ANNALES

DE

L'INSTITUT FOURIER

Shigeharu TAKAYAMA

Singularities of Narasimhan-Simha type metrics on direct images of relative pluricanonical bundles

Tome 66, nº 2 (2016), p. 753-783.

<http://aif.cedram.org/item?id=AIF_2016__66_2_753_0>

Ann. Inst. Fourier, Grenoble
66, 2 (2016) 753-783

SINGULARITIES OF NARASIMHAN-SIMHA TYPE METRICS ON DIRECT IMAGES OF RELATIVE PLURICANONICAL BUNDLES

by Shigeharu TAKAYAMA

Abstract. — We study singularities of the Narasimhan-Simha Hermitian metric on the direct image of a relative pluricanonical bundle. The upper bound relates to log-canonical thresholds.

Résumé. — Nous étudions les singularités des métriques hermitiennes de Narasimhan-Simha sur les images directes des fibrés pluricanoniques relatifs. La majoration est liée aux seuils log-canoniques.

1. Introduction

We continue the study of Narasimhan-Simha type Hermitian metrics on direct image sheaves of relative pluricanonical bundles. For a projective surjective holomorphic map $f : X \to Y$ of complex manifolds with connected fibers, a direct image sheaf $f_*(mK_{X/Y})$ admits a natural, possibly singular, Hermitian metric g_m, called the m-th Narasimhan-Simha Hermitian metric. In our work with M. Păun, we established the definition of g_m, even when the map f has singularities and even when $f_*(mK_{X/Y})$ is merely torsion free (it is a sort of an extension from the set of regular values of f to the whole Y), and proved the curvature semi-positivity in the sense of Griffiths of g_m (see [26, 1,1, 5.1.2]). Here we are interested in the asymptotic behavior of the metric g_m around the set of critical values of the fibration. Our main result is as follows.

Keywords: singularities of Narasimhan-Simha metric, relative pluricanonical bundle, Bergman kernel metric, nef vector bundle.
Math. classification: 14D06, 32J25.

Theorem 1.1. — *Let X be a normal complex space with only canonical singularities at worst, Y a complex manifold, and let $f : X \to Y$ be a projective surjective holomorphic map with connected fibers and with $\dim X - \dim Y = n$. Let m be a positive integer such that $f_*(mK_{X/Y})$ is non-zero. We endow $f_*(mK_{X/Y})$ with the m-th Narasimhan-Simha Hermitian metric g_m, via a resolution of singularities $\alpha : X' \to X$ and an isomorphism $f_*(mK_{X/Y}) \cong f'_*(mK_{X'/Y})$ with the m-th Narasimhan-Simha Hermitian metric on $f'_*(mK_{X'/Y})$ obtained in* [26, 1.1]*, where $f' = f \circ \alpha : X' \to Y$ is the composition. Let $0 \in Y$ be a point, and let X_0 be the scheme theoretic fiber of it.*

(1) Suppose $\dim Y = 1$ and $Y = \{t \in \mathbb{C};\ |t| < 1\}$ is a disc with our special point 0 as the origin. Let r_0 be the log-canonical threshold of the pair (X, X_0), namely

$$r_0 = \sup\{r \geqslant 0;\ \text{the pair } (X, rX_0) \text{ is log-canonical}\}.$$

Then, for every $u \in H^0(Y, f_*(mK_{X/Y}))$, there exists a constant $A_u > 0$ such that

$$g_m(u,u)(t) \leqslant \left(\frac{1}{|t|^{2(1-r_0)}}(-\log|t|)^n A_u\right)^m$$

holds for any $t \in Y' \setminus 0$, where $Y' \subset Y$ is a smaller disc independent of u. In particular if the pair (X, X_0) is log-canonical, i.e., $r_0 = 1$, then $g_m(u,u)(t)$ is at most $(-\log|t|)^{nm}$ growth as $t \to 0$.

(2) Suppose $\dim Y \geqslant 1$, and suppose that

- *(2.i) there exists a general (germ of) smooth curve $C \subset Y$ passing through 0 such that $X_C := X \times_Y C$ is normal and canonical singularities at worst, and that*
- *(2.ii) $F := f_*(mK_{X/Y})$ is locally free on an open neighborhood U of 0 in Y.*

Let h be the induced singular Hermitian metric on $\mathcal{O}_{\mathbb{P}(F|_U)}(1)$ of $\mathbb{P}(F|_U)$ from g_m via the quotient $\pi^*(F|_U) \to \mathcal{O}_{\mathbb{P}(F|_U)}(1)$, where $\pi : \mathbb{P}(F|_U) \to U$ is the projection. Its local weight φ, i.e., $h = e^{-\varphi}$ locally, is plurisubharmonic by the Griffiths semi-positivity of g_m. Let r_{0C} be the log-canonical threshold of (X_C, X_0) along X_0. Then the Lelong number of φ is at most $2(1-r_{0C})m$ at every point $P \in \pi^{-1}(0) \subset \mathbb{P}(F|_U)$, i.e., $\nu(\varphi, P) \leqslant 2(1-r_{0C})m$.

Here are additional explanations on the statement. We actually meant that the direct image sheaf of the relative m-canonical sheaf $mK_{X/Y} := j_*(K_{X_{\mathrm{reg}}}^{\otimes m}) \otimes f^*K_Y^{\otimes(-m)}$, where $j : X_{\mathrm{reg}} \to X$ is the open immersion of the smooth locus. (We are sorry to mix additive and multiplicative notations, caused by the presence of singularities and by the non-algebraicity of varieties.) As we shall see (in 3.2 and 4.1), the Narasimhan-Simha Hermitian metric g_m is independent of the choice of $\alpha : X' \to X$. In 1.1(1), we know that $0 < r_0 \leqslant 1$, $r_0 \in \mathbb{Q}$ and "sup" is in fact "max". For example, if X is smooth and $\operatorname{Supp} X_0$ is normal crossing with its prime decomposition $X_0 = \sum b_i X_{0i}$, then the log-canonical threshold of the pair (X, X_0) is $r_0 = (\max_i b_i)^{-1}$ (refer to [21, 9.3.12] for log-canonical thresholds, for example). Here in 1.1(2.i), a curve $C \subset Y$ is *general*, if there exists a point $t \in C$ such that the fiber X_t of f has canonical singularities at worst and the fiber X'_t of $f' : X' \to Y$ is smooth. Note that in the case $\dim Y = 1$, i.e., in (1), these conditions (2.i) and (2.ii) are automatically satisfied. In the case of $\dim Y > 1$, the simplest setting will be as follows. We suppose that X is smooth and the set of critical values of f is a non-zero divisor D_Y, and let $f^* D_Y = \sum b_i B_i$ be the prime decomposition. Suppose further that the divisor $\sum B_i$ is f-relative normal crossing over a general point of D_Y. Then, for every general point $0 \in D$, we can see (2.i) and (2.ii) are satisfied, and then we can conclude $\nu(\varphi, P) \leqslant 2(1 - r_0)m$ with $r_0 = (\max_i b_i)^{-1}$, where the maximum is taken among all components B_i for which $f(B_i)$ contains 0. Another possible setting for 1.1(2) is the case of weakly semi-stable reduction $f : X \to Y$ ([1]). This setting is seemingly quite technical, but it is natural and useful. We will discuss a bit around 1.2, 1.3 and the proof of 1.3 in §5.

We recall briefly how g_m is obtained on $f_*(mK_{X/Y})$ being X itself is smooth. Let $Y_0 \subset Y$ be the open subset over which f is smooth. For every $t \in Y_0$, the sheaf $f_*(mK_{X/Y})$ is locally free around t by Siu's invariance of plurigenera [30] (see also [25]), and its fiber is $f_*(mK_{X/Y})_t = H^0(X_t, mK_{X_t})$, where $X_t = f^{-1}(t)$ is the fiber. On each X_t, we construct a Bergman kernel type singular Hermitian metric $B_{m,t}$ on $-mK_{X_t}$ as a natural generalization of the usual Bergman kernel metric, and then with respect to a singular Hermitian metric $h_{m-1,t} = (B_{m,t}^{-1})^{(m-1)/m}$ on $L_{m-1,t} = (m-1)K_{X_t}$, the vector space $H^0(X_t, mK_{X_t})$ admits a natural Hermitian form $g_{m,t}$ given by $g_{m,t}(u_t, v_t) = \int_X (-1)^{n^2/2} u_t \wedge \overline{v}_t \, h_{m-1,t}$, where $u_t, v_t \in H^0(X_t, mK_{X_t}) = H^0(X_t, K_{X_t} + L_{m-1,t})$ are regarded as $L_{m-1,t}$-valued holomorphic n-forms. In this way, we obtain a metric g_m^o on $f_*(mK_{X/Y})|_{Y_0}$. When K_{X_t} is ample, this construction is due to Narasimhan-Simha [23]. In [26], we showed this metric extends as a singular Hermitian metric on $f_*(mK_{X/Y})$ with Griffiths semi-positive curvature. (See [28], [26, §2] for a general discussion on singular Hermitian metrics on vector bundles or on torsion free sheaves.) To obtain such an extension, we needed to show that g_m^o is bounded from below by a positive constant, like $g_m^o(u,u)(t) \geqslant c_0 > 0$

([26, 3.3.11]), which is an estimate from the opposite side in 1.1(1). Though it plays no roles in this paper, an extension of the relative Bergman kernel type metric $B_{m,X/Y}$ on $-mK_{X/Y}$ from $f^{-1}(Y_0)$ to X was a crucial step, for this the $L^{2/m}$-version of Ohsawa-Takegoshi's theorem was proved and applied (this part is due to [3]). In a classical term of complex analysis, the weight function φ $(= \varphi_u) := -\frac{1}{m}\log g_m(u,u)$, so that $g_m(u,u) = e^{-m\varphi}$, might be discussed. An upper bound $\varphi \leqslant c_0'$ on Y_0 is given in [26, 3.3.11]. While 1.1(1) gives a lower bound $\varphi(t) \gtrsim 2(1-r_0)\log|t| - n\log(-\log|t|)$.

When $m = 1$, the metric g_1 is the so-called canonical L^2-metric on $f_*(K_{X/Y})$ given by $\int_{X_t}(-1)^{n^2/2}u_t \wedge \overline{v}_t$ on each smooth fiber. In the situation of 1.1(1), if X is smooth, $\operatorname{Supp} X_0$ is normal crossing and $m = 1$, it has a Hodge theoretic interpretation or foundation, and hence there are more precise and informative results. Motivated by the classification theory and the moduli theory of varieties, there are many former researches in this direction, including the case $m \geqslant 1$ ([12], [11], [14], [15], [19], [2], [22], [32], [34], ... for the metric aspect, and more and more for the algebraic aspect, especially by Viehweg [33]), however not so many for the upper bounds as in 1.1 (see [34, §7] for an adjoint type bundle case). Among them, we are influenced by Kawamata [16], [17], who first focused a role of the log-canonical threshold in the asymptotic of fiberwise integrals.

Theorem 1.1 is also motivated by the following recent work by Fujino.

Theorem 1.2 ([10, 1.6]). — *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ has a good minimal model. Then there exists a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with the following property. Let X' be any resolution of the main component of $X \times_Y Y'$ sitting in the following commutative diagram:*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\tau} & Y \end{array}$$

Then $f'_*(mK_{X'/Y'})$ is a nef locally free sheaf for every positive integer m.

Recall that a locally free sheaf E (of finite rank) on a smooth projective variety is called *nef*, if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$. (Fujino uses a terminology "semi-positive" instead of nef.) We note that, if a line bundle L on a smooth projective algebraic variety admits a singular Hermitian metric h with semi-positive curvature and its local weight φ (i.e., $h = e^{-\varphi}$ locally, and φ can be taken to be plurisubharmonic, psh for short) has zero Lelong numbers everywhere, then L is nef ([6, 6.4]). However the converse does not hold in general ([8, 1.7]). (The Lelong numbers are measuring the "big" algebraic singularities of a psh function, so it extracts an algebraic object from an analytic one, refer [7, Ch. 2].) We shall strengthen 1.2 by applying 1.1 in the following form. These type of results 1.1, 1.2, 1.3 must be fundamental in further studies of analytic theory and differential geometry of moduli.

Corollary 1.3. — *In 1.2, the m-th Narasimhan-Simha Hermitian metric g_m on the locally free sheaf $F' := f'_*(mK_{X'/Y'})$ has Griffiths semi-positive curvature, the induced singular Hermitian metric $h = e^{-\varphi}$ on $\mathcal{O}_{\mathbb{P}(F')}(1)$ of $\mathbb{P}(F')$ has semi-positive curvature, and the Lelong number of the local weight φ is zero everywhere on $\mathbb{P}(F')$. In particular $\mathcal{O}_{\mathbb{P}(F')}(1)$ is nef.*

Let us explain what is improved. As we already explained, the property of the metric $h = e^{-\varphi}$ in 1.3 is strictly stronger than the nefness of $\mathcal{O}_{\mathbb{P}(F')}(1)$ in general. Furthermore we stress the following point of view, which is our standing position from [26]. A Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is (by definition) a Finsler metric on a locally free sheaf E. A Hermitian metric on E defines a Finsler metric on E. A long standing conjecture by Griffiths asks that if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample, i.e., E is ample, then E admits a Hermitian metric with Griffiths positive curvature. We note that the nefness of $\mathcal{O}_{\mathbb{P}(F')}(1)$ in 1.3 is derived from a singular Hermitian metric on F', not only by a singular Hermitian metric on $\mathcal{O}_{\mathbb{P}(F')}(1)$.

In the course of the proof of 1.2, Fujino applies a weak semi-stable reduction theorem [1], which gives a nice intermediate fibration (other than $f' : X' \to Y'$ in 1.2), and then obtains the local freeness of the direct image by passing further to a good relative minimal model. To prove the nefness, he uses a covering trick and a result of Popa and Schnell [27] which is cohomological and based on vanishing theorems. We note that the total space of the weak semi-stable reduction is not smooth in general, but only canonical Gorenstein singularities. The behavior of the Narasimhan-Simha Hermitian metric g_m around the critical set of a fibration is not well-understood in general, which will be likely the singularities of the fibration. A weak semi-stable reduction has only mild singularities, called toroidal singularities, and hence we can expect the Narasimhan-Simha Hermitian metric g_m for such fibrations admits only mild singularities as well. This observation leads 1.3 as a corollary of 1.1.

