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Takato UEHARA

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RATIONAL SURFACE AUTOMORPHISMS WITH POSITIVE ENTROPY

by Takato UEHARA (*)

Abstract. — The aim of this paper is to construct rational surface automorphisms with positive entropy by means of the concept of orbit data. The concept enables us to introduce some mild and verifiable condition, and to show that if an orbit data satisfies the condition, then there exists an automorphism realizing the orbit data. Applying this result, we describe the set of entropy values of the rational surface automorphisms in terms of Weyl groups.

Résumé. — Le but de ce travail est de construire des automorphismes de surfaces rationnelles d'entropie positive au moyen de la notion de donnée d'orbite. Celle-ci nous permet d'introduire une condition faible et vérifiable, et de démontrer que si une donnée d'orbite satisfait cette condition, alors il existe un automorphisme qui réalise la donnée d'orbite. En appliquant ce résultat, nous décrivons l'ensemble des valeurs d'entropie des automorphismes de surfaces rationnelles du point de vue des groupes de Weyl.

1. Introduction

In this paper, we consider automorphisms on compact complex surfaces with positive entropy. According to a result of S. Cantat [5], a surface admitting an automorphism with positive entropy must be either a K3 surface, an Enriques surface, a complex torus or a rational surface. For rational surfaces, rather few examples had been known (see [5], Section 2). However, some rational surface automorphisms with invariant anticanonical curves have been constructed recently. Bedford and Kim [3, 4] found some examples of automorphisms by studying an explicit family of quadratic birational maps on $\mathbb{P}^2$, and then McMullen [11] gave a synthetic

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construction of many examples. More recently, Diller [6] sought automorphisms from quadratic maps that preserve a cubic curve by using the group law for the cubic curve. We stress the point that these automorphisms can be all obtained from quadratic birational maps. The aim of this paper is to construct yet more examples of rational surface automorphisms with positive entropy from general birational maps on $\mathbb{P}^2$ preserving a cuspidal cubic curve.

Let $F : X \to X$ be an automorphism on a rational surface X. From results of Gromov and Yomdin [8, 14], the *topological entropy* $h_{\mathrm{top}}(F)$ of F is calculated as $h_{\mathrm{top}}(F) = \log \lambda(F^*)$, where $\lambda(F^*)$ is the spectral radius of the action $F^* : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ on the cohomology group. Therefore, when handling the topological entropy of a map, we need to discuss its action on the cohomology group, which can be described as an element of a Weyl group acting on a Lorentz lattice. The *Lorentz lattice* $\mathbb{Z}^{1,N}$ is the lattice with the Lorentz inner product given by

$$\mathbb{Z}^{1,N} = \bigoplus_{i=0}^{N} \mathbb{Z} \cdot e_i, \qquad (e_i, e_j) = \begin{cases} 1 & (i = j = 0) \\ -1 & (i = j = 1, \dots, N) \\ 0 & (i \neq j). \end{cases}$$

For $N \geqslant 3$, the *Weyl group* $W_N \subset O(\mathbb{Z}^{1,N})$ is the group generated by $(\rho_i)_{i=0}^{N-1}$, where $\rho_i : \mathbb{Z}^{1,N} \to \mathbb{Z}^{1,N}$ is a reflection defined by

$$\rho_i(x) = x + (x, \alpha_i) \cdot \alpha_i, \qquad \alpha_i := \begin{cases} e_0 - e_1 - e_2 - e_3 & (i = 0) \\ e_i - e_{i+1} & (i \neq 0). \end{cases} \tag{1.1}$$

We call the W_N-translate $\Phi_N := \bigcup_{i=0}^{N-1} W_N \cdot \alpha_i$ the *root system* of W_N, and each element of Φ_N a *root*. On the other hand, if $\lambda(F^*) > 1$, then there is a blowup $\pi : X \to \mathbb{P}^2$ of N points $(p_1, \dots, p_N)$ with $N \geqslant 10$ (see [12]), which gives an expression of the cohomology group : $H^2(X;\mathbb{Z}) = \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_N]$, where H is the total transform of a line in $\mathbb{P}^2$, and E_i is the total transform of the exceptional divisor over p_i. Moreover, there is a natural marking isomorphism

$$\phi_\pi : \mathbb{Z}^{1,N} \to H^2(X;\mathbb{Z}),\ \phi_\pi(e_0) = [H],\ \phi_\pi(e_i) = [E_i]\ (i = 1, \dots, N). \tag{1.2}$$

It is known (see [13]) that there is a unique element $w \in W_N$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}^{1,N} & \xrightarrow{w} & \mathbb{Z}^{1,N} \\ {\scriptstyle \phi_\pi}\downarrow & & \downarrow{\scriptstyle \phi_\pi} \\ H^2(X;\mathbb{Z}) & \xrightarrow{F^*} & H^2(X;\mathbb{Z}). \end{array} \tag{1.3}$$

Then w is said to be *realized* by (π, F) (see also [11]). A question at this stage is whether a given element $w \in W_N$ is realized by some pair (π, F). McMullen [11] states that if w has spectral radius $\lambda(w) > 1$ and no periodic roots, that is,

$$w^k(\alpha) \neq \alpha \qquad (\alpha \in \Phi_N, \quad k \geqslant 1), \tag{1.4}$$

then w is realized by a pair (π, F). However, since the root system Φ_N is an infinite set when $N \geqslant 10$, it is rather difficult to see whether w has no periodic roots. Indeed, he shows condition (1.4) only for the so called Coxeter element. One of our interest is to introduce a more verifiable condition and to construct realizations of much more Weyl group elements.

Another interest is to find the entropy values of rational surface automorphisms. In general, the topological entropy of any automorphism $F : X \to X$ is expressed as $h_{\mathrm{top}}(F) = \log \lambda(w)$ for some $w \in W_N$ (see also Proposition 3.3). Namely, for the entropy values

$$\mathbb{E} := \{h_{\mathrm{top}}(F) \,|\, F : X \to X \text{ is a rational surface automorphism}\},$$

we have

$$\mathbb{E} \subset \log \Lambda := \{\log \lambda \,|\, \lambda \in \Lambda\},$$

where Λ is given by

$$\Lambda := \{\lambda(w) \geqslant 1 \,|\, w \in W_N,\ N \geqslant 3\}. \tag{1.5}$$

The entropy values of automorphisms having been found so far seem to be contained in a very thin subset of $\log \Lambda$. On the other hand, one of our main results bridges the gap between two sets $\mathbb{E}$ and $\log \Lambda$, which is stated as follows.

THEOREM 1.1. — *The logarithm of any value $\lambda \in \Lambda$ is realized by the entropy of some rational surface automorphism $F : X \to X$. In particular, we have*

$$\mathbb{E} = \log \Lambda.$$

We will show this theorem by introducing the concept of orbit data.

Now let us consider an n-tuple $\overline{f} = (f_1, \dots, f_n)$ of quadratic birational maps $f_\ell : \mathbb{P}^2_{\ell-1} \to \mathbb{P}^2_\ell$ with each $\mathbb{P}^2_\ell$ being a copy of $\mathbb{P}^2$ and $\mathbb{P}^2_0 = \mathbb{P}^2_n$.

Note that the inverse of any quadratic map is also a quadratic map and a quadratic map has three points of indeterminacy. So the indeterminacy sets of $f_\ell^{\pm 1}$ can be denoted by $I(f_\ell) = \{p_{\ell,1}^+, p_{\ell,2}^+, p_{\ell,3}^+\} \subset \mathbb{P}^2_{\ell-1}$ and $I(f_\ell^{-1}) = \{p_{\ell,1}^-, p_{\ell,2}^-, p_{\ell,3}^-\} \subset \mathbb{P}^2_\ell$ with a suitable matching of the indices (ℓ, j) between forward and backward indeterminacies to be specified later (see Section 3). Then we assume that the orbit of each backward indeterminacy point reaches some forward one. More precisely, with the notation $p_\iota^\pm = p_{i,j}^\pm$ for $\iota \in \mathcal{K}(n) := \{\iota = (i,j) \,|\, i = 1,2,\dots,n,\ j = 1,2,3\}$, suppose that there is a permutation σ of $\mathcal{K}(n)$ and a function $\mu : \mathcal{K}(n) \to \mathbb{Z}_{\geqslant 0}$ such that the following condition holds for any $\iota \in \mathcal{K}(n)$:

$$(1.6) \qquad p_\iota^m \neq p_{\iota'}^+ \quad (0 \leqslant m < \mu(\iota),\ \iota' \in \mathcal{K}(n)), \qquad p_\iota^{\mu(\iota)} = p_{\sigma(\iota)}^+,$$

where p_ι^m is defined inductively by

$$(1.7) \qquad p_\iota^0 := p_\iota^- \in \mathbb{P}^2_i, \qquad p_\iota^m := f_\ell(p_\iota^{m-1}) \in \mathbb{P}^2_\ell \quad (\ell \equiv i + m \bmod n).$$

Then, by blowing up the orbit segments $p_\iota^- = p_\iota^0, p_\iota^1, \dots, p_\iota^{\mu(\iota)} = p_{\sigma(\iota)}^+$ for $\iota \in \mathcal{K}(n)$, we can cancel all indeterminacy points of (f_ℓ) and (f_ℓ^{-1}). That is, if $\pi_\ell : X_\ell \to \mathbb{P}^2_\ell$ is a blowup of points p_ι^m with $0 \leqslant m \leqslant \mu(\iota)$ and $i + m \equiv \ell \bmod n$, then the birational maps $f_\ell : \mathbb{P}^2_{\ell-1} \to \mathbb{P}^2_\ell$ lift to biholomorphisms $F_\ell : X_{\ell-1} \to X_\ell$, whose composition gives an automorphism $F := F_n \circ \cdots \circ F_1 : X_0 \to X_n = X_0$. Now, we denote by $\kappa(\iota)$ the number of points among $p_\iota^0, p_\iota^1, \dots, p_\iota^{\mu(\iota)}$ lying on $\mathbb{P}^2_n$ or, in other words, $\kappa(\iota) = (\mu(\iota) + i - i_1 + 1)/n$ with $\sigma(\iota) = (i_1, j_1)$. It is easy to see that $\kappa(\iota) \geqslant 1$ provided $i_1 \leqslant i$. This observation leads us to the following definition.

Definition 1.2. — *An orbit data is a triplet $\tau = (n, \sigma, \kappa)$ consisting of*

- *a positive integer n,*
- *a permutation σ of $\mathcal{K}(n)$, and*
- *a function $\kappa : \mathcal{K}(n) \to \mathbb{Z}_{\geqslant 0}$ such that $\mu(\iota) = \kappa(\iota) \cdot n + i_1 - i - 1 \geqslant 0$.*

Definition 1.3. — *An n-tuple $\overline{f} = (f_1, \dots, f_n)$ of quadratic birational maps f_ℓ is called a realization of an orbit data τ if condition* (1.6) *holds for any $\iota \in \mathcal{K}(n)$.*

A question here is whether a given orbit data admits some realization.

To answer this, we consider a class of birational maps preserving a cuspidal cubic C on $\mathbb{P}^2$. Let $\mathcal{Q}(C)$ be the set of quadratic birational maps $f : \mathbb{P}^2 \to \mathbb{P}^2$ satisfying $f(C) = C$ and $I(f) \subset C^*$, where C^* is the smooth locus of C. The smooth locus C^* is isomorphic to the complex plane $\mathbb{C}$ and is preserved by any map $f \in \mathcal{Q}(C)$. Thus, the restriction $f|_{C^*}$ is an automorphism expressed as

$$f|_{C^*} : \mathbb{C} \to \mathbb{C}, \quad t \mapsto \delta(f) \cdot t + c(f)$$

for some $\delta(f) \in \mathbb{C}^\times$ and $c(f) \in \mathbb{C}$. For an n-tuple $\overline{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$, the *determinant* of $\overline{f}$ is defined by $\delta(\overline{f}) := \prod_{\ell=1}^{n} \delta(f_\ell)$.

Moreover, we take advantage of introducing the concept of orbit data to state a more verifiable condition than (1.4) in terms of a *finite* subset $\Gamma(\tau)$ of Φ_N, that is,

$$w_\tau^k(\alpha) \neq \alpha \qquad (\alpha \in \Gamma(\tau), \quad k \geqslant 1), \tag{1.8}$$

where w_τ is an element of W_N with $N := \sum_{\iota \in \mathcal{K}(n)} \kappa(\iota)$. The orbit data τ canonically determines $\Gamma(\tau)$ and w_τ, whose definitions will be given later (see Definitions 3.9 and 5.10). It will be also seen later that any element $w \in W_N$ is expressed as $w = w_\tau$ for some orbit data τ (see Proposition 3.12). Thus, once an orbit data τ with $w = w_\tau$ is fixed, the finiteness of $\Gamma(\tau)$ enables us to check easily that w satisfies condition (1.8). It should be noted that an expression for w in terms of factorization into a product of the generators $(\rho_i)_{i=0}^{N-1}$ yields the orbit data $\tau = (n, \sigma, \kappa)$, and the number of ρ_0 in the expression is the length n, where ρ_0 corresponds to the standard Cremona transformation. As an expression for a given element $w \in W_N$ is not unique, neither is an orbit data τ satisfying $w = w_\tau$.

Condition (1.8) is referred as the *realizability condition*, for reasons that become clear in the following theorem.

Theorem 1.4. — *Let τ be an orbit data with $\lambda(w_\tau) > 1$. Then, τ satisfies the realizability condition* (1.8) *if and only if there is a realization $\overline{f}_\tau = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ of τ such that $\delta(\overline{f}_\tau) = \lambda(w_\tau)$. The realization $\overline{f}_\tau \in \mathcal{Q}(C)^n$ of τ with $\delta(\overline{f}_\tau) = \lambda(w_\tau)$ is uniquely determined. Moreover, τ determines a blowup $\pi_\tau : X_\tau \to \mathbb{P}^2$ of N points on C^* in a canonical way, which lifts $f_\tau := f_n \circ \cdots \circ f_1$ to an automorphism $F_\tau : X_\tau \to X_\tau$:*

$$\begin{array}{ccc} X_\tau & \xrightarrow{F_\tau} & X_\tau \\ {\scriptstyle \pi_\tau}\downarrow & & \downarrow{\scriptstyle \pi_\tau} \\ \mathbb{P}^2 & \xrightarrow{f_\tau} & \mathbb{P}^2. \end{array}$$

Finally, the pair (π_τ, F_τ) realizes w_τ and F_τ has positive entropy $h_{\mathrm{top}}(F_\tau) = \log \lambda(w_\tau) > 0$.

As seen in Theorem 1.6, almost all orbit data satisfy the realizability condition (1.8). Furthermore, even if an orbit data τ does not satisfy the realizability condition (1.8), another orbit data $\check{\tau}$ with the same spectral radius, called the *sibling* of τ, does satisfy the condition.

Theorem 1.5. — *For any orbit data τ with $\lambda(w_\tau) > 1$, there is an orbit data $\check{\tau}$ satisfying $\lambda(w_\tau) = \lambda(w_{\check{\tau}}) > 1$ and the realizability condition* (1.8). *In particular, $\check{\tau}$ is realized by $\overline{f}_{\check{\tau}}$.*

Theorem 1.1 is a consequence of Theorem 1.5 and the fact that any element $w \in W_N$ is expressed as $w = w_\tau$ for some τ.

Finally, we give a sufficient condition for (1.8), which enables us to see clearly that almost all orbit data are realized, and to obtain an estimate for the entropy.

Theorem 1.6. — *Assume that an orbit data $\tau = (n, \sigma, \kappa)$ satisfies*

(1) $n \geqslant 2$,

(2) $\kappa(\iota) \geqslant 3$ *for any* $\iota \in \mathcal{K}(n)$, *and*

(3) *if* $\iota \neq \iota'$ *satisfy* $i_m = i'_m$ *and* $\kappa(\sigma^m(\iota)) = \kappa(\sigma^m(\iota'))$ *for any* $m \geqslant 0$, *then* $\iota' \neq \sigma^m(\iota)$ *for any* $m \geqslant 0$, *where* $\sigma^m(\iota) = (i_m, j_m)$ *and* $\sigma^m(\iota') = (i'_m, j'_m)$.

Then the orbit data τ satisfies $2^n - 1 < \lambda(w_\tau) < 2^n$ and the realizability condition (1.8). *In particular, F_τ has positive entropy* $\log(2^n - 1) < h_{\mathrm{top}}(F_\tau) < \log 2^n$.

Diller [6] constructs, by studying single quadratic maps preserving C, automorphisms with positive entropy realizing orbit data $\hat{\tau} = (1, \hat{\sigma}, \hat{\kappa})$. As is seen in Example 5.13, there is an orbit data τ such that F_τ is not topologically conjugate to the iterates of $F_{\hat{\tau}}$ that Diller constructs for any $\hat{\tau} = (1, \hat{\sigma}, \hat{\kappa})$. Moreover, the element w_τ determined by this orbit data τ admits periodic roots and thus does not satisfy condition (1.4).

Any quadratic map in $\mathcal{Q}(C)$ is determined completely, up to a linear conjugation, by the configuration of the three indeterminacy points, which lie on the smooth locus $C^* \cong \mathbb{C}$, and the map is degenerate to a linear one when the indeterminacy points are collinear (see Lemma 4.2 and Remark 4.3). Hence for an orbit data $\tau = (n, \sigma, \kappa)$, the $3n$ conditions $p_\iota^{\mu(\iota)} = p^+_{\sigma(\iota)}$ determine $3n$ indeterminacy points $\{p_\iota^+\}_{\iota \in \mathcal{K}(n)}$ and an n-tuple $\overline{f} = (f_1, \dots, f_n)$. Our investigations on the existence of a realization are divided into two steps. The first step is to check that τ admits a tentative realization, namely, each map f_i is indeed quadratic (see also Definition 4.1). Proposition 4.13 states that a tentative realization $\overline{f}$ of τ exists if and only if w_τ has no periodic roots in a subset $\Gamma_1(\tau)$ of Φ with n elements. When τ does not satisfy the condition, another orbit data $\check{\tau} = (\check{n}, \check{\sigma}, \check{\kappa})$ with $\check{n} < n$ satisfies $\lambda(w_{\check{\tau}}) = \lambda(w_\tau)$ and admits a tentative realization. The second step is to check that the tentative realization $\overline{f}$ is indeed a realization of τ, that is, check whether an orbit $\{p_\iota^m\}_{m \geqslant 0}$ satisfies $p_\iota^m \neq p_{\iota'}^+$ before reaching $p^+_{\sigma(\iota)}$.

Proposition 5.7 shows that $\overline{f}$ is a realization of τ if and only if w_τ has no periodic roots in $\Gamma_2(\tau) = \Gamma(\tau) \setminus \Gamma_1(\tau)$. Even if τ does not satisfy the condition, $\overline{f}$ is a realization of another orbit data $\check{\tau}$. Note that the configuration of the orbits $\{p_\iota^0, \dots, p_\iota^{\mu(\iota)}\}_{\iota \in \mathcal{K}(n)}$, which are blown up to yield an automorphism, is closely related to the eigenvalue problem of w_τ (see Proposition 4.7). Then the sibling $\check{\tau}$ of τ determines essentially the same configuration of the blown up points as τ. On the other hand, under the assumptions in Theorem 1.6, Proposition 4.15 gives an estimate for the spectral radius $\lambda(w_\tau)$ and shows the absence of periodic roots in $\Gamma_1(\tau)$, and then Proposition 5.9 guarantees the absence of periodic roots in $\Gamma_2(\tau)$, which proves Theorem 1.6. We notice that most of the methods in this paper may be applicable for rational surface automorphisms in positive characteristic, *except one crucial point* where the first dynamical degree is related to the determinant of birational maps preserving the cuspidal cubic curve, which does not make any sense in positive characteristic at least as it appears.

This article is organized as follows. After a preliminary study in Section 2, Section 3 is devoted to developing a method for constructing a rational surface automorphism from a realization of τ. In Section 4, we discuss the existence of a tentative realization of τ, and in Section 5, we investigate whether it is indeed a realization and prove Theorems 1.1 and 1.4–1.6. Finally, Propositions 4.15 and 5.9 are proved in Section 6.

Remark 1.7. — In the sequel, we will often use the following notations for a given orbit data $\tau = (n, \sigma, \kappa)$.

(1) For $\iota = (i, j) \in \mathcal{K}(n)$, put $\iota_m = (i_m, j_m) = \sigma^m(\iota)$. We will also use a similar notation for another orbit data (e.g. for $\iota' = (i', j') \in \mathcal{K}(n)$, put $\iota'_m = (i'_m, j'_m) = \sigma^m(\iota')$).

(2) For each $\iota \in \mathcal{K}(n)$, put $\bar{\sigma}(\iota) := \sigma^k(\iota)$, where $k \geqslant 0$ is determined by the relations $\kappa(\sigma^\ell(\iota)) = 0$ for $0 \leqslant \ell < k$, and $\kappa(\sigma^k(\iota)) \geqslant 1$.

