ANNALES

DE

L'INSTITUT FOURIER

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A NEW FAMILY OF ALGEBRAS WHOSE REPRESENTATION SCHEMES ARE SMOOTH

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ABSTRACT. — We give a necessary and sufficient smoothness condition for the scheme parameterizing the n-dimensional representations of a finitely generated associative algebra over an algebraically closed field. In particular, our result implies that the points $M \in \operatorname{Rep}_A^n(k)$ satisfying $\operatorname{Ext}_A^2(M, M) = 0$ are regular. This generalizes well-known results on finite-dimensional algebras to finitely generated algebras.

RÉSUMÉ. — Dans cet article, nous fournissons une condition nécessaire et suffisante pour la lissité du schéma qui paramétrise les représentations n-dimensionelles d'une algèbre associative, engendrée par un nombre fini d'éléments sur un corps algébriquement clos. En particulier, notre résultat implique que les points $M \in \operatorname{Rep}_A^n(k)$ satisfaisant $\operatorname{Ext}_A^2(M, M) = 0$ sont réguliers. Ceci généralise aux algèbres engendrées par un nombre fini d'éléments des résultats connus sur les algèbres de dimension finie.

1. Introduction

Let A be a finitely generated associative k-algebra with k an algebraically closed field. Let $V_n(A)$ be the commutative k-algebra representing the functor from commutative algebras to sets

$$\mathcal{C}_k \to \mathsf{Set} : B \mapsto \operatorname{Hom}_{\mathcal{N}_k}(A, M_n(B))$$

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of the n-dimensional representations of A over B, (see Section 2.2). The scheme Rep_A^n of the linear representations of dimension n of A is defined to be $\text{Spec}\, V_n(A)$.

Formally smooth (or quasi-free) algebras provide a generalization of the notion of free algebra, since they behave like a free algebra with respect to nilpotent extensions. The definition goes back to J. Cuntz and D. Quillen and it was inspired by the Grothendieck's definition of formal smoothness given in the commutative setting, see [14, 19.3.1]. See also [13, 19] and [18, 4.1]. For further details, see 4.3.

It is well-known that if A is formally smooth then Rep_A^n is smooth (see [13, Proposition 19.1.4] and [18, Proposition 6.3]). If A is finite-dimensional then it is formally smooth if and only if it is hereditary (see Theorem 4.6) and, therefore Rep_A^n is smooth for all n if and only if A is hereditary (see [3, Proposition 1]).

For infinite-dimensional algebras the picture is more complex, e.g. there are hereditary algebras which are not formally smooth (see Remark 4.7). It is therefore interesting to find other sufficient (or necessary) conditions on A which ensure Rep_A^n to be smooth.

Let M be an A-module in $\text{Rep}_A^n(k)$. It is well-known that the linear space $\text{Ext}_A^2(M, M)$ contains the obstructions in extending the infinitesimal deformations of M to the formal ones. For this reason an algebra A such that $\text{Ext}_A^2(M, M) = 0$, for all $M \in \text{Rep}_A^n(k)$ and $n \geqslant 1$, will be called *finitely unobstructed*.

It has been proved by Geiss and de la Peña (see [9, 10]) that, when A is finite-dimensional, finitely unobstructed implies that Rep_A^n is smooth.

We underline that any hereditary algebra is finitely unobstructed but the converse is not true, e.g. the universal enveloping algebra of a finite-dimensional semisimple Lie algebra is finitely unobstructed but not hereditary if the dimension of the underlying Lie algebra is greater than one.

The proof given in [9, 10] is based on the analysis of the local geometry of Rep_A^n, and it specifically relies on the upper semicontinuity of certain dimension functions arising from the bar resolution of A. As we observe in the last section of this paper, their approach remains valid if one assumes that A is finitely presented or bimodule coherent.

We follow here a different path, namely, we study the smoothness problem via the adjunction

$$(1.1) \qquad \text{Hom}_{\mathcal{C}_k}(V_n(A), B) \xrightarrow{\cong} \text{Hom}_{\mathcal{N}_k}(A, M_n(B))$$

The adjunction (1.1) allows us to use the Harrison cohomology of $V_n(A)$ instead of the Hochschild cohomology of A. The Harrison cohomology of a commutative k-algebra is the symmetric part of its Hochschild cohomology, and it has been proved by Harrison [15] that an affine ring R is regular if and only if its second Harrison cohomology vanishes.

This is our main result.

THEOREM. — *Let A be a finitely generated k-algebra, let $f : V_n(A) \to k$ be a k-algebra map and let $\rho : A \to M_n(k)$ be the algebra map that corresponds to f through the adjunction above. Then there is a linear embedding of $\mathrm{Harr}^2(V_n(A), {}_f k)$ into $H^2(A, {}_\rho M_n(k)_\rho)$. As a consequence, $M \in \mathrm{Rep}_A^n$ is a regular point whenever $\mathrm{Ext}_A^2(M, M) = 0$.*

We have thus extended the known results on smoothness to infinite-dimensional finitely generated algebras.

We remark that the above embedding is not an isomorphism in general. We give a counterexample by using 2-Calabi Yau algebras (Remark 3.5).

The paper goes as follows.