The organization of this paper is as follows. In §2, we discuss the Bergman kernel type metrics and Narasimhan-Simha Hermitian forms on normal varieties. We especially discuss them on varieties with canonical singularities in §3. These parts are aiming to give a foundation, and hence they are more than enough to obtain the results stated in the introduction. We prove 1.1(1) in §4. Our method of proof is very direct, namely we compute the fiberwise integral by hand following the definition of the Bergman kernel and Narasimhan-Simha type metrics. Then in §5, we prove 1.1(2) which is deduced from 1.1(1) and a simple reduction argument to a general curve section, and prove 1.3 along the line explained in the last paragraph.

Acknowledgements. — It is our pleasure to acknowledge our gratitudes to Mihai Păun and Osamu Fujino for valuable discussions. We also thank the referee for their comments to improve the presentations of the paper. This research is supported by Grant-in-Aid for Scientific Research (B)23340013.

2. Construction of Bergman and Narasimhan-Simha type metrics

We recall basic constructions of Bergman kernel type metrics and Narasimhan-Simha Hermitian forms on complex manifolds, and extend them for normal varieties. These materials are first introduced in geometric setting by Kobayashi [18] and developed further in [23], [29], [2] and so on. We update them with some attentions on singularities of varieties and metrics.

Let us first mention a general remark to construct a singular Hermitian metric. If L is a line bundle on a complex manifold X with a non-zero section $s \in H^0(X, L)$. Then, by taking a C^∞ Hermitian metric h on L, $\frac{h}{|s|_h^2}$ defines a singular Hermitian metric on L. It is independent of a reference metric h and hence often written as $\frac{1}{|s|^2}$. Moreover it has semi-positive curvature, namely $\sqrt{-1}\,\overline{\partial}\partial \log \frac{1}{|s|^2}$ is a closed semi-positive $(1,1)$-current ([7, (3.13)]). We refer to say that $|s|^2$ is *log-psh*. As this example already shows, a singular Hermitian metric is defined on stalks not on fibers of the line bundle.

Definition-Notation 2.1. — Let X be a complex manifold of $\dim X = n$, which may be non-compact.

(1) We take a reference C^∞ volume form dV on X (which can be seen as a C^∞ Hermitian metric on $\wedge^n T_X$), and regard the inverse $h := dV^{-1}$ as a C^∞ Hermitian metric on K_X. On a local coordinate $(U, z = (z_1, \dots, z_n))$, the correspondence of local frames is given by identifying $c_n dz \wedge d\overline{z}$ and

$dz \otimes d\overline{z}$, where $dz = dz_1 \wedge \ldots \wedge dz_n$, $d\overline{z} = d\overline{z_1} \wedge \ldots \wedge d\overline{z_n}$ and $c_n = (-1)^{n^2/2}$. Let m be a positive integer. Then for every $u \in H^0(X, mK_X)$, the squared pointwise length function $|u|^2_{h^m}$ is a semi-positive C^∞-function on X, and

$$|u|^2 := |u|^2_{h^m} h^{-m} \quad \left(\text{or } |u|^2_{h^m} = \left|\frac{u \wedge \overline{u}}{dV^m}\right|\right)$$

is a C^∞, but possibly degenerate, Hermitian metric on $-mK_X$. As we explained above, if u is non-zero, $\frac{1}{|u|^2}$ defines a singular Hermitian metric with semi-positive curvature on mK_X. We then also obtain $|u|^{2/m} = |u|^{2/m}_{h^m} dV$, which is a continuous semi-positive (n,n)-form on X. On every local coordinate $(U, z = (z_1, \ldots, z_n))$, we write $u = \widetilde{u} \cdot (dz)^{\otimes m}$ with $\widetilde{u} \in H^0(U, \mathcal{O}_X)$. Then $|\widetilde{u}|^{2/m} c_n dz \wedge d\overline{z}$ glue together and in fact define the continuous semi-positive (n,n)-form $|u|^{2/m}$ on X.

(2) We set

$$\|u\|_m = \left(\int_X |u|^{2/m}\right)^{m/2}$$

for every $u \in H^0(X, mK_X)$, and set

$$V_m = \{u \in H^0(X, mK_X);\ \|u\|_m < \infty\}.$$

This $\|u\|_m$ is called an *$L^{2/m}$-pseudo-norm* on $V_m \subset H^0(X, mK_X)$. We see $\|tu\|_m = |t|\|u\|_m$ for every $u \in V_m$ and every constant $t \in \mathbb{C}$.

(3) We shall suppose $V_m \neq 0$ after this. For every $x \in X$, we set

$$B_{h^m}(x) = \sup\left\{|u(x)|^2_{h^m};\ u \in H^0(X, mK_X), \|u\|_m \leqslant 1\right\}.$$

We then obtain a function B_{h^m} on X (which is continuous in fact, see 2.2), and set

$$B_m = B_{h^m} h^{-m} = B_{h^m} (dV)^m.$$

Here B_{h^m} depends on h, but B_m does not. For this reason, we may denote as

$$B_m(x) = \sup\left\{|u(x)|^2;\ u \in H^0(X, mK_X), \|u\|_m \leqslant 1\right\}$$

for every $x \in X$. This B_m defines a singular Hermitian metric on $-mK_X$, and is called the *canonical $L^{2/m}$-metric*, or *m-th Bergman kernel metric*. If we write as $B_m = e^{\varphi_m}$ on every local chart, then φ_m is psh as a sup-limit of psh functions $\log|u|^2$ (φ_m is in fact upper-semi-continuous by 2.2). Thus the singular Hermitian metric B_m has semi-negative curvature. We would refer to say that B_m is log-psh.

(4) We also obtain a natural a singular Hermitian metric

$$h_{m-1} = (B_m^{-1})^{(m-1)/m}$$

on $L_{m-1} := (m-1)K_X$. Using h_{m-1}, we can put a natural Hermitian form on $V_m \subset H^0(X, mK_X)$, called the *m-th Narasimhan-Simha Hermitian form* g_{mNS} (we may say the mNS Hermitian form g_m for short), by

$$g_m(u,v) = g_{mNS}(u,v) = \int_X c_n u \wedge \overline{v}\, h_{m-1},$$

which is nothing but the canonical L^2-metric on $H^0(X, K_X + L_{m-1})$ with respect to h_{m-1}, where we regard u, v as L_{m-1}-valued holomorphic n-forms. We shall see in 2.2 that g_m is defined on V_m.

The next lemma is well-known classically, and may be needless to say. The last assertion shows that we can endow a natural pseudo-norm $\|u\|_m$ and a norm $g_m(u,u)^{1/2}$ on $V_m \subset H^0(X, mK_X)$, and they satisfy a simple relation by definition.

LEMMA 2.2. — *In 2.1, suppose that* $V_m \neq 0$.

(1) Let $x_0 \in X$. Then there exists $u_0 \in H^0(X, mK_X)$ with $\|u_0\|_m = 1$ and $|u_0(x_0)|^2 = B_m(x_0)$.

(2) The metric B_{h^m} is continuous on X, in particular $\log B_{h^m}$ *is upper-semi-continuous on X. We will say B_m is continuous, and* $\log B_m$ *is upper-semi-continuous respectively.*

(3) For every $u, v \in V_m$, an (n,n)-form $c_n u \wedge \overline{v}\, h_{m-1}$ is continuous on X.

(4) For every $u \in V_m$, an inequality $\|u\|_{g_m} := g_m(u,u)^{1/2} \leqslant \|u\|_m$ holds.

Proof. — (0) We denote by $\mathcal{F}_m = \{u \in H^0(X, mK_X); \|u\|_m \leqslant 1\}$. Let $x_0 \in X$ be a point, and take a local coordinate $(U, z = (z_1, \dots, z_n))$ containing a polydisc P centered at x_0 as a relatively compact set. We take $u \in \mathcal{F}_m$, and write $u = \widetilde{u} \cdot (dz)^{\otimes m}$ with $\widetilde{u} \in H^0(U, \mathcal{O}_X)$ and $|u|^{2/m} = |\widetilde{u}|^{2/m} c_n dz \wedge d\overline{z}$ on U. By the mean value inequality for a phs function $|\widetilde{u}|^{2/m}$, we obtain a local uniform bound

$$|\widetilde{u}(x_0)|^{2/m} \leqslant \frac{1}{\operatorname{vol} P} \int_P |\widetilde{u}|^{2/m} \frac{c_n}{2^n} dz \wedge d\overline{z} \leqslant \frac{1}{2^n \operatorname{vol} P} \int_X |u|^{2/m} \leqslant \frac{1}{2^n \operatorname{vol} P},$$

where $\frac{c_n}{2^n} dz \wedge d\overline{z}$ is the real Euclidean volume form and the $\operatorname{vol} P$ is the real Euclidean volume of P. We note that the bound $\frac{1}{2^n \operatorname{vol} P}$ is uniform in $u \in \mathcal{F}_m$ and also in $m > 0$. The Cauchy's estimate also implies that all $u_\lambda \in \mathcal{F}_m$ are equi-continuous on every compact set in U. As a consequence, $\mathcal{F}_m$ forms a normal family on U. (We note that this equi-continuity depends on m, not uniform in m.) Thus, on every compact subset in X, $\mathcal{F}_m$ forms a normal family. In particular, any sequence $\{u_j\}$ in $\mathcal{F}_m$ contains a subsequence $\{u_{j'}\}$, which converges uniformly on every compact set in X. We then obtain a limit $u_0 = \lim u_{j'} \in H^0(X, mK_X)$ with $\|u_0\|_{m,X} \leqslant 1$. We also note that all $u_\lambda \in \mathcal{F}_m$ are equi-continuous on every compact set in X.

(1) By definition of B_m, there exists a sequence $\{u_j\}$ in $\mathcal{F}_m$ such that $\lim_{j\to\infty} |u_j(x_0)|^2 = B_m(x_0)$. By the argument in (0), we can find a limit $u_0 \in \mathcal{F}_m$ of a subsequence of $\{u_j\}$ such that $|u_0(x_0)|^2 = B_m(x_0)$. We can choose u_0 so that $\|u_0\|_m = 1$.

(2) We take a point $x_0 \in X$ and a coordinate neighborhood (U, z) of x_0. By the equi-continuity, for any $\varepsilon > 0$, there exists a neighborhood $V \subset U$ of x_0 such that for any $u \in \mathcal{F}_m$ and any $z \in V$, $-\varepsilon < |u(z)|^2 - |u(x_0)|^2 < \varepsilon$ holds. We take $u_0 \in \mathcal{F}_m$ such that $|u_0(x_0)|^2 = B_m(x_0)$. Then from the left hand side inequality, we have, for any $z \in V, B_m(x_0) - \varepsilon < |u_0(z)|^2 \leqslant B_m(z)$. On the other hand, we take $z \in V$ and take $u_z \in \mathcal{F}_m$ such that $|u_z(z)|^2 = B_m(z)$. Then from the right hand side inequality, we have $|u_z(z)|^2 < |u_z(x_0)|^2 + \varepsilon \leqslant |u_0(x_0)|^2 + \varepsilon$, i.e., $B_m(z) < B_m(x_0) + \varepsilon$. These show the continuity of B_m at x_0.

(3) Let $X_m = \{x \in X;\ B_{h^m}(x) \neq 0\}$, which is a non-empty Zariski open subset of X. It is clear by (2) that $c_n u \wedge \overline{v}\, h_{m-1}$ is continuous on X_m. We define temporary $c_n u \wedge \overline{v}\, h_{m-1}(x) = 0$ for every $x \notin X_m$ (for any $u, v \in V_m$), and show that $c_n u \wedge \overline{v}\, h_{m-1}$ is continuous on X. If we could prove that $c_n u \wedge \overline{u}\, h_{m-1}$ is continuous for any $u \in V_m$, we can conclude $c_n u \wedge \overline{v}\, h_{m-1}$ is continuous for any $u, v \in V_m$, by considering $c_n(u+v) \wedge (\overline{u+v})\, h_{m-1}$, $c_n(u+iv) \wedge (\overline{u+iv})\, h_{m-1}$ and so on. We then consider $c_n u \wedge \overline{u}\, h_{m-1}$ for $u \in V_m$ with $\|u\|_m = 1$ without loss of generalities. We regard $u \otimes \overline{u}$ and B_m as sections of a complex (not holomorphic) line bundle $(K_X \otimes \overline{K_X})^{\otimes m}$, and then we can regard $(u \otimes \overline{u})/B_m$ as a $\mathbb{C}$-valued function on X. We see that

$$c_n u \wedge \overline{u}\, h_{m-1} = \left| \frac{u \otimes \overline{u}}{B_m} \right|^{(m-1)/m} |u|^{2/m}$$

as semi-positive (n, n)-forms on X. We note that $|(u \otimes \overline{u})/B_m| \leqslant 1$ by definition of B_m (strictly speaking this holds on X_m). At each $x \notin X_m$, it has to be $u(x) = 0$ by definition of B_m. Thus we obtain the continuity of $c_n u \wedge \overline{u}\, h_{m-1}$ at $x \notin X_m$ by the continuity of $|u|^{2/m}$.

(4) We obtained in (3) that $c_n u \wedge \overline{u}\, h_{m-1} \leqslant |u|^{2/m}$ on X. Then by integration, we have $\|u\|^2_{g_m} = \int_X c_n u \wedge \overline{u}\, h_{m-1} \leqslant \|u\|^2_m$. □

Remarks 2.3. — We still keep the notations in 2.1, and denote by $\mathcal{F}_m = \{u \in H^0(X, mK_X);\ \|u\|_m \leqslant 1\}$.

(1) By the Hölder inequality, we have $u \otimes v \in \mathcal{F}_{m+\ell}$ if $u \in \mathcal{F}_m$ and $v \in \mathcal{F}_\ell$, where m, ℓ are positive integers. In particular by 2.2(1), $B_m \cdot B_\ell \leqslant B_{m+\ell}$ (resp. $B_m^\ell \leqslant B_{m\ell}$) as a singular Hermitian metric on $-(m+\ell)K_X$ (resp. $-(m\ell K_X)$). However we note that each $\mathcal{F}_m$ is not a vector space.