2. Preliminary

In this section, we give a preliminary study of birational maps between complex surfaces, mainly in order to clarify the meaning of the equality in (1.6), since p_ι^m may be an infinitely near point on $\mathbb{P}^2$. To this end, from a birational map $f : X \to X$ on a compact surface X, we build up a new surface map $f_\sim : X_\sim \to X_\sim$. Although the surface $X_\sim$, which may be regarded as the set of all proper and infinitely near points on X, becomes noncompact and rather larger than X, it gives certain nice properties to the map $f_\sim$ as is mentioned below. These properties are used in our arguments of this article.

Throughout this paper, by a surface we mean a smooth irreducible projective surface over the complex numbers. Let X be a surface, and consider a pair $(x, \pi : \hat{X} \to X)$ of a point $x \in \hat{X}$ and a proper modification $\pi : \hat{X} \to X$. Two pairs (x_1, π_1) and (x_2, π_2) with $\pi_i : X_i \to X$ are said to be *equivalent*, denoted by $(x_1, \pi_1) \sim (x_2, \pi_2)$, if $\pi_1^{-1} \circ \pi_2 : X_2 \to X_1$ is locally biholomorphic at x_2 and $x_1 = \pi_1^{-1} \circ \pi_2(x_2)$. Let $X_\sim$ be the set of all equivalence classes of (x, π). The equivalence class of (x, π) is denoted by $[x, \pi]$. Then, $x \in X$ can be identified with $[x, \mathrm{id}_X] \in X_\sim$, which is said to be *proper*. Moreover $X_\sim$ is equipped with the topology generated by

$$\{U_\sim \mid [x, \pi : \hat{X} \to X] \in X_\sim \text{ and } U \subset \hat{X} \text{ is an open neighbourhood of } x\},$$

where $U_\sim := \{[y, \pi] \in X_\sim \mid y \in U\}$. We notice that $U_\sim$ is identified with U, via $y \mapsto [y, \pi]$, which gives $X_\sim$ the complex structure induced from that on X, namely, $X_\sim$ becomes a (noncompact) complex surface.

Let $f : X \to Y$ be a birational map with its inverse $f^{-1} : Y \to X$. Moreover assume that $Y \subset \mathbb{P}^N$ and $f(X)$ is contained in no hyperplane of $\mathbb{P}^N$, so that f^*H is a curve on X for any hyperplane $H \subset \mathbb{P}^N$. Put

$$I(f) := \{x \in X \mid x \in f^*H \text{ for any hyperplane } H \subset \mathbb{P}^N\}.$$

We call $I(f)$ the *indeterminacy set* of f, on which f is not defined.

Remark 2.1. — Let $f : X \to Y$ be a birational map and $x \in X$ be a point. Then f is locally holomorphic at x if and only if $x \notin I(f)$. Moreover, f is locally biholomorphic at x if and only if $x \notin I(f)$ and $f(x) \notin I(f^{-1})$.

For a curve $C \subset X$, let $\nu_{[x,\pi]}(C)$ be the multiplicity $\nu_x(\pi^{-1}(C))$ of $\pi^{-1}(C)$ at x and put

$$C_\sim := \{[x, \pi] \in X_\sim \mid \nu_{[x,\pi]}(C) \geqslant 1\} \subset X_\sim,$$

where $\pi^{-1}(C) := \overline{\pi^{-1}(C \setminus I(f))}$ is the strict transform of C. Note that the definition of multiplicity is well-defined by virtue of the following lemma.

Lemma 2.2. — *If (x_1, π_1) and (x_2, π_2) are equivalent, then we have $\nu_{x_1}(\pi_1^{-1}(C)) = \nu_{x_2}(\pi_2^{-1}(C))$.*

Proof. — For $i = 1, 2$, assume that $U_i \subset X_i$ are open subsets containing x_i such that $\pi_2^{-1} \circ \pi_1 : U_1 \to U_2$ is a biholomorphism with $x_2 = \pi_2^{-1} \circ \pi_1(x_1)$. Since $U := \pi_1(U_1) = \pi_2(\pi_2^{-1} \circ \pi_1(U_1)) = \pi_2(U_2) \subset X$, one may assume $I(\pi_i^{-1}) \cap U \subset \{x\}$ by taking sufficiently small U_i, where

$x := \pi_1(x_1) = \pi_2(x_2) \in X$. Note that $\nu_{x_i}(\pi_i^{-1}(C))$ equals the multiplicity of $\overline{\pi_i^{-1}(C \cap U \setminus \{x\})} \subset U_i$ at x_i, and that $\overline{\pi_2^{-1}(C \cap U \setminus \{x\})} = \overline{\pi_2^{-1} \circ \pi_1(\pi_1^{-1}(C \cap U \setminus \{x\}))} = \pi_2^{-1} \circ \pi_1(\overline{\pi_1^{-1}(C \cap U \setminus \{x\})})$. Hence, we have $\nu_{x_1}(\pi_1^{-1}(C)) = \nu_{x_2}(\pi_2^{-1}(C))$ as $\pi_2^{-1} \circ \pi_1 : U_1 \to U_2$ is a biholomorphism. □

In what follows, two proper modifications $\pi_1 : X_1 \to X$ and $\pi_2 : X_2 \to X$ are identified if $\pi_1^{-1} \circ \pi_2 : X_2 \to X_1$ is an isomorphism. Under the identification, a blowup $\pi_x : X_x \to X$ of a point $x \in X$ is uniquely determined. Then, we have the following proposition.

Proposition 2.3 ([1, 2]). — *Let $f : X \to Y$ be a birational morphism of surfaces. Assume that $y \in I(f^{-1})$. Then, f factorizes as*

$$f : X \xrightarrow{\hat{f}} Y_y \xrightarrow{\pi_y} Y,$$

where $\pi_y : Y_y \to Y$ is the blowup of the point y, and $\hat{f} : X \to Y_y$ is a birational morphism. Moreover there exists a sequence of blowups $\pi_i : Y_i \to Y_{i-1}$ of points with $Y_0 = Y$ and $Y_m = X$ such that $f = \pi_1 \circ \cdots \circ \pi_m$.

Proposition 2.4. — *Let $\pi_x : X_x \to X$ be the blowup of a point $x \in X$. Then there exists an isomorphism $(X_x)_\sim \to X_\sim \setminus \{[x, \mathrm{id}_X]\}$, given by $[z, \pi] \mapsto [z, \pi_x \circ \pi]$.*

Proof. — First we notice that $[z, \pi_x \circ \pi] \neq [x, \mathrm{id}_X]$ since $x \in I((\pi_x \circ \pi)^{-1})$. Now we construct the inverse of $(X_x)_\sim \to X_\sim \setminus \{[x, \mathrm{id}_X]\}$. Take an element $[z, \hat{\pi}] \in X_\sim \setminus \{[x, \mathrm{id}_X]\}$. Then we may assume that $x \in I(\hat{\pi}^{-1})$. Indeed, if $x = \hat{\pi}(z)$, then $\hat{\pi}$ is not locally biholomorphic at z as $[z, \hat{\pi}] \neq [x, \mathrm{id}_X]$, which means that $x \in I(\hat{\pi}^{-1})$. On the other hand, if $x \neq \hat{\pi}(z)$ and $x \notin I(\hat{\pi}^{-1})$, that is, $\hat{\pi}^{-1}$ is locally biholomorphic at x, then put $\bar{\pi} := \hat{\pi} \circ \pi_y$ with $y := \hat{\pi}^{-1}(x)$. Since $z \neq y$, $\bar{z} := \pi_y^{-1}(z)$ satisfies $[\bar{z}, \bar{\pi}] = [z, \hat{\pi}]$ and $x \in I(\bar{\pi}^{-1})$. Hence we can assume that $x \in I(\hat{\pi}^{-1})$. Proposition 2.3 says that $\hat{\pi}$ factorizes as $\hat{\pi} = \pi_x \circ \pi$ for some proper modification π, which yields inverse of $(X_x)_\sim \to X_\sim \setminus \{[x, \mathrm{id}_X]\}$. Therefore the proposition is established. □

For two pairs $[x_i, \pi_i : X_i \to X]$, we put $[x_1, \pi_1] < [x_2, \pi_2]$ if $\pi_1^{-1} \circ \pi_2 : X_2 \to X_1$ is locally *holomorphic* (not necessarily locally biholomorphic) at x_2 and $x_1 = \pi_1^{-1} \circ \pi_2(x_2)$. Note that the definition is independent of the choice of $(x_i, \pi_i) \in [x_i, \pi_i]$. Proposition 2.3 shows that $[x_1, \pi_1]$ is proper on $X_\sim$ if and only if there is no point $[x, \pi] \neq [x_1, \pi_1]$ with $[x, \pi] < [x_1, \pi_1]$. We write $[x_1, \pi_1] \approx [x_2, \pi_2]$ if either $[x_1, \pi_1] < [x_2, \pi_2]$ or $[x_1, \pi_1] > [x_2, \pi_2]$.

Definition 2.5. — *A finite subset $K \subset X_\sim$ is called a cluster if $[x_1, \pi_1] \in K$ for any $[x_1, \pi_1] < [x_2, \pi_2]$ with $[x_2, \pi_2] \in K$.*

Proposition 2.6. — *Let C be a curve on X, and C_1, C_2 be curves having no irreducible component in common. Then we have the following.*

(1) $\nu_{[x_1,\pi_1]}(C) \geqslant \nu_{[x_2,\pi_2]}(C)$ *for* $[x_1,\pi_1] < [x_2,\pi_2]$.

(2) $(C_1, C_2) = \sum_{[x,\pi]\in(C_1)_\sim\cap(C_2)_\sim} \nu_{[x,\pi]}(C_1)\cdot\nu_{[x,\pi]}(C_2)$, *where* $(\cdot,\cdot)$ *is the intersection form on* X.

(3) $(C_1)_\sim \cap (C_2)_\sim$ *is a cluster.*

Proof. — Assertion (1) is a consequence of Proposition 2.3 and the fact that $\nu_{z_1}(C) \geqslant \nu_{z_2}(\pi_x^{-1}(C))$ for any blowup $\pi_x : X_x \to X$ of a point x and for any points $z_2 \in \hat{X}$ with $z_1 = \pi(z_2) \in X$. In order to prove assertion (2), we use the fact that $(\pi_x^{-1}(C_1), \pi_x^{-1}(C_2)) = (C_1, C_2) - \nu_x(C_1)\cdot\nu_x(C_2)$ for the blowup $\pi_x : X_x \to X$ of a point x. If $(C_1, C_2) = 0$, then it follows that $\nu_x(C_1)\cdot\nu_x(C_2) = 0$ for any $x \in X$ and thus $\nu_{[x,\pi]}(C_1)\cdot\nu_{[x,\pi]}(C_2) = 0$ for any $[x,\pi] \in X_\sim$, which yields $(C_1)_\sim \cap (C_2)_\sim = \emptyset$. Hence assertion (2) holds when $(C_1, C_2) = 0$. On the other hand, if $(C_1, C_2) > 0$, then there is a point $x \in X$ such that $\nu_{[x,\mathrm{id}]}(C_1)\cdot\nu_{[x,\mathrm{id}]}(C_2) = \nu_x(C_1)\cdot\nu_x(C_2) > 0$ and thus $(\pi_x^{-1}(C_1), \pi_x^{-1}(C_2)) = (C_1, C_2) - \nu_{[x,\mathrm{id}]}(C_1)\cdot\nu_{[x,\mathrm{id}]}(C_2) < (C_1, C_2)$. Hence by replacing X with X_x, namely, $X_\sim$ with $(X_x)_\sim = X_\sim \setminus \{[x,\mathrm{id}]\}$ by Proposition 2.4, and C_i with $\pi_x^{-1}(C_i)$, we can repeat this argument finitely many times to yield assertion (2). Finally we notice that $(C_1)_\sim \cap (C_2)_\sim$ is a finite set by assertion (2). Assertion (1) shows that $(C_1)_\sim \cap (C_2)_\sim$ becomes a cluster, which yields assertion (3). □

For $[z_0, \pi_0] \in X_\sim$, an element $[z_m, \pi_m] \in X_\sim$ satisfying $\pi_m = \pi_0 \circ \pi_{z_0} \circ \cdots \circ \pi_{z_{m-1}}$ and $z_0 = \pi_{z_0} \circ \cdots \circ \pi_{z_{i-1}}(z_i)$ for $1 \leqslant i \leqslant m$ is called a *point in the m-th infinitesimal neighbourhood of* $[z_0, \pi_0]$ or a *point infinite near to* $[z_0, \pi_0]$. A point in the 0-th infinitesimal neighbourhood of $[z_0, \pi_0]$ is interpreted as $[z_0, \pi_0]$ itself. If $[z_0, \pi_0]$ is proper, $[z_m, \pi_m]$ is called an *m-th infinitely near point on X* or an *m-th point* for short. Note that if $[x_1, \pi_1] < [x_2, \pi_2]$, then $[x_2, \pi_2]$ is a point in the m-th infinitesimal neighbourhood of $[x_1, \pi_1]$ for some $m \in \mathbb{Z}_{\geqslant 0}$ by Proposition 2.3.

Next we construct a proper modification $\pi_K : X_K \to X$ from a cluster $K \subset X_\sim$. Put $K = \{[x_0, \pi_0], [x_1, \pi_1], \dots [x_{m-1}, \pi_{m-1}]\}$ so that if $[x_i, \pi_i] < [x_j, \pi_j]$ for $i \neq j$ then $[x_i, \pi_i] \neq [x_j, \pi_j]$ and $i < j$. We also put $Y_0 := X$ and $\nu_0 := \mathrm{id}_{Y_0} : Y_0 \to Y_0$. For $k \in \{0, \dots, m-1\}$, let y_k be a point of Y_k inductively given by the relation

$$(y_k, \nu_0 \circ \nu_1 \circ \cdots \circ \nu_k) \in [x_k, \pi_k], \tag{2.1}$$

and let $\nu_{k+1} : Y_{k+1} \to Y_k$ be the blowup of $y_k \in Y_k$. It should be noted that y_k is determined uniquely. Indeed, when $k = 0$, π_0 is locally biholomorphic at x_0 as $[x_0, \pi_0]$ is proper, and hence y_0 is given by $y_0 = \nu_0^{-1} \circ \pi_0(x_0) = \pi_0(x_0)$. Moreover, under the assumption that $y_0, \dots, y_{k-1}$ are already given by (2.1), the point $[x_k, \pi_k] \in X_\sim \setminus \{[x_0, \pi_0], \dots [x_{k-1}, \pi_{k-1}]\} \cong (Y_k)_\sim$ is proper on $(Y_k)_\sim$ and y_k is also determined uniquely in a similar argument.

The proper modification $\pi_K : X_K \to X$ is given by the composition

$$\pi_K : X_K = Y_m \xrightarrow{\nu_m} Y_{m-1} \xrightarrow{\nu_{m-1}} \cdots \xrightarrow{\nu_2} Y_1 \xrightarrow{\nu_1} Y_0 = X,$$

called *the blowup of the cluster* K. The definition of π_K is independent of the choice of ordering in the cluster K. Moreover, the total transform $E_k = (\nu_{k+1} \circ \cdots \circ \nu_m)^*(E'_k)$ of the exceptional curve E'_k of ν_k is called the *exceptional divisor of* π_K *over the point* $[x_{k-1}, \pi_{k-1}]$.

Now let X and Y be projective surfaces and $f : X \to Y$ be a birational map. Assume that $Y \subset \mathbb{P}^N$ and $f(X)$ is contained in no hyperplane of $\mathbb{P}^N$, so that f^*H is a curve on X for any hyperplane $H \subset \mathbb{P}^N$. Put

$$I(f)_\sim := \{[x, \pi] \in X_\sim \mid [x, \pi] \in (f^*H)_\sim \text{ for any hyperplanes } H \subset \mathbb{P}^N\}.$$

It follows from Proposition 2.6 that the set $I(f)_\sim$ is a cluster. For proper modifications $\pi : \hat{X} \to X$ and $\nu : \hat{Y} \to Y$, put $f_{\nu,\pi} := \nu^{-1} \circ f \circ \pi : \hat{X} \to \hat{Y}$.

Proposition 2.7 ([1, 2]). — *For any birational map $f : X \to Y$ of surfaces, put $\pi_0 := \pi_{I(f)_\sim} : X_0 := X_{I(f)_\sim} \to X$ and $\nu_0 := \pi_{I(f^{-1})_\sim} : Y_0 := Y_{I(f^{-1})_\sim} \to Y$. Then, the map $f_{\nu_0,\pi_0} : X_0 \to Y_0$ is a biholomorphism.*

Proposition 2.8. — *For any $[x, \pi : \hat{X} \to X] \in X_\sim \setminus I(f)_\sim$, there is a unique element $[y, \nu : \hat{Y} \to Y] \in Y_\sim \setminus I(f^{-1})_\sim$ such that $f_{\nu,\pi}$ is locally biholomorphic at $x \in \hat{X}$ and $y = f_{\nu,\pi}(x) \in \hat{Y}$ for some (and any) $(x, \pi) \in [x, \pi]$ and $(y, \nu) \in [y, \nu]$.*

Proof. — First we prove the existence of $[y, \nu] \in Y_\sim \setminus I(f^{-1})_\sim$. Let $f_0 := f_{\nu_0,\pi_0} : X_0 \to Y_0$ be the biholomorphism given in Proposition 2.7, and let $\hat{\pi} : \hat{X} \to X_0$ be a proper modification with $(x_0, \pi_0 \circ \hat{\pi}) \in [x, \pi] \in X_\sim \setminus I(f)_\sim \cong (X_0)_\sim$ (see Proposition 2.4). Since $f_0 \circ \hat{\pi} : \hat{X} \to Y_0$ is a birational morphism, Proposition 2.3 shows that there is a proper modification $\hat{\nu} : \hat{Y} \to Y_0$ such that $\hat{\nu}^{-1} \circ f_0 \circ \hat{\pi} : \hat{X} \to \hat{Y}$ is a biholomorphism. As $f_{\nu,\pi_0 \circ \hat{\pi}} = \hat{\nu}^{-1} \circ f_0 \circ \hat{\pi}$ with $\nu := \nu_0 \circ \hat{\nu}$, we can see that $[y, \nu] \in (Y_0) \cong Y_\sim \setminus I(f^{-1})_\sim$ for $y := f_{\nu,\pi_0 \circ \hat{\pi}}(x)$ is a desired element.

Next we prove the uniqueness of $[y, \nu]$. For $i = 1, 2$, assume that there are $(y_i, \nu_i : \hat{Y}_i \to Y)$ such that $f_{\nu_i,\pi} : \hat{X} \to \hat{Y}_i$ are locally biholomorphic at x and $y_i = f_{\nu_i,\pi}(x)$. Then $\nu_2^{-1} \circ \nu_1 = f_{\nu_2,\pi} \circ f_{\nu_1,\pi}^{-1} : \hat{Y}_1 \to \hat{Y}_2$ is locally biholomorphic at y_1 and $y_2 = f_{\nu_2,\pi} \circ f_{\nu_1,\pi}^{-1}(y_1) = \nu_2^{-1} \circ \nu_1(y_1)$, which means that $(y_1, \nu_1) \sim (y_2, \nu_2)$.

Finally, we assume that $(x_1, \pi_1 : \hat{X}_1 \to X) \sim (x_2, \pi_2 : \hat{X}_2 \to X)$ and $(y_1, \nu_1 : \hat{Y}_1 \to Y) \sim (y_2, \nu_2 : \hat{Y}_2 \to Y)$, and that $f_{\nu_1,\pi_1} : \hat{X}_1 \to \hat{Y}_1$ is locally biholomorphic at x_1 with $y_1 = f_{\nu_1,\pi_1}(x_1)$. Then $f_{\nu_2,\pi_2} = \nu_2^{-1} \circ f \circ \pi_2 = (\nu_2^{-1} \circ \nu_1) \circ f_{\nu_1,\pi_1} \circ (\pi_2^{-1} \circ \pi_1)^{-1}$ is locally biholomorphic with $y_2 = f_{\nu_2,\pi_2}(x_2)$, which means that the relation $[x, \pi] \mapsto [y, \nu]$ is well-defined. The proposition is established. □

By virtue of Proposition 2.8, we can define a map

$$f_\sim : X_\sim \setminus I(f)_\sim \to Y_\sim \setminus I(f^{-1})_\sim, \quad f_\sim([x, \pi]) = [y, \nu],$$

which is in fact a biholomorphism with the inverse $(f^{-1})_\sim$.

Remark 2.9. — Let $I_X = \{[x_1, \pi_1], \dots, [x_m, \pi_m]\} \subset X_\sim$ and $I_Y = \{[y_1, \nu_1], \dots, [y_m, \nu_m]\} \subset Y_\sim$ be clusters satisfying $I(f)_\sim \subset I_X$, $I(f^{-1})_\sim \subset I_Y$ and $f_\sim([x_k, \pi_k]) = [y_k, \nu_k]$ for $[x_k, \pi_k] \notin I(f)_\sim$, and let $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ be the blowups of I_X and I_Y respectively. Then the blowups lift $f : X \to Y$ to a biholomorphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$, and $\widetilde{f}$ sends E_k^X to E_k^Y, where E_k^X and E_k^Y are the exceptional divisors over $[x_k, \pi_k] \notin I(f)_\sim$ and $[y_k, \nu_k] \notin I(f^{-1})_\sim$ respectively.

