is resonant, therefore, by Lemma 5.1, $T^k(N_k\circ T^k\Phi)$ is resonant, which implies that $T^k(F_2\circ T^k\Phi)$ is resonant. Thus, keeping only the resonant parts in (5.4) and (5.5) and substituting them in the resonant part of (5.3), one has

$$a_k\cdot S = T^k(N_k\circ T^k\Phi)$$

Now this implies that T^kN_k itself is proportional to S: indeed, letting

$$\Phi_X(X,Y) = (T^k\Phi_1(X,Y),\ldots,T^k\Phi_d(X,Y))$$

and

$$\Phi_Y(X,Y) = (T^{k+1}\Phi_{d+1}(X,Y), \dots, T^{k+1}\Phi_{d+n}(X,Y))$$

note that Φ_X and Φ_Y are analytic) then developing in Taylor series with respect to the second set of variables Y, for all $0 \leqslant j \leqslant k$,

$$\begin{aligned}(5.6)\quad & T^j(N_k \circ (\Phi_X(X,Y), \Phi_Y(X,Y))) \\ &= \sum_{I\in\mathbb{N}^n, |I|\leqslant j} T^j \partial_I N_k(\Phi_X(X,Y), Y) \cdot (\Phi_Y(X,Y) - Y)^I\end{aligned}$$

where $|I| = i_1 + \dots + i_n$ for all $I = (i_1, \dots, i_n)$ and $\partial_I = \partial_{Y_1}^{i_1} \dots \partial_{Y_n}^{i_n}$ (recall that by Lemma 4.1, $f \mapsto \partial_I f \cdot (\Phi_Y(X,Y) - Y)^I$ increases the order by $|I|$). By a simple recurrence on the order of truncation j, one finds that

- $T^0 N_k = a_0 \cdot S$,
- for all $0 \leqslant j \leqslant k-1$,

$$\begin{aligned}(5.7)\quad & T^j N_k(X,Y) = a_j(X,Y) \cdot S(Y) \\ & - \sum_{|I|=1}^{j} T^j(\partial_I T^{j-|I|}(N_k(\Phi_X(X,Y), Y)) \cdot (\Phi_Y(X,Y) - Y)^I\end{aligned}$$

  and if by assumption $T^{j-1}N_k = b_{j-1}S$ for an analytic function b_{j-1}, then for all I in the sum, $T^{j-|I|}(N_k(\Phi_X(X,Y),Y))$ is proportional to S, therefore

$$T^j(\partial_I T^{j-|I|}(N_k(\Phi_X(X,Y), Y)) \cdot (\Phi_Y(X,Y) - Y)^I$$

  is also proportional to S since $(\Phi_Y(X,Y) - Y)^I$ is a scalar, and it is analytic in (X,Y).

Finally, $T^k N_k = b \cdot S$ for an analytic function b. □

5.2. Proof of Theorem 1.1

This section is dedicated to the proof of the main result. Let δ_0 be as in Section 3; assume that $\delta_0 \leqslant \min(1, C_\omega)$ where C_ω was defined in Section 3.

Theorem 5.5. — *Assume that the analytic vector field $S+R \in VF_{r_0,\delta_0}$ generating (3.1) is formally conjugate to $a \cdot S$, where the formal series a has constant part 1.*

There exists $\zeta_0 > 0$ depending only on Λ, ω, n, d such that if $||R||_{r_0,\delta_0} \leqslant \zeta_0$, then $S+R$ is holomorphically conjugate to a resonant vector field $b \cdot S$, where b is a holomorphic function.

5.2.1. Sequences of parameters

Let $(m_k)_{k\geqslant 0}$ and $(r_k)_{k\geqslant 0}$ be the sequences defined in the Assumption 3.1 (by Assumption 3.1.7, the sequence (r_k) has a positive limit as $k \to \infty$). For all $k \geqslant 0$, let

$$\delta_{k+1} = \delta_k g(m_k)^{-(17+10n)/m_k} \tag{5.8}$$

(recall that $m_k \geqslant 1$ and that $g(m_k)$ is positive). By the Assumption 3.1.2, the sequence δ_k has a positive limit $\delta_\infty \leqslant 1$ as $k \to +\infty$.

Notation. — For all $k \in \mathbb{N}$, we will hereafter denote by $\mathcal{O}_k$ the set $\mathcal{V}_{r_k} \times \mathcal{W}_{\delta_k}$.

Let $(\zeta_k)_{k\geqslant 0}$ be a real sequence such that

$$\zeta_0 < \frac{\delta_\infty}{8(n+d)(1+2C''_S(n+2)B)} \tag{5.9}$$

and

$$\zeta_{k+1} = \frac{2C''_S}{g(m_k)}\zeta_k$$

where C''_S is a fixed constant, depending only on n, d and S, which will appear in the computations below, and B is the Brjuno sum defined in the Assumption 3.1.2. As mentioned in Section 3, one can assume that for all $k \geqslant 0$,

$$\frac{2C''_S}{g(m_k)} < 1 \tag{5.10}$$

so that $(\zeta_k)_{k\geqslant 0}$ is a strictly decreasing sequence. Moreover, let

$$\eta_k = \sum_{j=0}^{k} \zeta_j$$

Notice that for all $k \geqslant 0$,

$$\eta_k = \zeta_0 + \sum_{j=1}^{k} \zeta_0 \prod_{l=0}^{j-1} \frac{2C''_S}{g(m_l)} \leqslant \zeta_0 + \sum_{j=1}^{k} \zeta_0 \frac{2C''_S}{g(m_{j-1})}$$

Using the Assumption 3.1.3, this implies that

$$\eta_k \leqslant \zeta_0 + \sum_{j=1}^{k} \zeta_0 \frac{2C''_S(n+2)\ln g(m_{j-1})}{m_{j-1}} \leqslant \zeta_0(1+2C''_S(n+2)B)$$

therefore, under the Assumption (5.9), $\eta_k \leqslant \frac{1}{8}$.