(2) For every $x \in X$, we set

$$B_{X,h}(x) = \sup{}^*\{|u(x)|_{h^m}^{2/m};\ m \in \mathbb{Z}_{>0}, u \in H^0(X, mK_X), \|u\|_m \leqslant 1\},$$

where $\sup^*$ is the upper-semi-continuous envelope of the pointwise sup-limit. We then obtain a bounded function $B_{X,h}$ on X, which is a consequence of the mean value property in the argument (0) of 2.2 (however the Cauchy's estimate depends on m, and hence we do not see the extremal property and the continuity of $B_{X,h}$ as in 2.2), and set

$$B_X = B_{X,h} h^{-1}.$$

This is a singular Hermitian metric on $-K_X$ and is log-psh (unless $B_X \equiv 0$). Moreover B_X^m is less singular than B_m for any $m > 0$. If X is smooth projective for example, then the canonical ring is finitely generated by [4], and hence the roles of B_X may be replaced by one $B_m^{1/m}$ for a sufficiently large $m > 0$.

We shall extend the construction above to normal spaces. In the rest of this section, we use the following notations.

Notation 2.4. — Let X be a normal complex space of $\dim X = n$, and let $j : X_{\mathrm{reg}} \to X$ be the open immersion of the smooth locus. We define the canonical sheaf as $\omega_X = j_* K_{X_{\mathrm{reg}}}$, and the m-canonical sheaf by $\omega_X^m = j_*(K_{X_{\mathrm{reg}}}^{\otimes m})$ for every integer $m > 0$. This is a reflexive sheaf of rank one on X, and satisfies $\omega_X^m = (\omega_X^{\otimes m})^{**}$, where $(\ )^{**}$ means the double dual. In particular, for any Zariski open subset $W \subset X$ with $\operatorname{codim}(X \setminus W) \geqslant 2$ ($W = X_{\mathrm{reg}}$ for example), the restriction map $H^0(X, \omega_X^m) \to H^0(W, \omega_X^m)$ is bijective ([24, Ch. 2, 1.1.12]). Thus for every $u \in H^0(X, \omega_X^m)$ and an integer $\ell > 0$, a power $(u|_{X_{\mathrm{reg}}})^\ell \in H^0(X_{\mathrm{reg}}, \omega_X^{m\ell})$ extends uniquely as an element of $H^0(X, \omega_X^{m\ell})$. We denote this extension by $u^{(\ell)} \in H^0(X, \omega_X^{m\ell})$.

We first explain an observation and a few conventions from [26, §2.4] on a notion of singular Hermitian metrics on torsion free or reflexive sheaves of rank one. When a variety is smooth, these are standard conventions in the theory of singular Hermitian metrics.

Remarks 2.5. — A singular Hermitian metric $h = e^{-\varphi}$ on a holomorphic line bundle on X has semi-positive curvature if its local weight φ is psh, and has semi-negative curvature if $-\varphi$ is psh respectively. A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is psh, if it is upper-semi-continuous and $\varphi \circ f$ is subharmonic (or $\varphi \circ f \equiv -\infty$) for any holomorphic map $f : \Delta \to X$ from the unit disc in $\mathbb{C}$ (refer [9, §5], especially [9, 5.3.1]).

(1) Let D be a $\mathbb{Q}$-Cartier divisor on X. Suppose there exist an integer $m > 0$ such that mD is Cartier and a singular Hermitian metric h on a line bundle $\mathcal{O}_X(mD)$. We then say that a symbol $h^{1/m}$ is a singular Hermitian metric on D. We can pull back $h^{1/m}$ by a dominant morphism $\alpha : X' \to X$, as a singular Hermitian metric on $\alpha^* D$.

(2) [26, 2.4.1, 2.4.2]. Let $\mathcal{L}$ be a torsion free sheaf of rank one on X. Let $X_0 \subset X_{\mathrm{reg}}$ be a Zariski open subset of codim $(X \setminus X_0) \geqslant 2$ such that $\mathcal{L}|_{X_0}$ is locally free ([24, Ch. 2, Corollary to 1.1.8]). Let h_0 be a singular Hermitian metric on a line bundle $\mathcal{L}|_{X_0}$, and suppose that the curvature current is semi-definite, let us say semi-negative. For every integer $m > 0$, h_0^m is a singular Hermitian metric on a line bundle $(\mathcal{L}^{\otimes m})^{**}|_{X_0}$. If $(\mathcal{L}^{\otimes m})^{**}$ is locally free, then h_0^m extends (uniquely) as a singular Hermitian metric on $(\mathcal{L}^{\otimes m})^{**}$ with semi-negative curvature by virtue of Hartogs' type extension of psh functions. Even if $(\mathcal{L}^{\otimes m})^{**}$ is not locally free, for any section $u \in H^0(U, (\mathcal{L}^{\otimes m})^{**})$ on an open subset $U \subset X$, $\log |u_{|U\cap X_0}|^2_{h_0^m}$ is psh on $U \cap X_0$, and hence it extends (uniquely) as a psh function on U by virtue of Hartogs' type extension again. From this point of view, we say h_0^m defines (or extends as) a singular Hermitian metric on a reflexive sheaf $(\mathcal{L}^{\otimes m})^{**}$ on X.

(3) As one can see, there are some overlaps in (1) and (2) above. If D is a Weil divisor on X, which is also $\mathbb{Q}$-Cartier, we can use both (1) and (2) above, namely (1) as a $\mathbb{Q}$-Cartier divisor, and (2) by considering the corresponding reflexive sheaf $\mathcal{L} = j_* \mathcal{O}_{X_{\mathrm{reg}}}(D|_{X_{\mathrm{reg}}})$.

(4) Let L be a Cartier divisor on X, which is also regarded as a line bundle on X, and let h be a C^∞ Hermitian metric on L. Suppose that there is a non-zero section $s \in H^0(X, L)$. Then for every rational number q, $\frac{1}{|s|^{2q}} = (\frac{h}{|s|_h^2})^q$ defines a singular Hermitian metric on qL.

Lemma 2.6. — *Let $\alpha : X' \to X$ be a resolution of singularities. Let L' be a line bundle on X' and let h' be a singular Hermitian metric with semi-positive (resp. semi-negative) curvature. Then the torsion free sheaf $\mathcal{L} = \alpha_* L'$ admits a singular Hermitian metric $\alpha_* h'$ with semi-positive (resp. semi-negative) curvature. The metric $\alpha_* h'$ will be called the push-forward of h'.*

Proof. — A precise definition of $\alpha_* h'$ will give a proof. Let $W \subset X$ be the maximum Zariski open subset where α is isomorphic on it. We note $\operatorname{codim}(X \setminus W) \geqslant 2$. We denote by $W' = \alpha^{-1}(W)$. Via the isomorphism $L'|_{W'} \cong \mathcal{L}|_W$ by α, $h'|_{W'}$ can be regarded as a singular Hermitian metric on a line bundle $\mathcal{L}|_W$, say $\alpha_*(h'|_{W'})$. It is clear $\alpha_*(h'|_{W'})$ has semi-positive curvature. Since $\operatorname{codim}(X \setminus W) \geqslant 2$, $\alpha_*(h'|_{W'})$ extends uniquely to a singular Hermitian metric on the torsion free sheaf $\mathcal{L}$ with semi-positive curvature in the sense of [26, §2.4], by [26, 2.4.1, 2.4.2] (or 2.5(2)). This is what we call $\alpha_* h$. □

We now extend 2.1 on our normal complex space X.

Definition-Notation 2.7. — Let $m > 0$ be an integer.

(1) For $u \in H^0(X, \omega_X^m)$, we set

$$\|u\|_m := \|u_{|X_{\mathrm{reg}}}\|_{m,X_{\mathrm{reg}}} = \left(\int_{X_{\mathrm{reg}}} |u_{|X_{\mathrm{reg}}}|^{2/m} \right)^{m/2},$$

and set

$$V_m = \{u \in H^0(X, \omega_X^m);\ \|u\|_m < \infty\}.$$

Here the last integral is the $L^{2/m}$-pseudo-norm on X_{reg} in 2.1. We see $\|u^{(\ell)}\|_{m\ell} = \|u\|_m^\ell$ for $u \in V_m$ and an integer $\ell > 0$. We will see $V_m = H^0(X, \omega_X^m)$ if X is compact and canonical singularities at worst.

(2) We have defined the m-th Bergman kernel metric $B_{m,X_{\mathrm{reg}}}$ and so on, provided $V_m \neq 0$. Since the singular Hermitian metric $B_{m,X_{\mathrm{reg}}}^{-1}$ on $\omega_X^m|_{X_{\mathrm{reg}}}$ has semi-positive curvature, it extends as a singular Hermitian metric $B_{m,X}^{-1}$ on a reflexive sheaf ω_X^m with semi-positive curvature (see 2.5, and 2.8 below). This $B_m = B_{m,X}$ is also called the *canonical $L^{2/m}$ metric*, or *m-th Bergman kernel metric* on $(\omega_X^m)^*$. (We can also obtain a singular Hermitian metric B_X on ω_X^* as an extension of $B_{X_{\mathrm{reg}}}$ in 2.3(2).)

(3) We also have a singular Hermitian metric h_{m-1} on a reflexive sheaf ω_X^{m-1} with semi-positive curvature, which is obtained as the extension of $(B_{m,X_{\mathrm{reg}}}^{-1})^{(m-1)/m}$ (which is defined on $K_{X_{\mathrm{reg}}}^{\otimes(m-1)}$). We then can define the *mNS Hermitian form*

$$g_m(u, v) = \int_{X_{\mathrm{reg}}} c_n u|_{X_{\mathrm{reg}}} \wedge \overline{v}|_{X_{\mathrm{reg}}} h_{m-1}|_{X_{\mathrm{reg}}}$$

for $u, v \in V_m$. In the last integral, $u|_{X_{\mathrm{reg}}}$ and $v|_{X_{\mathrm{reg}}}$ are regarded as line bundle $\omega_X^{m-1}|_{X_{\mathrm{reg}}}$-valued holomorphic n-forms on X_{reg}.

Remark 2.8. — This is a remark on 2.7(2). Here we suppose that there exists an integer $\ell > 0$ such that ω_X^ℓ is a locally free. In view of 2.5(3), there is another way to think $B_{m,X}^{-1}$ as a singular Hermitian metric of ω_X^m. By 2.1, we have a singular Hermitian metric $B_{m,X_{\mathrm{reg}}}^{-\ell}$ on $(\omega_X^\ell)^{\otimes m}|_{X_{\mathrm{reg}}}$ with semi-positive curvature. On every open subset $U \subset X$ such that $(\omega_X^\ell)^{\otimes m}$ is trivialized, i.e., $(\omega_X^\ell)^{\otimes m}|_U \cong U \times \mathbb{C}$, we can write as $B_{m,X_{\mathrm{reg}}}^{-\ell}|_{U_{\mathrm{reg}}} = e^{-\varphi_{\ell 0}}$ with a psh function $\varphi_{\ell 0}$ on U_{reg}. We can extend $\varphi_{\ell 0}$ as a psh function φ_ℓ on U by Hartogs type extension. Thus we obtain an extension of $B_{m,X_{\mathrm{reg}}}^{-\ell}$ as a singular Hermitian metric, say $\widetilde{B}_{m,X_{\mathrm{reg}}}^{-\ell}$ on a line bundle $(\omega_X^\ell)^{\otimes m}$ with semi-positive curvature, which is $e^{-\varphi_\ell}$ on U. Then the symbol $(\widetilde{B}_{m,X_{\mathrm{reg}}}^{-\ell})^{1/\ell}$ is a singular Hermitian metric $B_{m,X}^{-1}$ on ω_X^m.

Caution: We should not write as $B_{m,X}^{-1} = e^{-\frac{1}{\ell}\varphi_\ell}$ on U, because if ω_X^m itself is not locally free on U, $B_{m,X}^{-1}$ can not be expressed by a scaler valued function. While $(B_{m,X}^{-1})^\ell = e^{-\varphi_\ell}$ on U makes sense.

As an example of 2.6, we consider $mK_{X'}$ with the metric $B_{m,X'}^{-1}$ in the notation of 2.6. By 2.6, we have $\alpha_*(mK_{X'})$ with a singular Hermitian metric $\alpha_* B_{m,X'}^{-1}$. In general we only have an injection $\alpha_*(mK_{X'}) \to \omega_X^m$, which is isomorphic on a Zariski open $W \subset X$ with $\operatorname{codim}(X \setminus W) \geqslant 2$. Thus the metric $\alpha_* B_{m,X'}^{-1}$ on $\alpha_*(mK_{X'})$ can be regarded as a singular Hermitian metric on ω_X^m. We will see, if X has canonical singularities at worst, that $B_{m,X}^{-1}$ and $\alpha_* B_{m,X'}^{-1}$ on $\omega_X^m \cong \alpha_*(mK_{X'})$ coincide.

3. Bergman and Narasimhan-Simha type metrics on varieties with canonical singularities

We note that Bergman kernel type metrics and mNS Hermitian forms are birational invariant in an appropriate sense of compact complex manifolds (cf. [29]). The main point of the notion of canonical singularities is that we can obtain the same informations on pluricanonical forms from the regular part of the variety and from a smooth model of the variety. We confirm it is also the case of our Bergman kernel type metrics and mNS Hermitian forms. The aim of this section is to present the proof once for sure, although the results and the proofs are within expectations. For example, $B_{m,X'} = (\alpha^* B_{m,X})|s_{\ell E}|^{2m/\ell}$ in 3.2(2) would not be trivial without proof. Our set up in this section is as follows.

Set up 3.1. — Let X be a normal complex space with canonical singularities at worst. A normal complex space X has canonical singularities, if the canonical divisor K_X is $\mathbb{Q}$-Cartier and if for a (any) resolution of singularities $\alpha : X' \to X$, one has $K_{X'} \sim_{\mathbb{Q}} \alpha^* K_X + E$ with an α-exceptional effective $\mathbb{Q}$-divisor E (this is equivalent to saying that there is an α-exceptional effective divisor E' such that $\ell K_{X'} \sim \alpha^*(\ell K_X) + E'$, where $\ell > 0$ is the minimum integer such that ℓK_X is Cartier, and then $E = \frac{1}{\ell}E'$ ([20, 2.22])).