Hereafter, a surface $X_\sim$ is denoted simply by X, a point $[x, \pi]$ by x, a map $f_\sim$ by f, a curve $C_\sim$ by C, and a cluster $I(f)_\sim$ by $I(f)$ whenever no confusion arises.

Remark 2.10. — Under the above notations, we have $f(x_1) < f(x_2) \in Y$ for any $x_1 < x_2 \in X \setminus I(f)$. Hence, for an m-th point $x \in X \setminus I(f)$ on X, it is seen that $f(x)$ is an $(m - m_+ + m_-)$-th point on Y, where

$$m_+ := \max\{\ell \geqslant 0 \,|\, \exists (\ell - 1)\text{-th point } x_0 \in I(f) \text{ with } x \approx x_0\},$$

$$m_- := \max\{\ell \geqslant 0 \,|\, \exists (\ell - 1)\text{-th point } y_0 \in I(f^{-1}) \text{ with } f(x) \approx y_0\}.$$

In particular, for a proper point $x \in X \setminus I(f)$, the image $f(x)$ is also proper on Y if and only if there is no (proper) point $y \in I(f^{-1})$ with $f(x) \approx y$, where $f(x) \approx y$ is equivalent to $f(x) > y$ as $I(f^{-1})$ is a cluster.

3. Construction of Rational Surface Automorphisms

In this section, we develop a method for constructing a rational surface automorphism from a composition $f = f_n \circ \cdots \circ f_1 : \mathbb{P}^2 \to \mathbb{P}^2$ of quadratic birational maps $f_i : \mathbb{P}^2 \to \mathbb{P}^2$ and an orbit data τ. If τ is compatible with the maps $\overline{f} = (f_1, \dots, f_n)$, f lifts to an automorphism $F : X \to X$ through a blowup $\pi : X \to \mathbb{P}^2$. Moreover, we calculate the action $F^* : H^2(X; \mathbb{Z}) \to$

$H^2(X;\mathbb{Z})$ of the automorphism F, and also calculate a Weyl group element w_τ realized by the pair (π, F).

First we consider a rational surface X, that is, a surface birationally equivalent to $\mathbb{P}^2$, and an automorphism $F : X \to X$ of X. By theorems of Gromov and Yomdin, the *topological entropy of* F is given by $h_{\mathrm{top}}(F) = \log\lambda(F^*)$, where $\lambda(F^*)$ is the spectral radius of the action $F^* : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ on the cohomology group. In this paper, we are interested in the case where $F : X \to X$ has positive entropy $h_{\mathrm{top}}(F) > 0$ or, in other words, $\lambda(F^*) > 1$. Then, the surface X is characterized as follows (see [10, 12]).

Proposition 3.1. — *If X admits an automorphism $F : X \to X$ with $\lambda(F^*) > 1$, then there is a birational morphism $\pi : X \to \mathbb{P}^2$.*

It is known that any birational morphism $\pi : X \to \mathbb{P}^2$ is expressed as $\pi = \pi_K$ for some cluster $K = \{x_1, \dots, x_N\}$, where π_K is the blowup of K. Then $\pi : X \to \mathbb{P}^2$ gives an expression of the cohomology group: $H^2(X;\mathbb{Z}) \cong \mathrm{Pic}(X) = \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_N]$, where H is the total transform of a line in $\mathbb{P}^2$, and E_i is the exceptional divisor over the point x_i. The intersection form on $H^2(X;\mathbb{Z})$ is given by

$$\begin{cases} ([H],[H]) = 1 & \\ ([E_i],[E_j]) = -\delta_{i,j} & (i,j = 1,\dots,N) \\ ([H],[E_i]) = 0 & (i = 1,\dots,N), \end{cases}$$

where $\delta_{i,j}$ is the Kronecker delta. Therefore $H^2(X;\mathbb{Z})$ is isometric to the Lorentz lattice $\mathbb{Z}^{1,N}$ via the marking $\phi_\pi : \mathbb{Z}^{1,N} \to H^2(X;\mathbb{Z})$ given in (1.2). The marking ϕ_π is isometric and determined uniquely by $\pi : X \to \mathbb{P}^2$ in the sense that if ϕ_π and ϕ'_π are markings determined by π, then there is an element $\wp \in \langle \rho_1, \dots, \rho_{N-1} \rangle$, acting by a permutation on the basis elements $(e_1, \dots, e_N)$, such that $\phi_\pi = \phi'_\pi \circ \wp$, where ρ_i is given in (1.1). Moreover the Weyl group W_N plays an important role in our discussion as is mentioned in the following proposition (see [7, 9, 13]).

Proposition 3.2. — *For any birational morphism $\pi : X \to \mathbb{P}^2$ and any automorphism $F : X \to X$, there is a unique element $w \in W_N$ such that diagram* (1.3) *commutes.*

Thus, a pair (π, F) determines w uniquely, up to conjugacy by an element of $\langle \rho_1, \dots, \rho_{N-1} \rangle$. In this case, the element w is said to be *realized* by (π, F), and the entropy of F is expressed as $h_{\mathrm{top}}(F) = \log\lambda(w)$. Summing up these discussions, we have the following proposition.

Proposition 3.3. — *The entropy of any automorphism $F : X \to X$ on a rational surface X is given by $h_{\mathrm{top}}(F) = \log \lambda$ for some $\lambda \in \Lambda$, where Λ is given in* (1.5)*.*

Indeed, when $F : X \to X$ satisfies $\lambda(F^*) = 1$, the entropy of F is expressed as $h_{\mathrm{top}}(F) = \log \lambda(e)$ with the unit element $e \in W_N$.

Remark 3.4. — If $\pi : X \to \mathbb{P}^2$ is a blowup of N points with $N \leqslant 9$, and $F : X \to X$ is an automorphism, then one has $h_{\mathrm{top}}(F) = 0$ (see e.g. [11]).

Next we recall some properties of quadratic maps (see also [6]). Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a quadratic birational map on $\mathbb{P}^2$. It is known that f can be expressed as $f = l_- \circ g \circ l_+^{-1}$, where $l_+, l_- : \mathbb{P}^2 \to \mathbb{P}^2$ are linear transformations, and $g : \mathbb{P}^2 \to \mathbb{P}^2$ with $I(g^{\pm 1}) = \{p_1, p_2, p_3\}$ is a simple quadratic birational map given in exactly one of the following three cases:

Case 1: $g = g_1 : \mathbb{P}^2 \ni [x : y : z] \mapsto [yz : zx : xy] \in \mathbb{P}^2$,
$$\text{and} \quad \begin{cases} p_1 = [1 : 0 : 0] \\ p_2 = [0 : 1 : 0] \\ p_3 = [0 : 0 : 1], \end{cases}$$
Case 2: $g = g_2 : \mathbb{P}^2 \ni [x : y : z] \mapsto [xz : yz : x^2] \in \mathbb{P}^2$,
$$\text{and} \quad \begin{cases} p_1 = [0 : 1 : 0] \\ p_2 = [0 : 0 : 1] \\ p_3 > p_1, \end{cases}$$
Case 3: $g = g_3 : \mathbb{P}^2 \ni [x : y : z] \mapsto [x^2 : xy : y^2 + xz] \in \mathbb{P}^2$,
$$\text{and} \quad \begin{cases} p_1 = [0 : 0 : 1] \\ p_2 > p_1 \\ p_3 > p_2. \end{cases}$$

Let $\pi : X \to \mathbb{P}^2$ be the blowup of the cluster $\{p_1, p_2, p_3\}$, and let H be the total transform of a line in $\mathbb{P}^2$, L_1, L_2, L_3 be the strict transforms of the lines $x = 0, y = 0, z = 0$, respectively, and E_i be the exceptional divisor over the point p_i for $i = 1, 2, 3$. Then L_i is linearly equivalent to $H - E_j - E_k$ for $\{i, j, k\} = \{1, 2, 3\}$. The birational map g lifts to an automorphism $\widetilde{g} : X \to X$, which sends irreducible rational curves as follows:

Case 1: $\widetilde{g} = \widetilde{g}_1 : \ E_i \to L_i \quad (i \in \{1, 2, 3\})$,

Case 2: $\widetilde{g} = \widetilde{g}_2 : \begin{cases} E_1 - E_3 & \to & E_1 - E_3 \\ E_i & \to & L_i \quad (i \in \{2, 3\}), \end{cases}$

Case 3: $\widetilde{g} = \widetilde{g}_3 : \begin{cases} E_1 - E_2 & \to & E_1 - E_2 \\ E_2 - E_3 & \to & E_2 - E_3 \\ E_3 & \to & L_3, \end{cases}$

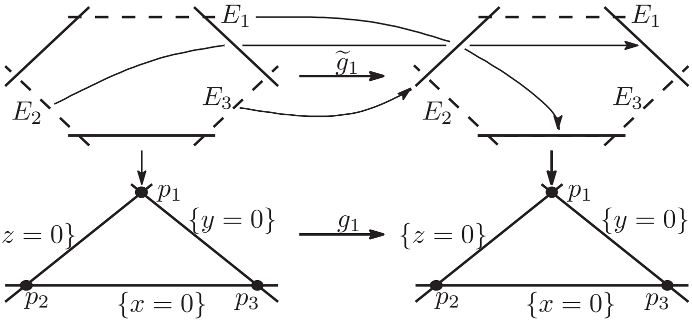

Figure 3.1. Geometry of $g_1 : \mathbb{P}^2 \to \mathbb{P}^2$

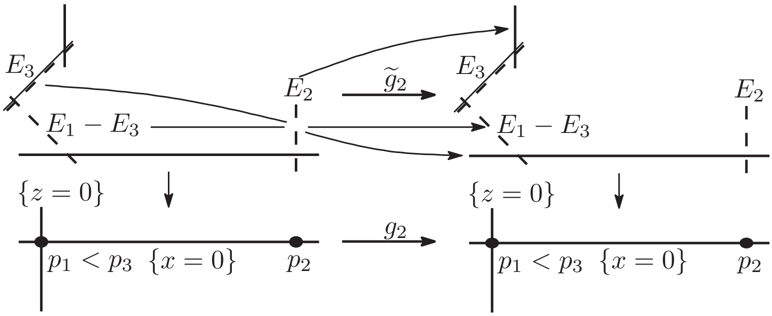

Figure 3.2. Geometry of $g_2 : \mathbb{P}^2 \to \mathbb{P}^2$

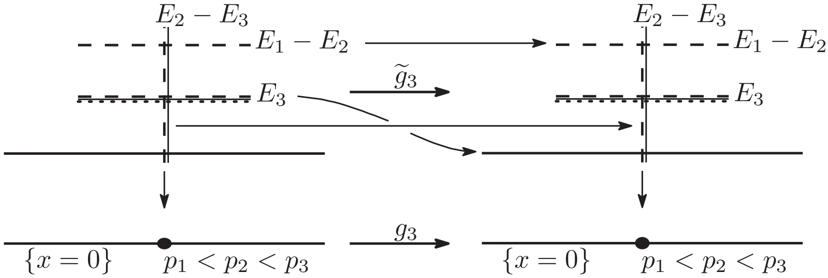

Figure 3.3. Geometry of $g_3 : \mathbb{P}^2 \to \mathbb{P}^2$

(see also Figures 3.1-3.3). Note that g sends a generic line to a conic passing through the three points p_1, p_2, p_3 in either case. Therefore, the action $\widetilde{g}^* : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ on the cohomology group $H^2(X;\mathbb{Z}) = \mathbb{Z}[H] \oplus$

$\mathbb{Z}[E_1] \oplus \mathbb{Z}[E_2] \oplus \mathbb{Z}[E_3]$ is given by

$$\widetilde{g}^* : \begin{cases} [H] & \mapsto 2[H] - \sum_{i=1}^{3}[E_i] \\ [E_i] & \mapsto [H] - [E_j] - [E_k] \quad (\{i,j,k\} = \{1,2,3\}). \end{cases} \tag{3.1}$$

For a general quadratic birational map $f = l_- \circ g \circ l_+^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$, put

$$p_i^{\pm} = l_{\pm}(p_1), \quad p_j^{\pm} = l_{\pm}(p_2), \quad p_k^{\pm} = l_{\pm}(p_3) \tag{3.2}$$

with $\{i,j,k\} = \{1,2,3\}$. Then one has

$$I(f^{\pm 1}) = \{p_1^{\pm}, p_2^{\pm}, p_3^{\pm}\}. \tag{3.3}$$

Moreover, formula (3.1) leads to the following lemma, which is stated in a general situation (see also Remark 2.9).

Lemma 3.5. — *In the above notations, assume that $I_{\pm} := \{p_1^{\pm}, \dots, p_N^{\pm}\}$ are clusters satisfying $f(p_i^+) = p_i^-$ for any $i = 4, \dots, N$. Let $\pi^{\pm} : X^{\pm} \to \mathbb{P}^2$ be the blowups of $I_{\pm}$, and let $H^{\pm} \subset X^{\pm}$ be the total transforms of lines in $\mathbb{P}^2$ under $\pi^{\pm}$, and $E_i^{\pm} \subset X^{\pm}$ be the exceptional divisors over the points $p_i^{\pm}$. Then, the quadratic birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ lifts to an isomorphism $\widetilde{f} : X^+ \to X^-$ and its cohomological action $\widetilde{f}^* : H^2(X^-;\mathbb{Z}) \to H^2(X^+;\mathbb{Z})$ is given by*

$$\widetilde{f}^* : \begin{cases} [H^-] & \mapsto 2[H^+] - \sum_{i=1}^{3}[E_i^+] \\ [E_i^-] & \mapsto [H^+] - [E_j^+] - [E_k^+] \quad (\{i,j,k\} = \{1,2,3\}) \\ [E_\ell^-] & \mapsto [E_\ell^+] \quad (\ell = 4, \dots, N). \end{cases} \tag{3.4}$$

Remark 3.6. — From here on, we assume that a quadratic birational map $f = l_- \circ g \circ l_+^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ lifts to $\widetilde{f} : X^+ \to X^-$ whose cohomological action is given as in (3.4). Then the points $p_i^{\pm}$ given in (3.3) are expressed as (3.2) for some $\{i,j,k\} = \{1,2,3\}$, and hence the indices of the forward indeterminacies are determined uniquely by those of the backward indeterminacies and vice versa. In particular, it follows that $p_i^+ < p_j^+$ if and only if $p_i^- < p_j^-$.

Next we turn our attention to a method for constructing rational surface automorphisms in a general context. Let $Y_1, \dots, Y_n$ be rational surfaces, and $\overline{f} := (f_1, \dots, f_n)$ be an n-tuple of birational maps $f_\ell : Y_{\ell-1} \to Y_\ell$ with $Y_0 := Y_n$. Let $I(f_\ell) = \{p_{\ell,1}^+, \dots, p_{\ell,\eta_+(\ell)}^+\} \subset Y_{\ell-1}$ and $I(f_\ell^{-1}) = \{p_{\ell,1}^-, \dots, p_{\ell,\eta_-(\ell)}^-\} \subset Y_\ell$ be the clusters of indeterminacy points, and let $\mathcal{K}_{\pm} := \{\iota = (i,j) \,|\, i = 1, \dots, n,\ j = 1, \dots, \eta_{\pm}(i)\}$ be the sets of indices. Then it turns out that the cardinalities of the sets $\mathcal{K}_{\pm}$ are the same, that is, $\sum_{\ell=1}^{n} \eta_+(\ell) = \sum_{\ell=1}^{n} \eta_-(\ell)$, since $Y_n = Y_0$. Moreover, for $m \geqslant 0$ and $\iota = (i,j) \in \mathcal{K}_-$, we inductively put

$$p_\iota^0 := p_\iota^- \in Y_i, \qquad p_\iota^m := f_\ell(p_\iota^{m-1}) \in Y_\ell \quad (\ell \equiv i+m \pmod{n}).$$

Note that a point p_ι^m is well-defined if $p_\iota^{m-1} \notin I(f_\ell)$. Moreover, let us introduce a *generalized orbit data* $\tau = (n, \sigma, \kappa)$ consisting of the integer $n \geqslant 1$, a bijection $\sigma : \mathcal{K}_- \to \mathcal{K}_+$ and a function $\kappa : \mathcal{K}_- \to \mathbb{Z}_{\geqslant 0}$ such that $\kappa(\iota) \geqslant 1$ provided $i_1 \leqslant i$, or in other words, a function κ satisfying $\mu(\iota) \geqslant 0$ for any $\iota \in \mathcal{K}_-$, where $\sigma(\iota) = \iota_1 = (i_1, j_1)$ and $\mu : \mathcal{K}_- \to \mathbb{Z}_{\geqslant 0}$ is given by

$$\mu(\iota) = \kappa(\iota) \cdot n + i_1 - i - 1 = \theta_{i,i_1-1}(\kappa(\iota)) \tag{3.5}$$

with

$$\theta_{i,i'}(k) := k \cdot n + i' - i. \tag{3.6}$$

Definition 3.7. — *Let $\overline{f}$ be an n-tuple of birational maps and $\tau = (n, \sigma, \kappa)$ be a generalized orbit data. Then $\overline{f}$ is called a realization of τ if the following condition holds for any $\iota \in \mathcal{K}_-$:*

$$p_\iota^m \neq p_{\iota'}^+ \quad (0 \leqslant m < \mu(\iota),\ \iota' \in \mathcal{K}_+), \qquad p_\iota^{\mu(\iota)} = p_{\sigma(\iota)}^+. \tag{3.7}$$

It should be noted that in condition (3.7), two points p_ι^m and $p_{\iota'}^+$ may satisfy $p_\iota^m \approx p_{\iota'}^+$. From a realization $\overline{f}$ of τ, we will construct an automorphism. So let us give the following lemma.

Lemma 3.8. — *There is an element $\iota^o \in \mathcal{K}_-$ such that $p_{\iota^o}^m$ is proper for any $0 \leqslant m \leqslant \mu(\iota^o)$.*

Proof. — Take an element $\iota^o = (i^o, j^o) \in \mathcal{K}_-$ such that $p_{\iota^o}^-$ is proper and

$$\mu(\iota^o) = \min\{\mu(\iota) \mid \iota \in \mathcal{K}_- \text{ and } p_\iota^- \text{ is proper}\}.$$

Then, we claim that $p_{\iota^o}^m$ is proper for any $0 < m \leqslant \mu(\iota^o)$. Indeed, assume the contrary that $p_{\iota^o}^{m-1}$ is proper but $p_{\iota^o}^m$ is not proper for some $0 < m \leqslant \mu(\iota^o)$. Then it follows from Remark 2.10 that there is a proper point p_ι^- such that $p_\iota^- < p_{\iota^o}^m$. The minimality of $\mu(\iota^o)$ yields $p_\iota^k \notin I(f_\ell)$ for any $0 \leqslant k < \mu(\iota^o) - m$, and so $p_\iota^{\mu(\iota^o)-m} < p_{\iota^o}^{\mu(\iota^o)}$ by Remark 2.10. Since $p_{\iota^o}^{\mu(\iota^o)} = p_{\sigma(\iota^o)}^+ \in I(f_{i_1^o})$ and $I(f_{i_1^o})$ is a cluster, $p_\iota^{\mu(\iota^o)-m}$ is also an element of $I(f_{i_1^o})$ and thus is equal to $p_{\sigma(\iota)}^+$. This means that $\mu(\iota) = \mu(\iota^o) - m < \mu(\iota^o)$, which contradicts the assumption that $\mu(\iota^o)$ is minimal. Thus, $p_{\iota^o}^m$ is proper for any $0 \leqslant m \leqslant \mu(\iota^o)$. □

For $\iota^o = (i^o, j^o) \in \mathcal{K}_-$ given as in Lemma 3.8, let $Y'_\ell \to Y_\ell$ be the blowups of distinct proper points $\{p_{\iota^o}^m \mid 0 \leqslant m \leqslant \mu(\iota^o),\ i^o + m \equiv \ell \pmod{n}\}$. These

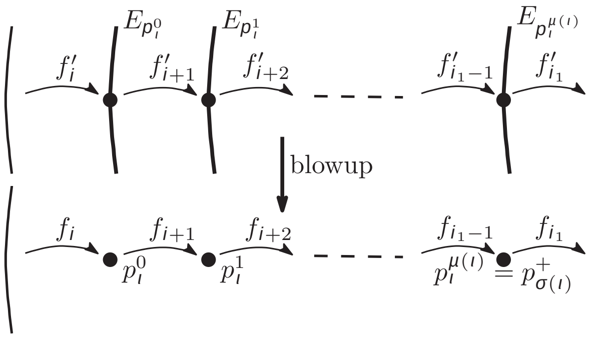

Figure 3.4. Blowup of indeterminacy points

blowups lift $f_\ell : Y_{\ell-1} \to Y_\ell$ to $f'_\ell : Y'_{\ell-1} \to Y'_\ell$ (see Figure 3.4). In this case, one has

(3.8)
$$I(f'_\ell) = \begin{cases} I(f_{i_1^o}) \setminus \{p^+_{\iota_1^o}\} & (\ell = i_1^o) \\ I(f_\ell) & (\ell \neq i_1^o), \end{cases}$$
$$I((f'_\ell)^{-1}) = \begin{cases} I(f^{-1}_{i^o}) \setminus \{p^-_{\iota^o}\} & (\ell = i^o) \\ I(f^{-1}_\ell) & (\ell \neq i^o). \end{cases}$$

We notice that $\overline{f}' = (f'_1, \dots, f'_n)$ also satisfies condition (3.7) for any $\iota \in \mathcal{K}'_- := \mathcal{K}_- \setminus \{\iota^o\}$, which means that $\overline{f}'$ is a realization of $\tau' = (n, \sigma|_{\mathcal{K}'_-}, \mu|_{\mathcal{K}'_-})$. One can therefore repeat the above argument by replacing $\overline{f}$ with $\overline{f}'$, $\mathcal{K}_-$ with $\mathcal{K}'_-$ and τ with τ'. In the end, from (3.8), the resulting map becomes a biholomorphism. Namely, let $\pi_\ell : X_\ell \to Y_\ell$ be the composition of the above blowups, that is, the blowup of the cluster $I_\ell := \{p_\iota^m \mid \iota \in \mathcal{K}_-, 0 \leqslant m \leqslant \mu(\iota), i + m \equiv \ell \pmod{n}\}$. Then the blowups π_ℓ lift $f_\ell : Y_{\ell-1} \to Y_\ell$ to biholomorphisms $F_\ell : X_{\ell-1} \to X_\ell$:

$$\begin{array}{ccc} X_{\ell-1} & \xrightarrow{F_\ell} & X_\ell \\ {\scriptstyle \pi_{\ell-1}}\downarrow & & \downarrow{\scriptstyle \pi_\ell} \\ Y_{\ell-1} & \xrightarrow{f_\ell} & Y_\ell, \end{array}$$

and $\pi_\tau := \pi_n : X_\tau \to Y$ also lifts $f := f_n \circ \cdots \circ f_1 : Y \to Y$ to the automorphism $F_\tau := F_n \circ \cdots \circ F_1 : X_\tau \to X_\tau$, where $Y := Y_0 = Y_n$ and $X_\tau := X_0 = X_n$.