5.2.2. Iteration step

Theorem 5.5 will be proved by an iteration of the following statement, the proof of which is postponed to the last section:

Proposition 5.6. — *Let $N_k, R_k \in VF_{r_k,\delta_k}$ with N_k resonant and R_k of order m_k. Assume that*

1. *$S + N_k + R_k$ is formally conjugate to the normal form $a \cdot S$ where a is a formal series;*
2. *$||N_k - S||_{r_k,\delta_k} \leqslant \eta_k$,*
3. *$||R_k||_{r_k,\delta_k} \leqslant \zeta_k$,*

then there exists an analytic diffeomorphism Φ_k on $\mathcal{O}_{k+1}$ with values in $\mathcal{O}_k$, conjugating $S+N_k+R_k$ to $S+N_{k+1}+R_{k+1}$ where N_{k+1} is resonant, R_{k+1} has order m_{k+1}, such that the following properties hold:

1. *$||N_{k+1} - S||_{r_{k+1},\delta_{k+1}} \leqslant \eta_{k+1}$,*
2. *$||R_{k+1}||_{r_{k+1},\delta_{k+1}} \leqslant \zeta_{k+1}$,*
3. *$||\Phi_k - Id||_{r_{k+1},\delta_{k+1}} \leqslant \zeta_{k+1}$,*
4. *the quantity*
$$\begin{aligned}&||D\Phi_k - I||_{r_{k+1},\delta_{k+1}}\\ &\qquad := \max\{||\partial_{X_j}(\Phi_k - Id)||_{r_{k+1},\delta_{k+1}}, ||\partial_{Y_{j'}}(\Phi_k - Id)||_{r_{k+1},\delta_{k+1}}\}\end{aligned}$$
is less than ζ_{k+1}.

Remark. — Under the assumptions of Theorem 5.5, the initial system satisfies the assumptions of Proposition 5.6 with $k = 0$, $N_0 = 0$, $R_0 = R$.

5.2.3. Convergence of the algorithm

We now proceed with the iteration of Proposition 5.6. For all $k \in \mathbb{N}$, let

$$\Psi_k := \Phi_0 \circ \cdots \circ \Phi_k \tag{5.11}$$

The map Ψ_k is well defined on $\mathcal{O}_{k+1}$ with values in $\mathcal{O}_0$; moreover, Ψ_k conjugates $S + N_k + R_k$ to $S + R$.

Lemma 5.7. — *The map Ψ_k is a holomorphic diffeomorphism from $\mathcal{O}_{k+1}$ to $\Psi_k(\mathcal{O}_{k+1})$.*

Proof. — The map Ψ_k is analytic on $\mathcal{O}_{k+1}$ by composition, since every Φ_j is analytic on $\mathcal{O}_{j+1}$. Moreover, for all k and all $(X,Y) \in \mathcal{O}_{k+1}$,

$$\begin{aligned}||D\Psi_k(X,Y) - I|| &= ||(D\Psi_{k-1} - I)\circ\Phi_k \cdot D\Phi_k + D\Phi_k - I||\\ &\leqslant ||(D\Psi_{k-1} - I)\circ\Phi_k \cdot D\Phi_k|| + \zeta_{k+1}\end{aligned} \tag{5.12}$$

which implies, by the property 4 of Proposition 5.6, that on $\mathcal{O}_{k+1}$,

$$||D\Psi_k(X,Y) - I|| \leqslant ||(D\Psi_{k-1} - I) \circ \Phi_k||(1+\zeta_{k+1}) + \zeta_{k+1} \tag{5.13}$$

Let $(u_n)_{n\in\mathbb{N}}$ be defined by $u_0 = \zeta_1$ and for all $n \geqslant 1$, $u_n = (1+\zeta_n)u_{n-1} + \zeta_n$ One computes easily that for all $n \geqslant 2$,

$$u_n = \zeta_1 \prod_{l=1}^{n}(1+\zeta_l) + \zeta_1 \prod_{l=2}^{n}(1+\zeta_l) + \cdots + \zeta_{n-2}(1+\zeta_{n-1})(1+\zeta_n) \\ + \zeta_{n-1}(1+\zeta_n) + \zeta_n$$

therefore $u_n \leqslant \eta_n \prod_{l=1}^{n}(1+\zeta_l) \leqslant \frac{1}{8}e^{\eta_n} \leqslant \frac{1}{8}e^{\frac{1}{8}}$. Now, assuming for all $(X,Y) \in \mathcal{O}_k$ that

$$||(D\Psi_{k-1} - I)(X,Y)|| \leqslant u_k$$

(which is true for $k=1$ by the property 4 of Proposition 5.6 with $k=0$), then on $\mathcal{O}_{k+1}$, the estimate (5.13) implies that

$$||D\Psi_k(X,Y) - I|| \leqslant u_k(1+\zeta_{k+1}) + \zeta_{k+1} = u_{k+1} \tag{5.14}$$

and one deduces by recurrence that for all k,

$$||D\Psi_k(X,Y) - I|| \leqslant \frac{1}{8}e^{\frac{1}{8}} < 1 \tag{5.15}$$

The map Ψ_k is therefore an analytic diffeomorphism from $\mathcal{O}_{k+1}$ to its image. □

Definition 5.8. — *Let $\mathcal{O}_\infty = \mathcal{V}_{r_\infty} \times \mathcal{W}_{\delta_\infty}$, where r_∞ is the limit of the sequence (r_k) and δ_∞ is the limit of the sequence (δ_k), or equivalently, $\mathcal{O}_\infty = \cap_k \mathcal{O}_k$.*

Proposition 5.9. — *The sequence $(\Psi_k)_{k\geqslant 1}$ defined in (5.11) has a subsequence which converges to a holomorphic diffeomorphism on $\mathcal{O}_\infty := \cap_{k\in\mathbb{N}}\mathcal{O}_k$.*

Proof. — A simple recurrence shows that $||Id - \Psi_k||_{r_\infty,\delta_\infty} \leqslant \eta_{k+1}$. Indeed,

$$Id - \Psi_k = (Id - \Psi_{k-1}) \circ \Phi_k + (Id - \Phi_k)$$

therefore

$$||Id - \Psi_k||_{r_{k+1},\delta_{k+1}} \leqslant ||Id - \Psi_{k-1}||_{r_k,\delta_k} + ||Id - \Phi_k||_{r_{k+1},\delta_{k+1}}$$

and one sees that if $||Id - \Psi_{k-1}||_{r_k,\delta_k} \leqslant \eta_k$, then $||Id - \Psi_k||_{r_{k+1},\delta_{k+1}} \leqslant \eta_{k+1}$. Thus (Ψ_k) is uniformly bounded on $\mathcal{O}_\infty$ in the analytic weighted norm, therefore there exists a subsequence (Ψ_{k_l}) which is a Cauchy sequence for this norm. In particular, (Ψ_{k_l}) converges uniformly on every compact subset of $\mathcal{O}_\infty$ to a map Ψ_∞ which is holomorphic on $\mathcal{O}_\infty$.