We allow in 3.1, X to be smooth and $\alpha : X' \to X$ to be non-isomorphic. We should say that, a normal complex space X has canonical singularities, if there exist a positive integer $\ell > 0$ such that ω_X^{ℓ} is locally free and a resolution of singularities $\alpha : X' \to X$ such that $\omega_{X'}^{\otimes \ell} \cong \alpha^*(\omega_X^{\ell}) \otimes \mathcal{O}_{X'}(E')$ for an α-exceptional effective divisor E' on X'.

We have defined in 2.7 the pseudo-norm, the Bergman kernel type metric and the mNS Hermitian forms on a normal variety. We compare them with those on a resolution X'. We will denote these objects by $\|u\|_{m,X}$ and $\|u'\|_{m,X'}$, $V_{m,X} \subset H^0(X, \omega_X^m)$ and $V_{m,X'} \subset H^0(X', mK_{X'})$, $B_{m,X}$ and $B_{m,X'}$, $h_{m-1,X}$ and $h_{m-1,X'}$, $g_{m,X}(u,v)$ and $g_{m,X'}(u',v')$, on X as in 2.7 and on X' as in 2.1 respectively. The main conclusion in this section is

Proposition 3.2. — *Let X be a normal complex space with canonical singularities at worst, and let $\alpha : X' \to X$ and $\ell K_{X'} \sim \alpha^*(\ell K_X) + \ell E$ be as in 3.1. Then for every integer $m > 0$, a natural pull-back homomorphism*

$$\alpha^* : H^0(X, \omega_X^m) \to H^0(X', mK_{X'})$$

is defined and an isomorphism of $\mathbb{C}$-vector spaces, moreover it is compatible with the natural $L^{2/m}$-pseudo-norms and the mNS Hermitian forms. More precisely;

(1) For every $u \in H^0(X, \omega_X^m)$, one has $u \in V_{m,X}$ if and only if $\alpha^* u \in V_{m,X'}$ and

$$\|u\|_{m,X} = \|\alpha^* u\|_{m,X'}.$$

In particular if X is compact, one has $\|u\|_{m,X} = \|\alpha^* u\|_{m,X'} < \infty$, i.e., $V_m = H^0(X, \omega_X^m)$.

(2) As a singular Hermitian metric on $-mK_{X'} \sim_{\mathbb{Q}} \alpha^*(-mK_X) - mE$, one has

$$B_{m,X'} = (\alpha^* B_{m,X})|s_{\ell E}|^{2m/\ell}$$

for an appropriate choice of a non-zero section $s_{\ell E} \in H^0(X', \ell E)$ (which is independent of m), where the pull back $\alpha^* B_{m,X}$ is in the sense of a singular Hermitian metric for a $\mathbb{Q}$-Cartier divisor $-mK_X$, and where we used a convention at the beginning of §2 for $|s_{\ell E}|^{2m/\ell}$.

(2') One has $\alpha_* B_{m,X'}^{-1} = B_{m,X}^{-1}$ as a singular Hermitian metric on $\alpha_*(mK_{X'}) \cong \omega_X^m$ in the sense of 2.6, and $\alpha_* h_{m-1,X'} = h_{m-1,X}$ as a singular Hermitian metric on $\omega_X^{m-1} \cong \alpha_*((m-1)K_{X'})$ as a result.

(3) One has

$$h_{m-1,X'} = (\alpha^* h_{m-1,X}) \frac{1}{|s_{\ell E}|^{2(m-1)/\ell}}$$

on X' as a singular Hermitian metric on $(m-1)K_{X'} \sim_{\mathbb{Q}} \alpha^*(m-1)K_X + (m-1)E$, and

$$g_{m,X}(u,v) = g_{m,X'}(\alpha^* u, \alpha^* v)$$

for every $u, v \in V_{m,X} \subset H^0(X, \omega_X^m)$.

It is known that there is an isomorphism $\alpha_*(mK_{X'}) \cong \omega_X^m$, and hence $H^0(X, \omega_X^m) \cong H^0(X', mK_{X'})$ for every integer $m > 0$. Because we would like to obtain exact values of integrations, we first recall this isomorphism in the level of differential forms as follows.

Preliminary 3.3. — (0) For the resolution of singularities $\alpha : X' \to X$ in 3.1, we let W be the Zariski open subset $W \subset X_{\mathrm{reg}}$ on which α is biholomorphic, and set $W' = \alpha^{-1}(W)$. We do not suppose $W = X_{\mathrm{reg}}$. In particular codim $(X \backslash W) \geqslant 2$ and $\alpha(E) \cap W = \emptyset$. We suppose ℓK_X is Cartier and regard it as a line bundle. We take a non-zero section $s_{\ell E} \in H^0(X', \ell E)$, whose zero divisor is an integral effective divisor ℓE.

Let $U \subset X$ be a small open subset with a nowhere vanishing section $\kappa^{(\ell)} \in H^0(U, \ell K_X)$. Let $U' = \alpha^{-1}(U)$. We will consider a frame $\alpha_B^* \kappa^{(\ell)} \in H^0(U', \alpha^* \ell K_X)$, where we use a different notation α_B^* for the pull-back as a general bundle valued object to avoid risk of confusions. The pull-back of functions and forms will be denoted by α^* as usual. We may also denote by α_F^* for the pull back of (pluricanonical-)forms if necessary.

(1) We regard $\kappa^{(\ell)}|_{U_{\mathrm{reg}}}$ as an ℓ-canonical form on U_{reg}, and pull it back $\alpha_F^*(\kappa^{(\ell)}|_{U_{\mathrm{reg}}})$, which is an ℓ-canonical form on $\alpha^{-1}(U_{\mathrm{reg}})$. Since both $\alpha_F^*(\kappa^{(\ell)}|_{U_{\mathrm{reg}}})$ and $\alpha_B^* \kappa^{(\ell)} \otimes s_{\ell E}$ are elements of $H^0(\alpha^{-1}(U_{\mathrm{reg}}), \ell K_{X'})$, the ratio $\alpha_F^*(\kappa^{(\ell)}|_{U_{\mathrm{reg}}})/(\alpha_B^* \kappa^{(\ell)} \otimes s_{\ell E})$ defines a meromorphic function on $\alpha^{-1}(U_{\mathrm{reg}})$, which turns out to be a nowhere vanishing holomorphic function on $\alpha^{-1}(U_{\mathrm{reg}}) \cap W'$ (as no zeros and no poles on $\alpha^{-1}(U_{\mathrm{reg}}) \cap W'$). As $H^0(\alpha^{-1}(U_{\mathrm{reg}}) \cap W', \mathcal{O}_{X'}) = H^0(U_{\mathrm{reg}} \cap W, \mathcal{O}_X) = H^0(U, \mathcal{O}_X)$ and $H^0(U, \mathcal{O}_X) = H^0(U', \mathcal{O}_{X'})$ in a natural way, we can conclude that there exists a nowhere vanishing holomorphic function $f^{(\ell)}$ on U' such that

$$\alpha_F^*(\kappa^{(\ell)}|_{U_{\mathrm{reg}}}) = f^{(\ell)} \cdot \alpha_B^* \kappa^{(\ell)} \otimes s_{\ell E}|_{\alpha^{-1}(U_{\mathrm{reg}})}.$$

We note that once $s_{\ell E} \in H^0(X', \ell E)$ is given, $f^{(\ell)}$ is independent of a local frame $\kappa^{(\ell)} \in H^0(U, \ell K_X)$. As a conclusion, the pull back as an ℓ-canonical form $\alpha^*(\kappa^{(\ell)}|_{U_{\mathrm{reg}}}) \in H^0(\alpha^{-1}(U_{\mathrm{reg}}), \ell K_{X'})$ extends (uniquely) to an ℓ-canonical form $f^{(\ell)} \cdot \alpha_B^*\kappa^{(\ell)} \otimes s_{\ell E}$ on U' vanishing along ℓE. We denote this map by

$$\alpha^*(= \alpha_F^*) : H^0(U, \ell K_X) \to H^0(U', \ell K_{X'}).$$

This is actually bijective. The inverse, say α_* is given by

$$\begin{aligned} H^0(U, \alpha_*(\ell K_{X'})) &= H^0(U \cap W, \alpha_*(\ell K_{X'})) = H^0(U' \cap W', \ell K_{X'}) \\ &\cong H^0(U' \cap W', \alpha^*(\ell K_X)) \\ &= H^0(U \cap W, \ell K_X) = H^0(U, \ell K_X). \end{aligned}$$

Here the middle isomorphism is given by dividing by $f^{(\ell)} s_{\ell E}$ (which has no zeros on $U' \cap W'$). (As long as X is normal, but not necessarily canonical, we have a (natural) map $H^0(U, \alpha_*(\ell K_{X'})) \to H^0(U, \ell K_X)$, which is merely injective.)

In the argument above, we remarked the independence of $f^{(\ell)}$ from a local frame $\kappa^{(\ell)} \in H^0(U, \ell K_X)$. In particular, if we take another small open set $V \subset X$ with $U \cap V \neq \emptyset$ and obtain $\alpha_F^*(\kappa_V^{(\ell)}|_{V_{\mathrm{reg}}}) = f_{V'}^{(\ell)} \cdot \alpha_B^*\kappa_V^{(\ell)} \otimes s_{\ell E}|_{\alpha^{-1}(V_{\mathrm{reg}})}$, then $f^{(\ell)} = f_{V'}^{(\ell)}$ on $\alpha^{-1}(U_{\mathrm{reg}} \cap V_{\mathrm{reg}})$ with some nowhere vanishing holomorphic function $f_{V'}^{(\ell)}$ on $V' = \alpha^{-1}(V)$, where $\kappa_V^{(\ell)} \in H^0(V, \ell K_X)$ is a local frame. Thus if we vary U so that a collection $\{U\}$ forms an open covering of X, then these $\{f_{U'}\}$ glue together and define a nowhere vanishing holomorphic function on X'. Hence by considering $f^{(\ell)} s_{\ell E} \in H^0(X', \ell E)$ in stead of $s_{\ell E}$ from the beginning, we may suppose that $f^{(\ell)} \equiv 1$ (if we prefer). Thus we obtain a natural global bijection

$$\alpha^*(= \alpha_F^*) : H^0(X, \ell K_X) \to H^0(X', \ell K_{X'}).$$

satisfying $\alpha_F^*(u|_{X_{\mathrm{reg}}}) = \alpha_B^* u \otimes s_{\ell E}|_{\alpha^{-1}(X_{\mathrm{reg}})}$. By a similar argument above, we can check that this α^* is well-defined (namely, glue together) when we vary open subsets U.

(2) We also obtain a natural bijection

$$\alpha^*(= \alpha_F^*) : H^0(X, m\ell K_X) \to H^0(X', m\ell K_{X'})$$

for every integer $m > 0$ satisfying $\alpha_F^*(u|_{X_{\mathrm{reg}}}) = (f^{(\ell)})^m \cdot \alpha_B^* u \otimes s_{\ell E}^m|_{\alpha^{-1}(X_{\mathrm{reg}})}$ for $u \in H^0(X, m\ell K_X)$, where we regard $u|_{X_{\mathrm{reg}}}$ as an $m\ell$-canonical form on X_{reg}, and where $f^{(\ell)} \in H^0(U', \mathcal{O}_{X'})$ is the same as in the case of $m = 1$ (it can be $f^{(\ell)} \equiv 1$ if we prefer). In fact, the first point is that on every small open set U with a frame $\kappa^{(\ell)} \in H^0(U, \ell K_X)$,

$$\alpha_F^*((\kappa^{(\ell)})^m|_{U_{\mathrm{reg}}}) = (f^{(\ell)})^m \cdot \alpha_B^*(\kappa^{(\ell)})^m \otimes s_{\ell E}^m|_{\alpha^{-1}(U_{\mathrm{reg}})}$$

holds, where we regard $(\kappa^{(\ell)})^m|_{U_{\mathrm{reg}}}$ as an $m\ell$-canonical form on U_{reg}. Then $\alpha_F^*((\kappa^{(\ell)})^m|_{U_{\mathrm{reg}}})$ extends holomorphically to X', which vanishes along $m\ell E$. We then proceed as in (1) to obtain the global isomorphism α_F^*.

(2') For general integer $m > 0$, we argue as follows. We take $u \in H^0(X, \omega_X^m)$, where ω_X^m is merely reflexive. We consider a power $u^{(\ell)} \in H^0(X, m\ell K_X)$ as in 2.4. Then by using (2), we have

$$\alpha_F^*(u|_{X_{\mathrm{reg}}})^\ell = \alpha_F^*(u^{(\ell)}|_{X_{\mathrm{reg}}}) = (f^{(\ell)})^m \cdot \alpha_B^* u^{(\ell)} \otimes s_{\ell E}^m|_{\alpha^{-1}(X_{\mathrm{reg}})}$$

and $\alpha_F^*(u^{(\ell)}|_{X_{\mathrm{reg}}})$ extends on X' holomorphically and vanishing along $m\ell E$ at least. In particular, as $\alpha_F^*(u|_{X_{\mathrm{reg}}})^\ell$ extends to X' holomorphically, $\alpha_F^*(u|_{X_{\mathrm{reg}}})$ itself extends to an element of $H^0(X', mK_{X'})$ by Riemann type extension (note that $mK_{X'}$ is a line bundle). Thus we obtain a homomorphism $\alpha^* : H^0(X, \omega_X^m) \to H^0(X', mK_{X'})$, which is clearly injective. To see α^* is surjective, we take an isomorphism $K_{W'} \cong \alpha^* K_W$ on W', induced by the isomorphism $\alpha : W' \to W$. Let $\beta_* : H^0(X', mK_{X'}) \to H^0(X, \omega_X^m)$ be the composition: $H^0(X', mK_{X'}) \to H^0(W', mK_{W'}) \cong H^0(W, mK_W) \cong H^0(X, \omega_X^m)$, where the middle isomorphism is induced by $K_{W'} \cong \alpha^* K_W$, and other two morphisms are the natural one (induced from the restriction maps). Let $u' \in H^0(X', mK_{X'})$. Then $\alpha^*\big(\beta_*(u')\big) \in H^0(X', mK_{X'})$, and moreover $(\alpha^*\big(\beta_*(u')\big)/u')|_{W'}$ is a nowhere vanishing holomorphic function, say τ, on W'. Since $H^0(W', \mathcal{O}_{W'}) = H^0(W, \mathcal{O}_W) = H^0(X, \mathcal{O}_X) = H^0(X', \mathcal{O}_{X'})$ in a natural manner, we have a nowhere vanishing holomorphic function $\tau' \in H^0(X', \mathcal{O}_{X'})$ such that $\tau'|_{W'} = \tau$. Then for $u'/\tau' \in H^0(X', mK_{X'})$, we have $\beta_*(u'/\tau') \in H^0(X, \omega_X^m)$ and $\alpha^*\big(\beta_*(u'/\tau')\big) = u'$, which implies α^* is surjective. As we see, this isomorphism α^* is not so clear at all, though it is clear if we take its ℓ-th power.

