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Carlos MATHEUS, Jean-Christophe YOCCOZ & David ZMIAIKOU

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## CORRIGENDUM TO “HOMOLOGY OF ORIGAMIS WITH SYMMETRIES”

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by Carlos MATHEUS,  
Jean-Christophe YOCCOZ & David ZMIAIKOU

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As it was kindly pointed out to us by Simion Filip, in Proposition 3.18 of [1] we gave a wrong description of the group  $Sp(W_a)$  when  $a$  is quaternionic.

More precisely, all facts claimed in [1] during the proof of this proposition (from pages 1149 to 1153) are correct *except* for the penultimate phrase of the argument:

“Moreover  $A$  preserves the symplectic form on  $W_a$  iff  $\iota^{-1} \circ A \circ \iota$  preserves the hermitian form

$$\sum_1^p \langle v_m, v'_m \rangle - \sum_{p+1}^{p+q} \langle v_m, v'_m \rangle$$

on  $V^{\ell_a}$ .”

As we are going to see in Section 1 below, after correcting the error pointed out above, one has that, when  $a$  is quaternionic, the group  $Sp(W_a)$  is not isomorphic to the unitary group  $U_{\mathbf{H}}(p, q)$  (as it was previously claimed in [1]), but rather to the group  $O(\ell_a, \mathbf{H})$  of matrices (with coefficients in  $\mathbf{H}$ ) satisfying  $A^\sharp A = I$ , where  $A^\sharp$  is the transpose matrix of  $\sigma(A)$  and  $\sigma(a + bi + cj + dk) = a - bi + cj + dk$  is a reversion on  $\mathbf{H}$ .

In particular, besides the discussion of  $Sp(W_a)$  when  $a$  is quaternionic undertaken in pages 1149 to 1153 in [1], all statements in [1] which were based on the description of  $Sp(W_a)$  in Proposition 3.18 in [1] must be changed. More concretely:

- at page 1135, lines 6 and 7, the phrase “a quaternionic unitary group  $U_{\mathbf{H}}(p, q)$  in the quaternionic case” must be replaced by “a quaternionic orthogonal group  $O(\ell, \mathbf{H})$  in the quaternionic case”;
- at page 1135, the second item of Theorem 1.4 must be replaced by “When  $V_a$  is complex, resp. quaternionic, and  $Sp(W_a)$  is isomorphic to an unitary group  $U_{\mathbf{C}}(p, q)$ , resp.  $O(\ell, \mathbf{H})$  with  $\ell$  odd, the multiplicity in  $W_a$  of the exponent 0 is at least  $|q - p| \dim_{\mathbf{R}}(V_a)$ , resp.  $\dim_{\mathbf{R}}(V_a)$ ;
- at page 1158, lines  $-5$  to  $-3$  must be replaced by: “Assume that  $a$  is *complex*, resp. *quaternionic*. From Propositions 3.17, resp. the new version of Proposition 3.18, there exists nonnegative integers  $p, q$  with  $p + q = \ell_a$  such that  $Sp(W_a)$  is isomorphic to  $U_{\mathbf{C}}(p, q)$ , resp.  $Sp(W_a)$  is isomorphic to  $O(\ell_a, \mathbf{H})$ .”;
- at page 1158, the statement of Proposition 4.11 must be replaced by “If  $a$  is complex, the multiplicity of the exponent 0 in  $W_a$  is at least  $|q - p| \dim_{\mathbf{R}}(V_a)$ . If  $a$  is quaternionic and  $\ell_a$  is odd, the multiplicity of the exponent 0 in  $W_a$  is at least  $\dim_{\mathbf{R}}(V_a)$ .”;
- at page 1159, the proof of Proposition 4.11 remains unchanged in the complex case, but in the case of  $a$  quaternionic and  $\ell_a$  odd, the new argument is the following. “By Corollary 4.10, the list of Lyapunov exponents in  $W_a$  has the form  $\theta_1 \geq \dots \geq \theta_{\ell_a}$  where each  $\theta_m$  appears with multiplicity  $\dim_{\mathbf{R}}(V_a)$ . By symplecticity, the list  $\theta_1 \geq \dots \geq \theta_{\ell_a}$  is symmetric with respect to 0. Therefore,  $\theta_{(\ell_a+1)/2} = 0$  when  $\ell_a$  is odd.”;
- at page 1173, the end of the phrase in Remark 5.13 must be simply “... isotypical components of complex type” (instead of “... isotypical components of complex or quaternionic type”) because there is no need for discussing the signature of Hermitian form in the quaternionic case.