Moreover, the estimate (5.15) is uniform in k. Therefore, Ψ_∞ is still injective on $\mathcal{O}_\infty$, thus it is a holomorphic diffeomorphism on $\mathcal{O}_\infty$. □

The sequence $(N_k) = (a_k S)$ is convergent in the topology of the analytic functions on $\mathcal{O}_\infty$, since $||N_{k+1} - N_k||_{r_\infty,\delta_\infty} = ||\bar{R}_{k,res}||_{r_\infty,\delta_\infty} \leqslant \zeta_k$ and since ζ_k is summable (there exists a constant $c < 1$ such that $\zeta_k \leqslant c^k \zeta_0$). Let N_∞ be the limit of (N_k) in this topology; then $N_\infty = a_\infty S$ for some resonant function a_∞ which is holomorphic on $\mathcal{O}_\infty$. The sequence R_k tends to 0 in the analytic topology on $\mathcal{O}_\infty$. Therefore Ψ_∞ conjugates $S + N_\infty$ to $S + R$.

This concludes the proof of Theorem 5.5 based on Proposition 5.6, which is proved in the next section.

6. Proof of Proposition 5.6

We now make all the assumptions of Proposition 5.6.

If $S + N_k + R_k$ is formally conjugate to the normal form $a \cdot S$, then by Proposition 5.4, there exists a formal series a_k with constant part 1 such that $N_k = a_k \cdot S$ and since N_k is analytic on $\mathcal{V}_{r_k} \times \mathcal{W}_{\delta_k}$, so is a_k. Moreover, if $||N_k - S||_{r_k,\delta_k} \leqslant \eta_k$, then by our definition of the norm of a vector field given in Section 2, for all $1 \leqslant j \leqslant d$,

$$|(a_k - 1)\omega_j|_{r_k,\delta_k} \leqslant \eta_k \tag{6.1}$$

therefore $|a_k - 1|_{r_k,\delta_k} \leqslant \frac{\eta_k}{\max_j |\omega_j|} \leqslant \frac{\delta_\infty}{8(n+d)C_\omega} \leqslant \frac{1}{2}$ (since it was assumed that $\delta_0 \leqslant C_\omega$).

6.1. Homological equation

As a first step in the construction of the diffeomorphism Φ_k, one has to solve the homological equation

$$\mathcal{L}G_k := [G_k, N_k] = \bar{R}_k$$

where $\bar{R}_k$ contains all non resonant monomials of R_k with degree between m_k and $m_{k+1} - 1$. For every vector field F, one has

$$\mathcal{L}F = [F, a_k S] = a_k[F, S] + F(a_k)S$$

(where $F(a_k)$ stands for the Lie derivative of a_k under F). Let $\mathcal{D} : F \mapsto a_k[F, S]$ and $\mathcal{N} : F \mapsto F(a_k)S$ which are defined and linear on the space of formal vector fields.

The operator $\mathcal{N}$ satisfies:

$$\mathcal{N}^2 F = \mathcal{N}(F(a_k)S) = F(a_k)S(a_k) = 0$$

(since a_k is resonant). Moreover,

$$\mathcal{D}\mathcal{N}F = \mathcal{D}(F(a_k)S) = a_k[F(a_k)S, S] = -a_k S(F(a_k))S$$

and

$$\mathcal{N}\mathcal{D}F = \mathcal{N}(a_k[F,S]) = -a_k S(F(a_k))S$$

so that $\mathcal{N}$ and $\mathcal{D}$ commute. Thus, formally

$$(\mathcal{D}+\mathcal{N})^{-1} = (I+\mathcal{D}^{-1}\mathcal{N})^{-1}\mathcal{D}^{-1} = \mathcal{D}^{-1}(I-\mathcal{N}\mathcal{D}^{-1})$$

where $\mathcal{D}^{-1}$ and $(\mathcal{D}^{-1})^2$ are defined, i.e. on non resonant formal vector fields. Therefore, the solution will be

$$G_k = \mathcal{D}^{-1}(I-\mathcal{N}\mathcal{D}^{-1})\bar{R}_k$$

since $\bar{R}_k$ is a non resonant vector field. Now for all $(P,Q) \in \mathbb{Z}^d \times \mathbb{N}^n$,

$$\begin{aligned}(6.2)\quad & ad_S\left(\sum_{j=1}^{d} e^{i\langle P,X\rangle} Y^Q \frac{\partial}{\partial X_j} + \sum_{j'=1}^{n} e^{i\langle P,X\rangle} Y^{Q+E_{j'}} \frac{\partial}{\partial Y_{j'}}\right)\\ &= (i\langle P,\omega\rangle + \langle Q,\Lambda\rangle)\left(\sum_{j=1}^{d} e^{i\langle P,X\rangle} Y^Q \frac{\partial}{\partial X_j} + \sum_{j'=1}^{n} e^{i\langle P,X\rangle} Y^{Q+E_{j'}} \frac{\partial}{\partial Y_{j'}}\right)\end{aligned}$$

therefore the operators ad_S and $a_k\cdot$ (the multiplication by a_k) preserve the equivalence classes $\mathcal{C}_c$ of indices $(P,Q) \in \mathbb{Z}^d \times \mathbb{N}^n$ such that $i\langle P,\omega\rangle + \langle Q,\Lambda\rangle = c$; moreover, $\mathcal{N}$ also preserves these equivalence classes since

$$\begin{aligned}(6.3)\quad & \sum_{j=1}^{d} e^{i\langle P,X\rangle} Y^Q \frac{\partial}{\partial X_j} + \sum_{j'=1}^{n} e^{i\langle P,X\rangle} Y^{Q+E_{j'}} \frac{\partial}{\partial Y_{j'}}\\ &= \left(\sum_{j=1}^{d} e^{i\langle P,X\rangle} Y^Q \frac{d}{dX_j} a_k(X,Y) + \sum_{j'=1}^{n} e^{i\langle P,X\rangle} Y^{Q+E_{j'}} \frac{d}{dY_{j'}} a_k(X,Y)\right) S(Y)\end{aligned}$$

Therefore,

$$(6.4)\quad (\mathcal{D}+\mathcal{N})F = R \iff F = \sum_c F_c \text{ and } \forall c \in \mathbb{C},$$

$$(\mathcal{D}+\mathcal{N})F_c = \sum_{(P,Q)\in\mathcal{C}_c}\left(\sum_{j=1}^{d} R_{j,P,Q}\frac{\partial}{\partial X_j} + \sum_{j'=1}^{n} R_{d+j',P,Q+E_{j'}} Y_{j'}\frac{\partial}{\partial Y_{j'}}\right) e^{i\langle P,X\rangle} Y^Q$$

This makes it possible to separate the homological equation into two parts: the low frequency part, and the high frequency part. This means distinguishing two cases for c:

(1) either $\mathcal{C}_c$ contains a couple (P,Q) with $|P| \leqslant m_k$,

(2) or all elements of $\mathcal{C}_c$ satisfy $|P| > m_k$

(the equivalence class $\mathcal{C}_0$ does not enter into the computation, since $\bar{R}_k$ is non resonant). We will denote by $\mathcal{I}_0$ the set of all non empty equivalence classes which are in the first case, and by $\mathcal{I}_\infty$ the set of all equivalence classes which are in the second case. Thus there is a decomposition

$$\bar{R}_k = \bar{R}_k^0 + \bar{R}_k^\infty$$

where $\bar{R}_k^0$ only has monomials $\bar{R}_{k,P,Q}$ such that (P,Q) belongs to an element of $\mathcal{I}_0$, and $\bar{R}_k^\infty$ only has monomials $\bar{R}_{k,P,Q}$ such that (P,Q) belongs to an element of $\mathcal{I}_\infty$, both being non resonant. Since $\bar{R}_k$ is truncated with respect to the degree in Y, the low frequency part $\bar{R}_k^0$ has monomials whose indices belong to a finite number of equivalence classes. One will solve separately $\mathcal{L}G_k^0 = \bar{R}_k^0$ and $\mathcal{L}G_k^\infty = \bar{R}_k^\infty$, then let $G_k = G_k^0 + G_k^\infty$. As shown above, the solutions are given by

$$G_k^0 = \mathcal{D}^{-1}(I - \mathcal{N}\mathcal{D}^{-1})\bar{R}_k^0$$

and

$$G_k^\infty = \mathcal{D}^{-1}(I - \mathcal{N}\mathcal{D}^{-1})\bar{R}_k^\infty$$

(since $\bar{R}_k^0$ and $\bar{R}_k^\infty$ are non resonant, the solutions are defined and they will be analytic under some assumptions on $\bar{R}_k^0, \bar{R}_k^\infty$). It remains to estimate G_k^0 and G_k^∞ in the analytic norm on $\mathcal{O}_{k+1}$.

Estimate of the operator $\mathcal{N}$

It is convenient first to give an estimate of $\mathcal{N}$ acting on $VF_{r'',\delta''}$ into VF_{r_3,δ_3}, where the positive parameters $r'' > r_3, \delta'' > \delta_3$ will be given later.

Applying $\mathcal{N}$ does not add small divisors. One has

$$
\begin{aligned}
(6.5)\quad & F(a_k)(X,Y)\\
&= \sum_{j=1}^{d} F_j(X,Y)\frac{d}{dX_j}a_k(X,Y) + \sum_{j'=1}^{n} F'_{j'}(X,Y)\frac{d}{dY_{j'}}a_k(X,Y)\\
&= \sum_{P,Q}(a_k)_{P,Q}\sum_{j=1}^{d}F_j(X,Y)iP_j e^{i\langle P,X\rangle}Y^Q\\
&\quad + \sum_{P,Q/Q_{j'}\geqslant 1}(a_k)_{P,Q}\sum_{j'=1}^{n}F'_{j'}(X,Y)Q_{j'}Y^{Q-E_{j'}}e^{i\langle P,X\rangle}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(6.6)\quad & |F(a_k)|_{r_3,\delta_3}\\
&\leqslant \sum_{P,Q} e^{r_3|P|}\hat{\mathrm{E}}\delta_3^{|Q|}\sum_{P',Q'}|(a_k)_{P',Q'}|\\
&\quad\cdot[\sum_j |P'_j||F_{j,P-P',Q-Q'}| + \sum_{j'}|Q'_{j'}||F_{j',P-P',Q+E_{j'}-Q'}|]\\
&\leqslant |a_k|_{r'',\delta''}||F||_{r'',\delta''}\sum_{P,Q}e^{r_3|P|}\hat{\mathrm{E}}\delta_3^{|Q|}\\
&\quad\cdot\sum_{P',Q'}e^{-|P'|r''}\delta''^{-|Q'|}e^{-|P-P'|r''}\delta''^{-|Q-Q'|}[\sum_j|P'_j|+\sum_{j'}|Q'_{j'}|\delta'']
\end{aligned}
$$

Let $r_4 = \frac{r_3+r''}{2}$ and $\delta_4 = \sqrt{\delta''\delta_3}$, in order to have $r''-r_4 = r_4 - r_3$ and $\delta''/\delta_4 = \delta_4/\delta_3$. Then

$$
\begin{aligned}
(6.7)\quad |F(a_k)|_{r_3,\delta_3} &\leqslant |a_k|_{r'',\delta''}||F||_{r'',\delta''}\sum_{P,Q}e^{r_3|P|}\hat{\mathrm{E}}\delta_3^{|Q|}\\
&\quad\cdot\sum_{P'}e^{-|P'|r_4}e^{-|P-P'|r''}e^{-|P'|(r''-r_4)}(|P'|+n)\\
&\quad\cdot\sum_{Q'}\delta_4^{-|Q'|}(\delta''/\delta_4)^{-|Q'|}\delta''^{-|Q-Q'|}(d+|Q'|\delta'')\\
&\leqslant |a_k|_{r'',\delta''}||F||_{r'',\delta''}\sum_{P,Q}e^{r_3|P|}\hat{\mathrm{E}}\delta_3^{|Q|}\\
&\quad\cdot\sum_{P'}e^{-|P|r_4}e^{-|P'|(r''-r_4)}(|P'|+n)\\
&\quad\cdot\sum_{Q'}\delta_4^{-|Q|}(\delta''/\delta_4)^{-|Q'|}(d+|Q'|\delta'')
\end{aligned}
$$

Now

$$\sum_{P'} e^{-|P'|(r''-r_4)}(|P'|+n) = \sum_{K\in\mathbb{N}}\sum_{|P'|=K}(K+n)e^{-K(r''-r_4)} \tag{6.8}$$
$$\leqslant \sum_{K\in\mathbb{N}} C(d)K^{d-1}(K+n)e^{-K(r''-r_4)}$$
$$\leqslant C(n,d)\int_0^\infty t^d e^{-t(r''-r_4)}dt$$
$$\leqslant \frac{C(n,d)}{(r''-r_4)^{d+1}}$$

and

$$\sum_{Q'}(\delta''/\delta_4)^{-|Q'|}(d+|Q'|\delta'') = \sum_{L\in\mathbb{N}}\sum_{|Q'|=L}(\delta''/\delta_4)^{-L}(d+L\delta'') \tag{6.9}$$
$$\leqslant C(n,d)\sum_{L\in\mathbb{N}} L^n(\delta''/\delta_4)^{-L}$$
$$\leqslant C(n,d)\int_0^\infty t^n(\delta''/\delta_4)^{-t}dt$$
$$\leqslant C(n,d)(\ln\delta''/\delta_4)^{-n-1}$$