(3) We take $u \in H^0(X, m\ell K_X)$. We here regard $u|_{X_{\mathrm{reg}}}$ as an $L_{(m-1)\ell} := (m-1)\ell K_X$-valued ℓ-canonical form on X_{reg} and pull it back, we denote it by $\alpha_A^*(u|_{X_{\mathrm{reg}}})$, which is an $\alpha^* L_{(m-1)\ell}$-valued ℓ-canonical from on $\alpha^{-1}(X_{\mathrm{reg}})$ ("A" stands for "adjoint"). Let U be a small open subset with a frame $\kappa^{(\ell)} \in H^0(U, \ell K_X)$ as before. We regard as $(\kappa^{(\ell)})^m|_{U_{\mathrm{reg}}} = (\kappa^{(\ell)}|_{U_{\mathrm{reg}}}) \cdot (\kappa^{(\ell)})^{m-1}$, where $\kappa^{(\ell)}|_{U_{\mathrm{reg}}}$ is an ℓ-canonical form on U_{reg} and $(\kappa^{(\ell)})^{m-1} \in H^0(U, (m-1)\ell K_X)$ is a local frame of a bundle $(m-1)\ell K_X$. Then

$$\begin{aligned}
\alpha_A^*((\kappa^{(\ell)})^m|_{U_{\mathrm{reg}}}) &= \alpha_F^*(\kappa^{(\ell)}|_{U_{\mathrm{reg}}}) \cdot \alpha_B^*(\kappa^{(\ell)})^{m-1} \\
&= f^{(\ell)} \cdot \alpha_B^* \kappa^{(\ell)} \otimes s_{\ell E}|_{\alpha^{-1}(U_{\mathrm{reg}})} \otimes \alpha_B^*(\kappa^{(\ell)})^{m-1} \\
&= f^{(\ell)} \cdot \alpha_B^*(\kappa^{(\ell)})^m \otimes s_{\ell E}|_{\alpha^{-1}(U_{\mathrm{reg}})},
\end{aligned}$$

where $f^{(\ell)} \in H^0(U', \mathcal{O}_{X'})$ is the same as in the case of $m = 1$ (it can be $f^{(\ell)} \equiv 1$ if we prefer). Here we used the relation $\alpha_F^*(\kappa^{(\ell)}|_{U_{\mathrm{reg}}}) = f^{(\ell)} \cdot$

$\alpha_B^*\kappa^{(\ell)}\otimes s_{\ell E}|_{\alpha^{-1}(U_{\rm reg})}$ in (1). We note as a consequence that $\alpha_F^*((\kappa^{(\ell)})^m|_{U_{\rm reg}}) = \alpha_A^*((\kappa^{(\ell)})^m|_{U_{\rm reg}}) \otimes s_{\ell E}^{m-1}|_{\alpha^{-1}(U_{\rm reg})}$. Then we proceed as before, and we obtain a natural bijection

$$\alpha_A^* : H^0(X, m\ell K_X) \to H^0(X', \ell K_{X'} + (m-1)\alpha^*(\ell K_X))$$

for every integer $m > 0$ satisfying $\alpha_F^*(u|_{X_{\rm reg}}) = \alpha_A^*(u|_{X_{\rm reg}})\otimes s_{\ell E}^{m-1}|_{\alpha^{-1}(X_{\rm reg})}$ for $u \in H^0(X, m\ell K_X)$.

Proof of Proposition 3.2. — We give a proof of 3.2 under the set-up and the notations in 3.3. We normalize $f^{(\ell)} \equiv 1$ in 3.3. We take an integer $m > 0$ with $V_m \neq 0$. We have already proved in 3.3(0)–(2') that there is a natural pull-back isomorphism $\alpha^* : H^0(X, \omega_X^m) \to H^0(X', mK_{X'})$.

(1) Let $u \in H^0(X, \omega_X^m)$. By a formula of change of variables, we can see

$$\int_{X_{\rm reg}} |u_{|X_{\rm reg}}|^{2/m} = \int_{\alpha^{-1}(X_{\rm reg})} |\alpha_F^*(u_{|X_{\rm reg}})|^{2/m}$$

holds, and thus we have $\|u\|_{m,X} = \|\alpha_F^* u\|_{m,X'}$. If X is compact, the last integral converges, as (thanks to 3.3(2')) the integrand extends continuously to X' which is compact.

(2) As we noticed in 2.8, it is safe to show $B_{m,X'}^\ell = (\alpha^* B_{m,X}^\ell)|s_{\ell E}|^{2m}$ as a singular Hermitian metric on a line bundle $-m\ell K_{X'} \sim \alpha^*(-m\ell K_X) - m\ell E$.

Every $u' \in H^0(X', mK_{X'})$ can be written as $u' = \alpha_F^* u$ for a unique $u \in H^0(X, \omega_X^m)$ with $\|u\|_{m,X} = \|u'\|_{m,X'}$ by (1). Moreover by 3.3, $\alpha_F^*(u^{(\ell)}|_{X_{\rm reg}}) = (\alpha_B^* u^{(\ell)}) \otimes s_{\ell E}^m|_{\alpha^{-1}(X_{\rm reg})}$. Then we have

$$|(u')^\ell|^2 = (\alpha^*|u^{(\ell)}|^2) \cdot |s_{\ell E}|^{2m}$$

as a singular Hermitian metric on $-m\ell K_{X'} \sim \alpha^*(-m\ell K_X) - m\ell E$. We can check this by taking reference C^∞-Hermitian metrics; $h_{\ell K}$ on a line bundle ℓK_X, $h_{\ell E}$ on a line bundle ℓE, and then $h_{\ell K'} = \alpha^* h_{\ell K} \cdot h_{\ell E}$ on $\ell K_{X'}$.

We take an arbitrary point $x' \in W' \subset X' \setminus E$ and set $x := \alpha(x') \in W$. Noting $\|u\|_{m,X} = \|u'\|_{m,X'}$, we take $\sup_{\|u'\|_{m,X'} \leqslant 1}$ and $\sup_{\|u\|_{m,X} \leqslant 1}$ in the equality above, we obtain $B_{m,X'}^\ell(x') = \alpha^* B_{m,X}^\ell(x) \cdot |s_{\ell E}(x')|^{2m}$. (We are not sure $B_{m,X}(x) = \sup_{\|u\|_{m,X}\leqslant 1} |u(x)|^2$ at $x \notin X_{\rm reg}$.) Thus we have $B_{m,X'}^\ell = (\alpha^* B_{m,X}^\ell)|s_{\ell E}|^{2m}$ on W'. On W', $\log |s_{\ell E}|^{2m}$ is pluriharmonic and hence $B_{m,X'}^\ell/|s_{\ell E}|^{2m}$ is log-psh on W' and uniformly bounded from above on W' (as $\alpha^* B_{m,X}^\ell$ is). Thus $B_{m,X'}^\ell/|s_{\ell E}|^{2m} = \alpha^* B_{m,X}^\ell$ holds on X' by the Riemann type extension for their local weights (being $B_{m,X}^\ell = e^{\varphi_\ell}$ as in 2.8 for example).

(2') It is enough to see $\alpha_*(B_{m,X'}^{-\ell}) = B_{m,X}^{-\ell}$ on W. On $W' = \alpha^{-1}(W) \cong W$, we have an isomorphism $m\ell K_{X'} \cong \alpha^*(m\ell K_X)$ via the map dividing by $s_{\ell E}^m$. Let $U \subset W$ be an open subset. Every $u' \in H^0(\alpha^{-1}(U), m\ell K_{X'})$ is written as $u' = \alpha_F^* u = \alpha_B^* u \otimes s_{\ell E}^m$ for a unique $u \in H^0(U, m\ell K_X)$. Thus $(B_{m,X'}^{-\ell})(u', v') = (\alpha_* B_{m,X}^{-\ell}) \frac{1}{|s_{\ell E}|^{2m}} (\alpha_B^* u \otimes s_{\ell E}^m, \alpha_B^* v \otimes s_{\ell E}^m) = B_{m,X}^{-\ell}(u, v)$.

(3) We denote by $h_{m-1} = (B_{m,X}^{-1})^{(m-1)/m}$, respectively $h'_{m-1} = (B_{m,X'}^{-1})^{(m-1)/m}$, the singular Hermitian metric on a $\mathbb{Q}$-Cartier divisor $L_{m-1} = (m-1)K_X$, respectively on $L'_{m-1} = (m-1)K_{X'} \sim_{\mathbb{Q}} \alpha^* L_{m-1} + (m-1)E$. By (2) above, we have $h'_{m-1} = (\alpha^* h_{m-1}) \frac{1}{|s_{\ell E}|^{2(m-1)/\ell}}$ on X'. Let us see $g_m(u, v) = g_{m,X'}(u', v')$ for $u, v \in V_{m,X}$ and $u' = \alpha_F^* u, v' = \alpha_F^* v \in V_{m,X'}$. We note by 3.3(3), on $\alpha^{-1}(X_{\mathrm{reg}})$ that

$$
\begin{aligned}
&c_n u' \wedge \overline{v'}\ h'_{m-1} \\
&\quad = \left((c_n u' \wedge \overline{v'}\ h'_{m-1})^{\ell} \right)^{1/\ell} \\
&\quad = \left(c_n \alpha_A^* u^{(\ell)} \otimes s_{\ell E}^{m-1} \wedge \overline{\alpha_A^* v^{(\ell)} \otimes s_{\ell E}^{m-1}}\ (\alpha^* h_{m-1}^{\ell}) \frac{1}{|s_{\ell E}|^{2(m-1)}} \right)^{1/\ell} \\
&\quad = \left(c_n \alpha_A^* u^{(\ell)} \wedge \overline{\alpha_A^* v^{(\ell)}}\ \alpha^* h_{m-1}^{\ell} \right)^{1/\ell} \\
&\quad = c_n \alpha_A^* u \wedge \overline{\alpha_A^* v}\ \alpha^* h_{m-1} = \alpha_F^* (c_n u \wedge \overline{v}\ h_{m-1}).
\end{aligned}
$$

These equalities make sense on $\alpha^{-1}(X_{\mathrm{reg}})$. In particular $\alpha_A^* u$ in the last two term means $\alpha_A^*(u|_{X_{\mathrm{reg}}})$ in which we regard $u|_{X_{\mathrm{reg}}}$ as a bundle $L_{m-1}|_{X_{\mathrm{reg}}}$-valued holomorphic n-form on X_{reg} and pull it back. Then by integration, we have $\int_{X_{\mathrm{reg}}} c_n u \wedge \overline{v}\, h_{m-1} = \int_{\alpha^{-1}(X_{\mathrm{reg}})} c_n u' \wedge \overline{v'}\, h'_{m-1}$ and also $g_{m,X'}(u', v') = g_m(u, v)$. $\square$

4. Singularities of Narasimhan-Simha metrics

We shall study Narasimhan-Simha Hermitian forms in a family of varieties, especially the upper bounds when the fibers approach to a degenerate fiber. We shall prove 1.1(1) in this section. Before doing so, we would like to make a remark on the definition of the m-th Narasimhan-Simha Hermitian metric g_m for a family of varieties with only canonical singularities.

Definition 4.1. — *Let $f : X \to Y$ be as in 1.1. We denote by $mK_{X/Y} := \omega_X^m \otimes f^* K_Y^{\otimes(-m)}$, $mK_X := \omega_X^m$ and so on for integers $m > 0$ by an abuse of notations. Let $Y_0 := \{t \in Y;$ the fiber X_t has canonical singularities at worst$\}$. For every $t \in Y_0$, we let $\varphi_t : H^0(Y, f_*(mK_{X/Y})) \to H^0(X_y, (mK_{X/Y})|_{X_t}) \cong H^0(X_t, mK_{X_t})$ be the natural homomorphism, where the last isomorphism is given by the adjunction formula ([20, 5.73]).*

The invariance of plurigenera including the case with canonical singularities ([30], [31]) shows that φ_t is surjective. We denote by $u_t = u(t) = \varphi_t(u)$ for $u \in H^0(Y, f_*(mK_{X/Y}))$ and $t \in Y_0$. We can define (by 2.7 and 3.2) the $L^{2/m}$-pseudo-norm

$$\|u\|_m(t) := \|u_t\|_{m,t},$$

where $\| \cdot \|_{m,t}$ is the $L^{2/m}$-pseudo-norm on $H^0(X_t, mK_{X_t})$, and the mNS Hermitian form

$$g_m(u,v)(t) := g_{m,t}(u_t, v_t),$$

where $g_{m,t}$ is the mNS Hermitian form on $H^0(X_t, mK_{X_t})$.