In paragraph 2.2 we recall the definition of Rep_A^n as the scheme parameterizing the n-dimensional representations of A.

In Section 3 we recall the Harrison cohomology which may be seen as the commutative version of the Hochschild cohomology. We prove that the regularity of a point in $\mathrm{Rep}_A^n = \mathrm{Spec}(V_n(A))$ is equivalent to the vanishing of $\mathrm{Harr}^2(V_n(A), {}_f k)$, for the k-algebra map $f : V_n(A) \to k$ associated to the point (see Theorem 3.3). Then Theorem 3.4 shows that there is a linear embedding of $\mathrm{Harr}^2(V_n(A), {}_f k)$ into $H^2(A, {}_\rho M_n(k)_\rho)$ and as a consequence, that $M \in \mathrm{Rep}_A^n$ is a regular point whenever $\mathrm{Ext}_A^2(M, M) = 0$.

Then, as said before, by using 2-Calabi Yau algebras, we exhibit an example which shows that the above embedding is not an isomorphism.

In Section 4 we present a list of examples and applications of the aforementioned results. To this aim, we first recall the notions of formally smooth and hereditary algebra. We mention the known result on the smoothness of Rep_A^n when A is formally smooth or hereditary to compare the notions of formally smoothness, hereditary, finitely unobstructed and we stress the difference between the finite and the infinite-dimensional case.

Afterward, we give the definition of finitely unobstructed algebra and we prove that if A is finitely unobstructed then Rep_A^n is smooth (see Corollary 4.2).

Then we produce examples of finitely unobstructed algebras (neither hereditary nor formally smooth) whose associate representation scheme is smooth (see Example 4.12).

In Section 5 we study the relationships between the deformation theory of $M \in \mathrm{Rep}_A^n(k)$, in the sense of Gerstenhaber, Geiss and de la Peña , and the deformation theory of $V_n(A)$ as usually defined in algebraic geometry.

In particular, by using the adjunction (1.1), we will see that there are no obstructions to the integrability of the infinitesimal deformations of M if and only if $\mathrm{Harr}^2(V_n(A), {}_f k) = 0$. Motivated by this fact we formulate the following conjecture.

CONJECTURE. — *The image of the embedding*

$$\mathrm{Harr}^2(V_n(A), {}_f k) \hookrightarrow \mathrm{Ext}^2(M, M)$$

contains the subspace of $\mathrm{Ext}^2(M, M)$ *of the obstructions to integrate the infinitesimal deformations of* M.

As a bonus, we further show that the approach to this smoothness problem developed for A finite-dimensional in [9, 10] works as well if A is finitely unobstructed and finitely presented or bimodule coherent.

2. Preliminaries

2.1. Notations

Unless otherwise stated we adopt the following notations:

- k is an algebraically closed field;
- $F = k\{x_1, \ldots, x_m\}$ is the associative free k-algebra on m letters;
- $A \cong F/J$ is a finitely generated associative k-algebra;
- $\mathcal{N}_-, \mathcal{C}_-$ and Set denote the categories of -algebras, commutative -algebras and sets, respectively;
- The term "A-module" indicates a left A-module. The categories of left -modules is denoted by -Mod. The full subcategory of modules having finite dimension over k will be denoted by -Mod_f;
- We write $\mathrm{Hom}_{\mathcal{A}}(B, C)$ for the morphisms from an object B to C in a category $\mathcal{A}$. If $\mathcal{A} = A$-Mod, then we will write $\mathrm{Hom}_A(-, -)$;
- A^{op} is the opposite algebra of A and $A^e := A \otimes A^{\mathrm{op}}$ is the envelope of A. It is an A-bimodule and a k-algebra. One can identify the category of the A-bimodules with A^e-Mod and we will do it thoroughly this paper;
- $\mathrm{Ext}_-^i(\,,\,)$ denotes the Ext groups on the category -Mod;
- $\mathrm{H}^i(A, -)$ is the Hochschild cohomology with coefficients in A^e-Mod.

2.2. The scheme of n-dimensional representations

The study of the affine scheme Rep_A^n of n-dimensional representations of an algebra A goes back to the early 1970's with work of M. Artin, P. Gabriel, C. Procesi and D. Voigt. See for example [8] and the references therein.

Denote by $M_n(B)$ the full ring of $n \times n$ matrices over B, with B a ring. If $f : B \to C$ is a ring homomorphism we denote by $M_n(f) : M_n(B) \to M_n(C)$ the homomorphism induced on matrices.

DEFINITION 2.1. — Let $A \in \mathcal{N}_k$, $B \in \mathcal{C}_k$. By an n-dimensional representation of A over B we mean a homomorphism of k-algebras $\rho : A \to M_n(B)$.

It is clear that this is equivalent to give an A-module structure on B^n. The assignment $B \mapsto \operatorname{Hom}_{\mathcal{C}_k}(A, M_n(B))$ defines a covariant functor

$$\mathcal{C}_k \longrightarrow \mathsf{Set}.$$

which is represented by a commutative k-algebra $V_n(A)$.