(here and below, $C(d)$ and $C(n,d)$ stand for generic constants which depend only on n,d) therefore

$$|F(a_k)|_{r_3,\delta_3} \leqslant C(n,d)|a_k|_{r'',\delta''}||F||_{r'',\delta''} \cdot \sum_{P,Q} e^{r_3|P|}\hat{\mathrm{E}}\delta_3^{|Q|}e^{-|P|r_4}(r''-r_4)^{-(d+1)}\delta_4^{-|Q|}(\ln\delta''/\delta_4)^{-n-1} \tag{6.10}$$

Thus,

$$|F(a_k)|_{r_3,\delta_3} \leqslant ||F||_{r'',\delta''}|a_k|_{r'',\delta''}\frac{C(n,d)}{(r''-r_3)^{2d+4}(\ln\delta_3-\ln\delta'')^{2n+4}}$$

Therefore,

$$|||\mathcal{N}|||_{C^\omega_{r'',\delta''}\to C^\omega_{r_3,\delta_3}} \leqslant |a_k|_{r'',\delta''}\frac{C(n,d)}{(r''-r_3)^{2d+4}(\ln\delta_3-\ln\delta'')^{2n+4}}C_\omega \tag{6.11}$$

Low frequency part

First consider the set $\mathcal{I}_0$ and let $\mathcal{C}_c \in \mathcal{I}_0$ with $c \neq 0$. Let

$$R_c(X,Y) = \sum_{(P,Q)\in\mathcal{C}_c}\left(\sum_{j=1}^{d}(\bar{R}_k)_{j,P,Q}\frac{\partial}{\partial X_j} + \sum_{j'=1}^{n}(\bar{R}_k)_{d+j',P,Q+E_{j'}}Y_{j'}\frac{\partial}{\partial Y_{j'}}\right)e^{i\langle P,X\rangle}Y^Q$$

The equation $\mathcal{D}F_c = R_c$ can be solved as follows:

$$F_c = \mathcal{D}^{-1}R_c = \sum_{(P,Q)\in\mathcal{C}_c}\frac{(R_c a_k^{-1})_{P,Q}}{c}e^{i\langle P,X\rangle}Y^Q \tag{6.12}$$

Now, since a_k is analytically close to 1 (i.e. $|a_k - 1|_{r_k,\delta_k} \leqslant \frac{1}{2}$), using the sub-multiplicativity of the weighted norm, $R_c a_k^{-1}$ is in VF_{r_k,δ_k}, with norm less than $2||R_c||_{r_k,\delta_k}$ and $R_c a_k^{-1}$ also has quasi-order m_k in Y. Thus, using the estimate (2.2), for all $1 \leqslant j \leqslant d+n$, $(R_c a_k^{-1})_{j,P,Q}$ has modulus less than $2||R_c||_{r_k,\delta_k}e^{-r_k|P|}\delta_k^{-|Q|}$. Therefore, letting $\delta = \delta_k, r = r_k$, for all $r' \in (0,r)$ and all $\delta' \in (0,\delta)$,

$$\begin{aligned} ||\mathcal{D}^{-1}R_c||_{r',\delta'} &\leqslant \frac{2}{c}||R_c||_{r,\delta}\sum_{Q,|Q|=m_k}^{m_{k+1}-1}(\frac{\delta'}{\delta})^{|Q|}\sum_{P/(P,Q)\in\mathcal{C}_c}e^{-(r-r')|P|} \\ &\leqslant \frac{2||R_c||_{r,\delta}}{c(r-r')^{d+1}}\sum_{Q,|Q|=m_k}^{m_{k+1}-1}(\frac{\delta'}{\delta})^{|Q|} \\ &\leqslant \frac{2||R_c||_{r,\delta}}{c(r-r')^{d+1}}(\frac{\delta'}{\delta})^{m_k}\frac{1}{(\ln\delta' - \ln\delta)^{n+1}} \end{aligned} \tag{6.13}$$

This implies the following estimate:

$$||\mathcal{D}^{-1}\bar{R}_k^0||_{r',\delta'} \leqslant \frac{2\sup_{c\in\mathcal{I}_0\setminus\{0\}}\{c^{-1}\}}{(r-r')^{d+1}}(\ln\frac{\delta'}{\delta})^{-(n+1)}||\bar{R}_k^0||_{r,\delta}(\frac{\delta'}{\delta})^{m_k} \tag{6.14}$$

(since $||\bar{R}_k^0||_{r,\delta} = \sum_{\mathcal{C}_c\in\mathcal{I}_0}||R_c||_{r,\delta}$). Applying $\mathcal{D}^{-1}$ once more on (6.12), one has

$$(\mathcal{D}^{-1})^2R_c = \sum_{(P,Q)\in\mathcal{C}_c}\frac{((\mathcal{D}^{-1}R_c)a_k^{-1})_{P,Q}}{c}e^{i\langle P,X\rangle}Y^Q$$

and since $(\mathcal{D}^{-1})R_c$ is in $C^{\omega}_{r',\delta'}$ and $|a_k - 1|_{r',\delta'} \leqslant \frac{1}{2}$, then for all $1 \leqslant j \leqslant d+n$,

$$|((\mathcal{D}^{-1}R_c)a_k^{-1})_{j,P,Q}| \leqslant ||(\mathcal{D}^{-1}R_c)a_k^{-1}||_{r',\delta'}e^{-|P|r'}\delta'^{-|Q|}$$

thus for all $r'' < r'$ and $\delta'' < \delta'$,

$$(6.15)\quad ||(\mathcal{D}^{-1})^2 R_c||_{r'',\delta''} \leqslant \frac{4\sup_{c\in\mathcal{I}_0\setminus\{0\}}\{c^{-1}\}^2}{(r-r')^{d+1}(r'-r'')^{d+1}}(\frac{\delta''}{\delta})^{m_k} \cdot \frac{1}{(\ln\delta'-\ln\delta)^{n+1}(\ln\delta''-\ln\delta')^{n+1}}||R_c||_{r,\delta}$$

therefore, by the Assumption 3.1.5,

$$(6.16)\quad ||(\mathcal{D}^{-1})^2 R_c||_{r'',\delta''} \leqslant \frac{4g(m_k)^2}{(r-r')^{d+1}(r'-r'')^{d+1}}(\frac{\delta''}{\delta})^{m_k} \cdot \frac{1}{(\ln\delta'-\ln\delta)^{n+1}(\ln\delta''-\ln\delta')^{n+1}}||R_c||_{r,\delta}$$