We now restrict our attention to the case where each component of $\overline{f} = (f_1, \dots, f_n)$ is a quadratic birational map with $Y_\ell = \mathbb{P}^2_\ell$ and τ is an original orbit data with $\mathcal{K}_+ = \mathcal{K}_- = \mathcal{K}(n)$. Note that f_ℓ satisfies the assumption in Lemma 3.5 with $I_+ = I_{\ell-1}$ and $I_- = I_\ell$. Therefore, the cohomological action of the biholomorphism $F_\ell : X_{\ell-1} \to X_\ell$ is expressed as in the form of (3.4), and that of the automorphism $F_\tau : X_\tau \to X_\tau$ can be calculated in terms of the composition $F_\tau^* = F_1^* \circ \cdots \circ F_n^*$. The Weyl group element $w_\tau \in W_N$ realized by (π_τ, F_τ) is given as follows.

Let $H \subset X_\tau$ be the total transform of a line in $\mathbb{P}^2$ under π_τ and $E_\iota^k \subset X_\tau$ be the exceptional divisor over p_ι^m, where $m = \theta_{i,0}(k)$. Then the cohomology group of X_τ is expressed as $H^2(X_\tau;\mathbb{Z}) = \mathbb{Z}[H] \oplus \big(\oplus_{\iota\in\mathcal{K}(n)} \oplus_{k=1}^{\kappa(\iota)} \mathbb{Z}[E_\iota^k]\big)$. Now, we consider the lattice

$$\mathbb{Z}^\tau := \mathbb{Z}e_0 \oplus \big(\oplus_{\iota\in\mathcal{K}(n)} \oplus_{k=1}^{\kappa(\iota)} \mathbb{Z}e_\iota^k \big) \cong \mathbb{Z}^{1,N} \qquad \Big(N = \sum_{\iota\in\mathcal{K}(n)} \kappa(\iota)\Big),$$

with the inner product given by

$$\begin{cases} (e_0, e_0) = 1 & \\ (e_\iota^k, e_\iota^k) = -1 & (\iota \in \mathcal{K}(n), \quad 1 \leqslant k \leqslant \kappa(\iota)) \\ (e_0, e_\iota^k) = (e_\iota^k, e_{\iota'}^{k'}) = 0 & ((\iota, k) \neq (\iota', k')). \end{cases}$$

Then an isomorphism $\phi_{\pi_\tau} : \mathbb{Z}^{1,N} \to H^2(X_\tau : \mathbb{Z})$ defined by $\phi_{\pi_\tau}(e_0) = [H]$ and $\phi_{\pi_\tau}(e_\iota^k) = [E_\iota^k]$ is the marking corresponding to π_τ. Moreover, for each $\iota \in \mathcal{K}(n)$, put $\bar\sigma(\iota) := \sigma^k(\iota)$, where $k \geqslant 0$ is determined by the relations $\kappa(\sigma^\ell(\iota)) = 0$ for $0 \leqslant \ell < k$, and $\kappa(\sigma^k(\iota)) \geqslant 1$. Then an automorphism $r_\tau : \mathbb{Z}^\tau \to \mathbb{Z}^\tau$ is defined by

$$r_\tau : \begin{cases} e_0 & \mapsto & e_0 & \\ e^1_{\bar\sigma(\iota_1)} & \mapsto & e_\iota^{\kappa(\iota)} & (\kappa(\iota) \geqslant 1) \\ e_\iota^k & \mapsto & e_\iota^{k-1} & (2 \leqslant k \leqslant \kappa(\iota)), \end{cases}$$

Note that the map $\iota \mapsto \bar\sigma(\iota_1) = \bar\sigma(\sigma(\iota))$ becomes a permutation of $\{\iota \in \mathcal{K}(n) \,|\, \kappa(\iota) \geqslant 1\}$, and so $e^1_{\bar\sigma(\iota_1)}$ is well-defined. The automorphism r_τ is an element of the subgroup $\langle \rho_1, \ldots, \rho_{N-1}\rangle \subset W_N$ generated by $\rho_1, \ldots, \rho_{N-1}$. On the other hand, for $1 \leqslant m \leqslant n$, an automorphism $q_m : \mathbb{Z}^\tau \to \mathbb{Z}^\tau$ is defined by

$$\text{(3.9)} \quad q_m : \begin{cases} e_0 & \mapsto & 2e_0 - \sum_{\ell=1}^3 e^1_{\bar\sigma(m,\ell)} & \\ e^1_{\bar\sigma(m,i)} & \mapsto & e_0 - e^1_{\bar\sigma(m,j)} - e^1_{\bar\sigma(m,k)} & (\{i,j,k\} = \{1,2,3\}) \\ e_\iota^k & \mapsto & e_\iota^k & \text{(otherwise)}. \end{cases}$$

The automorphism q_m is conjugate to ρ_0 under the action of $\langle \rho_1, \ldots, \rho_{N-1}\rangle$. We notice that if $i \neq j \in \{1,2,3\}$ then $\bar\sigma(m,i) \neq \bar\sigma(m,j)$. Indeed, assume the contrary that $\bar{\sigma}(m,i)=\bar{\sigma}(m,j)$. Let $k_i \geqslant 0$ be the integer determined by the relations $\sigma^{k_i}(m,i)=\bar{\sigma}(m,i)$ and $\kappa(\sigma^{\ell}(\iota))=0$ for $0\leqslant \ell < k_i$. One may assume that $k_i > k_j$ and thus $(m,j)=\sigma^k(m,i)$ with $k=k_i-k_j$. As $(m_\ell,i_\ell):=\sigma^\ell(m,i)$ satisfies $\kappa(m_\ell,i_\ell)=0$ for $0\leqslant \ell \leqslant k-1 \leqslant k_i-1$, we have $m=m_0<m_1<\cdots<m_k=m$, which is a contradiction.

Now we define the lattice automorphism $w_\tau : \mathbb{Z}^{1,N}\to\mathbb{Z}^{1,N}$.

Definition 3.9. — *For an orbit data τ, we define the lattice automorphism $w_\tau : \mathbb{Z}^\tau \to \mathbb{Z}^\tau$ by*

$$w_\tau := r_\tau \circ q_1 \circ \cdots \circ q_n : \mathbb{Z}^\tau \to \mathbb{Z}^\tau.$$

We sometimes write $w_\tau : \mathbb{Z}^{1,N}\to\mathbb{Z}^{1,N}$.

Indeed, it will be seen that $w_\tau \in W_N$ is realized by (π_τ, F_τ), that is, $\phi_{\pi_\tau}\circ w_\tau = F_\tau^*\circ\phi_{\pi_\tau} : \mathbb{Z}^{1,N}\to H^2(X_\tau;\mathbb{Z})$. Summing up these discussions, we have the following proposition.

Proposition 3.10. — *Assume that $\overline{f}$ is a realization of τ. Then the blowup $\pi_\tau : X_\tau \to \mathbb{P}^2$ of $N=\sum_{\iota\in\mathcal{K}(n)}\kappa(\iota)$ points $\{p_\iota^m \,|\, \iota=(i,j)\in\mathcal{K}(n),\, m=\theta_{i,0}(k), 1\leqslant k\leqslant\kappa(\iota)\}$ lifts $f=f_n\circ\cdots\circ f_1$ to the automorphism $F_\tau : X_\tau\to X_\tau$. Moreover, (π_τ,F_τ) realizes w_τ and F_τ has positive entropy $h_{\mathrm{top}}(F_\tau)=\log\lambda(w_\tau)>0$.*

Proof. — We will only show that (π_τ,F_τ) realizes the Weyl group element w_τ given in Definition 3.9. For the blowup $\pi_\ell : X_\ell\to\mathbb{P}^2_\ell$, let $H_\ell\subset X_\ell$ be the total transform of a line in $\mathbb{P}^2_\ell$ and, for $k\geqslant 1$, $E^k_{(i,j),\ell}\subset X_\ell$ be the exceptional divisor over the point $p^m_{i,j}$ with

$$m=\begin{cases}\theta_{i,\ell}(k-1) & (i\leqslant \ell)\\ \theta_{i,\ell}(k) & (i>\ell).\end{cases}$$

The cohomology group of X_ℓ admits an expression

$$H^2(X_\ell;\mathbb{Z})=\mathbb{Z}[H_\ell]\oplus\big(\oplus_{\iota\in\mathcal{K}(n)}\oplus_{k=1}^{\kappa(\iota,\ell)}\mathbb{Z}[E^k_{\iota,\ell}]\big),$$

where $\kappa(\iota,\ell)$ is the number of points among $p^0_\iota,p^1_\iota,\dots,p^{\mu(\iota)}_\iota$ lying on $\mathbb{P}^2_\ell$. Since the indeterminacy sets of $f_\ell^{\pm1}$ are expressed as $I(f_\ell^{-1})=\{p^0_{\ell,j}\,|\,j=1,2,3\}$ and $I(f_\ell)=\{p^{\mu(\sigma^{-1}(\ell,j))}_{\sigma^{-1}(\ell,j)}\,|\,j=1,2,3\}$, the action $F_\ell^*: H^2(X_\ell;\mathbb{Z})\to$

$H^2(X_{\ell-1};\mathbb{Z})$ is given by

$$F_\ell^* : \begin{cases} [H_\ell] & \mapsto & 2[H_{\ell-1}] - \sum_{j=1}^3 [E_{\sigma^{-1}(\ell,j),\ell-1}^{\kappa(\sigma^{-1}(\ell,j),\ell-1)}] \\ [E_{(\ell,i),\ell}^1] & \mapsto & [H_{\ell-1}] - [E_{\sigma^{-1}(\ell,j),\ell-1}^{\kappa(\sigma^{-1}(\ell,j),\ell-1)}] - [E_{\sigma^{-1}(\ell,k),\ell-1}^{\kappa(\sigma^{-1}(\ell,k),\ell-1)}] \\ & & \qquad (\{i,j,k\} = \{1,2,3\}) \\ [E_{(\ell,j),\ell}^k] & \mapsto & [E_{(\ell,j),\ell-1}^{k-1}] \qquad (j \in \{1,2,3\},\ k \geqslant 2) \\ [E_{(i,j),\ell}^k] & \mapsto & [E_{(i,j),\ell-1}^k] \qquad (i \neq \ell,\ j \in \{1,2,3\},\ k \geqslant 1) \end{cases}$$

(see Lemma 3.5). Now, for $\ell \geqslant 1$, an isomorphism $G_\ell : H^2(X_\ell;\mathbb{Z}) \to H^2(X_\tau;\mathbb{Z})$ is defined by $G_\ell([H_\ell]) = [H]$ and $G_\ell([E_{\iota,\ell}^k]) = [E_{\iota'}^{k'}]$, where $E_{\iota'}^{k'}$ is the exceptional divisor over $p_\iota^{m+n-\ell} \in \mathbb{P}_n^2$ if $E_{\iota,\ell}^k$ is the exceptional divisor over $p_\iota^m \in \mathbb{P}_\ell^2$. In this definition, p_ι^m should be interpreted as $p_{\sigma(\iota)}^{m-\mu(\iota)-1}$ provided $m > \mu(\iota)$. The isomorphism G_ℓ sends $[E_{\iota,\ell}^k]$ as

$$G_\ell([E_{\iota,\ell}^k]) = \begin{cases} [E_{\bar\sigma(\iota_1)}^1] & (\text{ if } i_1 > \ell \text{ and } k = \kappa(\iota,\ell)) \\ [E_\iota^{k+1}] & (\text{ otherwise, if } i > \ell) \\ [E_\iota^k] & (\text{ otherwise }). \end{cases}$$

Since $G_n = \mathrm{id}$, the action F_τ^* is expressed as

$$F_\tau^* = F_1^* \circ \cdots \circ F_n^* = (F_1^* \circ G_1^{-1}) \circ \widehat{F}_2^* \circ \cdots \circ \widehat{F}_n^*,$$

where $\widehat{F}_\ell^* = G_{\ell-1} \circ F_\ell^* \circ G_\ell^{-1}$. It should be noted that

$$\begin{aligned} G_{\ell-1}([E_{\sigma^{-1}(\ell,j),\ell-1}^{\kappa(\sigma^{-1}(\ell,j),\ell-1)}]) &= G_\ell([E_{(\ell,j),\ell}^1]) \\ &= \begin{cases} [E_{\bar\sigma(\sigma(\ell,j))}^1] & (\text{ if } \ell_1 > \ell \text{ and } \kappa((\ell,j),\ell) = 1) \\ [E_{(\ell,j)}^1] & (\text{ otherwise }) \end{cases} \\ &= [E_{\bar\sigma(\ell,j)}^1], \end{aligned}$$

since the conditions $\ell_1 > \ell$ and $\kappa((\ell,j),\ell) = 1$ are equivalent to saying that $\kappa(\ell,j) = 0$, where $\sigma(\ell,j) = (\ell_1,j_1)$. Therefore, for $\ell \geqslant 2$, one has

(3.10)

$$\widehat{F}_\ell^* : \begin{cases} [H] & \mapsto & 2[H] - \sum_{j=1}^3 [E_{\bar\sigma(\ell,j)}^1] \\ [E_{\bar\sigma(\ell,i)}^1] & \mapsto & [H] - [E_{\bar\sigma(\ell,j)}^1] - [E_{\bar\sigma(\ell,k)}^1] \qquad (\{i,j,k\} = \{1,2,3\}) \\ [E_{(i,j)}^k] & \mapsto & [E_{(i,j)}^k] \qquad (\text{ otherwise }), \end{cases}$$

as $G_{\ell-1}([E^{k-1}_{(\ell,j),\ell-1}]) = G_\ell([E^k_{(\ell,j),\ell}])$ and $G_{\ell-1}([E^k_{(i,j),\ell-1}]) = G_\ell([E^k_{(i,j),\ell}])$ when $i \neq \ell$. Finally, by observing that

$$F_1^* \circ G_1^{-1} : \begin{cases} [H] & \mapsto & 2[H] - \sum_{j=1}^3 [E^{\kappa(\sigma^{-1}(1,j))}_{\sigma^{-1}(1,j)}] \\ [E^1_{\bar\sigma(1,i)}] & \mapsto & [H] - [E^{\kappa(\sigma^{-1}(1,j))}_{\sigma^{-1}(1,j)}] - [E^{\kappa(\sigma^{-1}(1,k))}_{\sigma^{-1}(1,k)}] \\ & & \qquad (\{i,j,k\} = \{1,2,3\}) \\ [E^1_{\bar\sigma(\sigma(i,j))}] & \mapsto & [E^{\kappa(i,j)}_{(i,j)}] \quad (i_1 \neq 1) \\ [E^k_{(i,j)}] & \mapsto & [E^{k-1}_{(i,j)}] \quad (\text{otherwise}), \end{cases}$$

we define an isomorphism $G_0 : H^2(X_\tau;\mathbb{Z}) \circlearrowleft$ by

$$G_0 : \begin{cases} [H] & \mapsto & [H] \\ [E^{\kappa(\iota)}_\iota] & \mapsto & [E^1_{\bar\sigma(\iota_1)}] \quad (\kappa(\iota) \geqslant 1) \\ [E^k_\iota] & \mapsto & [E^{k+1}_\iota] \quad (1 \leqslant k \leqslant \kappa(\iota) - 1). \end{cases} \tag{3.11}$$

Then, $\widehat{F}_1^* = G_0 \circ F_1^* \circ G_1^{-1}$ satisfies (3.10) with $\ell = 1$, and F_τ^* satisfies $F_\tau^* = G_0^{-1} \circ \widehat{F}_1^* \circ \cdots \circ \widehat{F}_n^*$. From (3.10) and (3.11), one has $\widehat{F}_\ell^* = \phi_{\pi_\tau} \circ q_\ell \circ \phi_{\pi_\tau}^{-1}$ and $G_0^{-1} = \phi_{\pi_\tau} \circ r_\tau \circ \phi_{\pi_\tau}^{-1}$, which shows that w_τ is realized by (π_τ, F_τ). $\square$

We conclude this section by establishing the statement that a given element $w \in W_N$ can be expressed as $w = w_\tau$ for some orbit data τ. To this end, we spend a short while working with an element $w \in W_N$ not acting by a permutation on a non-empty subset of the basis elements $\{e_j\}_{j=1}^N$, namely, there is no element $e_j \in \{e_i\}$ with $w^\ell(e_j) \in \{e_i\}$ for any $\ell \geqslant 1$, and explain how to construct an orbit data τ with $w = w_\tau$ briefly. First, note that w can be expressed as

$$w = r \circ q_1 \circ \cdots q_n, \tag{3.12}$$

where r is a permutation of $\{e_j\}_{j=1}^N$ (see the proof of Proposition 3.12). Moreover, there are elements $\{e^1_{m,i}\}_{i=1}^3 \subset \{e_j\}_{j=1}^N$ such that q_m sends $\{e_j\}_{j=0}^N$ as

$$q_m : \begin{cases} e_0 & \mapsto & 2e_0 - \sum_{\ell=1}^3 e^1_{m,\ell} \\ e^1_{m,i} & \mapsto & e_0 - e^1_{m,j} - e^1_{m,k} \quad (\{i,j,k\} = \{1,2,3\}) \\ e_j & \mapsto & e_j \quad (\text{otherwise}). \end{cases}$$

We notice that it may happen that $e^1_{i,j} = e^1_{i',j'}$ when $i \neq i'$. For each $(i,j) \in \mathcal{K}(n)$, it turns out that there is a unique element (i',j') such that

$$e^1_{i',j'} = q_{i'-1} \circ \cdots \circ q_1 \circ r^{-1} \circ q_n \circ \cdots \circ q_1 \circ r^{-1} \circ q_n \circ \cdots \circ q_{i+1}(e^1_{i,j}) \tag{3.13}$$

with minimal length, where the length is the number of automorphisms q_m in (3.13). Roughly speaking, $e^1_{i,j}$ and $e^1_{i',j'}$ are regarded as backward and forward indeterminacy points respectively, and the length is the number of points from $e^1_{i,j}$ to $e^1_{i',j'}$. So we define $\sigma(i,j) = (i',j')$ and $\mu(i,j)$ to be the length in (3.13). Note that $\kappa(i,j)$ is the number of r^{-1} in (3.13), and σ becomes a permutation of $\mathcal{K}(n)$ because of the minimality of the length. Then $\tau = (n,\sigma,\kappa)$ is an orbit data satisfying $w = w_\tau$.