LEMMA 2.2 ([20, Ch.4, §1]). — For all $A \in \mathcal{N}_k$ and $\rho : A \to M_n(B)$ a linear representation, there exist $V_n(A) \in \mathcal{C}_k$ and a representation $\eta_A : A \to M_n(V_n(A))$ such that $\rho \mapsto M_n(\rho) \circ \eta_A$ gives an isomorphism

$$(2.1) \qquad \operatorname{Hom}_{\mathcal{C}_k}(V_n(A), B) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{N}_k}(A, M_n(B))$$

for all $B \in \mathcal{C}_k$.

If $A = F$, one has that $V_n(F) := k[\xi_{lij}]$, the polynomial ring in variables $\{\xi_{lij} : i, j = 1, \ldots, n, l = 1, \ldots, m\}$ over k. If $A = F/J$ finitely generated k-algebra, one defines $V_n(A) := k[\xi_{lij}]/I$ where I is the ideal of $V_n(F)$ generated by the $n \times n$ entries of $f(\xi_1, \ldots, \xi_m)$, f runs over the elements of J and ξ_l is the matrix (ξ_{lij}). Therefore $V_n(A)$ is an affine ring (i.e. a finitely generated algebra with identity) when A is a finitely generated k-algebra.

DEFINITION 2.3. — We write Rep_A^n to denote $Spec\, V_n(A)$. It is considered as a k-scheme. The map

$$\eta_A : A \to M_n(V_n(A)), \quad a_l \longmapsto \xi_l^A := (\xi_{lij} + I).$$

is called the universal n-dimensional representation.

Examples 2.4. — (i) By construction, if $A = F$, then $\operatorname{Rep}_F^n(k) = M_n(k)^m$. If $A = F/J$, the B-points of Rep_A^n can be described as follows:

$$\operatorname{Rep}_A^n(B) = \{(X_1, \ldots, X_m) \in M_n(B)^m : f(X_1, \ldots, X_m) = 0 \text{ for all } f \in J\};$$

(ii) If $A = \mathbb{C}[x, y]$, $\text{Rep}_A^n(\mathbb{C}) = \{(M_1, M_2) \in M_2(\mathbb{C})^2, M_1 M_2 = M_2 M_1\}$ is the *commuting scheme*, see [22].

Remark 2.5. — Note that Rep_A^n may be quite complicated. It is not reduced in general and it seems to be hopeless to describe the coordinate ring of its reduced structure. The scheme Rep_A^n is also known as *the scheme of n-dimensional A-modules*.

3. The main result

We prove our main result using Harrison cohomology. Given a commutative ring R and an R-module N, we denote by $\text{Harr}^*(R, N)$ the Harrison cohomology group i.e. the group $\mathcal{E}^*(R, N)$ introduced in [15]. Harrison cohomology can be seen as a commutative version of Hochschild cohomology. For further details the reader is referred to [23, section 9.3], where $\text{Harr}^2(R, N)$ is denoted by $\text{H}_s^2(R, N)$.

The following standard result establishes a link between $\text{Ext}_A^i(M, M)$ and the Hochschild cohomology of A with coefficients in $\text{End}_k(M)$.

THEOREM 3.1 ([4, Corollary 4.4]). — *We have*
$$\text{Ext}_A^i(M, M) \cong \text{H}^i(A, \text{End}_k(M)).$$

For every algebra map $f : B \to A$ and $N \in A\text{-Mod}$, denote by $_f N$ the corresponding left B-module structure on N. A similar notation is used on the right. In particular, if $N \in A^e\text{-Mod}$, the notation $_f N_f$ means that N is regarded as a B^e-module via f.

PROPOSITION 3.2. — *The following assertions are equivalent for $A \in \mathcal{N}_k$ and for every $M \in A^e\text{-Mod}$:*

(1) $\text{H}^2(A, M) = 0$;
(2) *Let $f : A \to B$ be an algebra map and let $p : E \to B$ be a Hochschild extension of $B \in \mathcal{N}_k$ with kernel N such that $_f N_f = M$ (here N, being an ideal of square zero, is endowed with its canonical B-bimodule structure). Then f has a lifting i.e. there is an algebra map $\overline{f} : A \to E$ such that $p \circ \overline{f} = f$.*

$$0 \longrightarrow N \overset{i}{\longrightarrow} E \underset{\sigma}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

with A mapping via f to B and $\overline{f}$ to E.

Proof. — The proof is the same of [23, Proposition 9.3.3] for our specific M. However, we recall a different proof of (1) implies (2) that will be needed in the proof of Theorem 3.4. Let $\omega : B \otimes B \to N$ be the Hochschild 2-cocycle associated to the Hochschild extension E of B by M. Then $\overline{\omega} := \omega \circ (f \otimes f) : A \otimes A \to {}_f N_f = M$ is a Hochschild 2-cocycle so that we can consider the Hochschild extension $A \oplus_{\overline{\omega}} M$ of A by M, see [23, page 312]. Since $\overline{\omega}$ is, by assumption, a 2-coboundary, then the latter extension is trivial i.e. there is an algebra map $s : A \to A \oplus_{\overline{\omega}} M$ which is a right inverse of the canonical projection. Composing s with the algebra map $A \oplus_{\overline{\omega}} M \to E : (a, m) \mapsto \sigma f(a) + i(m)$ yields the required map $\overline{f}$. $\qquad \square$

Let R be a commutative noetherian ring. Recall that a point $p \in \operatorname{Spec} R$ is *regular* if the localization R_p of R at p is a regular local ring i.e. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim R_p$, where $\mathfrak{m}$ is the unique maximal ideal of R_p and $\dim R_p$ is its Krull dimension. The ring R is said to be *regular* if the localization at every prime ideal is a regular local ring.