Using now the estimate (6.11), for the low frequencies of $\bar{R}_k$, one has, for some constant C' which only depends on n,d,

$$(6.17)\quad ||\mathcal{D}^{-1}(I-\mathcal{N}\mathcal{D}^{-1})R_c||_{r_3,\delta_3} \leqslant \frac{C'g(m_k)^2}{[(r-r')(r'-r'')]^{d+1}}\frac{(\frac{\delta''}{\delta})^{m_k}|a_k|_{r'',\delta''}}{[(\ln(\delta'/\delta))(\ln(\delta''/\delta'))]^{n+1}} \cdot \frac{C_\omega||R_c||_{r,\delta}}{(r''-r_3)^{2d+4}(\ln(\delta_3/\delta''))^{2n+4}}$$

therefore

$$(6.18)\quad ||G_k^0||_{r_3,\delta_3} \leqslant \frac{C'g(m_k)^2}{[(r-r')(r'-r'')]^{d+1}}\frac{(\frac{\delta''}{\delta})^{m_k}|a_k|_{r'',\delta''}}{[(\ln(\delta'/\delta))(\ln(\delta''/\delta'))]^{n+1}} \cdot \frac{C_\omega||\bar{R}_k^0||_{r,\delta}}{(r''-r_3)^{2d+4}(\ln(\delta_3/\delta''))^{2n+4}}$$

High frequency part

If $\mathcal{C}_c$ is in the second case (which implies that $c\neq 0$), by the Assumption 3.1, one has the estimate

$$(6.19)\quad ||\mathcal{D}^{-1}R_c||_{r',\delta'} \leqslant \frac{2}{c}||R_c||_{r,\delta}\sum_{Q,|Q|\geqslant m_k}(\frac{\delta'}{\delta})^{|Q|}\sum_{P/(P,Q)\in\mathcal{C}_c}e^{-(r-r')|P|}$$
$$\leqslant 2||R_c||_{r,\delta}\sum_{Q,|Q|\geqslant m_k}(\frac{\delta'}{\delta})^{|Q|}\sum_{P/(P,Q)\in\mathcal{C}_c}e^{-(r-r'-\epsilon_k)|P|}$$

(the computation is similar to (6.13)) thus

$$\begin{aligned}(6.20)\quad & ||\mathcal{D}^{-1}\bar{R}_k^{\infty}||_{r',\delta'}\\ &\leqslant 2\sum_{\mathcal{C}_c\in\mathcal{I}_\infty}||R_c||_{r,\delta}\sum_{Q,|Q|\geqslant m_k}(\frac{\delta'}{\delta})^{|Q|}\sum_{P/(P,Q)\in\mathcal{C}_c}e^{-(r-r'-\epsilon_k)|P|}\\ &\leqslant 2||\bar{R}_k^{\infty}||_{r,\delta}\sum_{Q,|Q|\geqslant m_k}(\frac{\delta'}{\delta})^{|Q|}\sum_{P,|P|>m_k}e^{-(r-r'-\epsilon_k)|P|}\\ &\leqslant \frac{2||\bar{R}_k^{\infty}||_{r,\delta}}{(r-r'-\epsilon_k)^{d+1}}(\frac{\delta'}{\delta})^{m_k}\frac{e^{-(r-r'-\epsilon_k)m_k}}{(\ln\delta'-\ln\delta)^{n+1}}\end{aligned}$$

(as in (6.8) and (6.9), the sums on P and Q were estimated by means of an integral). Applying $\mathcal{D}^{-1}$ once more, one gets

$$\begin{aligned}(6.21)\quad & ||(\mathcal{D}^{-1})^2\bar{R}_k^{\infty}||_{r'',\delta''}\\ &\leqslant \frac{4||\bar{R}_k^{\infty}||_{r,\delta}}{[(r-r'-\epsilon_k)(r'-r''-\epsilon_k)]^{d+1}}(\frac{\delta''}{\delta})^{m_k}\frac{e^{-(r-r'-\epsilon_k)m_k}e^{-(r'-r''-\epsilon_k)m_k}}{[(\ln(\delta'/\delta)\ln(\delta''/\delta')]^{n+1}}\\ &\leqslant \frac{4||\bar{R}_k^{\infty}||_{r,\delta}}{[(r-r'-\epsilon_k)(r'-r''-\epsilon_k)]^{d+1}}(\frac{\delta''}{\delta})^{m_k}\frac{e^{-(r-r''-2\epsilon_k)m_k}}{[(\ln(\delta'/\delta)\ln(\delta''/\delta')]^{n+1}}\end{aligned}$$

Finally, using the estimate (6.11),

$$\begin{aligned}(6.22)\quad & ||\mathcal{N}(\mathcal{D}^{-1})^2\bar{R}_k^{\infty}||_{r_3,\delta_3}\\ &\leqslant \frac{4||\bar{R}_k^{\infty}||_{r,\delta}(\delta''/\delta)^{m_k}}{[(r-r'-\epsilon_k)(r'-r''-\epsilon_k)]^{d+1}}\frac{e^{-(r-r''-2\epsilon_k)m_k}|a_k|_{r'',\delta''}}{[(\ln(\delta'/\delta)\ln(\delta''/\delta')]^{n+1}}\\ &\qquad\cdot\frac{C_{n,d}C_\omega}{(r''-r_3)^{2d+4}(\ln\delta_3-\ln\delta'')^{2n+4}}\end{aligned}$$

which implies that

$$\begin{aligned}(6.23)\quad & ||G_k^{\infty}||_{r_3,\delta_3}\\ &\leqslant \frac{C_{n,d,S}||\bar{R}_k^{\infty}||_{r,\delta}(\delta''/\delta)^{m_k}}{[(r-r'-\epsilon_k)(r'-r''-\epsilon_k)]^{d+1}}\frac{e^{-(r-r''-2\epsilon_k)m_k}}{[(\ln(\delta'/\delta)\ln(\delta''/\delta')]^{n+1}}\\ &\qquad\cdot[(r''-r_3)^{2d+4}(\ln\delta_3-\ln\delta'')^{2n+4}]^{-1}\end{aligned}$$

where $C_{n,d,S}$ depends on n,d,S but not on the step of iteration.