Example 3.11. — Consider the element $w \in W_5$ given by

$$w : \begin{cases} e_0 \mapsto 3e_0 - 2e_1 - e_2 - e_3 - e_4 - e_5 \\ e_1 \mapsto 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 \\ e_i \mapsto e_0 - e_1 - e_{7-i} \qquad (i \in \{2,3,4,5\}). \end{cases}$$

It can be checked that w is expressed as $w = r \circ q_1 \circ q_2$, where

$$r : e_2 \longleftrightarrow e_4, \quad e_3 \longleftrightarrow e_5, \quad e_i \circlearrowleft (i = 0,1),$$

and $(e_{1,1}, e_{1,2}, e_{1,3}) = (e_1, e_2, e_3)$, $(e_{2,1}, e_{2,2}, e_{2,3}) = (e_1, e_4, e_5)$. Then it is seen that

$$\begin{aligned} &e_{2,1} = e_{1,1}, \quad e_{1,k} = r^{-1}(e_{2,k}), \quad (k \in \{1,2,3\}), \\ &e_{2,j} \neq e_{1,k}, \quad e_{1,j} \neq r^{-1} \circ q_2(e_{1,k}), \quad e_{2,k} = q_1 \circ r^{-1} \circ q_2(e_{1,k}), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (j \in \{1,2,3\},\ k \in \{2,3\}). \end{aligned}$$

This means that w admits an expression $w = w_\tau$, where $\tau = (2,\sigma,\kappa)$ is given by

$$\begin{cases} \sigma : (1,k) \mapsto (2,k) \mapsto (1,k), \quad (k \in \{1,2,3\}), \\ \kappa(1,1) = 0, \qquad \kappa(i,j) = 1 \text{ (otherwise)}, \\ (\mu(1,1) = \mu(2,1) = \mu(2,2) = \mu(2,3) = 0, \quad \mu(1,2) = \mu(1,3) = 2). \end{cases}$$

Proposition 3.12. — *For any $w \in W_N$, there is an orbit data $\tau = (n,\sigma,\kappa)$ such that $w = w_\tau : \mathbb{Z}^{1,N} \to \mathbb{Z}^{1,N}$ under some identification $\{e_j \mid 1 \leqslant j \leqslant N\} = \{e^k_\iota \mid \iota \in \mathcal{K}(n),\ 1 \leqslant k \leqslant \kappa(\iota)\}$ with $N = \sum_{\iota \in \mathcal{K}(n)} \kappa(\iota)$.*

Proof. — First we prove that any element w admits expression (3.12). Since w is an element of W_N, it can be expressed as

$$w = r_0 \cdot \rho_0 \cdot r_1 \cdots \rho_0 \cdot r_{m-1} \cdot \rho_0 \cdot r_m,$$

where r_ℓ is a permutation of $\{e_j\}_{j=1}^N$. The expression can be written as

$$\begin{aligned} w = r \cdot &\big\{(r_1 \cdots r_m)^{-1} \cdot \rho_0 \cdot (r_1 \cdots r_m)\big\} \\ &\cdots \big\{(r_{m-1} \cdot r_m)^{-1} \cdot \rho_0 \cdot (r_{m-1} \cdot r_m)\big\} \cdot \big\{r_m^{-1} \cdot \rho_0 \cdot r_m\big\}, \end{aligned}$$

where $r := r_0 \cdots r_m$ is also a permutation of $\{e_j\}$. By putting $q_i := (r_i \cdots r_m)^{-1} \cdot \rho_0 \cdot (r_i \cdots r_m)$ and $e^1_{i,j} := (r_i \cdots r_m)^{-1}(e_j)$ for $i = 1, \dots, m$ and $j = 1, 2, 3$, we have expression (3.12).

Next, under the assumption that w does not act by a permutation on a non-empty subset of $\{e_j\}_{j=1}^N$, we show that the orbit data τ constructed above realizes w. Put

$$e^{k+1}_{i,j} := q_n \circ \cdots \circ q_1 \circ r^{-1} \circ q_n \circ \cdots \circ q_1 \circ r^{-1} \circ q_n \circ \cdots \circ q_{i+1}(e^1_{i,j})$$

for $1 \leqslant k \leqslant \kappa(i,j) - 1$, where the number of r^{-1} in the righthand side is k. Since $e^1_{\sigma(i,j)} = e^1_{i,j}$ when $\kappa(i,j) = 0$, one has $e^1_{\bar{\sigma}(i,j)} = e^1_{i,j}$, which shows q_m sends $\{e^k_\iota\}$ as in (3.9). Moreover it follows from the minimality of the length in (3.13) that $e^k_{i,j} = r \circ q_1 \circ \cdots \circ q_n(e^{k+1}_{i,j}) = r \circ q_1 \circ \cdots \circ q_{n-1}(e^{k+1}_{i,j}) = \cdots = r(e^{k+1}_{i,j})$, and also that $e^{\kappa(i,j)}_{i,j} = r \circ q_1 \circ \cdots \circ q_{i'-1}(e^1_{\sigma(i,j)}) = r(e^1_{\sigma(i,j)}) = r(e^1_{\bar{\sigma}(\sigma(i,j))})$ for $\kappa(i,j) \geqslant 1$, which means that $r = r_\tau$. Now we claim that $\{e_j\} = \{e^k_\iota\}$, that is, any element $e_m \in \{e_j\}$ can be expressed as $e_m = e^k_\iota$ for some $\iota \in \mathcal{K}(n)$ and $1 \leqslant k \leqslant \kappa(\iota)$. Indeed, assume the contrary that $e_m \neq e^k_\iota$ for any $\iota \in \mathcal{K}(n)$ and $1 \leqslant k \leqslant \kappa(\iota)$. Then e_m satisfies $w^\ell(e_m) \notin \{e^1_\iota\}$ and thus $w^\ell(e_m) \in \{e_j\}$ for all $\ell \geqslant 1$, which is a contradiction. Moreover, we can easily check that $e^k_\iota \neq e^{k'}_{\iota'}$ for any $(\iota, k) \neq (\iota', k')$ with $1 \leqslant k \leqslant \kappa(\iota)$ and $1 \leqslant k' \leqslant \kappa(\iota')$, which shows that $N = \sum_{\iota \in \mathcal{K}(n)} \kappa(\iota)$. These observations show that w is expressed as $w = w_\tau$.

Finally we consider a general element $w \in W_N$. Then w can be expressed as $w = w_1 \cdot \hat{w}$, where w_1 does not act by a permutation on a non-empty subset of $\{e_j\}_{j=1}^N$ and hence admits an expression $w_1 = w_{\check{\tau}}$ for some orbit data $\check{\tau} = (m, \check{\sigma}, \check{\kappa})$, and $\hat{w}$ is a permutation of $\{e_j\}$, sending the basis elements, after reordering $\{e_j\}$, as

$$\hat{w} : \begin{cases} e_0 & \mapsto \quad e_0 \\ e^k_\iota & \mapsto \quad e^k_\iota \qquad (\iota \in \mathcal{K}(m), 1 \leqslant k \leqslant \check{\kappa}(\iota)) \\ e^k_{m+2i,1} & \mapsto \quad e^{k-1}_{m+2i,1} \quad (i = 1, \dots, \ell, \quad k \in \mathbb{Z}/\hat{\kappa}(i)\mathbb{Z}) \end{cases}$$

for some $\ell \geqslant 0$ and $\hat{\kappa} : \{1, \dots, \ell\} \to \mathbb{Z}_{\geqslant 1}$. Now, by composing automorphisms of the form $q_{m+2i-1} \circ q_{m+2i}$ with $e^1_{m+2i-1,j} = e^1_{m+2i,j}$ for all $i = 1, 2, 3$, which are also permutations of $\{e_j\}_{j=1}^N$, we add elements $e^1_{m+2i,1}$

and construct $\hat{w}$. Namely, $\tau = (n, \sigma, \kappa)$ is defined by $n := m + 2\ell$ and

$$\sigma(i,j) := \begin{cases} (i+1,j) & \\ \quad \begin{pmatrix} \text{either} & j=1 \text{ and } i = m+1, m+3, \dots, m+2\ell-1, \\ \text{or} & j=2,3 \text{ and } i = m, m+1, \dots, m+2\ell-1 \end{pmatrix} & \\ (i-1,j) & (j=1 \text{ and } i=m+2, m+4, \dots, m+2\ell) \\ \check{\sigma}(m,j) & (j=2,3 \text{ and } i=m+2\ell) \\ \check{\sigma}(i,j) & (\text{otherwise}), \end{cases}$$

$$\kappa(i,j) := \begin{cases} 0 \begin{pmatrix} \text{either} & j=1 \text{ and } i = m+1, m+3, \dots, m+2\ell-1, \\ \text{or} & j=2,3 \text{ and } i = m, m+1, \dots, m+2\ell-1 \end{pmatrix} & \\ \hat{\kappa}((i-m)/2) & (j=1 \text{ and } i=m+2, m+4, \dots, m+2\ell) \\ \check{\kappa}(m,j) & (j=2,3 \text{ and } i=m+2\ell) \\ \check{\kappa}(i,j) & (\text{otherwise}). \end{cases}$$

A straightforward calculation shows that $w = w_{\check{\tau}} \cdot \hat{w}$ can be expressed as $w = w_\tau = r_\tau \circ q_1 \circ \cdots \circ q_n$. Thus the proposition is established. ☐

4. Tentative Realizability

As mentioned in the previous section, an automorphism can be constructed in terms of a realization of an orbit data. At this stage, of particular interest is the existence of such a realization. In this and next sections, we investigate this existence by restricting our attention to birational maps preserving a cuspidal cubic. The aim of this section is to define a concept of tentative realization of an orbit data τ, which is a necessary condition for realization, and to show that the tentative realization of τ exists under the condition that some finitely many roots determined by τ are not periodic ones of the Weyl group element w_τ (see condition (4.9)).

In this section, we mainly consider the smooth points of the cuspidal cubic, which is also described in a more general context as follows. Let X be a surface, C be a curve in X, and x be a proper point of the smooth locus C^* of C. Moreover, put $(X_0, C_0^*, x_0) := (X, C^*, x)$, and for $m > 0$, inductively determine (X_m, C_m^*, x_m) from the blowup $\pi_m : X_m \to X_{m-1}$ of $x_{m-1} \in C_{m-1}^*$, the strict transform C_m^* of C_{m-1}^* under π_m, and a unique point $x_m \in C_m^* \cap E_m$, where E_m is the exceptional curve of π_m. The unique point x_m is called the *point in the m-th infinitesimal neighbourhood of x on C^**, or an *m-th point on C^**. Moreover, if a cluster I consists of proper or infinitely near points on C^*, then we say that I is *a cluster in C^**.

Now let C be a cubic curve on $\mathbb{P}^2$ with a cusp singularity. In what follows, a coordinate on $\mathbb{P}^2$ is chosen so that $C = \{[x : y : z] \in \mathbb{P}^2 \,|\, yz^2 = x^3\} \subset \mathbb{P}^2$ with a cusp $[0 : 1 : 0]$. Then the smooth locus $C^* = C \setminus \{[0 : 1 : 0]\}$ is parametrized as $\mathbb{C} \ni t \mapsto [t : t^3 : 1] \in C^*$. We denote by $\mathcal{B}(C)$ the set of birational self-maps f of $\mathbb{P}^2$ such that $f(C) := \overline{f(C \setminus I(f))} = C$ and $I(f) \subset C^*$, and denote by $\mathcal{Q}(C) \subset \mathcal{B}(C)$ and $\mathcal{L}(C) \subset \mathcal{B}(C)$ the subsets consisting of the quadratic maps in $\mathcal{B}(C)$ and of the linear maps in $\mathcal{B}(C)$, respectively. Any map $f \in \mathcal{B}(C)$ restricted to C^* is an automorphism of C^* expressed as

$$f|_{C^*} : C^* \ni [t : t^3 : 1] \mapsto [\delta(f) \cdot t + c(f) : (\delta(f) \cdot t + c(f))^3 : 1] \in C^*,$$

for some $\delta(f) \in \mathbb{C}^\times$ and $c(f) \in \mathbb{C}$. The value $\delta(f)$ is called the *determinant* of f. It is independent of the choice of coordinates. Moreover, when $f \in \mathcal{Q}(C)$, it turns out that the indeterminacy cluster $I(f^{-1})$ is also contained in C^* (see Lemma 4.2).

We give the following definition for an n-tuple $\overline{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ of quadratic birational maps f_i preserving C.

Definition 4.1. — *An n-tuple $\overline{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ is called a tentative realization of an orbit data $\tau = (n, \sigma, \kappa)$ if $p_\iota^{\mu(\iota)} \approx p^+_{\sigma(\iota)}$ for any $\iota \in \mathcal{K}(n)$, where p_ι^m is given in (1.7) with f_ℓ restricted to C^* and thus is well-defined, as $f_\ell|_{C^*}$ is an automorphism.*

We should note that a realization $\overline{f}$ of τ is of course a tentative realization of τ, and thus the existence of a tentative realization is of interest to us.

Now, we describe a quadratic birational map $f \in \mathcal{Q}(C)$ in terms of the behavior of $f|_{C^*}$. The following proposition states that the configuration of $I(f^{-1})$ on C^* and the determinant $\delta(f)$ of f determine the map $f \in \mathcal{Q}(C)$ uniquely (see [6, 11]).

Lemma 4.2. — *A birational map f belongs to $\mathcal{Q}(C)$ if and only if there exists $d \in \mathbb{C}^\times$ and $b = (b_\ell)_{\ell=1}^3 \in \mathbb{C}^3$ with $b_1 + b_2 + b_3 \neq 0$ such that f can be expressed as $f = f_{d,b}$, where $f_{d,b} \in \mathcal{Q}(C)$ is a unique map determined by the following properties.*

(1) $\delta(f_{d,b}) = d$.
(2) $p_\ell^- \approx [b_\ell : b_\ell^3 : 1] \in C^*$ *for* $I(f_{d,b}^{-1}) = \{p_1^-, p_2^-, p_3^-\}$.

Moreover, the map $f_{d,b} \in \mathcal{Q}(C)$ satisfies the following.

(1) $c(f_{d,b}) = -\frac{1}{3}(b_1 + b_2 + b_3) \in \mathbb{C}^\times$.
(2) $p_\ell^+ \approx [a_\ell : a_\ell^3 : 1] \in C^*$ *for* $I(f_{d,b}) = \{p_1^+, p_2^+, p_3^+\}$, *where* $a_\ell := \dfrac{1}{d}\left\{b_\ell - \dfrac{2}{3}(b_1 + b_2 + b_3)\right\}$.

Remark 4.3. — The quadratic map $f_{d,b}([x:y:z]) = [f_1 : f_2 : f_3]$ mentioned in Lemma 4.2 is explicitly written as

$$\begin{cases} f_1 = (d/3)\{(\nu_1^2 - 3\nu_2)x^2 + \nu_1\nu_3 z^2 - 3xy + 2\nu_1 yz - (\nu_1\nu_2 - 3\nu_3)zx\} \\ f_2 = (d/3)^3\{\nu_1(\nu_1^3 - 9\nu_1\nu_2 + 27\nu_3)x^2 - 27y^2 + \nu_1^3\nu_3 z^2 + 9(2\nu_1^2 - 3\nu_2)xy \\ \qquad + (8\nu_1^3 - 27\nu_1\nu_2 + 27\nu_3)yz - \nu_1^2(\nu_1\nu_2 - 9\nu_3)zx\} \\ f_3 = \nu_1 x^2 + \nu_3 z^2 - yz - \nu_2 zx, \end{cases}$$

where $\nu_\ell = \nu_\ell(d,b)$ is given by

$$\nu_1 = a_1 + a_2 + a_3, \quad \nu_2 = a_1a_2 + a_2a_3 + a_3a_1, \quad \nu_3 = a_1a_2a_3.$$

In a similar manner, any linear map $f \in \mathcal{L}(C)$ is determined uniquely by the determinant $\delta(f)$ of f (see [6]).

Lemma 4.4. — *For any $d \in \mathbb{C}^\times$, there is a unique linear map $f \in \mathcal{L}(C)$ such that $\delta(f) = d$. In particular, the map $f \in \mathcal{L}(C)$ with $\delta(f) = 1$ is the identity. Moreover, for any $f \in \mathcal{L}(C)$, the automorphism $f|_{C^*}$ restricted to C^* is given by*

$$f|_{C^*} : [t : t^3 : 1] \mapsto [\delta(f)\cdot t : (\delta(f)\cdot t)^3 : 1].$$

Next, let us consider the composition $f = f_n \circ f_{n-1} \circ \cdots \circ f_1 : \mathbb{P}^2 \to \mathbb{P}^2$ of quadratic birational maps $\overline{f} = (f_1, \ldots, f_n) \in \mathcal{Q}(C)^n$. Put $I(f_i^{\pm 1}) = \{p_{i,1}^\pm, p_{i,2}^\pm, p_{i,3}^\pm\}$ and

$$\check{p}_{i,j}^+ := f_1^{-1}|_C \circ \cdots \circ f_{i-1}^{-1}|_C(p_{i,j}^+), \qquad \check{p}_{i,j}^- := f_n|_C \circ \cdots \circ f_{i+1}|_C(p_{i,j}^-) \tag{4.1}$$

(see Figure 4.1). Then it is easy to see that $I(f^{\pm 1}) \subset \{\check{p}_{i,j}^\pm \mid (i,j) \in \mathcal{K}(n)\}$. Moreover, let $\delta(\overline{f})$ be the *determinant* of $\overline{f}$ defined by $\delta(\overline{f}) = \prod_{i=1}^n \delta(f_i)$ or, in other words, $\delta(\overline{f}) = \delta(f)$.

Proposition 4.5. — *Let $\overline{f} = (f_1, \ldots, f_n) \in \mathcal{Q}(C)^n$ be an n-tuple of quadratic birational maps in $\mathcal{Q}(C)$ with $d = \delta(\overline{f}) \neq 1$, and let $\check{p}_{i,j}^\pm$ be the points given in* (4.1)*. Then there is a unique pair (v, s) of values $v = (v_\iota)_{\iota \in \mathcal{K}(n)} \in \mathbb{C}^{3n}$ and $s = (s_i)_{i=1}^n \in (\mathbb{C}^\times)^n$ satisfying*

$$v_{i,1} + v_{i,2} + v_{i,3} = -\sum_{k=1}^{i-1} s_k + (d-2)\cdot s_i - d\sum_{k=i+1}^{n} s_k, \quad (1 \leqslant i \leqslant n), \tag{4.2}$$

such that the composition $f = f_n \circ \cdots \circ f_1$ satisfies

(1) $f|_{C^*} : [t + \frac{1}{3}c_s : (t + \frac{1}{3}c_s)^3 : 1] \mapsto [d\cdot t + \frac{1}{3}c_s : (d\cdot t + \frac{1}{3}c_s)^3 : 1]$ *with*

$$c_s := \sum_{k=1}^{n} s_k, \tag{4.3}$$

(2) $\check{p}^-_{i,j} \approx [v_{i,j} + \frac{1}{3}c_s : (v_{i,j} + \frac{1}{3}c_s)^3 : 1] \in C^*$,
(3) $\check{p}^+_{i,j} \approx [u_{i,j} + \frac{1}{3}c_s : (u_{i,j} + \frac{1}{3}c_s)^3 : 1] \in C^*$, *where*

$$u_{i,j} := \frac{1}{d}\big\{v_{i,j} - (d-1)\cdot s_i\big\}. \tag{4.4}$$

Conversely, for any $d \in \mathbb{C} \setminus \{0,1\}$, $v \in \mathbb{C}^{3n}$ *and* $s \in (\mathbb{C}^\times)^n$ *satisfying equation* (4.2)*, there exists an* n*-tuple* $\overline{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ *such that the composition* $f = f_n \circ \cdots \circ f_1$ *satisfies* $\delta(f) = d$ *and conditions (1)–(3). Moreover,* $\overline{f}$ *is determined uniquely by* (d, v, s) *in the sense that if* $\overline{f} = (f_1, \dots, f_n)$ *and* $\overline{f}' = (f'_1, \dots, f'_n)$ *are determined by* (d, v, s)*, then there are linear maps* $g_1, \dots, g_{n-1} \in \mathcal{L}(C)$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{P}^2_0 \arrow[r, "f'_1"] \arrow[d, equal] & \mathbb{P}^2_1 \arrow[r, "f'_2"] \arrow[d, "g_1"'] & \cdots \arrow[r, "f'_{n-1}"] & \mathbb{P}^2_{n-1} \arrow[r, "f'_n"] \arrow[d, "g_{n-1}"'] & \mathbb{P}^2_n \arrow[d, equal] \\
\mathbb{P}^2_0 \arrow[r, "f_1"] & \mathbb{P}^2_1 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{n-1}"] & \mathbb{P}^2_{n-1} \arrow[r, "f_n"] & \mathbb{P}^2_n.
\end{tikzcd}$$

Proof. — From Lemma 4.2, each map $f_i \in \mathcal{Q}(C)$ is given by $f_i = f_{d_i,(b_{i,j})}$ for some $d_i \in \mathbb{C}^\times$ and $(b_{i,j})_{j=1}^3 \in \mathbb{C}^3$ with $b_i := b_{i,1} + b_{i,2} + b_{i,3} \neq 0$. Then the maps f_i and f restricted to C^* can be expressed as $f_i|_{C^*}([t : t^3 : 1]) = [y_i(t) : y_i(t)^3 : 1]$ and $f|_{C^*}([t : t^3 : 1]) = [y(t) : y(t)^3 : 1]$, respectively, where y_i, $y : \mathbb{C} \to \mathbb{C}$ are given by

$$y_i(t) = d_i \cdot t - \frac{1}{3}b_i,$$

and $y := y_n \circ y_{n-1} \circ \cdots \circ y_1$. Now we put $\check{d}_i := d_{i+1} \cdot d_{i+2} \cdots d_n$, $a_{i,j} := (b_{i,j} - \frac{2}{3}b_i)/d_i$ and

$$\begin{aligned}
\check{a}_{i,j} &:= y_1^{-1} \circ \cdots \circ y_{i-1}^{-1}(a_{i,j}),\\
\check{b}_{i,j} &:= y_n \circ \cdots \circ y_{i+1}(b_{i,j}),\\
\check{b}_i &:= \check{b}_{i,1} + \check{b}_{i,2} + \check{b}_{i,3}.
\end{aligned}$$