After these preliminaries, we complete this note by explaining how to change the statement and proof of Proposition 3.18 in [1] to get a correct description of  $Sp(W_a)$  when  $a$  is quaternionic.

## 1. Corrections to the statement and proof of Proposition 3.18 in [1]

For the sake of convenience of the reader, instead of just giving a list of punctual changes, we indicated below how to completely rewrite the

content of pages 1149 to 1153 in [1] in order to obtain a correct description of  $Sp(W_a)$  when  $a$  is quaternionic.

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\* \*

Consider finally the case where  $a$  is quaternionic. We fix an isomorphism between  $D_a$  and  $\mathbf{H}$ . We equip  $V_a$  with the structure of a right vector space over  $\mathbf{H}$  by setting

$$vz = \bar{z}v, \quad z \in \mathbf{H}, v \in V_a.$$

Here,  $\bar{z} = a - bi - cj - dk$  is the conjugate of the quaternion  $z = a + bi + cj + dk$ .

Recall that an hermitian form on a right vector space  $V$  over  $\mathbf{H}$  is a map  $H : V \times V \rightarrow \mathbf{H}$  which satisfies

$$\begin{aligned} H(v, v_1z_1 + v_2z_2) &= H(v, v_1)z_1 + H(v, v_2)z_2, \quad \forall v, v_1, v_2 \in V, z_1, z_2 \in \mathbf{H}, \\ H(v, v') &= \overline{H(v', v)}, \quad \forall v, v' \in V. \end{aligned}$$

Writing  $H = H_0 + H_i i + H_j j + H_k k$ , the  $\mathbf{R}$ -bilinear form  $H_0$  is symmetric, and the  $\mathbf{R}$ -bilinear forms  $H_i, H_j, H_k$  are alternate. They are related by

$$H_0(v, v') = H_i(v, v' i) = H_j(v, v' j) = H_k(v, v' k).$$

For  $v, v' \in V, z \in \mathbf{H}$ , we have  $H(vz, v'z) = \bar{z}H(v, v')z$ , hence

$$\begin{aligned} H_i(vi, v'i) &= H_i(v, v'), \\ H_i(vj, v'j) &= -H_i(v, v'), \\ H_i(vk, v'k) &= -H_i(v, v'). \end{aligned}$$

Equivalently, one has, for  $v, v' \in V, z \in \mathbf{H}$

$$H_i(v, v'z) = H_i(v\sigma(z), v'),$$

where  $\sigma$  is the reversion  $\sigma(a + bi + cj + dk) = a - bi + cj + dk$ .

Conversely, if a  $\mathbf{R}$ -bilinear alternate form  $H_i$  satisfy these relations, one defines an hermitian form by

$$H(v, v') = H_i(v, v') + H_i(v, v')i + H_i(v, v'k)j - H_i(v, v'j)k.$$

On the irreducible  $\text{Aut}(M)$ -module  $V_a$ , there exists, up to a positive real scalar, a unique positive definite  $\text{Aut}(M)$ -invariant hermitian form  $\langle, \rangle$  (obtained as usual by averaging over the group an arbitrary positive definite hermitian form). The space of  $\text{Aut}(M)$ -invariant  $\mathbf{R}$ -bilinear forms on  $V_a$  is 4-dimensional, generated by the four components of  $\langle, \rangle$ .

PROPOSITION 3.18. — *There exists an isomorphism of  $\text{Aut}(M)$ -modules  $\iota : V_a^{\ell_a} \rightarrow W_a$  such that the bilinear form  $\{\iota(v_1, \dots, v_{\ell_a}), \iota(v'_1, \dots, v'_{\ell_a})\}$  is the  $i$ -component of the hermitian form  $\sum_1^{\ell_a} \langle v_m, v'_m \rangle$ .*

*An element of  $Sp(W_a)$  is of the form  $\iota^{-1} \circ A \circ \iota(v_1, \dots, v_{\ell_a}) = (V_1, \dots, V_{\ell_a})$  with  $V_m = \sum_n v_n a_{n,m}$ ,  $a_{n,m} \in \mathbf{H}$ . The map  $A \mapsto (a_{m,n})$  is an isomorphism from  $Sp(W_a)$  onto the group  $O(\ell_a, \mathbf{H})$  of matrices satisfying  $A^\sharp A = I$ , where  $A^\sharp$  is the transpose matrix of  $\sigma(A)$ .*