The following result is a variant of [15, Corollary 20].

THEOREM 3.3. — *Let $f : V_n(A) \to k$ be a k-point of Rep_A^n. Then f is a regular point of $\operatorname{Rep}_A^n(k)$ if and only if $\operatorname{Harr}^2(V_n(A), {}_f k) = 0$.*

Proof. — Set $\mathfrak{m} := \ker(f)$ and $R := V_n(A)$. Note that k is a perfect field, as it is algebraically closed. Moreover, since A is f.g., then R is an affine ring (as observed after Lemma 2.2) and hence we can apply [15, Corollary 20] to get that f is regular if and only if $\operatorname{Harr}^2(R, R/\mathfrak{m}) = 0$. We conclude by observing that ${}_f k = \operatorname{im} f \cong R/\ker f = R/\mathfrak{m}$ as left R-modules. $\qquad \square$

THEOREM 3.4. — *Let A be a f.g. k-algebra, let $f : V_n(A) \to k$ be a k-algebra map and let $\rho : A \to M_n(k)$ be the algebra map that corresponds to f through (2.1). Then there is a linear embedding of $\operatorname{Harr}^2(V_n(A), {}_f k)$ into $H^2(A, {}_\rho M_n(k)_\rho)$. As a consequence, $M \in \operatorname{Rep}_A^n$ is a regular point whenever $\operatorname{Ext}_A^2(M, M) = 0$.*

Proof. — Each $M \in \operatorname{Rep}_A^n$ is of the form $M \cong {}_\rho(k^n)$ for some $\rho : A \to M_n(k)$ as in the statement. By Theorem 3.1, we have $\operatorname{Ext}_A^2(M, M) \cong H^2(A, \operatorname{End}_k(M)) \cong H^2(A, {}_\rho M_n(k)_\rho)$. Thus the last assertion of the statement follows by Theorem 3.3 once proved the embedding of $\operatorname{Harr}^2(V_n(A), {}_f k)$ into $H^2(A, {}_\rho M_n(k)_\rho)$. Let us construct it explicitly. The idea of the proof of this fact is inspired by [13, Proposition 19.1.4] where the functor $M_n(-)$ is applied to a commutative extension with nilpotent kernel. Set $B := V_n(A)$ and let $\omega : B \otimes B \to {}_f k$ be a Harrison 2-cocycle. Consider the Hochschild

extension associated to ω

$$(3.1) \qquad 0 \longrightarrow {}_fk \xrightarrow{\ i\ } B_\omega \underset{\sigma}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

where, for brevity, we set $B_\omega := B \oplus_\omega k$. Set $S := M_n(k)$ and apply the exact functor $S \otimes (-)$ to (3.1) to obtain the Hochschild extension

$$(3.2) \quad 0 \longrightarrow S \otimes {}_fk \xrightarrow{S \otimes i} S \otimes B_\omega \underset{S \otimes \sigma}{\overset{S \otimes p}{\rightleftarrows}} S \otimes B \longrightarrow 0$$

Here $S \otimes {}_fk$ is a bimodule over $S \otimes B$ via $(s \otimes b)(s' \otimes l)(s'' \otimes b'') = ss's'' \otimes blb'' = ss's'' \otimes f(b)lf(b'')$, for every $s, s', s'' \in S, l \in k, b, b'' \in B$. Now, let E be either ${}_fk$, B_ω or B and apply the canonical isomorphism $S \otimes E \to M_n(E) : (k_{ij}) \otimes e \mapsto (k_{ij}e)$ to (3.2) to obtain the Hochschild extension

$$(3.3) \qquad 0 \longrightarrow N \xrightarrow{\ i_n\ } M_n(B_\omega) \underset{\sigma_n}{\overset{p_n}{\rightleftarrows}} M_n(B) \longrightarrow 0$$

where we set $p_n := M_n(p), \sigma_n = M_n(\sigma), i_n := M_n(i)$ and N is $M_n(k)$ regarded as a bimodule over $M_n(B)$ via $(b_{is})(l_{tj}) = (\sum_s b_{is}l_{sj})$ and $(l_{tj})(b_{is}) = (\sum_j l_{tj}b_{js})$ for every $(b_{is}) \in M_n(B)$ and $(l_{tj}) \in M_n(k)$. Thus

$$(b_{is})(l_{tj}) = (\sum_s b_{is}l_{sj}) = (\sum_s f(b_{is})l_{sj}) = f_n((b_{is})) \cdot (l_{tj})$$