Then it follows that $\check{p}^{+}_{i,j} \approx [\check{a}_{i,j} : \check{a}^3_{i,j} : 1]$, $\check{p}^{-}_{i,j} \approx [\check{b}_{i,j} : \check{b}^3_{i,j} : 1]$ and $d = \check{d}_0$. A straightforward calculation shows that

$$
\begin{aligned}
y(t) &= d \cdot t - \frac{1}{3}\sum_{k=1}^{n} 2^{k-1} \cdot \check{b}_k, \\
\check{b}_{i,j} &= \check{d}_i \cdot b_{i,j} - \frac{1}{3}\sum_{k=i+1}^{n} \check{d}_k \cdot b_k \\
&= \check{d}_i \cdot b_{i,j} - \frac{d-1}{3}\sum_{k=i+1}^{n} s_k, \\
\check{a}_{i,j} &= \frac{1}{d}\Big(\check{d}_i \cdot b_{i,j} - \check{d}_i \cdot b_i + \frac{1}{3}\sum_{k=1}^{i} \check{d}_k \cdot b_k\Big) \\
&= \frac{1}{d}\Big\{\check{d}_i \cdot b_{i,j} - (d-1) \cdot s_i + \frac{d-1}{3}\sum_{k=1}^{i} s_k\Big\},
\end{aligned}
$$

where $s_i := \check{d}_i \cdot b_i/(d-1) \neq 0$. If we put

$$
v_{i,j} := \check{d}_i \cdot b_{i,j} - \frac{1}{3}\Big(\sum_{k=1}^{i} s_k + d \cdot \sum_{k=i+1}^{n} s_k\Big),
$$

then we have

$$
\begin{aligned}
v_{i,1} + v_{i,2} + v_{i,3} &= \check{d}_i b_i - \Big(\sum_{k=1}^{i} s_k + d\sum_{k=i+1}^{n} s_k\Big) \\
&= -\sum_{k=1}^{i-1} s_k + (d-2)s_i - d\sum_{k=i+1}^{n} s_k,
\end{aligned}
$$

which shows that equation (4.2) holds. Moreover, since $\check{b}_i = \check{d}_i \cdot b_i - (d-1) \cdot \sum_{k=i+1}^{n} s_k = (d-1)\cdot\{s_i - \sum_{k=i+1}^{n} s_k\}$ and thus $\sum_{k=1}^{n} 2^{k-1}\cdot\check{b}_k = (d-1)\cdot c_s$, the map $y(t) = d \cdot t - (d-1) \cdot c_s/3$ has the unique fixed point $c_s/3$ under the assumption that $d \neq 1$. Finally, we have

$$\begin{aligned}
\check{b}_{i,j} &= \check{d}_i \cdot b_{i,j} - \frac{d-1}{3}\sum_{k=i+1}^{n} s_k \\
&= v_{i,j} + \frac{1}{3}\Big(\sum_{k=1}^{i} s_k + d\cdot \sum_{k=i+1}^{n} s_k\Big) - \frac{d-1}{3}\sum_{k=i+1}^{n} s_k \\
&= v_{i,j} + \frac{1}{3}c_s,
\end{aligned}$$

$$\begin{aligned}
d\cdot \check{a}_{i,j} &= \check{d}_i \cdot b_{i,j} - (d-1)\cdot s_i + \frac{d-1}{3}\sum_{k=1}^{i} s_k \\
&= v_{i,j} + \frac{1}{3}\Big(\sum_{k=1}^{i} s_k + d\cdot \sum_{k=i+1}^{n} s_k\Big) - (d-1)\cdot s_i + \frac{d-1}{3}\sum_{k=1}^{i} s_k \\
&= v_{i,j} - (d-1)\cdot s_i + \frac{d}{3}c_s.
\end{aligned}$$

Thus conditions (1)–(3) hold.

Conversely, for any $d \neq 1$, (s_i) and $(v_{i,j})$ satisfying (4.2), the maps $(f_i) = (f_{d_i,(b_{i,j})})$ with

$$d_1 \cdots d_n = d,$$

$$b_{i,j} = \frac{1}{d_{i+1}\cdots d_n}\Big\{v_{i,j} + \frac{1}{3}\Big(\sum_{k=1}^{i} s_k + d\cdot \sum_{k=i+1}^{n} s_k\Big)\Big\}$$

give the birational map $f = f_n \circ \cdots \circ f_1$ satisfying $\delta(f) = d$ and conditions (1)–(3). Moreover, assume that there are two n-tuples $\overline{f} = (f_1, \ldots, f_n)$ and $\overline{f}' = (f_1', \ldots, f_n')$ in $\mathcal{Q}(C)^n$ such that $f = f_n \circ \cdots \circ f_1$ and $f' = f_n' \circ \cdots \circ f_1'$ satisfy $\delta(f) = \delta(f') = d$ and conditions (1)–(3) for (d, v, s). Put $g_i := f_{i+1}^{-1} \circ \cdots \circ f_n^{-1} \circ f_n' \circ \cdots \circ f_{i+1}' : \mathbb{P}^2_i \to \mathbb{P}^2_i$. Then one has $f_i' = g_i^{-1} \circ f_i \circ g_{i-1}$, where $g_n = \mathrm{id}$. It follows from condition (2) that $I(f_n^{-1}) = I((f_n')^{-1})$, which means that $g_{n-1} = f_n^{-1} \circ f_n'$ is an automorphism of $\mathbb{P}^2_{n-1}$ preserving C, and thus $g_{n-1} \in \mathcal{L}(C)$. In a similar manner, under the assumption that $g_i \in \mathcal{L}(C)$ for some $1 \leqslant i \leqslant n$, condition (2) shows that $I(f_i^{-1}) = g_i I((f_i')^{-1})$ and hence $g_{i-1} = f_i^{-1} \circ g_i \circ f_i' \in \mathcal{L}(C)$. Moreover, it follows from condition (1) that the determinant of g_0 is given by $\delta(g_0) = \delta(f') \cdot \delta(f)^{-1} = 1$, which means that $g_0 = \mathrm{id}$ (see Lemma 4.4). This completes the proof. □

Remark 4.6. — From the definition of $\check{p}_\iota^{\pm}$ given in (4.1), we have the following relations:

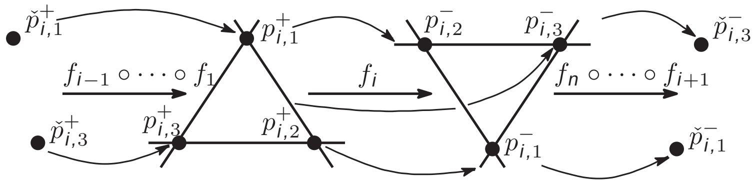

Figure 4.1. The points $\check{p}^+_{i,j} \in I(f)$ and $\check{p}^-_{i,j} \in I(f^{-1})$

(1) $f|_C^k(\check{p}^-_\iota) \approx \check{p}^-_{\iota'}$, or equivalently $d^k \cdot v_\iota = v_{\iota'}$, if and only if $p^m_\iota \approx p^-_{\iota'}$, where $m = \theta_{i,i'}(k)$ with $\theta_{i,i'}(k)$ given in (3.6).

(2) $f|_C^k(\check{p}^-_\iota) \approx \check{p}^+_{\iota'}$, or equivalently $d^k \cdot v_\iota = u_{\iota'}$, if and only if $p^m_\iota \approx p^+_{\iota'}$, where $m = \theta_{i,i'-1}(k+1)$. In particular, it follows from (3.5) that $p^{\mu(\iota)}_\iota \approx p^+_{\sigma(\iota)}$ if and only if $f|_C^{\kappa(\iota)-1}(\check{p}^-_\iota) \approx \check{p}^+_{\sigma(\iota)}$.

Indeed, for example, assertion (1) can be established from the relation

$$\begin{aligned} p^-_{\iota'} &= f^{-1}_{i'+1}|_C \circ \cdots \circ f^{-1}_n|_C(\check{p}^-_{\iota'}) \\ &\approx f^{-1}_{i'+1}|_C \circ \cdots \circ f^{-1}_n|_C(f|_C^k(\check{p}^-_\iota)) \\ &= f^{-1}_{i'+1}|_C \circ \cdots \circ f^{-1}_n|_C \circ (f|_C)^k \circ f_n|_C \circ \cdots \circ f_{i+1}|_C(p^-_\iota) \\ &= p^{\theta_{i,i'}(k)}_\iota. \end{aligned}$$

Assume that there is a tentative realization $\overline{f}$ of τ. Then, the relation $p^{\mu(\iota)}_\iota \approx p^+_{\sigma(\iota)}$ yields $f|_C^{\kappa(\iota)-1}(\check{p}^-_\iota) \approx \check{p}^+_{\sigma(\iota)}$ and thus $d^{\kappa(\iota)-1} \cdot v_\iota = u_{\sigma(\iota)} = u_{\iota_1}$ for any $\iota \in \mathcal{K}(n)$. Hence, from (4.4), the pair $(v, s) \in \mathbb{C}^{3n} \times (\mathbb{C}^\times)^n$ satisfies

$$v_{\iota_1} = d^{\kappa(\iota)} \cdot v_\iota + (d-1) \cdot s_{i_1} \qquad (\iota = (i,j) \in \mathcal{K}(n)), \tag{4.5}$$

which is equivalent, when d is not a root of unity, to the expression

$$v_\iota = v_\iota(d) = -\frac{d^{\varepsilon_{|\iota|}} \cdot (d-1)}{d^{\varepsilon_{|\iota|}} - 1}\left(d^{-\varepsilon_1} \cdot s_{i_1} + d^{-\varepsilon_2} \cdot s_{i_2} + \cdots + d^{-\varepsilon_{|\iota|}} \cdot s_{i_{|\iota|}}\right), \tag{4.6}$$

where $|\iota| := \#\{\iota_k \,|\, k \geqslant 0\}$ and $\varepsilon_\ell := \varepsilon_\ell(\iota) = \sum_{k=0}^{\ell-1} \kappa(\iota_k)$. Conversely, if there is a pair $(d, v, s) \in (\mathbb{C} \backslash \{0,1\}) \times \mathbb{C}^{3n} \times (\mathbb{C}^\times)^n$ satisfying (4.2) and (4.5), then, by virtue of Proposition 4.5, there is a tentative realization $\overline{f} \in \mathcal{Q}(C)$ of τ with $\delta(\overline{f}) = d$. Therefore, it is important to consider whether the system of equations

$$\begin{cases} v_{i,1} + v_{i,2} + v_{i,3} = -\displaystyle\sum_{k=1}^{i-1} s_k + (d-2) \cdot s_i - d \sum_{k=i+1}^{n} s_k, \ (1 \leqslant i \leqslant n) \\ v_{\iota_1} = d^{\kappa(\iota)} \cdot v_\iota + (d-1) \cdot s_{i_1} \qquad (\iota \in \mathcal{K}(n)) \end{cases} \tag{4.7}$$

for $(d, v, s) \in (\mathbb{C} \setminus \{0,1\}) \times \mathbb{C}^{3n} \times (\mathbb{C}^\times)^n$ can admit solutions, which are closely related to the eigenvalue problem of the Weyl group element w_τ. Namely, as is mentioned in the following proposition, solutions of (4.7) will appear in the coefficients of eigenvectors of w_τ.

Proposition 4.7. — *Let τ be an orbit data, and d be a complex number different from 0 and 1. Then, a vector $y \neq 0$ in $\mathbb{Z}^\tau \otimes_{\mathbb{Z}} \mathbb{C}$ expressed as*

$$y = v_0 \cdot e_0 + \sum v_\iota^k \cdot e_\iota^k \in \mathbb{Z}^\tau \otimes_{\mathbb{Z}} \mathbb{C}$$

is an eigenvector of w_τ corresponding to the eigenvalue d if and only if there is a pair $(v, s) \neq (0,0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ satisfying equations (4.7) such that the following conditions hold:

(1) *$v_\iota^k = d^{k-1} \cdot v_\iota$ for any $\iota \in \mathcal{K}(n)$ and $1 \leqslant k \leqslant \kappa(\iota)$.*
(2) *$v_0 = c_s$, where c_s is given in (4.3).*

Moreover, for the eigenvector y corresponding to the eigenvalue d, a pair $(v, s) \neq (0,0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ satisfying equations (4.7) and conditions (1)–(2) is uniquely determined.

Proof. — Assume that y is an eigenvector corresponding to d. It is easy to see that the coefficient of e_ι^k in $w_\tau(y)$ is v_ι^{k+1} for any $1 \leqslant k \leqslant \kappa(\iota) - 1$. Hence, one has $v_\iota^{k+1} = d \cdot v_\iota^k$, or $v_\iota^k = d^{k-1} \cdot v_\iota^1$. Moreover, we determine $(v, s) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ as follows. Put $v_{n,i} = v_{n,i}^1$ for $i \in \{1,2,3\}$, and $s_n = (v_0 + v_n)/(d-1)$, where $v_n := v_{n,1} + v_{n,2} + v_{n,3}$. For $1 \leqslant \ell \leqslant n-1$, assume that $v_{j,i}$ and s_j with $j \geqslant \ell + 1$ are already determined. Then, put

$$v_{\ell,i} = \begin{cases} v_{\ell,i}^1 & (\kappa(\ell,i) \geqslant 1) \\ v_{(\ell,i)_1} - (d-1) \cdot s_{\ell_1} & (\kappa(\ell,i) = 0), \end{cases}$$

where $\sigma(\ell,i) = (\ell,i)_1 = (\ell_1, i_1)$ (note that $\ell_1 \geqslant \ell + 1$ if $\kappa(\ell,i) = 0$), and

$$s_\ell = \begin{cases} (v_0 + v_\ell)/(d-1) + \sum_{k=\ell+1}^n s_k & (\ell \geqslant 2) \\ v_0 - \sum_{k=2}^n s_k & (\ell = 1), \end{cases}$$

where $v_\ell := v_{\ell,1} + v_{\ell,2} + v_{\ell,3}$. At this stage, it is easily checked that (v, s) satisfies conditions (1)–(2), equation (4.2) for $i \geqslant 2$ and equation (4.5) for $\kappa(\iota) = 0$. Now we claim that the following relation holds for $1 \leqslant \ell \leqslant n+1$:

$$\begin{aligned} q_\ell \circ \cdots \circ q_n(y) = v^\ell \cdot e_0 + \sum_{\substack{i<\ell \\ \kappa(\iota) \geqslant 1}} v_\iota \cdot e_\iota^1 + \sum_{\substack{i<\ell\leqslant i_1 \\ \kappa(\iota)=0}} v_\iota \cdot e_{\bar\sigma(\iota)}^1 \\ + \sum_{\substack{\ell \leqslant i \\ \kappa(\sigma^{-1}(\iota)) \geqslant 1}} \Big\{ v_\iota - (d-1) \cdot s_i \Big\} \cdot e_{\bar\sigma(\iota)}^1 + \sum_{k \geqslant 2} d^{k-1} \cdot v_\iota \cdot e_\iota^k, \end{aligned} \tag{4.8}$$

where $v^\ell := \sum_{k=1}^{\ell-1} s_k + d\sum_{k=\ell}^{n} s_k$. Indeed, if $\ell = n+1$, the relation is trivial. Assume that the relation holds when $\ell+1 \geqslant 2$. Then the automorphism q_ℓ changes only the coefficients $v^{\ell+1}$ and $v_{\ell,i}$ in $q_{\ell+1} \circ \cdots \circ q_n(y)$ as follows:

$$q_\ell\Big(v^{\ell+1}\cdot e_0+\sum_{i=1}^{3} v_{\ell,i}\cdot e^1_{\bar\sigma(\ell,i)}\Big) = \big(2v^{\ell+1}+v_\ell\big)\cdot e_0-\sum_{i=1}^{3}\big(v^{\ell+1}+v_{\ell,j}+v_{\ell,k}\big)\cdot e^1_{\bar\sigma(\ell,i)},$$

where $\{i,j,k\} = \{1,2,3\}$. Therefore, when $\ell \geqslant 2$, equation (4.8) holds from the facts that $2v^{\ell+1} + v_\ell = v^\ell$, $v^{\ell+1} + v_{\ell,j} + v_{\ell,k} = (d-1)\cdot s_\ell - v_{\ell,i}$, and that $\big\{v_{\ell,i} - (d-1)\cdot s_\ell\big\}\cdot e^1_{\bar\sigma(\ell,i)} = v_{\sigma^{-1}(\ell,i)}\cdot e^1_{\bar\sigma(\sigma^{-1}(\ell,i))}$ if $\kappa(\sigma^{-1}(\ell,i)) = 0$. Moreover, since the coefficient of e_0 in $w_\tau(y)$ is $d\cdot v_0$ and e_0 is fixed by r_τ, the coefficient of e_0 in $q_1\circ\cdots\circ q_n(y)$ is expressed as $2v^2 + v_1 = d\cdot v_0 = v^1$, which yields $v_1 = (d-2)\cdot s_1 - d\sum_{k=2}^{n} s_k$. Thus, the coefficient of $e_{1,i}$ in $q_1\circ\cdots\circ q_n(y)$ is given by $v_{1,i} - (d-1)\cdot s_1$ and equation (4.8) holds when $\ell = 1$. The claim follows from these observations. In particular, (v,s) satisfies equation (4.2) for $i = 1$.

By the above claim, we have

$$q_1\circ\cdots\circ q_n(y) = v^1\cdot e_0 + \sum_{\kappa(\iota)\geqslant 1}\Big\{v_{\iota_1} - (d-1)\cdot s_{i_1}\Big\}\cdot e^1_{\bar\sigma(\iota_1)} + \sum_{k\geqslant 2} d^{k-1}\cdot v_\iota\cdot e^k_\iota.$$

Thus, the coefficient of e_0 in $w_\tau(y)$ is $v^1 = d\cdot c_s$. Similarly, the coefficient of $e_\iota^{\kappa(\iota)}$ in $w_\tau(y)$ is given by $v_{\iota_1} - (d-1)\cdot s_{i_1}$. This means that $v_{\iota_1} - (d-1)\cdot s_{i_1} = d\cdot v_\iota^{\kappa(\iota)} = d^{\kappa(\iota)}\cdot v_\iota$ and that (v,s) satisfies equation (4.5) for $\kappa(\iota)\geqslant 1$. It should be noted that $v \neq 0$, since if $v = 0$ then one has $s = 0$ from (4.5) and so $y = 0$, which is a contradiction.

Moreover, the above argument shows that, for the eigenvector y, a pair (v,s) is determined uniquely by equation (4.2) for $i \geqslant 2$, equation (4.5) for $\kappa(\iota) = 0$, condition (1) for $k = 1$ and condition (2), which gives the uniqueness of (v,s).

Conversely, we can easily check that, for a pair $(v,s) \neq (0,0) \in \mathbb{C}^{3n}\times\mathbb{C}^n$ satisfying equations (4.7), the vector y given by conditions (1)–(2) is an eigenvector of w_τ corresponding to d. The proof is complete. □

Let $w : \mathbb{Z}^{1,N} \to \mathbb{Z}^{1,N}$ be a lattice automorphism in W_N. It is known that the characteristic polynomial $\chi_w(t)$ of w can be expressed as

$$\chi_w(t) = \begin{cases} R_w(t) & (\lambda(w) = 1) \\ R_w(t)S_w(t) & (\lambda(w) > 1), \end{cases}$$

where $R_w(t)$ is a product of cyclotomic polynomials, and $S_w(t)$ is a Salem polynomial, namely, the minimal polynomial of a Salem number. Here, a *Salem number* is an algebraic integer $\delta > 1$ such that its conjugates include

$\delta^{-1} < 1$ and the conjugates other than $\delta^{\pm 1}$ lie on the unit circle (see [11]). Therefore, if w satisfies $\lambda(w) > 1$ and d satisfies $S_w(d) = 0$, then there is a unique eigenvector, up to constant multiple, corresponding to d. Moreover, an eigenvalue d with $|d| > 1$ is unique and is a Salem number $d = \lambda(w) > 1$.

To simplify the notation, we put $S_\tau(t) := S_{w_\tau}(t)$ and $\lambda(\tau) := \lambda(w_\tau)$. Then, the following corollary of Proposition 4.7 can be established.