6.2. Choice of the parameters

We make the following choice of parameters:

$$\delta=\delta_k;\delta''=\delta g(m_k)^{\frac{-D''}{m_k}};\ \delta'=\sqrt{\delta''\delta};\ \delta_3={\delta''}^2/\delta'\text{ and }\delta_4=\delta_{k+1}$$

where $D'' = 10 + 6n$, and

$$r' = r - \frac{1}{4}(r_k - r_{k+1})\ ;\ r'' = r - \frac{1}{2}(r_k - r_{k+1});$$
$$r_3 = r - \frac{3}{4}(r_k - r_{k+1})\ ;\ r_4 = r_{k+1} \tag{6.24}$$

This choice implies that $r - r' = r' - r'' = r'' - r_3$ and

$$|\ln(\delta'/\delta)| = |\ln(\delta''/\delta')| = |\ln(\delta_3/\delta'')| = \frac{1}{2}|\ln(\delta''/\delta)| = \frac{D''}{2m_k}\ln g(m_k)$$

as well as

$$\ln\delta_3 - \ln\delta_4 = (n+2)\frac{\ln g(m_k)}{m_k}$$

Applied to estimate (6.18) on the low frequency part G_k^0, this choice gives

$$||G_k^0||_{r_3,\delta_3} \leqslant \frac{C'|a_k|_{r'',\delta''}g(m_k)^2(\frac{\delta''}{\delta})^{m_k}}{(r-r')^{4d+6}[\frac{D''}{2m_k}\ln g(m_k)]^{4n+6}}C_\omega||\bar{R}_k||_{r,\delta} \tag{6.25}$$

Now by Assumption 3.1.6,

$$(r-r')^{-4d-6} = [\frac{1}{4}(r_k - r_{k+1})]^{-4d-6} \leqslant 4^{4d+6}(r_k - r_{k+1} - \epsilon_k)^{-4d-6} \leqslant g(m_k)$$

Therefore,

$$||G_k^0||_{r_3,\delta_3} \leqslant \frac{C''_S g(m_k)^3(\frac{\delta''}{\delta})^{m_k}}{[\frac{D''}{2m_k}\ln g(m_k)]^{4n+6}}||\bar{R}_k||_{r,\delta} \tag{6.26}$$

where C''_S only depends on n, d, S. By Assumption 3.1.3,

$$||G_k^0||_{r_3,\delta_3} \leqslant C''_S g(m_k)^{3+4n+6-D''}||\bar{R}_k||_{r,\delta} \leqslant \frac{C''_S}{g(m_k)}||\bar{R}_k||_{r,\delta} \tag{6.27}$$

since $D'' = 10 + 6n$. By the smallness Assumption 3 on $\bar{R}_k$ and the Assumption (5.10) on C''_S and g, and by the definition of the sequence (ζ_k),

$$||G_k^0||_{r_3,\delta_3} \leqslant \frac{\zeta_{k+1}}{2} \tag{6.28}$$

Concerning high frequencies, when applied to estimate (6.23), the choice of parameters implies

$$\begin{aligned} ||G_k^\infty||_{r_3,\delta_3} \leqslant {} & \frac{C_{n,d,S}||\bar{R}_k^\infty||_{r,\delta}(\delta''/\delta)^{m_k}}{(r-r'-\epsilon_k)^{2(d+1)}} \\ & \cdot \frac{e^{-(r-r''-2\epsilon_k)m_k}}{[\frac{D''}{2m_k}\ln g(m_k)]^{4(n+1)+2n+4}}(\frac{1}{4}(r_k - r_{k+1}))^{-2d-4} \end{aligned} \tag{6.29}$$

Again by Assumption 3.1.3 and 3.1.6, this implies

$$(6.30)\quad ||G_k^\infty||_{r_3,\delta_3} \leqslant C_{n,d,S}||\bar{R}_k^\infty||_{r,\delta}g(m_k)^{1+4(n+1)+2n+4} \cdot (\delta''/\delta)^{m_k}e^{-(r-r''-2\epsilon)m_k}$$

Therefore, since $D'' \geqslant 2+4(n+1)+2n+4$ and since $(\delta''/\delta)^{m_k} \leqslant g(m_k)^{-D''}$,

$$(6.31)\qquad ||G_k^\infty||_{r_3,\delta_3} \leqslant \frac{C_{n,d,S}}{g(m_k)}||\bar{R}_k^\infty||_{r,\delta}$$

which implies, using the Assumption 3 and the definition of the sequence (ζ_k), that

$$||G_k^\infty||_{r_3,\delta_3} \leqslant \frac{\zeta_{k+1}}{2}$$

The following estimate of G_k is finally obtained:

$$(6.32)\qquad ||G_k||_{r_3,\delta_3} \leqslant \zeta_{k+1}$$

Analytic bounds on the differential DG_k of G_k will also be necessary. For all $1 \leqslant j \leqslant d$,

$$(6.33)\quad \begin{aligned} ||\partial_{X_j}G_k||_{r_4,\delta_4} &= \sum_{P,Q\geqslant m_k} ||iP_jG_{k,P,Q}||e^{r_4|P|}\delta_4^Q \\ &\leqslant (\frac{\delta_4}{\delta_3})^{m_k}||G_k||_{r_3,\delta_3}(r_3-r_4)^{-(d+2)}(\ln\delta_3-\ln\delta_4)^{-(n+1)} \end{aligned}$$

Similarly, for all $1 \leqslant j' \leqslant n$,

$$(6.34)\quad \begin{aligned} ||\partial_{Y_{j'}}G_k||_{r_4,\delta_4} &= \sum_{P,Q\geqslant m_k} ||Q_{j'}G_{k,P,Q}||e^{r_4|P|}\delta_4^{Q-1} \\ &\leqslant (\frac{\delta_4}{\delta_3})^{m_k}||G_k||_{r_3,\delta_3}(r_3-r_4)^{-(d+1)}(\ln\delta_3-\ln\delta_4)^{-(n+2)} \end{aligned}$$

Therefore (using the estimate (6.32), the Assumption 3.1.6 and the choice of δ_4),

$$(6.35)\quad \begin{aligned} ||DG_k||_{r_4,\delta_4} &:= \max_{j,j'}\{||\partial_{X_j}G_k||_{r_4,\delta_4}, ||\partial_{Y_{j'}}G_k||_{r_4,\delta_4}\} \\ &\leqslant (\frac{\delta_4}{\delta_3})^{m_k}g(m_k)^{n+2}\zeta_{k+1} \leqslant \zeta_{k+1} \end{aligned}$$