where $f_n := M_n(f)$ i.e. $N = {}_{f_n}(M_n(k))$. Let $\eta = \eta_A : A \to M_n(V_n(A)) = M_n(B)$ be the universal n-dimensional representation of Definition 2.3. Hence ${}_\eta N = {}_\eta({}_{f_n}(M_n(k))) = {}_{(f_n \circ \eta)}M_n(k) = {}_\rho M_n(k)$, where we used that $f_n \circ \eta = M_n(f) \circ \eta = \rho$ which holds by definition of ρ. A similar argument applies to the right so that we get ${}_\eta N_\eta = {}_\rho M_n(k)_\rho$. Let $\omega_n : M_n(B) \otimes M_n(B) \to N$ be the Hochschild 2-cocycle associated to the Hochschild extension (3.3). Then $\overline{\omega_n} := \omega_n \circ (\eta \otimes \eta)$ is a Hochschild 2-cocycle so that we can consider the assignment

$$\alpha : \mathrm{Harr}^2(V_n(A), {}_fk) \to H^2(A, {}_\eta N_\eta) : [\omega] \mapsto [\overline{\omega_n}].$$

This is a well-defined map. In fact, if $[\omega] = 0$, then we can choose σ to be an algebra map from the very beginning and hence σ_n is an algebra map so that $\omega_n = 0$. Suppose $\alpha([\omega]) = 0$. Then $\overline{\omega_n}$ is a 2-coboundary. This condition guarantees, by the proof of Proposition 3.2, that there is a k-algebra map $\overline{\lambda} : A \to M_n(B_\omega)$ such that $p_n \circ \overline{\lambda} = \eta_A$. This map corresponds, via (2.1), to an algebra map $\lambda : B \to B_\omega$ such that $p \circ \lambda = \mathrm{Id}_B$. This means that the Hochschild extension (3.1) is trivial whence $[\omega] = 0$. Thus α is injective. $\quad \square$

Remark 3.5. — The map $\mathrm{Harr}^2(V_n(A), {}_fk) \hookrightarrow H^2(A, {}_\rho M_n(k)_\rho)$ is not an isomorphism in general. Furthermore the condition $\mathrm{Ext}_A^2(M, M) = 0$

is not necessarily satisfied by regular points in Rep_A^n. There is indeed the following counterexample.

Let A be a 2-Calabi Yau algebra, see [12, Definition 3.2.3] for details. It has been proven by Bocklandt that such an algebra has simple modules and that these modules are regular points in Rep_A^n (see [2, Section 7.1]). Therefore, for a simple $M \in \mathrm{Rep}_A^n(k)$ one has $\mathrm{Harr}^2(V_n(A), {}_f k) = 0$. On the other hand, since A is 2-Calabi Yau, one has $\mathrm{Ext}_A^2(M, M) \cong \mathrm{Ext}_A^0(M, M) \cong \mathrm{End}_A(M) \cong k$, for all $M \in \mathrm{Rep}_A^n$.

The referee pointed out to our attention the following example. Consider the (4-dimensional) preprojective algebra Π for a quiver of type A_2. An elementary calculation shows that for the two 1-dimensional simple Π-modules S_1 and S_2 one has $\mathrm{Ext}_\Pi^1(S_i, S_i) = 0$ but $\mathrm{Ext}_\Pi^2(S_i, S_i) \cong k$. Thus Rep_Π^n has two smooth points with non-trivial obstructions.

4. Examples and Applications

Next aim is to introduce and investigate the notion of finitely unobstructed algebra. We will give several examples of such algebras. Moreover we will analyze the relationship between finitely unobstructed, formally smooth and hereditary algebras to better understand the influence of the structure of A on the smoothness of Rep_A^n.

4.1. Finitely unobstructed algebras

DEFINITION 4.1. — *Let A be a k-algebra. Given $n \in \mathbb{N}$, we say that A is n-finitely unobstructed, if $\mathrm{Ext}_A^2(M, M) = 0$, for every $M \in \mathrm{Rep}_A^n(k)$. We say that A is finitely unobstructed, if it is n-finitely unobstructed for every $n \in \mathbb{N}$.*

COROLLARY 4.2. — *The scheme Rep_A^n is smooth for all n-finitely unobstructed k-algebra A.*

Proof. — It follows by Theorem 3.4. □

We recast here some basic concepts in order to list examples and applications of the results proven in Section 3.

4.2. Hereditary algebras

Recall that the *projective dimension* $\mathrm{pd}(M)$ of an $M \in A\text{-}\mathsf{Mod}$ is the minimum length of a projective resolution of M.

DEFINITION 4.3. — *The global dimension of a ring A, denoted with $gd(A)$, is the supremum of the set of projective dimensions of all (left) A-modules. If $gd(A) \leqslant 1$, then A is called* hereditary.

It holds that $gd(A) \leqslant d$ if and only if $\mathrm{Ext}_A^{d+1}(M, N) = 0$, for all $M, N \in A$-mod, see [4, Proposition 2.1, page 110].

4.3. Formally smooth algebras

For further readings on these topics see [13, 12].

DEFINITION 4.4 (Definition 3.3. [7]). — *An $A \in \mathcal{N}_k$ is said to be formally smooth (or quasi-free), if it satisfies the equivalent conditions:*

(i) *any homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{N}_k}(A, R/N)$ where N is a nilpotent (two-sided) ideal in an algebra $R \in \mathcal{N}_k$, can be lifted to a homomorphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{N}_k}(A, R)$ that commutes with the projection $R \to R/N$;*

(ii) $\mathrm{H}^2(A, M) = 0$ *for any $M \in A^e$-Mod;*

(iii) *the kernel Ω_A^1 of the multiplication $A \otimes A \to A$ is a projective A^e-module.*

Remark 4.5. — When A is commutative Ω_A^1 is nothing but the module of the Kähler differentials (see [13, Section 8]).