Corollary 4.8. — *Assume that d is not a root of unity and there is a solution $(v, s) \neq (0, 0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ of equations* (4.7)*. Then, d is a root of $S_\tau(t) = 0$, and v and s are nonzero. Conversely, if d is a root of $S_\tau(t) = 0$, then there is a unique solution $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ of equations* (4.7)*, up to a constant multiple.*

Proof. — First assume that d is not a root of unity and there is a solution $(v, s) \neq (0, 0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ of (4.7). Then d is a root of $S_\tau(t) = 0$ from Proposition 4.7. Moreover v and s are nonzero. Indeed, if $v = 0$ then $s = 0$ from (4.5), and if $s = 0$ then $v = 0$ from (4.6). Conversely, if d is a root of $S_\tau(t) = 0$, then a solution $(v, s) \neq (0, 0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ of (4.7), which satisfies $v \neq 0$ and $s \neq 0$ from the above argument, is unique, as an eigenvector corresponding to d is unique. □

For a root d of $S_\tau(t) = 0$, let $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ be a solution of (4.7) as is mentioned in Corollary 4.8. If s satisfies $s_\ell \neq 0$ for any $\ell = 1, \ldots, n$, then there is a tentative realization $\overline{f} \in \mathcal{Q}(C)^n$ of τ such that $\delta(\overline{f}) = d$ from the above argument. Moreover, the composition $f = f_n \circ \cdots \circ f_1$ is unique up to conjugacy by a linear map in $\mathcal{L}(C)$, as a solution of (4.7) is unique up to a constant multiple. Summing up these discussions, we have the following proposition.

Proposition 4.9. — *Let τ be an orbit data with $\lambda(\tau) > 1$, d be a root of $S_\tau(t) = 0$ and $s \neq 0$ be the unique solution of equations* (4.7) *(see Corollary 4.8). Then s satisfies $s_\ell \neq 0$ for any $1 \leqslant \ell \leqslant n$ if and only if there is a tentative realization $\overline{f}$ of τ such that $\delta(\overline{f}) = d$. Moreover, the tentative realization $\overline{f}$ of τ is uniquely determined in the sense that if there are two tentative realizations $\overline{f} = (f_1, \ldots, f_n)$ and $\overline{f}' = (f'_1, \ldots, f'_n)$ of τ such that $\delta(\overline{f}) = \delta(\overline{f}') = d$, then there are linear maps $g_1, \ldots, g_n \in \mathcal{L}(C)$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{P}^2_0 \arrow[r, "f'_1"] \arrow[d, "g_0"'] & \mathbb{P}^2_1 \arrow[r, "f'_2"] \arrow[d, "g_1"'] & \cdots \arrow[r, "f'_{n-1}"] & \mathbb{P}^2_{n-1} \arrow[r, "f'_n"] \arrow[d, "g_{n-1}"'] & \mathbb{P}^2_n \arrow[d, "g_n"'] \\
\mathbb{P}^2_0 \arrow[r, "f_1"] & \mathbb{P}^2_1 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{n-1}"] & \mathbb{P}^2_{n-1} \arrow[r, "f_n"] & \mathbb{P}^2_n,
\end{tikzcd}$$

where $g_0 := g_n$.

Remark 4.10. — As is seen in Proposition 4.9, the tentative realization $\overline{f}$ of τ with $\delta(\overline{f}) = d$ is unique. However, when $p_\iota^- \approx p_{\iota'}^-$ for some $\iota \neq \iota' \in \{(i,1),(i,2),(i,3)\}$, there remains an ambiguity about how to label indeterminacy points, namely, about a choice between $p_\iota^\pm < p_{\iota'}^\pm$ and $p_\iota^\pm > p_{\iota'}^\pm$ (see also Remark 3.6).

Proposition 4.9 raises a question as to whether, for a given orbit data τ, the solution s of (4.7) satisfies $s_\ell \neq 0$ for any $\ell = 1,\dots,n$. This question can be solved in terms of the absence of periodic roots. To see this, we need some preliminaries.

Let $w \in W_N$ be a general Weyl group element with $\lambda(w) > 1$. Then there is a direct sum decomposition of the real vector space:

$$\mathbb{R}^{1,N} := \mathbb{Z}^{1,N} \otimes_{\mathbb{Z}} \mathbb{R} = V_w \oplus V_w^c,$$

such that the decomposition is preserved by w, and $S_w(t)$ and $R_w(t)$ are the characteristic polynomials of $w|_{V_w}$ and $w|_{V_w^c}$, respectively. We notice that V_w^c is the orthogonal complement of V_w with respect to the Lorentz inner product. Moreover, let ℓ_w be the the minimal positive integer satisfying $d^{\ell_w} = 1$ for any root d of the equation $R_w(t) = 0$. Then we have the following lemma.

Lemma 4.11. — *Assume that $\delta = \lambda(w) > 1$, and let d be an eigenvalue of w that is not a root of unity. Then, for a vector $z \in \mathbb{Z}^{1,N}$, the following are equivalent.*

(1) $(z, y_d) = 0$, *where y_d is the eigenvector of w corresponding to d.*
(2) $z \in V_w^c \cap \mathbb{Z}^{1,N}$.
(3) *z is a periodic vector of w with period ℓ_w.*
(4) *z is a periodic vector of w with some period k.*

Proof. — (1) $\Longrightarrow$ (2). First, we notice that y_d can be chosen so that $y_d \in \mathbb{Z}^{1,N} \otimes_{\mathbb{Z}} \mathbb{Z}[d]$. The coefficient of e_i in y_d, and thus that in $y_{d'}$ for any conjugate d', are expressed as $(y_d)_i = v_i(d)$ and $(y_{d'})_i = v_i(d')$ for some $v_i(x) \in \mathbb{Z}[x]$. Since $z = (z_i) \in \mathbb{Z}^{1,N}$ and so $(z, y_x) = z_0 \cdot v_0(x) - \sum_{i \neq 0} z_i \cdot v_i(x) \in \mathbb{Z}[x]$, we have $(z, y_{d'}) = 0$ from the relation $(z, y_d) = 0$. Thus it follows that $z \in V_w^c \cap \mathbb{Z}^{1,N}$.

(2) $\Longrightarrow$ (3). For any eigenvalues d, d', we have

$$(y_d, y_{d'}) = (w(y_d), w(y_{d'})) = d \cdot d' \cdot (y_d, y_{d'}),$$

which means that $(y_d, y_{d'}) = 0$ if $d \cdot d' \neq 1$. In particular, one has $(y_\delta, y_\delta) = (y_{1/\delta}, y_{1/\delta}) = 0$. Moreover, since $y_\delta, y_{1/\delta} \in \mathbb{R}^{1,N}$ are linearly independent over $\mathbb{R}$, $(y_\delta, y_{1/\delta})$ is nonzero, and thus either $(y_\delta + y_{1/\delta}, y_\delta + y_{1/\delta})$

or $(y_\delta - y_{1/\delta}, y_\delta - y_{1/\delta})$ is positive. As $\mathbb{R}^{1,N}$ has signature $(1,N)$ and V_w has signature $(1,s)$ for some $s \geqslant 1$, V_w^c is negative definite. This shows that $w|_{V_w^c}$ has finite order. Since any eigenvalue d of $w|_{V_w^c}$ satisfies $d^{\ell_w} = 1$, we have $w^{\ell_w}(z) = z$.

(4) $\Longrightarrow$ (2). Assume that $w^k(z) = z$ for some $k \geqslant 1$. We express z as $z = z' + z''$ for some $z' \in V_w$ and $z'' \in V_w^c$, and then express z' as $z' = \sum_{S_w(d)=0} z_d \cdot y_d$ for some $z_d \in \mathbb{C}$. Under the assumption that $w^k(z) = z$, one has $\sum_{S_w(d)=0} z_d \cdot y_d = z' = w^k(z') = \sum_{S_w(d)=0} d^k \cdot z_d \cdot y_d$. This means that $z_d = d^k \cdot z_d$ for any d with $S_w(d) = 0$. Since d is not a root of unity, z_d is zero for any d. Therefore, we have $z' = 0$ and $z = z'' \in V_w^c$, and the assertion is established.

Assertions (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are obvious. Therefore the lemma is established. $\square$

Now, for an orbit data τ, let $P(\tau)$ be the set of periodic roots with period ℓ_{w_τ}, that is,

$$P(\tau) := \left\{\alpha \in \Phi_N \,\middle|\, w_\tau^{\ell_{w_\tau}}(\alpha) = \alpha\right\}.$$

Moreover, we define a finite subset of the root system by

$$\Gamma_1(\tau) := \left\{\alpha_\ell^c \,\middle|\, \ell = 1, \dots, n\right\} \subset \Phi_N,$$

where α_ℓ^c is given by

$$\alpha_\ell^c := q_n \circ \cdots \circ q_{\ell+1}(e_0 - e^1_{\bar\sigma(\ell,1)} - e^1_{\bar\sigma(\ell,2)} - e^1_{\bar\sigma(\ell,3)}).$$

Lemma 4.12. — *Let d be a root of $S_\tau(t) = 0$ and $s = (s_\ell)$ be the solution of* (4.7)*. Then for each $1 \leqslant \ell \leqslant n$, α_ℓ^c belongs to $P(\tau)$ if and only if $s_\ell = 0$.*

Proof. — Assume $\alpha_\ell^c \in P(\tau)$, which is equivalent to saying that $(\alpha_\ell^c, y_d) = 0$ from Lemma 4.11. By (4.8), we have

$$\begin{aligned}
(\alpha_\ell^c, y_d) &= (e_0 - e^1_{\bar\sigma(\ell,1)} - e^1_{\bar\sigma(\ell,2)} - e^1_{\bar\sigma(\ell,3)}, q_{\ell+1} \circ \cdots \circ q_n(y_d)) \\
&= \Big(\sum_{k=1}^{\ell} s_k + d \sum_{k=\ell+1}^{n} s_k\Big) + \sum_{i=1}^{3} v_{\ell,i} \\
&= \Big(\sum_{k=1}^{\ell} s_k + d \sum_{k=\ell+1}^{n} s_k\Big) + \Big(-\sum_{k=1}^{\ell-1} s_k + (d-2)s_\ell - d \sum_{k=\ell+1}^{n} s_k\Big) \\
&= (d-1)s_\ell.
\end{aligned}$$

As $d \neq 1$, the equation $(\alpha_\ell^c, y_d) = 0$ is equivalent to saying that $s_\ell = 0$. $\square$

Propositions 4.13, 4.14 and 4.15 mentioned below run parallel with Theorems 1.4–1.6 in terms of condition (4.9) (see Proposition 4.13). Namely, Proposition 4.13 states that τ admits a tentative realization if and only if τ satisfies condition (4.9), Proposition 4.14 shows that the sibling $\check{\tau}$ of any orbit data satisfies the condition, and finally Proposition 4.15 gives a sufficient condition for (4.9).

Proposition 4.13. — *Let τ be an orbit data with $\lambda(\tau) > 1$ and d be a root of $S_\tau(t) = 0$. Then, τ satisfies the condition*

$$\Gamma_1(\tau) \cap P(\tau) = \emptyset, \tag{4.9}$$

if and only if there is a tentative realization $\overline{f} \in \mathcal{Q}(C)^n$ of τ such that $\delta(\overline{f}) = d$. Moreover, the tentative realization $\overline{f} \in \mathcal{Q}(C)^n$ of τ with $\delta(\overline{f}) = d$ is uniquely determined.

Proof. — This proposition is an immediate consequence of Proposition 4.9 and Lemma 4.12. □

Proposition 4.14. — *For any orbit data $\tau = (n, \sigma, \kappa)$ with $\lambda(\tau) > 1$, there is a data $\check{\tau} = (\check{n}, \check{\sigma}, \check{\kappa})$ with $\check{n} \leqslant n$ such that $\check{\tau}$ satisfies condition* (4.9) *and $\lambda(\tau) = \lambda(\check{\tau})$.*

Proof. — Let d be a root of $S_\tau(t) = 0$ and $(v, s) \in \mathbb{C} \times (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ be the solution of (4.7) for τ as in Corollary 4.8. If $s_\ell \neq 0$ for any $\ell = 1, \dots, n$, then putting $\check{\tau} = \tau$ leads to the proposition. Otherwise, assume that $s_\ell = 0$ for some ℓ. Then we put $\check{n} := n - 1$, and for any $\iota \in \mathcal{K}(\check{n}) \cong \{(i,j) \in \mathcal{K}(n) \,|\, i \neq \ell\}$, choose $\nu(\iota)$ so that $i_1 = \cdots = i_{\nu(\iota)-1} = \ell$ but $i_{\nu(\iota)} \neq \ell$. A new orbit data $\check{\tau} = (\check{n}, \check{\sigma}, \check{\kappa})$ is defined by $\check{\sigma}(\iota) := \sigma^{\nu(\iota)}(\iota)$ and $\check{\kappa}(\iota) := \sum_{k=0}^{\nu(\iota)-1} \kappa(\sigma^k(\iota))$ for any $\iota \in \mathcal{K}(\check{n})$. Then, since $v_{\iota_1} = d^{\kappa(\iota)} \cdot v_\iota + (d-1) \cdot s_{i_1}$ and $s_\ell = 0$, we have $v_{\iota_{\check{\sigma}}} = d^{\check{\kappa}(\iota)} \cdot v_\iota + (d-1) \cdot s_{i_{\check{\sigma}}}$ for any $\iota \in \mathcal{K}(\check{n})$, where $\check{\sigma}(\iota) = \iota_{\check{\sigma}} = (i_{\check{\sigma}}, j_{\check{\sigma}})$. Moreover, as v satisfies (4.2), the new vector $\check{v} = (v_\iota)_{\iota \in \mathcal{K}(\check{n})}$ also satisfies (4.2) with $n = \check{n}$ and $\check{s} = (s_1, \dots, s_{\ell-1}, s_{\ell+1}, \dots, s_n) \neq 0$. Hence, $(d, \check{v}, \check{s})$ is a solution of (4.7) for $\check{\tau}$, which means that $S_\tau(t) = S_{\check{\tau}}(t)$ and thus $\lambda(\tau) = \lambda(\check{\tau})$.

Therefore, either $\check{s}_\ell \neq 0$ for any ℓ, or we can repeat the above argument to eliminate $\check{s}_\ell = 0$ from $\check{s}$. Since each step reduces n by 1, $\check{\tau}$ satisfies $\check{s}_\ell \neq 0$ for any ℓ after finitely many steps. □

Proposition 4.15. — *For any orbit data τ satisfying conditions* (1) *and* (2) *in Theorem 1.6, there is an estimate $2^n - 1 < \lambda(\tau) < 2^n$. Moreover, τ satisfies condition* (4.9).

The proof of this proposition is given in Section 6.

5. Realizability

The aim of this section is to construct a realization of an orbit data τ and to establish our main theorems. In the previous section, we construct a tentative realization of $\overline{f} \in \mathcal{Q}(C)^n$ of τ under condition (4.9). However, $\overline{f}$ does not necessarily become a realization of τ. We give two such examples after stating a preliminary lemma. To this end, for $\iota = (i,j), \iota' = (i',j') \in \mathcal{K}(n)$, let us define a root $\alpha^k_{\iota,\iota'}$ by

$$\alpha^k_{\iota,\iota'} := q_n \circ \cdots \circ q_{i'+1}(e^{k+1}_{\bar{\sigma}(\iota)} - e^1_{\bar{\sigma}(\iota')}) \in \Phi_N.$$

Lemma 5.1. — *Assume that an orbit data τ satisfies condition* (4.9). *Then, for a tentative realization $\overline{f}$ of τ mentioned in Proposition 4.13, the following are equivalent.*

(1) $p^m_\iota \approx p^-_{\iota'}$ *for* $m \geqslant 0$.
(2) $p^{m+\ell}_\iota \approx p^\ell_{\iota'}$ *for some* $\ell \geqslant 0$.
(3) $p^{m+\ell}_\iota \approx p^\ell_{\iota'}$ *for any* $\ell \geqslant 0$.
(4) $d^k \cdot v_\iota = v_{\iota'}$, *where* $m = \theta_{i,i'}(k) \geqslant 0$.
(5) $\alpha^k_{\iota,\iota'} \in P(\tau)$.

Proof. — One can easily check that assertions (1)–(4) are equivalent (see also Remark 4.6). Moreover, by virtue of (4.8), the condition $\alpha^k_{\iota,\iota'} \in P(\tau)$, or $(\alpha^k_{\iota,\iota'}, y_d) = 0$ from Lemma 4.11, is equivalent to saying that

$$0 = (\alpha^k_{\iota,\iota'}, y_d) = (e^{k+1}_{\bar{\sigma}(\iota)} - e^1_{\bar{\sigma}(\iota')}, q_{i'+1} \circ \cdots \circ q_n(y_d)) = d^k \cdot v_\iota - v_{\iota'},$$

where the last equality follows from the fact that the coefficients of $e^{k+1}_{\bar{\sigma}(\iota)}$ and $e^1_{\bar{\sigma}(\iota')}$ in $q_{i'+1} \circ \cdots \circ q_n(y_d)$ are $d^k \cdot v_\iota$ and $v_{\iota'}$ respectively, since $\theta_{i,i'}(k) \geqslant 0$ (see also (4.8))). The lemma is established. □

Remark 5.2. — If $\overline{f}$ is a realization of τ and $p^m_\iota \approx p^-_{\iota'}$ for a positive integer $m > 0$, then it follows that $p^m_\iota > p^-_{\iota'}$ and $p^{m+\ell}_\iota > p^\ell_{\iota'}$ for any $\ell \geqslant 0$ with $m + \ell \leqslant \mu(\iota)$ and $\ell \leqslant \mu(\iota')$ (see Remark 2.10).

Example 5.3. — Consider the orbit data $\tau = (2, \sigma, \kappa)$ given by

$$\begin{cases} \sigma : (1,1) \mapsto (1,2) \mapsto (2,2) \mapsto (2,1) \mapsto (1,1), \quad (1,3) \mapsto (2,3) \mapsto (1,3) \\ \kappa(1,1) = \kappa(2,2) = 4,\ \kappa(1,2) = \kappa(1,3) = 0,\ \kappa(2,1) = 1,\ \kappa(2,3) = 3 \\ (\mu(1,1) = \mu(2,2) = 7,\ \mu(1,2) = \mu(1,3) = 0,\ \mu(2,1) = 0,\ \mu(2,3) = 4). \end{cases}$$

Then, equations (4.7) for τ admit a solution (d, v, s) with $d = \lambda(\tau) \approx 1.582$, which is the unique root of $t^6 - t^4 - 2t^3 - t^2 + 1 = 0$ in $|t| > 1$, $v = (v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}) \approx (1, 7.269, 8.048, 0, 1, 1.779)$ and $s = (s_1, s_2) \approx (1.717, -10.765)$. Proposition 4.9 assures that there is a tentative

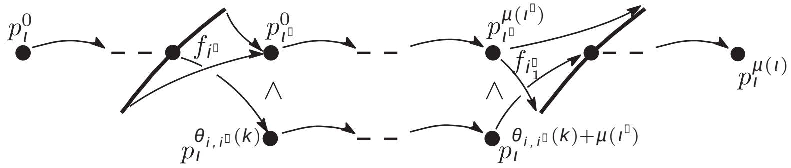

Figure 5.1. Orbit segments starting at two indeterminacy points

realization $\overline{f} = (f_1, f_2) \in \mathcal{Q}(C)^2$ of τ, all the indeterminacy points of which are proper as $v_{k,i} \neq v_{k,j}$ for any $i \neq j$. However, $\overline{f}$ is not a realization of τ. Indeed, assume the contrary that $\overline{f}$ is a realization of τ. The fact that $v_{1,1} = v_{2,2}$ and Lemma 5.1 yield $p^1_{1,1} \approx p^-_{2,2}$, which means that $p^1_{1,1} = f_2(p^-_{1,1}) > p^-_{2,2}$ and thus $p^\ell_{1,1} > p^{\ell-1}_{2,2}$ for each $\ell = 1, \dots, \mu(1,1) = 7$ (see Remark 5.2). On the other hand, since $p^{\mu(1,1)}_{1,1} \approx p^+_{\sigma(1,1)} = p^+_{1,2}$ and $p^+_{1,2}$ is proper, one has $p^{\mu(1,1)}_{1,1} \neq p^+_{1,2}$ but $p^{\mu(1,1)-1}_{2,2} = p^+_{1,2}$. Hence, $\overline{f}$ is not a realization of τ. This argument implies that there should not be a periodic root $\alpha^k_{\iota,\iota'} \in P(\tau)$ with $m = \theta_{i,i'}(k) > 0$ and $\mu(\iota) < m + \mu(\iota')$ (see Figure 5.1). We remark that the solutions of (4.7) for τ are the same as the ones for another orbit data $\check{\tau} = (2, \check{\sigma}, \kappa)$ given by

$$\check{\sigma} : (1,\ell) \mapsto (2,\ell) \mapsto (1,\ell), \quad (\ell \in \{1,2,3\}),$$

and that $\overline{f}$ is a realization of $\check{\tau}$. In particular, one has $\lambda(\check{\tau}) = \lambda(\tau) > 1$.