6.3. Change of variables

Let G_k be the vector field on $\mathcal{O}_{k+1}$ with order m_k+1 defined in Section 6.1, and let $\Phi_k = Id + G_k$ (this is a slight abuse of notation: we identify the vector $G_k(X,Y) \in \mathbb{C}^d \times \mathbb{C}^n$ with its projection on $\mathbb{C}^d/\mathbb{Z}^d \times \mathbb{C}^n$). Then

$$||\Phi_k - Id||_{r_{k+1},\delta_{k+1}} = ||G_k||_{r_{k+1},\delta_{k+1}} \leqslant \zeta_{k+1}$$

therefore Property 3 holds. One sees that Φ_k is a diffeomorphism from $\mathcal{O}_{k+1}$ to $\mathcal{O}_k$: indeed, if $(X,Y) \in \mathcal{O}_{k+1}$, then for all $1 \leqslant j \leqslant d$,

$$|\Phi_{k,j}(X,Y)| \leqslant |X_j| + \zeta_{k+1} \leqslant r_{k+1} + \zeta_{k+1} \tag{6.36}$$

and the Assumption 3.1.6 implies that $|\Phi_{k,j}(X,Y)| \leqslant r_k$, as long as $g(m_0)$ is large enough as a function of d, S, which is not a restrictive assumption. For all $1 \leqslant j' \leqslant n$,

$$|\Phi_{k,j'}(X,Y)| \leqslant |Y_{j'}| + \zeta_{k+1} \leqslant \delta_{k+1} + \zeta_{k+1} \tag{6.37}$$

therefore Φ_k has values in $\mathcal{O}_k$ if it can be shown that

$$\frac{2C_S''\zeta_k}{g(m_k)} \leqslant \delta_k - \delta_{k+1} = \delta_{k+1}(g(m_k)^{\frac{17+10n}{m_k}} - 1) \tag{6.38}$$

Since by the Assumption (5.9), $\zeta_0 \leqslant \frac{1}{2C_S''}\delta_\infty$, then in particular $2C_S''\zeta_k \leqslant \delta_{k+1}$; moreover,

$$g(m_k)^{\frac{17+10n}{m_k}} - 1 \geqslant \frac{17+10n}{m_k} \ln g(m_k) \geqslant \frac{17+10n}{n+2} g(m_k)^{-1} \geqslant g(m_k)^{-1}$$

which implies (6.38). Therefore Φ_k has values in $\mathcal{O}_k$.

By the estimate (6.35), for all $(X,Y) \in \mathcal{O}_{k+1}$, the spectrum of $D\Phi_k(X,Y)$ cannot contain 0 since

$$|D\Phi_k(X,Y)| \geqslant 1 - 2\zeta_{k+1}.$$

Therefore Φ_k is injective on $\mathcal{O}_{k+1}$ and Φ_k^{-1} is defined on $\Phi_k(\mathcal{O}_{k+1}) \subset \mathcal{O}_k$. Property 4 also comes from the estimate (6.35).

Moreover, Φ_k satisfies

$$\Phi_k^*(N_{k+1} + R_{k+1}) = N_k + R_k$$

where

$$N_{k+1} = N_k + T^{m_{k+1}-1} R_k - \bar{R}_k$$

and

$$(6.39)\quad \begin{aligned}&(I+DG_k)R_{k+1}\\ &\quad = -DG_k(T^{m_{k+1}-1}R_k-\bar{R}_k)+(R_k-T^{m_{k+1}-1}R_k)+DR_kG_k\\ &\qquad +\sum_{|I|\geqslant 2}\frac{1}{I_1!\dots I_{d+n}!}\partial_I(R_k+N_k)\cdot G_k^I\end{aligned}$$

Thus, under the Property 1 of Assumption 3.1, R_{k+1} has order m_{k+1}. Moreover, for any vector field F of order m_k and any index $I\in\mathbb{N}^{d+n}$,

$$(6.40)\quad \begin{aligned}&||\partial_I F||_{r_{k+1},\delta_{k+1}}\\ &\quad =\sum_{P,|Q|\geqslant m_k}\left|(iP_1)^{I_1}\dots(iP_d)^{I_d}\frac{Q_1!}{(Q_1-I_{d+1})!}\cdots\frac{Q_n!}{(Q_n-I_{d+n})!}F_{P,Q}\right|\\ &\qquad\qquad \cdot e^{|P|r_{k+1}}\delta_{k+1}^{|Q|-|I|}\\ &\quad\leqslant(\delta_{k+1}/\delta_k)^{m_k}(r_k-r_{k+1})^{-i_1-\dots-i_d-d}\\ &\qquad\qquad\cdot(\ln\delta_k-\ln\delta_{k+1})^{-i_{d+1}-\dots-i_{d+n}-n}||F||_{r_k,\delta_k}\\ &\quad\leqslant(\delta_{k+1}/\delta_k)^{m_k}(C(n,d)g(m_k))^{|I|+1+n}||F||_{r_k,\delta_k}\end{aligned}$$

which, applied to N_k and R_k, implies

$$(6.41)\quad \begin{aligned}&||(I+DG_k)R_{k+1}||_{r_{k+1},\delta_{k+1}}\\ &\leqslant\zeta_k\zeta_{k+1}+(\delta_{k+1}/\delta_k)^{m_{k+1}}\zeta_k+C(n,d)(\delta_{k+1}/\delta_k)^{m_k}g(m_k)^{n+2}\zeta_k\zeta_{k+1}\\ &\quad+(\delta_{k+1}/\delta_k)^{m_k}\sum_{|I|\geqslant 2}\frac{1}{I_1!\dots I_{d+n}!}(C(n,d)g(m_k))^{|I|+1+n}\zeta_{k+1}^{|I|}(\zeta_k+\eta_k)\\ &\leqslant\zeta_k\zeta_{k+1}+\frac{1}{10}\zeta_{k+1}+\zeta_k\zeta_{k+1}\\ &\quad+(\delta_{k+1}/\delta_k)^{m_k}g(m_k)^{n+2}\sum_{|I|\geqslant 2}\frac{1}{I_1!\dots I_{d+n}!}(C(S,n,d)\zeta_k)^{|I|-1}\zeta_{k+1}\\ &\leqslant\frac{1}{2}\zeta_{k+1}\end{aligned}$$

Now

$$(6.42)\quad \begin{aligned}&||(I+DG_k)R_{k+1}||_{r_{k+1},\delta_{k+1}}\\ &\qquad\geqslant||R_{k+1}||_{r_{k+1},\delta_{k+1}}-||DG_kR_{k+1}||_{r_{k+1},\delta_{k+1}}\\ &\qquad\geqslant(1-\zeta_{k+1})||R_{k+1}||_{r_{k+1},\delta_{k+1}}\end{aligned}$$

which, together with (6.41), finally implies that $||R_{k+1}||_{r_{k+1},\delta_{k+1}}\leqslant\zeta_{k+1}$, whence Property 2.

The vector field N_{k+1} satisfies the following estimate:

$$||N_{k+1} - S||_{r_{k+1},\delta_{k+1}} \leqslant ||N_k - S||_{r_k,\delta_k} + ||R_k||_{r_k,\delta_k} \leqslant \eta_k + \zeta_k \leqslant \eta_{k+1} \tag{6.43}$$

whence Property 1. This concludes the proof of Proposition 5.6. □

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