The proof of this proposition relies on the following three lemmas:

LEMMA 3.19. — *Let  $G$  be a finite group, and let  $W$  be an isotypic  $G$ -module of quaternionic type. Let  $B$  be an alternate non-degenerate  $G$ -invariant bilinear form on  $W$ . Then, there exists a non-zero vector  $v \in W$  and  $g \in G$  such that  $B(v, g.v) \neq 0$ .*

*Proof.* — Assume that the conclusion of the lemma does not hold. Then, one has  $B(v, g.v') + B(v', g.v) = 0$  for all  $v, v' \in W$ . As  $B$  is alternate and  $G$ -invariant, one has  $B(v, g^2.v') = B(v, v')$  for all  $v, v' \in W$ ,  $g \in G$ . As  $B$  is non-degenerate, this implies that  $g^2.v' = v'$  for all  $v' \in W$ ,  $g \in G$ . Thus  $G$  acts through a group where all non trivial elements are of order 2. Such a group is abelian and  $W$  cannot be quaternionic. □

LEMMA 3.20. — *Under the hypotheses of the lemma above, one can write*

$$W = V_1 \oplus \dots \oplus V_\ell,$$

where  $V_1, \dots, V_\ell$  are irreducible  $G$ -modules which are orthogonal for  $B$ .

*Proof.* — This is an immediate induction on the multiplicity  $\ell$  of  $W$ . From the lemma above, one can find  $v \in W$  such that the restriction of  $B$  to the irreducible  $G$ -module  $V_1$  generated by  $v$  is nonzero. Because  $V_1$  is irreducible and  $B$  is  $G$ -invariant, this restriction is non-degenerate. Then, the  $B$ -orthogonal  $W'$  of  $V_1$  in  $W$  is  $G$ -invariant and satisfies  $W = V_1 \oplus W'$ . We conclude by applying to  $W'$  the induction hypothesis. □

LEMMA 3.21. — *Let  $b$  be an alternate  $\text{Aut}(M)$ -invariant nonzero  $\mathbf{R}$ -bilinear form on  $V_a$ . There exists  $u \in \mathbf{H}^*$  such that the form  $b_u(v, v') := b(vu, v'u)$  is the  $i$ -component  $B_i$  of  $\langle \cdot, \cdot \rangle$ .*

*Proof.* — Any nonzero alternate  $\text{Aut}(M)$ -invariant  $\mathbf{R}$ -bilinear form  $b$  on  $V_a$  is non-degenerate. This allows to define an adjoint map  $\sigma_b : \mathbf{H} \rightarrow \mathbf{H}$  through  $b(v, v'a) = b(v\sigma_b(a), v')$  (the  $\mathbf{R}$ -endomorphism  $\sigma_b(a)$  of  $V_a$  belongs to  $\mathbf{H}$  as it commutes with the action of  $\text{Aut}(M)$ ). The map  $\sigma_b$  is a  $\mathbf{R}$ -linear involution satisfying  $\sigma_b(aa') = \sigma_b(a')\sigma_b(a)$  and  $\sigma_b(a) = a$  for

$a \in \mathbf{R}$ . Therefore  $\sigma_b$  preserves the set of quaternions  $a$  such that  $a^2 = -1$ , which is nothing else than the purely imaginary quaternions of norm 1. Observe that, for the  $i$ -component  $B_i$  of the hermitian scalar product on  $V_a$ , one has that  $\sigma_{B_i}$  is the reversion  $\sigma$ .

As any nonzero alternate  $\text{Aut}(M)$ -invariant  $\mathbf{R}$ -bilinear form  $b$  on  $V_a$  is a linear combination of the imaginary components  $B_i, B_j, B_k$  of  $\langle \cdot, \cdot \rangle$  and thus can be deformed to  $B_i$  through nonzero alternate  $\text{Aut}(M)$ -invariant  $\mathbf{R}$ -bilinear forms, we conclude that:

- there exists a unique (up to sign) quaternion  $a_0$  of norm 1 such that  $\sigma_b(a_0) = -a_0$  and, moreover,  $a_0$  is purely imaginary;
- $\sigma_b$  is the identity on a hyperplane  $H$  of  $\mathbf{H}$  containing  $\mathbf{R}$  and the purely imaginary quaternions in  $H$  are those satisfying  $za_0 + a_0z = 0$ .