We take a resolution of singularities $\alpha : X' \to X$ and let $f' = f \circ \alpha : X' \to Y$ be the composition. Let $Y_0' := \{t \in Y;$ the fiber X_t' of f' is smooth$\}$. By 3.2, for every $t \in Y_0 \cap Y_0'$, we have the same $L^{2/m}$-pseudo-norm $\|u\|_{m,t}$ and the mNS Hermitian form $g_{m,t}(u,v)$ on $H^0(X_t, mK_{X_t}) \cong H^0(X_t', mK_{X_t'})$. Thus we first obtain the same (namely natural) m-th Narasimhan-Simha Hermitian metric on $f_*'(mK_{X'/Y}) = f_*(mK_{X/Y})$ over $Y_0 \cap Y_0'$. Then by [26, 5.1.2], *we have a unique extension as a singular Hermitian metric on $f_*'(mK_{X'/Y})$. As a result, we obtain our g_m on $f_*(mK_{X/Y})$ in 1.1. Another smooth model of X will give another metric on $f_*(mK_{X/Y})$. However they coincide on a non-empty Zariski open subset of Y by the discussion above. Thus their extensions must coincide too. (The uniqueness of the extension is a consequence of the uniqueness of the Riemann type and the Hartogs type extension for psh functions.)*

In the rest of this section, we devote ourself to the proof of 1.1(1). Yoshikawa [34, §7] treats some special cases with $m = 1$ and with conditions on singularities on X and (X, X_0). Our method here is similar to [34], however more involved because of the generality of our setting.

Preliminary 4.2. — As a preliminary for the proof of 1.1(1), we here consider a fiberwise integral near a possible degenerate fiber of f. We will use the notations in 1.1(1) and 3.1.

(1) We take a log-resolution of singularities $\alpha : X' \to X$ of the pair (X, X_0). We denote by $f' = f \circ \alpha : X' \to Y$, $X_0' = f'^*(0) = \sum \widetilde{b}_j \widetilde{B}_j$ (all $\widetilde{b}_j$ are positive integers) and $\operatorname{Supp} X_0' = \sum \widetilde{B}_j$. We let

$$\widetilde{b} = \max_j \widetilde{b}_j.$$

As in 3.1, we let $\ell > 0$ be the smallest integer such that ℓK_X is Cartier and $\ell K_{X'} \sim \alpha^*(\ell K_X) + \ell E$ for an α-exceptional effective integral divisor ℓE on X'. We write $E = \sum \widetilde{k}_j \widetilde{B}_j + \Delta$ with rational numbers $\widetilde{k}_j \geqslant 0$ and an effective $\mathbb{Q}$-divisor Δ on X' without common components with

X'_0. Recall that r_0 is the log-canonical threshold of the pair (X, X_0). As $K_{X'} \sim_{\mathbb{Q}} \alpha^*(K_X + rX_0) + \sum(\widetilde{k}_j - r\widetilde{b}_j)\widetilde{B}_j + \Delta$, we see

$$r_0 = \min_j(\widetilde{k}_j + 1)/\widetilde{b}_j.$$

Since (X, X_0) is canonical outside X_0, no components of Δ provide a log-canonical singularity of (X, X_0). (The number r_0 satisfies $\widetilde{k}_j - r_0\widetilde{b}_j \geqslant -1$ for all j and $\widetilde{k}_j - r_0\widetilde{b}_j = -1$ for some j. This is almost the definition of r_0 in our setting.) By shrinking Y, we may suppose f' is smooth over $Y \setminus 0$.

(2) We take a point $x'_0 \in X'_0$, and put $x_0 = \alpha(x'_0) \in X_0$. We take an open neighborhood W of x_0 in X with a local frame $\kappa^{(\ell)} \in H^0(W, \ell K_X)$. We take a local coordinate $z = (z_1, \dots, z_n, z_{n+1})$ of X' centered at x'_0, which is defined on an open subset containing a polydisc $U = \{z \in \mathbb{C}^{n+1};\ |z_i| \leqslant 1 \text{ for any } 1 \leqslant i \leqslant n+1\}$ and $\alpha(U) \subset W$. We may assume that

$$t = f'(z) = c(x'_0) z_1^{b_1} z_2^{b_2} \cdots z_{n+1}^{b_{n+1}},$$
$$\alpha^*\kappa^{(\ell)} = \delta(z) z_1^{\ell k_1} \cdots z_{n+1}^{\ell k_{n+1}} (dz_1 \wedge \ldots \wedge dz_{n+1})^{\otimes \ell}$$

on U, where b_j are non-negative integers, $c(x'_0) > 0$ is a constant, k_j are non-negative rational numbers such that all ℓk_j are integers, and $\delta(z) \in H^0(U, \mathcal{O}_U)$ is a holomorphic function whose zero divisor is $\ell\Delta|_U$ (cf. 3.3(1)). Moreover we may assume that

$$b_1/(k_1+1) \leqslant b_2/(k_2+1) \leqslant \ldots \leqslant b_{n+1}/(k_{n+1}+1)$$

holds. Note that $\max_j b_j \leqslant \widetilde{b}$ and

$$0 < b_{n+1}/(k_{n+1}+1) \leqslant \max_j(\widetilde{b}_j/(\widetilde{k}_j+1)) = 1/r_0.$$

As in a calculus Lemma 4.3 below, we find non-negative integers J_0 and J_1 (lengths of indexes) with $J_0 + J_1 \leqslant n$ so that $b_j = 0$ for every $j \leqslant J_0$, $b_j/(k_j+1) = b_{n+1}/(k_{n+1}+1)$ for every $j \geqslant n - J_1 + 1$, and $0 < b_j/(k_j + 1) < b_{n+1}/(k_{n+1}+1)$ for other $J_0 < j < n - J_1 + 1$ (if it exists). Since $c(x'_0) = f'(1, \dots, 1) \in Y = \{|t| < 1\}$, we have $c(x'_0) < 1$.

(3) We take an arbitrary point $t \in f'(U) \setminus 0$ and $|t| < \min\{c(x'_0), e^{-1}\}$ in this paragraph. Here we note that $f'(U)$ contains an open neighborhood of $0 \in Y$ as f' is an open mapping (as f' is flat). Let $\mathrm{pr} : (\mathbb{C}^{n+1} \supset)\ U \ni z \mapsto (z_1, \dots, z_n)$ be a projection. We set $U_t = \mathrm{pr}\,(X'_t \cap U)$ and

$$U_t^* = \left\{ \begin{matrix} (z_1, \dots, z_n) \in \mathbb{C}^n; & |z_j| \leqslant 1 \text{ for every } j \leqslant J_0, \\ & |t/c(x'_0)|^{1/b_j} \leqslant |z_j| \leqslant 1 \text{ for every } J_0 < j \leqslant n \end{matrix} \right\}.$$

We see $U_t \subset U_t^*$. We can regard $X'_t \cap U \subset \mathbb{C}^{n+1}$ as the graph of (or the domain of uniformizing) the multi-valued holomorphic function

$$z_{n+1} = \left(\frac{t}{c(x'_0)z_1^{b_1}\cdots z_n^{b_n}}\right)^{1/b_{n+1}}$$

defined on $U_t \subset \mathbb{C}^n$. Since $t \neq 0$, the projection $X'_t \cap U \to U_t$ is an unramified covering, and hence $X'_t \cap U$ is a union of b_{n+1}-branches: $X'_t \cap U = \bigcup_{i=1}^{b_{n+1}} U_{ti}$, where each pair U_{ti} and $U_{t(i+1)}$ (and also $U_{tb_{n+1}}$ and U_{t1}) intersect along their collars ($U_{ti} \cap U_{t(i+1)}$ is a real $2n-1$ dimensional manifold). By an appropriate (somewhat standard) choice of U_{ti}; distinguished by a b_{n+1}-th root of unity multiplications, we can take $(U_t, (z_1, \dots, z_n))$ as a local coordinate of U_{ti} via the projection $\mathrm{pr} : U_{ti} \to U_t$ on each branch U_{ti}. By 4.3, we have

$$\begin{aligned} I_t &:= \int_{U_t^*} (|z_1|^{k_1 - b_1(k_{n+1}+1)/b_{n+1}} \cdots |z_n|^{k_n - b_n(k_{n+1}+1)/b_{n+1}})^2 dV_n \\ &= 2^{J_1}\pi^n \prod_{J_0 < j < n-J_1+1} \frac{1 - |t/c(x'_0)|^{2((k_j+1)/b_j - (k_{n+1}+1)/b_{n+1})}}{k_j + 1 - b_j(k_{n+1}+1)/b_{n+1}} \\ &\qquad \times \prod_{n-J_1+1 \leqslant j \leqslant n} (-\log|t/c(x'_0)|^{1/b_j}). \end{aligned}$$

Here the first product is less than $(\ell b_{n+1})^n \leqslant (\widetilde{\ell b})^n$, since each factor is estimated as

$$\begin{aligned} \frac{1 - |t/c(x'_0)|^{2((k_j+1)/b_j - (k_{n+1}+1)/b_{n+1})}}{k_j + 1 - b_j(k_{n+1}+1)/b_{n+1}} &< \frac{1}{k_j + 1 - b_j(k_{n+1}+1)/b_{n+1}} \\ &= \frac{\ell b_{n+1}}{\ell(b_{n+1}(k_j+1) - b_j(k_{n+1}+1))} \\ &\leqslant \ell b_{n+1}, \end{aligned}$$

where we note that $|t/c(x'_0)| < 1$ and $\ell(b_{n+1}(k_j+1) - b_j(k_{n+1}+1)) \in \mathbb{Z}_{>0}$ (it is positive and an integer). The second product is less than $(-\log|t|)^{J_1}$, since each factor is estimated as

$$-\log|t/c(x'_0)|^{1/b_j} = \frac{1}{b_j}(-\log|t| + \log c(x'_0)) < -\log|t|,$$

where we note that $b_j \in \mathbb{Z}_{>0}$, $\log c(x'_0) < 0$ and $-\log|t| > 1$. Hence we have for example

$$I_t < (2\pi)^n (\widetilde{\ell b})^n (-\log|t|)^n.$$

(4) Since f' is proper, we can cover a neighborhood ${f'}^{-1}(Y')$ of X'_0 for a smaller disc $Y' \subset Y$ by a finite number of coordinate neighborhoods U_λ

as above, namely we choose a disc Y' so that

$$Y' \subset \{t \in Y;\ t \in f'(U_\lambda), |t| < \min\{c(x'_\lambda), e^{-1}\}\}$$

for every U_λ with center x_λ as above. We will indicate this open covering of ${f'}^{-1}(Y')$ as $\{U_\lambda\}_\lambda$ if we need.

We now prove Theorem 1.1(1).

Proof of Theorem 1.1(1). — We still keep the notations in 4.2. For example, a resolution $\alpha : X' \to X$, $f' : X' \to Y$, and $\ell K_{X'} \sim \alpha^*(\ell K_X) + \ell E$. We also take a finite open covering $\{U_\lambda\}_\lambda$ of ${f'}^{-1}(Y')$ in 4.2(4) such that each $U = U_\lambda$ enjoying properties in 4.2(2). We also take a small open subset $W(= W_\lambda) \subset X$ such that $\alpha(U) \subset W$ and with a local frame $\kappa^{(\ell)} \in H^0(W, \ell K_X)$ of a line bundle ℓK_X.

(1) We first consider the pseudo-norm in the following special situation. Let $m > 0$ be an integer. Let $u \in H^0(Y, f_*(m\ell K_{X/Y}))$ and suppose u vanishes along $m\ell X_0$ (regarded as an element of $H^0(X, m\ell K_{X/Y})$), i.e. u is divided by $f^*t^{m\ell}$. Then we shall show that there exists a constant $A_{\widetilde{u}} > 0$ depending on $u/(f^*t^{m\ell})$ such that

$$\|u(t)\|_{m\ell}^{2/(m\ell)} \leqslant |t|^{2r_0}(-\log|t|)^n A_{\widetilde{u}}$$

holds for any $t \in Y' \setminus 0$.

(1.1) In view of $H^0(Y, f_*(m\ell K_{X/Y})) = \mathrm{Hom}_{\mathcal{O}_Y}(m\ell K_Y, f_*(m\ell K_X))$, our u gives $u(dt^{\otimes m\ell}) \in H^0(Y,\ f_*(m\ell K_X)) = H^0(X, m\ell K_X)$. As we suppose u vanishes along $m\ell X_0$, we have $u(dt^{\otimes m\ell})|_W = f^*t^{m\ell} \cdot \widetilde{u} \cdot (\kappa^{(\ell)})^m$ with some $\widetilde{u} \in H^0(W, \mathcal{O}_X)$. Let us denote by $u' = \alpha^*\big(u(dt^{\otimes m\ell})\big) \in H^0(X', m\ell K_{X'})$, which is an $m\ell$-canonical form on X' vanishing along $m\ell E$ (see 3.2). We denote by $\widetilde{u}' = (\alpha|_U)^*\widetilde{u} \in H^0(U, \mathcal{O}_{X'})$. We note $f'(z) = {f'}^*t$, and recall $\alpha^*\kappa^{(\ell)} = \delta(z) z_1^{\ell k_1} \cdots z_{n+1}^{\ell k_{n+1}}\ (dz_1 \wedge \ldots \wedge dz_{n+1})^{\otimes \ell}$. Then

$$\begin{aligned} u'|_U &= {f'}^*t^{m\ell} \cdot \widetilde{u}' \cdot (\alpha^*\kappa^{(\ell)})^m \\ &= \widetilde{u}'\big(\delta(z) z_1^{\ell k_1} \cdots z_{n+1}^{\ell k_{n+1}} (f'(z) dz_1 \wedge \ldots \wedge dz_{n+1})^{\otimes \ell}\big)^{\otimes m} \end{aligned}$$

Noting a relation ${f'}^*dt = f'(z) \sum_{i=1}^{n+1} b_i \frac{dz_i}{z_i}$, we can write as

$$\begin{aligned} f'(z) dz_1 \wedge \ldots \wedge dz_{n+1} &= (b_{n+1}^{-1} z_{n+1} dz_1 \wedge \ldots \wedge dz_n) \wedge (f'(z) b_{n+1} \frac{dz_{n+1}}{z_{n+1}}) \\ &= (b_{n+1}^{-1} z_{n+1} dz_1 \wedge \ldots \wedge dz_n) \wedge {f'}^*dt\,. \end{aligned}$$

We let

$$\sigma = \widetilde{u}'(z)\big(b_{n+1}^{-\ell}\delta(z) z_1^{\ell k_1} \ldots z_n^{\ell k_n} z_{n+1}^{\ell(k_{n+1}+1)}\big)^m \big(dz_1 \wedge \ldots \wedge dz_n\big)^{\otimes m\ell},$$

which is an element of $H^0(U,(\Omega^n_{X'})^{\otimes m\ell})$. Then $u'|_U$ is the image of $\sigma\otimes(f'^*dt^{\otimes m\ell})$ under the naturally induced homomorphism $(\Omega^n_{X'})^{\otimes m\ell}\otimes f^*K_Y^{\otimes m\ell}\to K_{X'}^{\otimes m\ell}$ restricted to U. We note that for any $t\in Y'\setminus 0$, the restriction $\sigma|_{X'_t\cap U}\in H^0(X'_t\cap U, m\ell K_{X'_t})$ for $t\in f'(U)\setminus 0$ is well-defined for u' and u (independent of dt), and a collection $\{\sigma|_{X'_t\cap U_\lambda}\}_\lambda$, for the open covering $\{U_\lambda\}_\lambda$ of ${f'}^{-1}(Y')$, glues together and simply recovers $u'_t=\alpha^*u_t\in H^0(X'_t, m\ell K_{X'_t})$.