If we substitute $A \in \mathcal{C}_k$ and $\mathrm{Hom}_{\mathcal{C}_k}(A, -)$ in Definition 4.4, we obtain the classical definition of regularity in the commutative case (see [18, Proposition 4.1]). On the other hand, if we ask for a commutative algebra A to be formally smooth in the category $\mathcal{N}_k$ we obtain regular algebras of dimension $\leqslant 1$ only (see [7, Proposition 5.1]). Thus, if $X = \mathrm{Spec} A$ is an affine smooth scheme, then A is not formally smooth unless $\dim X \leqslant 1$.

4.4. Implications and equivalences

Let us collect the following, well-known, characterizations of finite-dimensional hereditary algebras.

THEOREM 4.6. — *Let A be a finite-dimensional algebra over k. The following assertions are equivalent:*

(1) *A is formally smooth;*

(2) $\mathrm{H}^2(A, N) = 0$ *for every $N \in A^e$-Mod_f;*

(3) *A is finitely unobstructed;*
(4) Rep_A^n *is smooth for every* $n \in \mathbb{N}$;
(5) *A is hereditary.*

Proof. — (1) $\Rightarrow$ (2) is trivial. Implication (2) $\Rightarrow$ (3) follows by Theorem 3.1, while (3) $\Rightarrow$ (4) follows by Corollary 4.2. Finally (4) $\Rightarrow$ (5) is [3, Proposition 1] and (5) $\Rightarrow$ (1) follows from [5, Proposition 0.6].   □

Remark 4.7. — Let A be an infinite-dimensional finitely generated algebra over k. Let us check the following chain of implications

$$(1) \Rightarrow (5) \Rightarrow (3) \Rightarrow (4) \qquad \text{but} \qquad (1) \nLeftarrow (5) \nLeftarrow (3) \nLeftarrow (4)$$

where the notations are the same of Theorem 4.6.

(1) $\Rightarrow$ (5). This is [7, Proposition 5.1].

(1) $\nLeftarrow$ (5). Consider $A^1 = \mathbb{C}[x, \delta]/ < x\delta - \delta x = 1 >$ the first Weyl algebra. It is an example of a hereditary but not formally smooth algebra, since it can be proved that $\text{H}^2(A, A^e) \neq 0$ (see [1, Proposition 3]). This is due to William Crawley-Boevey (personal communication).

(5) $\Rightarrow$ (3). This is trivial.

(5) $\nLeftarrow$ (3). Let $U := U(\mathfrak{g})$ be the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$. *Whitehead's second lemma* (see e.g. [23, Corollary 7.8.12, page 246]) asserts that, in characteristic zero, $\text{H}^2_{Lie}(\mathfrak{g}, N) = 0$ for every $\mathfrak{g}$-module N of finite-dimension over k. In particular, for every $M \in A\text{-Mod}_f$, we obtain $\text{H}^2_{Lie}(\mathfrak{g}, \text{End}_k(M)) = 0$. By [23, Exercise 7.3.5, page 226], we have that

$$\text{H}^*_{Lie}(\mathfrak{g}, \text{End}_k(M)) \cong \text{Ext}^*_U(M, M).$$

Therefore, $\text{Ext}^2_U(M, M) = 0$. If $\mathfrak{g}$ is finite-dimensional, then U is finitely generated and, thus, U is finitely unobstructed. Now, U has global dimension $\dim_k(\mathfrak{g})$, see [23, Exercise 7.7.2, page 241], and we are done.

(3) $\Rightarrow$ (4). This is Corollary 4.2.

(3) $\nLeftarrow$ (4). Remark 3.5 shows that there might exist regular points M in Rep_A^n with $\text{Ext}^2_A(M, M) \neq 0$.

Remark 4.8. — The implication (1) $\Rightarrow$ (4) was already known, see [13, Proposition 19.1.4], [18, Proposition 6.3].

Remark 4.7 and the argument on $U = U(\mathfrak{g})$, contained in the proof thereby, together imply smoothness of Rep_U^n. This result was known, see e.g. the comment by Le Bruyn in [17].

4.5. Unobstructed Algebras

We now list some examples and results in case A is finitely generated but not necessarily finite-dimensional.

In the remaining part of the section k can be any field.

Example 4.9. — We have seen in the Remark 4.7 that $U(\mathfrak{g})$ is finitely unobstructed for a semisimple Lie algebra $\mathfrak{g}$.

More generally, in [24, Theorem 0.2], there is a characterization of all finite-dimensional Lie algebras $\mathfrak{g}$ over a field k of characteristic zero such that their second cohomology with coefficients in any finite-dimensional module vanishes. Such a Lie algebra is one of the following: (i) a one-dimensional Lie algebra; (ii) a semisimple Lie algebra; (iii) the direct sum of a semisimple Lie algebra and a one-dimensional Lie algebra. Note that a one-dimensional Lie algebra $\mathfrak{g}$ is not semisimple as $[\mathfrak{g}, \mathfrak{g}] = 0 \neq \mathfrak{g}$ (cf. [16, Corollary at page 23]). The same argument as above shows that the universal enveloping algebras of all of these Lie algebras are finitely unobstructed.