Next we relate, for  $\bar{u}u = 1$ ,  $\sigma_{b_u}$  to  $\sigma_b$ . For  $v, v' \in V_a$ , we have

$$\begin{aligned} b_u(v, v'a) &= b(vu, v'au) \\ &= b(vu, v'u\bar{u}au) \\ &= b(vu\sigma_b(\bar{u}au), v'u) \\ &= b(vu\sigma_b(\bar{u}au)\bar{u}u, v'u) \\ &= b_u(vu\sigma_b(\bar{u}au)\bar{u}, v'), \end{aligned}$$

and thus  $\sigma_{b_u}(a) = u\sigma_b(\bar{u}au)\bar{u}$ . Hence,  $\sigma_{b_u} = \sigma_{B_i}$  iff  $\bar{u}iu = \pm a_0$ . As  $a_0$  is purely imaginary of norm 1, it is possible to choose  $u$  such that  $\bar{u}iu = a_0$ . Then we have  $\sigma_{b_u}(i) = -i$  and therefore  $\sigma_{b_u} = \sigma$ .

It follows that there exists  $c \in \mathbf{R}^*$  such that  $b_u$  is the  $i$ -component of  $c\langle \cdot, \cdot \rangle$ . Replacing  $u$  by  $c^{-1/2}u$  if  $c > 0$ , by  $|c|^{-1/2}ju$  if  $c < 0$ , we obtain the required conclusion.  $\square$

*Proof of Proposition 3.18.* — According to Lemma 3.20 above, one can choose an isomorphism of  $\text{Aut}(M)$ -modules  $\iota_0 : V_a^{\ell_a} \rightarrow W_a$  such that the symplectic form is written in a diagonal way as

$$\{\iota_0(v_1, \dots, v_{\ell_a}), \iota_0(v'_1, \dots, v'_{\ell_a})\} = \sum_1^{\ell_a} b^{(m)}(v_m, v'_m),$$

for some alternate nonzero  $\text{Aut}(M)$ -invariant  $\mathbf{R}$ -bilinear forms  $b^{(m)}$  on  $V_a$ . According to Lemma 3.21, one can find nonzero quaternions  $u_1, \dots, u_{\ell_a}$  such that, setting  $\iota(v_1, \dots, v_{\ell_a}) = \iota_0(v_1u_1, \dots, v_{\ell_a}u_{\ell_a})$ , the alternate form

$$\{\iota(v_1, \dots, v_{\ell_a}), \iota(v'_1, \dots, v'_{\ell_a})\}$$

is the  $i$ -component  $\sum_1^{\ell_a} B_i(v_m, v'_m)$  of the hermitian form  $\sum_1^{\ell_a} \langle v_m, v'_m \rangle$ , which proves the first assertion of the proposition. For the second assertion,

any automorphism  $A$  of the  $\text{Aut}(M)$ -module  $W_a$  is of the form  $\iota^{-1} \circ A \circ \iota(v_1, \dots, v_{\ell_a}) = (V_1, \dots, V_{\ell_a})$  with  $V_m = \sum_n v_n a_{n,m}$ ,  $a_{n,m} \in \mathbf{H}$ . Moreover we have

$$\begin{aligned} \sum_m B_i(V_m, V'_m) &= \sum_m \sum_n \sum_{n'} B_i(v_n a_{n,m}, v'_{n'} a_{n',m}) \\ &= \sum_n \sum_{n'} B_i(v_n \sum_m a_{n,m} \sigma(a_{n',m}), v'_{n'}) \end{aligned}$$

This is equal to  $\sum_m B_i(v_m, v'_m)$  for all  $v_m, v'_m$  iff one has, for all  $n, n'$

$$\sum_m a_{n,m} \sigma(a_{n',m}) = \delta_{n,n'}.$$

This means that the matrix associated to  $A$  satisfies  $AA^\sharp = I$ . □

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Carlos MATHEUS  
Université Paris 13  
Sorbonne Paris Cité  
LAGA, CNRS (UMR 7539)  
93430, Villetaneuse (France)  
matheus.cmss@gmail.com

Jean-Christophe YOCCOZ  
Collège de France (PSL)  
3 rue d'Ulm  
75005 Paris (France)  
jean-c.yoccoz@college-de-france.fr

David ZMIAIKOU  
Sanger Institute, Genome Campus  
Hinxton  
Cambridge CB10 1HH (UK)  
david.zmiaikou@gmail.com