This is a side remark. Since we supposed that u vanishes along $m\ell X_0$ (regarded as an element of $H^0(X,m\ell K_{X/Y})$), i.e. u is divided by $f^*t^{m\ell}$, σ can be taken from $H^0(U,(\Omega^n_{X'})^{\otimes m\ell})$. However in general it is merely $\sigma\in H^0(U\setminus X'_0,(\Omega^n_{X'})^{\otimes m\ell})$. As we will see in (2) below, the assumption that u is divided by $f^*t^{m\ell}$ is not essential, just for convenience of computation.

(1.2) We set $A_{\widetilde{u}}:=\sup_W|\widetilde{u}|^{2/m\ell}$. We can suppose that $A_{\widetilde{u}}<+\infty$ being everything is defined on a larger open subset containing W. We have $\sup_U|\widetilde{u}'|^{2/m\ell}\leqslant A_{\widetilde{u}}$. We also set $A_\Delta=\sup_U|\delta(z)|^2(<+\infty)$. Then for every $t\in Y'\setminus 0$, we have

$$\begin{aligned}
&\int_{X'_t\cap U}|u'_t|^{\frac{2}{m\ell}}=\sum_{i=1}^{b_{n+1}}\int_{U_{ti}}|\sigma_{|X'_t}|^{\frac{2}{m\ell}}\\
&=\sum_{i=1}^{b_{n+1}}\int_{U_{ti}}b_{n+1}^{-2}|\widetilde{u}'|^{\frac{2}{m\ell}}|\delta(z)|^{\frac{2}{\ell}}\left|z_1^{\ell k_1}\cdots z_n^{\ell k_n}z_{n+1}^{\ell(k_{n+1}+1)}\right|^{\frac{2}{\ell}}\bigwedge_{j=1}^n\sqrt{-1}dz_j\wedge d\overline{z_j}\\
&=\frac{1}{b_{n+1}}\int_{U_t}\left|z_1^{k_1}\cdots z_n^{k_n}\right|^2\left|\frac{t}{c(x'_0)z_1^{b_1}\cdots z_n^{b_n}}\right|^{2(k_{n+1}+1)/b_{n+1}}|\widetilde{u}'|^2|\delta(z)|^{\frac{2}{\ell}}2^n dV_n\\
&\leqslant|t|^{2(k_{n+1}+1)/b_{n+1}}\,I_t\,A_{\widetilde{u}}A_\Delta 2^n/c(x'_0)^{2(k_{n+1}+1)/b_{n+1}}.
\end{aligned}$$

Here I_t is the integral in 4.2(3) (recall $U_t\subset U_t^*$). As $|t|<1$ and $r_0=\min_j(\widetilde{k}_j+1)/\widetilde{b}_j$, we have $|t|^{2(k_{n+1}+1)/b_{n+1}}\leqslant|t|^{2r_0}$. If we set $r_1=\max_j(\widetilde{k}_j+1)/\widetilde{b}_j$, we have $c(x'_0)^{-2(k_{n+1}+1)/b_{n+1}}\leqslant c(x'_0)^{-2r_1}$ as $0<c(x'_0)<1$. Hence by 4.2(3), we have

$$\int_{X'_t\cap U}|u'_t|^{2/(m\ell)}\leqslant|t|^{2r_0}(-\log|t|)^nA_{\widetilde{u}}A_\Delta(4\pi\ell\widetilde{b})^nc(x'_0)^{-2r_1}.$$

The latter part $A_\Delta(4\pi\ell\widetilde{b})^nc(x'_0)^{-2r_1}$ is determined by X,f and may be by $\alpha:X'\to X$. We then have our assertion in (1) by combining the last estimate on each $U=U_\lambda$ of the finite open covering of ${f'}^{-1}(Y')$ and

$$\|u\|_{m\ell}^{2/(m\ell)}(t)=\|u'_t\|_{m\ell}^{2/(m\ell)}=\int_{X'_t}|\sigma_{|X'_t}|^{2/(m\ell)}\leqslant\sum_{U_\lambda}\int_{X'_t\cap U_\lambda}|\sigma_{|X'_t}|^{2/(m\ell)}$$

for any $t \in Y' \setminus 0$.

(2) We consider the general case. We take $u \in H^0(Y, f_*(mK_{X/Y})) = H^0(X, mK_{X/Y})$. We consider a power $u^{(\ell)} \in H^0(X, m\ell K_{X/Y}) = H^0(Y, f_*(m\ell K_{X/Y}))$ (explained in 2.7). We can apply (1) for $(f^*t^{m\ell})u^{(\ell)}$ to obtain

$$\|u^{(\ell)}\|_{m\ell}^{2/(m\ell)}(t) = \frac{1}{|t|^2}\|(f^*t^{m\ell})u^{(\ell)}\|_{m\ell}^{2/(m\ell)}(t) \leqslant \frac{1}{|t|^2}|t|^{2r_0}(-\log|t|)^n A_{u^{(\ell)}}$$

for any $t \in Y' \setminus 0$. As ℓ is given by X, we may say $A_{u^{(\ell)}}$ is given by u, and hence denote it by A_u. Noting $\|u^{(\ell)}\|_{m\ell}(t) = \|u\|_m^\ell(t)$ (2.7(1)), we have $\|u\|_m^{2/m}(t) \leqslant \frac{1}{|t|^2}|t|^{2r_0}(-\log|t|)^n A_u$. Combining with an inequality $g_m(u,u)(t) \leqslant \|u\|_m^2(t)$ for any $t \neq 0$ (which follows from a fiberwise estimate 2.2(4) with 3.2), we have our assertion. □

Lemma 4.3. — *Let a_j $(j = 1, 2, \dots, n)$ be real numbers, and J_0, J_1 be non-negative integers such that $J_0 + J_1 \leqslant n$. Assume $a_j = 0$ for every $j \leqslant J_0$, $a_j = -1$ for every $j \geqslant n - J_1 + 1$, and $-1 < a_j < 0$ for other $J_0 < j < n - J_1 + 1$ (if it exists). In particular $J_0 = \sharp\{j;\ a_j = 0\}$, $J_1 = \sharp\{j;\ a_j = -1\}$, possibly $J_0 = 0$ or $J_1 = 0$. Let $0 < \varepsilon_j < 1$ be a real number for each $j = J_0 + 1, J_0 + 2, \dots, n$, and let $U^* \subset \mathbb{C}^n$ be the set of points $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ such that*

$$U^* = \{|z_j| \leqslant 1 \text{ for every } j \leqslant J_0, \varepsilon_j \leqslant |z_j| \leqslant 1 \text{ for every } J_0 < j \leqslant n\}.$$

Then

$$\int_{U^*}\prod_{j=1}^n |z_j|^{2a_j}dV_n = 2^{J_1}\pi^n \prod_{J_0<j<n-J_1+1}\frac{1-\varepsilon_j^{2(1+a_j)}}{1+a_j}\prod_{n-J_1+1\leqslant j\leqslant n}(-\log\varepsilon_j)$$

holds, where $dV_n = \bigwedge_{j=1}^n(\sqrt{-1}/2)dz_j \wedge d\overline{z_j}$ is the standard real Euclidean volume form in $\mathbb{C}^n$.

Proof. — Straight forward from Fubini's theorem. □

Remark 4.4. — The computation, we have done in the proof of 1.1(1) above, is itself elementary. It will be adapted to any proper holomorphic mapping between complex manifolds with connected fibers.

5. In the case of higher dimensional base

We shall prove 1.1(2) and 1.3 in this section. We first state a general result which is valid for singular Hermitian vector bundles (and we refer to [28], [26, §2.2, §2.3] for a general discussion about it). Let $X \subset \mathbb{C}^n$ be a domain containing the origin, and let $E = X \times \mathbb{C}^r$ be a trivial vector bundle of rank $r > 1$. Let $\pi : \mathbb{P}(E)(= X \times \mathbb{P}^{r-1}) \to X$ be the projective space bundle and let $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological bundle (i.e., the universal quotient line bundle $\pi^* E \to L$). Let h be a singular Hermitian metric on E with Griffiths semi-positive curvature. Let e^{φ} be the induced singular Hermitian metric on the dual L^* as the sub-bundle $L^* \to \pi^*(E^*, h^*)$. By a general theory ([26, 2.3.4]), e^{φ} has a semi-negative curvature current, i.e., the local weight φ is psh.

Lemma 5.1. — *Suppose that (i) there exists a smooth curve $(C, t) \subset X$ passing through $0 \in X$ with a coordinate t and $t = 0$ corresponding to $0 \in X$, and $0 < \det h < +\infty$ on $C \setminus 0$ (in particular the restriction $h|_C$ is well-defined as a singular Hermitian metric on $E|_C$), and that (ii) there exist real numbers $\ell \geqslant 0$ and $k > 0$, such that, for every $u \in H^0(C, E|_C)$, there exists a constant $A_u > 0$ such that $|u|_h^2(t) < \frac{1}{|t|^{2\ell}}(-\log|t|)^k A_u$ holds for any $t \neq 0$. Then the Lelong number of φ is bounded by $\nu(\varphi, P) \leqslant 2\ell$ at any point $P \in \pi^{-1}(0) \subset \mathbb{P}(E)$.*

Proof. — We shall show two inequalities $\nu(\varphi, P) \leqslant \nu(\varphi|_{\pi^{-1}(C)}, P)$ and $\nu(\varphi|_{\pi^{-1}(C)}, P) \leqslant 2\ell$.

(1) It is well-known that the Lelong numbers only increase by a slice, as long as its restriction is well-defined. For example, if ψ is a psh function on a neighborhood of the origin O in $\mathbb{C}^n$ with coordinate $z = (z_1, \dots, z_n)$, then the Lelong number of ψ at O is characterized by $\nu(\psi, O) = \sup\{\gamma \geqslant 0;\ \psi(z) \leqslant \gamma \log|z| + O(1) \text{ around the origin}\}$, where $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$ and $O(1)$ is a bounded term ([7, (2.8)]). This property deduces $\nu(\varphi, P) \leqslant \nu(\varphi|_{\pi^{-1}(C)}, P)$.

(2) To prove $\nu(\varphi|_{\pi^{-1}(C)}, P) \leqslant 2\ell$, we can suppose $\dim X = 1$ and C is X itself. We take a standard basis $\{e_i\}_{i=1}^r$, where $e_i = (\dots, 0, 1, 0, \dots)$ (the i-th entry is 1 and others are 0), of the vector space $\mathbb{C}^r$, and regard also $\{e_i\}_{i=1}^r$ as the global basis of E (so that $E = \oplus_{i=1}^r \mathcal{O}_X e_i$). With respect to this basis, we write $h = (h_{i\bar{j}})_{1 \leqslant i,j \leqslant r}$ as a matrix valued measurable function on X, where $h_{i\bar{j}} = h(e_i, e_j)$. By our assumption, for every $i = 1, \dots, r$, there exists a constant $A_i > 0$ such that $h_{i\bar{i}}(t) = |e_i|_h^2(t) < \frac{1}{|t|^{2\ell}}(-\log|t|)^k A_i$ holds for any $t \neq 0$. We set $A_h = \max_{i=1,\dots,r} A_i$ and $\alpha(t) := \frac{1}{|t|^{2\ell}}(-\log|t|)^k A_h$ as a function on t. We also have $|h_{i\bar{j}}(t)| \leqslant (h_{i\bar{i}}(t) h_{j\bar{j}}(t))^{1/2} < \alpha(t)$ for any i, j. Let $\lambda_r(t) > 0$ be the largest eigen-value of the matrix $h(t)$. Then we see $\lambda_r(t) \leqslant r \max_{i,j} |h_{i\bar{j}}(t)| < r\alpha(t)$, and then the smallest eigen-value of $h^*(t) = {}^t h(t)^{-1}$ is $\lambda_r(t)^{-1} > 1/(r\alpha(t))$. (We supposed $t \neq 0$.)