The proof of the following result is analogous to [7, Proposition 5.3(4)].

PROPOSITION 4.10. — *Let A and S be finitely unobstructed algebras over a field k. If $\mathrm{Ext}^1_S(M, M) = 0$ for every $M \in S\text{-Mod}_f$, then $S \otimes A$ is finitely unobstructed.*

Remark 4.11. — Since $gd\,k[x_1, \ldots, x_n] = n$, from Remark 4.7 it follows that the algebra $k[x_1, \ldots, x_n]$ is not formally smooth for $n > 1$.
In general, the tensor product of two formally smooth algebras is not formally smooth. Indeed, in the setting of Proposition 4.10, if both A and S are finitely generated algebras over k, then, by [4, Proposition 7.4], we have $\mathrm{pd}(S \otimes A) = \mathrm{pd}(S) + \mathrm{pd}(A)$, where $\mathrm{pd}(\Lambda)$ denotes the projective dimension of a k-algebra Λ regarded as a bimodule over itself. Since $\mathrm{pd}(A) \leqslant n$ if and only if $\mathrm{H}^{n+1}(A, N) = 0$ for every $N \in A^e\text{-Mod}$, we get that the algebra $S \otimes A$ is not formally smooth unless $\mathrm{pd}(S) + \mathrm{pd}(A) \leqslant 1$ i.e. unless S and A are both formally smooth and at least one of them is separable.

By using Proposition 4.10, we can give new examples of algebras whose associated representation scheme is smooth.

Examples 4.12. — 1) Let A be a finitely unobstructed algebra and let S be a separable algebra (see [7, above Proposition 3.2]), that is $\mathrm{H}^i(S, N) = 0$ for every $i > 0$ and for every S-bimodule N. By Theorem 3.1, we get $\mathrm{Ext}^i_S(M, M) = 0$ for every $i > 0$ and for every $M \in S\text{-Mod}$. By Proposition 4.10, we get that $S \otimes A$ is finitely unobstructed. As a particular case,

when $\mathrm{char}(k) = 0$, we have that $M_n(A) \cong M_n(k) \otimes A$ and the group A-ring $A[G] \cong k[G] \otimes A$, for every finite group G, are finitely unobstructed as the matrix ring $M_n(k)$ and the group algebra $k[G]$ are separable in characteristic zero (see [7, Example of page 271]).

2) Let A be finitely unobstructed algebra and S a separable algebra, then $\mathrm{Rep}^n_{A \otimes S}$ is smooth. This follows from Proposition 4.10 and example 1).

3) Let $\mathfrak{g}$ be a semisimple Lie algebra and assume $\mathrm{char}(k) = 0$. As observed in Example 4.9, $U := U(\mathfrak{g})$ is finitely unobstructed. Moreover Whitehead's first lemma [23, Corollary 7.8.10] ensures that $\mathrm{H}^1_{Lie}(\mathfrak{g}, N) = 0$ for every $\mathfrak{g}$-module N of finite dimension over k so that, by the same argument used in Example 4.9 for the second group of cohomology, we obtain $\mathrm{Ext}^1_U(M, M) = 0$, for every $M \in U\text{-Mod}_f$. Thus, by Proposition 4.10, we get that $U \otimes A$ is finitely unobstructed if A is.

4) In analogy with [7, Proposition 5.3(5)], we have that the direct sum of finitely unobstructed algebras is finitely unobstructed too.

LEMMA 4.13. — *Assume that* $\mathrm{H}^2(A, N) = 0$ *for some* $N \in A^e\text{-Mod}$ *and let* Ω^1_A *be as in Definition 4.4. Then* Ω^1_A *is projective with respect to any surjective morphism of* A^e-modules with kernel N.

Proof. — One gets that $\mathrm{Ext}^1_{A^e}(\Omega^1_A, N) \cong \mathrm{Ext}^2_{A^e}(A, N) = \mathrm{H}^2(A, N) = 0$ analogously to [7, Proposition 3.3]. The conclusion follows by applying the long exact sequence of $\mathrm{Ext}^*_{A^e}(\Omega^1_A, -)$ to the exact sequence formed by any surjective morphism of A^e-modules and its kernel N. $\qquad\square$

The proof of the following result is similar to [7, Proposition 5.3(3)].

PROPOSITION 4.14. — *Let* A *be a finitely unobstructed algebra over a field* k. *Then the tensor algebra* $T_A(P)$ *is finitely unobstructed for every* $P \in A^e\text{-Mod}$ *which is projective with respect to any surjective morphism in* A^e-Mod *with kernel* $\mathrm{End}_k(M)$ *for every* $M \in A\text{-Mod}_f$.

Example 4.15. — Let A be a finitely unobstructed algebra over a field k. By Theorem 3.1, Lemma 4.13 and Proposition 4.14, we have that $T_A(\Omega^1_A)$ is finitely unobstructed. The latter, by [7, Proposition 2.3], identifies with ΩA, the DG-algebra of noncommutative differential forms on A.

5. Deformations

In this section we would like to analyze the relationships between the results of Section 3 and the theory of deformations of module structures.

DEFINITION 5.1. — *Let $M \in \operatorname{Rep}_A^n(k)$ and let $\mu : A \to \operatorname{End}_k(M)$ be the associated linear representation. For $(R, \mathfrak{m})$ a local commutative k-algebra, an R-deformation of M is an element $\widetilde{M} \in \operatorname{Rep}_A^n(R)$ whose associate linear representation $\widetilde{\mu} : A \to \operatorname{End}_R(\widetilde{M})$ verifies $\alpha \circ \widetilde{\mu} = \mu$ where $\alpha : \operatorname{End}_R(\widetilde{M}) \to \operatorname{End}_k(M)$ is the morphism of k-algebras indueced by the projection $R \to R/\mathfrak{m} \cong k$.*

When $R = k[\epsilon] := k[t]/(t^2)$, the ring of dual numbers or $R = k[\![t]\!]$, the ring of formal power series, then an R-deformation will be called infinitesimal *or* formal, *respectively.*

For the general theory on deformations of finite-dimensional modules see [9] and [11].

Remark 5.2. —   It is well-known that the obstructions in extending the infinitesimal deformations of M to formal deformations are in $\operatorname{Ext}_A^2(M, M)$ (see for example [9, 3.6 and 3.6.1]).

The theory of local and global deformations of algebraic schemes is an ample and well-established domain of modern algebraic geometry. Sernesi wrote an excellent treatise on this topic [21], and we address the interested reader to it.

We just recall some facts we need to develop our analysis. Let X be a scheme over k, let $x \in X(k)$ be a k-point of X and let $(R, \mathfrak{m})$ be a local commutative k-algebra.

DEFINITION 5.3. — *An R-deformation of X at x is an R-point x_R of X such that the restriction $\operatorname{Spec} k \to \operatorname{Spec} R$ maps x to x_R. When $R = k[\epsilon]$ or $R = k[\![t]\!]$, then an R-deformation will be called* infinitesimal *or* formal, *respectively.*

LEMMA 5.4. — *Let $R \in \mathcal{C}_k$ and let $x : R \to k$ be a rational point of $X = \operatorname{Spec} R$. Then, for all local $S \in \mathcal{C}_k$, there is a bijection*

$$\{S - \text{deformations of } X \text{ at } x\} \cong \operatorname{Hom}_{\mathcal{C}_k}(R_x, S)$$

where R_x denotes the localization of R at $\mathfrak{m}_x := \ker x$.

Proof. — Let $\alpha : R \to S$ be such that $x = \pi_S \alpha$, where $\pi_S : S \to S/\mathfrak{m}_S \cong k$ is the canonical projection with $\mathfrak{m}_S$ the maximal ideal of S. Then if $a \in R - \ker x$, it follows that $\alpha(a)$ is invertible in S and, therefore, by universality, there is a unique morphism $\alpha_x : R_x \to S$ such that $\alpha_x j_R^x = \alpha$, where $j_R^x : R \to R_x$ is the canonical map, and hence $x = \pi_S \alpha_x j_R^x$.

On the other hand, given a morphism $\beta : R_x \to S$ one has that $x = \pi_S \beta j_R^x$ thus giving the unique S-deformation βj_R^x of X at x. It is, indeed,

trivial that $\ker x \subset \ker(\pi_S \beta j_R^x)$. If $r \in \ker(\pi_S \beta j_R^x)$ then $\beta(j_R^x(a)) \in \mathfrak{m}_S$ and, therefore, $j_R^x(a) \in \mathfrak{m}_{R_x}$. Thus $a \in \ker x$. $\qquad\square$

The adjunction in Lemma 2.2 gives the dictionary to describe deformations of A-modules in terms of deformations at points of Rep_A^n. The following result complements Theorem 3.4.

PROPOSITION 5.5. — Let $A \in \mathcal{N}_k$ be finitely generated. Let $M \in \mathrm{Rep}_A^n(k)$ and let $f : V_n(A) \to k$ be the associated point. Then M is regular if and only if, for all finite-dimensional local commutative k-algebras S, T, a surjective homomorphism of k-algebras $S \to T$ induces a surjection

$$\{S - \text{deformations of } M\} \longrightarrow \{T - \text{deformations of } M\}.$$

Proof. — This follows for example from [19, Proposition, page 151]. $\quad\square$

Remark 5.6. — Geiss and de la Peña proved that, if A is a finite-dimensional algebra, then $M \in \mathrm{Rep}_A^n(k)$ is regular if $\mathrm{Ext}_A^2(M, M) = 0$, see [9, 10]. A careful analysis of their argument shows that it is easy to adapt their proof if one supposes that A is bimodule coherent in the sense of [12, 3.5.1], since in this case each finite-dimensional A-module admits a projective resolution by finitely generated projectives. If A is finitely presented, an argument involving cones, similar to [6, Lemma 4.3], allows to extend the above mentioned argument to this situation.

Acknowledgement

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