We take a point $P \in \mathbb{P}(E)$ such that $\pi(P) = 0 \in X$, and take a local coordinate $(U, (t, w))$ of $\mathbb{P}(E) = X \times \mathbb{P}^{r-1}$ centered at P. We regard $\varphi \in$

$L^1_{\mathrm{loc}}(U,\mathbb{R})$ as a psh function (not as a collection of functions satisfying a gluing condition). Let $\nu_0 = \nu(\varphi, P)$ be the Lelong number of φ at P. We suppose $\nu_0 > 2\ell$. We take a constant $\varepsilon > 0$ such that $\nu_0 - \varepsilon > 2\ell + \varepsilon$. Then there exist a small neighborhood $U' \subset U$ of P and a constant $M_1 > 0$ such that $\varphi(t,w) < (\nu_0 - \varepsilon)\log(|t|^2 + |w|^2)^{1/2} + M_1$ holds on U' ([7, (2.8)]), in particular

$$e^{\varphi(t,w)} < e^{M_1}(|t|^2 + |w|^2)^{(\nu_0-\varepsilon)/2} < e^{M_1}(|t|^2 + |w|^2)^{(2\ell+\varepsilon)/2}$$

on U', where $|w| = (\sum_{i=1}^{r-1}|w_i|^2)^{1/2}$. On the other hand, there exists a constant $M_2 > 0$ depending on a choice of a local trivialization of L (which is irrelevant to compare asymptotic as $(t,w) \to P$ or $t \to 0$) such that, for any $t \neq 0$ and any $(t,w) \in \pi^{-1}(t) \cap U$, we have

$$e^{\varphi(t,w)} \geqslant M_2\lambda_r(t)^{-1} > \frac{|t|^{2\ell}}{(-\log|t|)^k}\frac{M_2}{rA_h}.$$

In particular, if $|t|$ is small but $t \neq 0$ and if $w = 0$, we have

$$\frac{|t|^{2\ell}}{(-\log|t|)^k}\frac{M_2}{rA_h} \;<\; e^{\varphi(t,w)} < e^{M_1}|t|^{2\ell+\varepsilon}.$$

This is impossible. $\square$

Remark 5.2. — Under the same assumption in 5.1, we take a non-zero section $\xi \in H^0(X, E^*)$ of the dual. By definition of the Griffiths semi-negativity of the curvature of the dual metric h^*, $\psi := \log|\xi|^2_{h^*}$ is psh on X ([26, 2.2.2]). If ξ is non-zero at 0 (for example $\xi = e_i^*$ a part of the dual local frame), then the Lelong number of ψ at 0 is bounded by $\nu(\psi, 0) \leqslant 2\ell$. In fact, by the proof above, we see $|\xi|^2_{h^*}(t) \geqslant c_\xi\lambda_r(t)^{-1} > c_\xi/(r\alpha(t)) = (c_\xi/r)|t|^{2\ell}(-\log|t|)^{-k}A_h^{-1}$ holds around 0 on the curve C, where $c_\xi > 0$ is a constant depending on ξ. Then $\log|\xi|^2_{h^*}(z)$ can not be smaller (i.e., negative) than $\log|z|^{2\ell+\varepsilon}$ around 0 for any given $\varepsilon > 0$, where $z = (z_, \dots, z_n)$ is a local coordinate of X centered at 0. That means $\nu(\psi, 0) \leqslant 2\ell$. (If we allow ξ vanishes at 0, the Lelong number can be arbitrary large. It does not make sense.)

We are ready to prove Theorem 1.1(2) and Corollary 1.3.

Proof of Theorem 1.1(2). — Since our assertion is local, we can suppose that $Y \subset \mathbb{C}^m$ is a ball centered at 0 with a coordinate $z = (z_1, \dots, z_m)$, and $C = \{z_2 = \ldots = z_m = 0\}$ in Y and $t = z_1|_C$ is a coordinate of C. We let $Y_1 = \{y \in Y;$ the fiber X_y of f has canonical singularities at worst, and the fiber X'_y of f' is smooth$\}$. Since C is general, by shrinking Y if necessary, we can suppose that $C \setminus 0 \subset Y_1$. In particular, any fiber X_t of f_C, but the central fiber X_0, has canonical singularities at worst.

We take $u \in H^0(C, F|_C)$, which can be seen as an element of $H^0(C, f_{C*}(mK_{X_C/C}))$ by the base change morphism. In fact, as we are working locally around $0 \in Y$ (Y is Stein and F is locally free), we have a natural exact sequence $0 \to H^0(Y, F \otimes \mathcal{I}_C) \to H^0(Y, F) \to H^0(C, F|_C) \to 0$, where $\mathcal{I}_C \subset \mathcal{O}_Y$ is the defining ideal sheaf of C. If we let $b_C : H^0(Y, F) = H^0(Y, f_*(mK_{X/Y})) \to H^0(C, f_{C*}(mK_{X/Y}|_{X_C}))$ be the base change morphism, then $H^0(C, F|_C) \to H^0(C, f_{C*}(mK_{X/Y}|_{X_C}))$, given by $u \mapsto b_C(\widetilde{u})$ for any choice of $\widetilde{u} \in H^0(Y, F)$ mapped to u (i.e., $\widetilde{u}|_C = u$) is well-defined.

Moreover as long as $t \neq 0$, the mNS Hermitian metric g_m for F at $t \in (C \subset) Y$ and the mNS Hermitian metric, say $g_{m,C}$, for $f_{C*}(mK_{X_C/C})$ at $t \in C$ is the same, which is nothing but the mNS Hermitian form on $H^0(X_t, mK_{X_t}) = H^0(X'_t, mK_{X'_t})$, where $X'_t = {f'}^{-1}(t)$ is a smooth model of X_t (this is what we call "the base change property" of the mNS Hermitian metric [26, 5.1.2]). By 1.1(1) for $f_C : X_C \to C$, we have

$$\|u\|^2_{g_m}(t) = g_{m,C}(u,u)(t) \leqslant \left(\frac{1}{|t|^{2(1-r_{0C})}}(-\log|t|)^n A_u\right)^m$$

for any $t \neq 0$ sufficiently small. Thus the assumptions in 5.1 are satisfied with $\ell = (1 - r_{0C})m$ and $k = nm$, and 1.1(2) is proved. □

Proof of Corollary 1.3. — We first review a brief outline of Fujino's argument in the proof of [10, 1.6], which is needed to our proof. We use the notations in 1.2, namely the one in [10, 1.6].

Step 1. By [1, Theorem 0.3], there exist a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y', a normal projective rational Gorenstein (i.e., Gorenstein canonical, see [20, 5.24]) variety X', which is birational to the main component of $X \times_Y Y'$, and a morphism $f^\dagger : X^\dagger \to Y'$ a so-called weak semi-stable reduction of $f : X \to Y$ ([1, Definition 0.1]). For our purpose, we may suppose our resolution $X' \to X \times_Y Y'$ is also a resolution of $X^\dagger$. Since $X^\dagger$ has canonical singularities at worst, we have

$$f'_*(mK_{X'/Y'}) \cong f^\dagger_*(mK_{X^\dagger/Y'}).$$

Step 2. By the assumption of 1.2, the geometric generic fiber of $f^\dagger : X^\dagger \to Y'$ has a good minimal model. Therefore, it has a relative good minimal model $\widetilde{f} : \widetilde{X} \to Y'$ by applying [13, 2.12]. Thus we obtain a diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\widetilde{f}"'] & X^{\dagger} \arrow[l, "\text{birat'l}"'] \arrow[d, "f^{\dagger}"'] & X' \arrow[l] \arrow[r] \arrow[d, "f'"'] & X \times_Y Y' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y' \arrow[r, equal] & Y' \arrow[r, equal] & Y' \arrow[r, equal] & Y' \arrow[r, "\tau"'] & Y
\end{tikzcd}$$

Here the map $X^{\dagger} \dashrightarrow \widetilde{X}$ is merely birational, not necessarily regular. Since $\widetilde{X}$ has canonical singularities at worst, we have

$$f^{\dagger}_{*}(mK_{X^{\dagger}/Y'}) \cong \widetilde{f}_{*}(mK_{\widetilde{X}/Y'}).$$

Step 3. Fujino shows that $\widetilde{f}_{*}(mK_{\widetilde{X}/Y'})$ is locally free.

We note that these three direct images $f'_{*}(mK_{X'/Y'})$, $f^{\dagger}_{*}(mK_{X^{\dagger}/Y'})$ and $\widetilde{f}_{*}(mK_{\widetilde{X}/Y'})$ admit the same mNS Hermitian metric g_m by 4.1. We shall apply 1.1(2) to $f^{\dagger} : X^{\dagger} \to Y'$. We check the conditions (2.i) and (2.ii) at every point $0 \in Y'$. The condition (2.ii); the local freeness of $f^{\dagger}_{*}(mK_{X^{\dagger}/Y'})$ is already obtained above. The condition (2.i) for any given point $0 \in Y'$ follows from the mildness of the weak semi-stable reduction. In fact, if we take general very ample Cartier divisors $H_1, H_2, \ldots, H_{k-1}$, where $k = \dim Y$, such that $C = H_1 \cap H_2 \cap \ldots \cap H_{k-1}$ is a smooth projective curve passing through 0, then by [1, Lemma 6.2], the induced morphism $f^{\dagger}_{C} : X^{\dagger}_{C} = X^{\dagger} \times_{Y'} C \to C$ is a weak semi-stable reduction. In particular, $X^{\dagger}_{C}$ has only rational Gorenstein (i.e., Gorenstein canonical) singularities (see [1, Lemma 6.1]). Moreover as the morphism $f^{\dagger}_{C}$ is toroidal by definition of the weak semi-stability, it is a log-canonical morphism ([5, 11.4.24]), in particular the log-canonical threshold of the pair $(X^{\dagger}_{C}, (f^{\dagger}_{C})^{*}(0))$ is zero. Thus we can conclude 1.3 via 1.1(2). $\square$

BIBLIOGRAPHY

[1] D. Abramovich & K. Karu, “Weak semistable reduction in characteristic 0”, *Invent. Math.* **139** (2000), no. 2, p. 241-273.

[2] B. Berndtsson & M. Păun, “Bergman kernels and the pseudoeffectivity of relative canonical bundles”, *Duke Math. J.* **145** (2008), no. 2, p. 341-378.

[3] ———, “Bergman kernels and subadjunction”, `http://arxiv.org/abs/1002.4145v1`, 2010.

[4] C. Birkar, P. Cascini, C. D. Hacon & J. McKernan, “Existence of minimal models for varieties of log general type”, *J. Amer. Math. Soc.* **23** (2010), no. 2, p. 405-468.

[5] D. A. Cox, J. B. Little & H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011, xxiv+841 pages.

[6] J.-P. Demailly, "Regularization of closed positive currents and intersection theory", *J. Algebraic Geom.* **1** (1992), no. 3, p. 361-409.

[7] ———, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012, viii+231 pages.

[8] J.-P. Demailly, T. Peternell & M. Schneider, "Pseudo-effective line bundles on compact Kähler manifolds", *Internat. J. Math.* **12** (2001), no. 6, p. 689-741.

[9] J. E. Fornæss & R. Narasimhan, "The Levi problem on complex spaces with singularities", *Math. Ann.* **248** (1980), no. 1, p. 47-72.

[10] O. Fujino, "Direct images of relative pluricanonical bundles", to appear in Algebr. Geom., `http://arxiv.org/abs/1409.7437v3`.

[11] T. Fujita, "On Kähler fiber spaces over curves", *J. Math. Soc. Japan* **30** (1978), no. 4, p. 779-794.

[12] P. A. Griffiths, "Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping", *Inst. Hautes Études Sci. Publ. Math.* (1970), no. 38, p. 125-180.

[13] C. D. Hacon & C. Xu, "Existence of log canonical closures", *Invent. Math.* **192** (2013), no. 1, p. 161-195.

[14] Y. Kawamata, "Characterization of abelian varieties", *Compositio Math.* **43** (1981), no. 2, p. 253-276.

[15] ———, "Kodaira dimension of algebraic fiber spaces over curves", *Invent. Math.* **66** (1982), no. 1, p. 57-71.

[16] ———, "Subadjunction of log canonical divisors for a subvariety of codimension 2", in *Birational algebraic geometry (Baltimore, MD, 1996)*, Contemp. Math., vol. 207, Amer. Math. Soc., Providence, RI, 1997, p. 79-88.

[17] ———, "Subadjunction of log canonical divisors. II", *Amer. J. Math.* **120** (1998), no. 5, p. 893-899.

[18] S. Kobayashi, "Geometry of bounded domains", *Trans. Amer. Math. Soc.* **92** (1959), p. 267-290.

[19] J. Kollár, "Subadditivity of the Kodaira dimension: fibers of general type", in *Algebraic geometry, Sendai, 1985*, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, p. 361-398.

[20] J. Kollár & S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original, viii+254 pages.

[21] R. Lazarsfeld, *Positivity in algebraic geometry. I, II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, 49, Springer-Verlag, Berlin, 2004.

[22] C. Mourougane & S. Takayama, "Extension of twisted Hodge metrics for Kähler morphisms", *J. Differential Geom.* **83** (2009), no. 1, p. 131-161.

[23] M. S. Narasimhan & R. R. Simha, "Manifolds with ample canonical class", *Invent. Math.* **5** (1968), p. 120-128.

[24] C. Okonek, M. Schneider & H. Spindler, *Vector bundles on complex projective spaces*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2011, Corrected reprint of the 1988 edition, With an appendix by S. I. Gelfand, viii+239 pages.

[25] M. Păun, "Siu's invariance of plurigenera: a one-tower proof", *J. Differential Geom.* **76** (2007), no. 3, p. 485-493.

[26] M. Păun & S. Takayama, "Positivity of twisted relative pluricanonical bundles and their direct images", `http://arxiv.org/abs/1409.5504`.

[27] M. Popa & C. Schnell, "On direct images of pluricanonical bundles", http://arxiv.org/abs/1405.6125v2.

[28] H. Raufi, "Singular hermitian metrics on holomorphic vector bundles", http://arxiv.org/abs/1211.2948v3.

[29] F. Sakai, "Kodaira dimensions of complements of divisors", in *Complex analysis and algebraic geometry*, Iwanami Shoten, Tokyo, 1977, p. 239-257.

[30] Y.-T. Siu, "Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type", in *Complex geometry (Göttingen, 2000)*, Springer, Berlin, 2002, p. 223-277.

[31] S. Takayama, "On the invariance and the lower semi-continuity of plurigenera of algebraic varieties", *J. Algebraic Geom.* **16** (2007), no. 1, p. 1-18.

[32] H. Tsuji, "Curvature semipositivity of relative pluricanonical systems", http://arxiv.org/abs/math/0703729v2.

[33] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995, viii+320 pages.

[34] K.-I. Yoshikawa, "Singularities and analytic torsion", http://arxiv.org/abs/1007.2835v1.

Manuscrit reçu le 4 décembre 2014,
révisé le 18 juin 2015,
accepté le 10 septembre 2015.

Shigeharu TAKAYAMA
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo, 153-8914 (Japan)
taka@ms.u-tokyo.ac.jp