Example 5.4. — Consider the orbit data $\tau = (1, \sigma, \kappa)$ given by

$$\begin{cases} \sigma : (1,1) \mapsto (1,2) \mapsto (1,1), \quad (1,3) \mapsto (1,3) \\ \kappa(1,1) = \kappa(1,2) = 4, \quad \kappa(1,3) = 3. \end{cases}$$

Then, equations (4.7) for τ admit a solution (d, v, s) with $d = \lambda(\tau) \approx 1.582$, which is the unique root of $t^6 - t^4 - 2t^3 - t^2 + 1 = 0$ in $|t| > 1$, $v = (v_{1,1}, v_{1,2}, v_{1,3}) \approx (-0.190, -0.190, -0.338)$ and $s = (s_1) = (1)$. Proposition 4.9 assures that there is a tentative realization $\overline{f} = (f_1) \in \mathcal{Q}(C)$ of τ, whose indeterminacy points satisfy $p^\pm_{1,1} \approx p^\pm_{1,2}$. However, $\overline{f}$ is not a realization of τ. Indeed, assume the contrary that $\overline{f}$ is a realization of τ. If it is assumed that $p^\pm_{1,2} > p^\pm_{1,1}$ (see Remark 4.10), then it follows that $p^\ell_{1,2} > p^\ell_{1,1}$ for any $\ell = 0, \dots, 3$. On the other hand, since $p^3_{1,1} \approx p^3_{1,2} \approx p^+_{1,1} \approx p^+_{1,2}$ and $p^+_{1,2} > p^+_{1,1}$, one has $p^3_{1,1} = p^+_{1,1}$ and $p^3_{1,2} = p^+_{1,2}$. The case $p^\pm_{1,1} > p^\pm_{1,2}$ is the same. Hence, $\overline{f}$ is not a realization of τ. This argument implies that there should not be a periodic root $\alpha^0_{(1,i),(1,j)} \in P(\tau)$ with $\mu(1,i) = \mu(1,j)$

and $(1, j) = \sigma(1, i)$ for some $i \neq j$. We remark that the solutions of (4.7) for τ are the same as the ones for another orbit data $\check{\tau} = (2, \check{\sigma}, \kappa)$ given by

$$\check{\sigma} : (1, \ell) \mapsto (1, \ell), \quad (\ell \in \{1, 2, 3\}),$$

and that $\overline{f}$ is a realization of $\check{\tau}$. In particular, one has $\lambda(\check{\tau}) = \lambda(\tau) > 1$.

Based on the above two examples, we give the following definition.

Definition 5.5. — *We define subsets $\Gamma_2^1(\tau)$, $\Gamma_2^2(\tau)$ and $\Gamma_2(\tau)$ of the set*

$$\overline{\Gamma}_2(\tau) = \left\{\alpha_{\iota,\iota'}^k \,\middle|\, \iota = (i, j), \iota' = (i', j') \in \mathcal{K}(n), 0 \leqslant \theta_{i,i'}(k) \leqslant \mu(\iota)\right\} \subset \Phi_N.$$

(1) *Let $\Gamma_2^1(\tau)$ be the set of all roots $\alpha_{\iota,\iota'}^k \in \overline{\Gamma}_2(\tau)$ with $\theta_{i,i'}(k) > 0$ such that $\mu(\sigma^\ell(\iota)) = \mu(\sigma^\ell(\iota')) + \delta_{\ell,0} \cdot \theta_{i,i'}(k)$ for $\ell = 0, \dots, h-1$ and $\mu(\sigma^h(\iota)) < \mu(\sigma^h(\iota')) + \delta_{h,0} \cdot \theta_{i,i'}(k)$ with some $h \geqslant 0$, where $\delta_{i,j}$ stands for the Kronecker delta.*

(2) *Let $\Gamma_2^2(\tau)$ be the set of all roots $\alpha_{\iota,\iota'}^k \in \overline{\Gamma}_2(\tau)$ with $k = 0$, $i = i'$ and $j \neq j'$ such that $\mu(\sigma^\ell(\iota)) = \mu(\sigma^\ell(\iota'))$ for any $\ell \geqslant 0$ and $\iota' = \sigma^h(\iota)$ for some $h \geqslant 1$.*

(3) *The set $\Gamma_2(\tau)$ is defined by the union $\Gamma_2(\tau) := \Gamma_2^1(\tau) \cup \Gamma_2^2(\tau)$.*

Example 5.6. — Consider τ and $\check{\tau}$ given in Example 5.4. Then we have

$$\Gamma_2^1(\tau) = \{e_i^k - e_j^1 \,|\, i, j \in \{1, 2, 3\}, k = 2, 3, 4\}, \quad \Gamma_2^2(\tau) = \{e_1^1 - e_2^1, e_2^1 - e_1^1\},$$
$$\Gamma_2^1(\check{\tau}) = \Gamma_2^1(\tau), \qquad \Gamma_2^2(\check{\tau}) = \emptyset.$$

Moreover, it can be checked that w_τ admits periodic roots $e_1^1 - e_2^1$ and $e_2^1 - e_1^1$ in $\Gamma_2(\tau)$, and $w_{\check{\tau}}$ admits no periodic roots in $\Gamma_2(\check{\tau})$.

From the above preliminaries, we have the following three propositions, which also run parallel with Theorems 1.4–1.6 in a similar way to Propositions 4.13, 4.14 and 4.15.

Proposition 5.7. — *Let τ be an orbit data satisfying $\lambda(\tau) > 1$ and condition (4.9), and $\overline{f} \in \mathcal{Q}(C)^n$ be the tentative realization of τ such that $\delta(\overline{f}) = d$ is a root of $S_\tau(t) = 0$ as is mentioned in Proposition 4.13. Then, $\overline{f}$ is a realization of τ if and only if τ satisfies the condition*

$$\Gamma_2(\tau) \cap P(\tau) = \emptyset. \tag{5.1}$$

Proof. — First, assume that $\overline{f} \in \mathcal{Q}(C)^n$ is a realization of τ, and also assume the contrary that there is a root $\alpha_{\iota,\iota'}^k \in \Gamma_2(\tau) \cap P(\tau) \neq \emptyset$. If $\alpha_{\iota,\iota'}^k \in \Gamma_2^1(\tau)$, then Lemma 5.1 shows that $p_\iota^m \approx p_{\iota'}^-$ with $m = \theta_{i,i'}(k) > 0$ and thus $p_\iota^m > p_{\iota'}^-$ (see Remark 5.2). Let $h \geqslant 0$ be the integer given in item (1) of Definition 5.5. When $h = 0$, the relation $p_{\sigma(\iota)}^+ = p_\iota^{\mu(\iota)} > p_{\iota'}^{\mu(\iota)-m}$ means that

$p_{\iota'}^{\mu(\iota)-m} = p_{\iota''}^{+}$ for some ι'' and thus $\mu(\iota') = \mu(\iota)-m$, since the indeterminacy set is a cluster. But this is impossible because $\mu(\iota) - m < \mu(\iota')$. In a similar manner, when $h \geqslant 1$, it follows that $p_{\sigma^\ell(\iota)}^{\pm} > p_{\sigma^\ell(\iota')}^{\pm}$ for each $1 \leqslant \ell \leqslant h$. Moreover, the relation $p_{\sigma^{h+1}(\iota)}^{+} = p_{\sigma^h(\iota)}^{\mu(\sigma^h(\iota))} > p_{\sigma^h(\iota')}^{\mu(\sigma^h(\iota))}$ means that $\mu(\sigma^h(\iota)) = \mu(\sigma^h(\iota'))$, which is a contradiction. Hence we have $\Gamma_2^1(\tau) \cap P(\tau) = \emptyset$. On the other hand, if $\alpha_{\iota,\iota'}^k \in \Gamma_2^2(\tau)$, then the relation $<$ can not be well defined. Indeed, assuming that $p_\iota^- < p_{\iota'}^-$, one has $p_{\sigma^\ell(\iota)}^{\pm} < p_{\sigma^\ell(\iota')}^{\pm}$ for any $\ell \geqslant 0$. Since $\sigma^h(\iota) = \iota'$ and $\mu(\sigma^\ell(\iota)) = \mu(\sigma^\ell(\iota'))$ for any $\ell \geqslant 0$, it is easily seen that ι' satisfies either $\sigma^h(\iota') = \iota$, or $\sigma^h(\iota') = \iota''$ and $\sigma^h(\iota'') = \iota$ with $\{\iota, \iota', \iota''\} = \{(i,1),(i,2),(i,3)\}$. The former case is impossible because it follows in this case that $p_{\iota'}^- < p_\iota^-$, which contradicts the assumption that $p_\iota^- < p_{\iota'}^-$. The latter case is also impossible, because it follows in this case that $p_{\iota'}^- < p_{\iota''}^-$ and $p_{\iota''}^- < p_\iota^-$, which is also a contradiction. Similarly, the assumption $p_\iota^- > p_{\iota'}^-$ yields a contradiction. Therefore, we have $\Gamma_2^2(\tau) \cap P(\tau) = \emptyset$. These observations show that condition (5.1) holds.

Conversely, assume that condition (5.1) holds. For two pairs of indeterminacy points $p_{k,i}^{\pm}$ and $p_{k,j}^{\pm}$ satisfying $p_{k,i}^{\pm} \approx p_{k,j}^{\pm}$, we fix $p_{k,i}^{\pm} < p_{k,j}^{\pm}$ if either $\mu(k,i) < \mu(k,j)$, or $\mu(k,i) = \mu(k,j)$ and $p_{\sigma(k,i)}^{\pm} < p_{\sigma(k,j)}^{\pm}$ (see Remark 4.10). In other words, when there is $0 \leqslant h \leqslant \infty$ such that $\mu(\sigma^\ell(k,i)) = \mu(\sigma^\ell(k,j))$ for $0 \leqslant \ell < h$ and $\mu(\sigma^h(k,i)) < \mu(\sigma^h(k,j))$, we fix $p_{\sigma^\ell(k,i)}^{\pm} < p_{\sigma^\ell(k,j)}^{\pm}$ for each $0 \leqslant \ell < h$. This is well-defined because of the absence of roots in $\Gamma_2^2(\tau) \cap P(\tau)$. Now in order to check condition (1.6), for a given $\iota \in \mathcal{K}(n)$, by putting

$$Q_\iota^+(m) := \begin{cases} \{p_{\iota'}^+ \,|\, p_{\iota'}^+ \approx p_\iota^m\} & (0 \leqslant m \leqslant \mu(\iota) - 1) \\ \{p_{\iota'}^+ \,|\, p_{\iota'}^+ \lneqq p_{\sigma(\iota)}^+\} & (m = \mu(\iota)), \end{cases}$$

$$Q_\iota^-(m) := \begin{cases} \{p_{\iota'}^- \,|\, p_{\iota'}^- \lneqq p_\iota^-\} & (m = 0) \\ \{p_{\iota'}^- \,|\, p_{\iota'}^- \approx p_\iota^m\} & (1 \leqslant m \leqslant \mu(\iota)), \end{cases}$$

we will show the condition

$$\begin{aligned} &\sum_{k=0}^{m} (\#Q_\iota^-(k) - \#Q_\iota^+(k)) \geqslant 0 \quad (0 \leqslant m < \mu(\iota)), \\ (5.2) \qquad &\sum_{k=0}^{\mu(\iota)} (\#Q_\iota^-(k) - \#Q_\iota^+(k)) = 0. \end{aligned}$$

In our situation, it should be noted that

$$\#Q_\iota^{\pm}(m) = \max\{\ell \geqslant 0 \,|\, \text{there is an } (\ell-1)\text{-th point } p_{\iota'}^{\pm} \text{ with } p_\iota^m \approx p_{\iota'}^{\pm}\}.$$

Hence, for each $0 \leqslant m \leqslant \mu(\iota)-1$, if p_ι^m is an ℓ-th point with $\ell \geqslant \#Q_\iota^+(m)$, which means that $p_\iota^m \neq p_{\iota'}^+$ for any $\iota' \in \mathcal{K}(n)$, then p_ι^{m+1} is a $(\#Q_\iota^-(m+1) - \#Q_\iota^+(m)+\ell)$-th point (see Remark 2.10). As p_ι^0 is a $\#Q_\iota^-(0)$-th point, p_ι^m is a $(\sum_{k=0}^m \#Q_\iota^-(k) - \sum_{k=0}^{m-1} \#Q_\iota^+(k))$-th point with $(\sum_{k=0}^m \#Q_\iota^-(k) - \sum_{k=0}^{m-1} \#Q_\iota^+(k)) \geqslant \#Q_\iota^+(m)$ under condition (5.2), which shows that $p_\iota^m \neq p_{\iota'}^+$ for any $0 \leqslant m \leqslant \mu(\iota)-1$ and $\iota' \in \mathcal{K}(n)$. Finally, since $p_\iota^{\mu(\iota)} \approx p_{\sigma(\iota)}^+$ are $(\sum_{k=0}^{\mu(\iota)} \#Q_\iota^-(k) - \sum_{k=0}^{\mu(\iota)-1} \#Q_\iota^+(k)) = \#Q_\iota^+(\mu(\iota))$-th points, we have $p_\iota^{\mu(\iota)} = p_{\sigma(\iota)}^+$.

Condition (5.2) is a consequence of the following two assertions:

(1) For any $p_{\iota'}^- \in Q_\iota^-(m_1)$, there is a unique m_2 with $m_1 \leqslant m_2 \leqslant \mu(\iota)$ and $p_{\sigma(\iota')}^+ \in Q_\iota^+(m_2)$.

(2) For any $p_{\iota'}^+ \in Q_\iota^+(m_2)$, there is a unique m_1 with $0 \leqslant m_1 \leqslant m_2$ and $p_{\sigma^{-1}(\iota')}^- \in Q_\iota^-(m_1)$.

Indeed, these two assertions lead to the bijection

$$\begin{aligned}\mathcal{F}_\iota : \{(m_1, p_{\iota'}^-) \,|\, 0 \leqslant m_1 \leqslant \mu(\iota), p_{\iota'}^- \in Q_\iota^-(m_1)\} \\ \to \{(m_2, p_{\iota'}^+) \,|\, 0 \leqslant m_2 \leqslant \mu(\iota), p_{\iota'}^+ \in Q_\iota^+(m_2)\}\end{aligned}$$

defined by $\mathcal{F}_\iota(m_1, p_{\iota'}^-) = (m_2, p_{\sigma(\iota')}^+)$. Since $m_1 \leqslant m_2$, the bijection $\mathcal{F}_\iota$ yields condition (5.2).

To finish the proof of the proposition, we only prove assertion (1) as assertion (2) can be treated in a similar manner. For uniqueness, assuming that there are $m_2 < m$ such that $p_{\sigma(\iota')}^+ \in Q_\iota^+(m_2) \cap Q_\iota^+(m)$, namely $p_{\sigma(\iota')}^+ \approx p_\iota^{m_2} \approx p_\iota^m$, one has $\alpha_{\iota,\iota}^k \in \Gamma_2^1(\tau) \cap P(\tau)$ with $m - m_2 = n \cdot k > 0$, which is a contradiction.

Next we show the existence of m_2. Assume that $p_{\iota'}^- \in Q_\iota^-(m_1)$ for some $0 \leqslant m_1 \leqslant \mu(\iota)$, which means that $\alpha_{\iota,\iota'}^{k_1} \in P(\tau)$ with $m_1 = \theta_{i,i'}(k_1) \geqslant 0$. If $\mu(\iota) > \mu(\iota') + m_1$, then one has $p_{\sigma(\iota')}^+ \in Q_\iota^+(\mu(\iota') + m_1)$. On the other hand, if $\mu(\iota) < \mu(\iota') + m_1$, then the root $\alpha_{\iota,\iota'}^{k_1}$ belongs to $\Gamma_2^1(\tau) \cap P(\tau)$, which is a contradiction. Finally, suppose that $\mu(\iota) = \mu(\iota') + m_1$, or in other words $p_{\sigma(\iota)}^\pm \approx p_{\sigma(\iota')}^\pm$. If $m_1 = 0$, then the assumption $p_\iota^- > p_{\iota'}^-$ leads to $p_{\sigma(\iota)}^\pm > p_{\sigma(\iota')}^\pm$ and thus $p_{\sigma(\iota')}^+ \in Q_\iota^+(\mu(\iota))$. On the other hand, if $m_1 > 0$, then we also have $p_{\sigma(\iota')}^+ \in Q_\iota^+(\mu(\iota))$. Indeed, assume the contrary that $p_{\sigma(\iota')}^+ \notin Q_\iota^+(\mu(\iota))$ or $p_{\sigma(\iota')}^+ > p_{\sigma(\iota)}^+$. Fix $1 \leqslant h \leqslant \infty$ satisfying $\mu(\sigma^\ell(\iota)) = \mu(\sigma^\ell(\iota'))$ for $1 \leqslant \ell < h$ and $\mu(\sigma^h(\iota)) < \mu(\sigma^h(\iota'))$. If $h = \infty$, then the existence of $L > 0$ with $\sigma^L = \mathrm{id}$ shows that $\mu(\iota) = \mu(\sigma^L(\iota)) = \mu(\sigma^L(\iota')) = \mu(\iota')$, which is a contradiction. If $h < \infty$, then the root $\alpha_{\iota,\iota'}^{k_1}$ belongs to $\Gamma_2^1(\tau) \cap P(\tau)$, which is also a contradiction. Assertion (1) is established by combining all these observations. Therefore the proof of the proposition is complete. $\square$

Proposition 5.8. — *Let τ be an orbit data satisfying $\lambda(\tau) > 1$ and condition* (4.9)*, and $\overline{f}$ be a tentative realization mentioned in Proposition 4.13. Then, there is an orbit data $\check{\tau}$ such that $\lambda(\tau) = \lambda(\check{\tau})$ and $\overline{f}$ is a realization of $\check{\tau}$. In particular, $\check{\tau}$ satisfies condition* (5.1).

Proof. — Let d be a root of $S_\tau(t) = 0$, $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ be a solution of (4.7) as in Corollary 4.8, and u_ι be given in (4.4). If $\Gamma_2(\tau) \cap P(\tau) = \emptyset$, then putting $\check{\tau} = \tau$ leads to the proposition. Otherwise, we make a decomposition

$$P_2(\tau) := \Gamma_2(\tau) \cap P(\tau) = P_2^1(\tau) \cup P_2^2(\tau) \cup P_2^3(\tau),$$

and divide the proof into three steps, where $P_2^\ell(\tau)$ is given by

$$P_2^\ell(\tau) := \begin{cases} \{\alpha^k_{\iota',\iota''} \in P_2(\tau) \,|\, \iota' = \iota''\} & (\ell = 1) \\ \{\alpha^k_{\iota',\iota''} \in P_2(\tau) \,|\, \iota' \neq \iota'', \mu(\iota'') - \mu(\iota') + \theta_{i',i''}(k) > 0\} & (\ell = 2) \\ \{\alpha^k_{\iota',\iota''} \in P_2(\tau) \,|\, \iota' \neq \iota'', \mu(\iota'') - \mu(\iota') + \theta_{i',i''}(k) = 0\} & (\ell = 3). \end{cases}$$

It should be noted that $\mu(\iota'') - \mu(\iota') + \theta_{i',i''}(k)$ turns out to be nonnegative provided $\alpha^k_{\iota',\iota''} \in \Gamma_2(\tau)$, and that $\mu(\iota'') - \mu(\iota') + \theta_{i',i''}(k) = 0$ if and only if $p^\pm_{\sigma(\iota')} \approx p^\pm_{\sigma(\iota'')}$. Now we fix a root $\alpha^k_{\iota',\iota''} \in P_2(\tau)$, which means that $d^k \cdot v_{\iota'} = v_{\iota''}$ and $p^m_{\iota'} \approx p^-_{\iota''}$ with $0 \leqslant m = \theta_{i',i''}(k) \leqslant \mu(\iota')$ by Lemma 5.1, and put $m' = \mu(\iota'') - \mu(\iota') + m \geqslant 0$.

Step 1. — First suppose that $\alpha^k_{\iota',\iota'} \in P_2^1(\tau)$, which belongs to $\Gamma_2^1(\tau)$ and thus satisfies $\theta_{i',i'}(k) > 0$ or $k \geqslant 1$. As $d^k \cdot v_{\iota'} = v_{\iota'}$ and d is not a root of unity, we have $v_{\iota'} = 0$ and thus $d^\ell \cdot v_{\iota'} = v_{\iota'}$ for any $\ell \geqslant 0$. Hence, it follows that $p^0_{\iota'} \approx p^n_{\iota'} \approx \cdots \approx p^{k''\cdot n}_{\iota'}$ and $p^+_{\sigma(\iota')} \approx p^{\mu(\iota')}_{\iota'} \approx p^{\mu(\iota')-k''\cdot n}_{\iota'}$, where $k'' \geqslant 1$ is chosen so that $0 \leqslant \mu(\iota') - k'' \cdot n < n$. A new orbit data $\check{\tau} = (n, \sigma, \check{\kappa})$ is defined by $\check{\kappa}(\iota') := \kappa(\iota') - k''$, and $\check{\kappa}(\iota) := \kappa(\iota)$ if $\iota \neq \iota'$. Then $\alpha^k_{\iota',\iota'}$ is not defined for $\check{\tau}$. The relation $d^{\check{\kappa}(\iota)-1} \cdot v_\iota = u_{\sigma(\iota)} = u_{\iota_1}$ shows that $v_{\iota_1} = d^{\check{\kappa}(\iota)} \cdot v_\iota + (d-1) \cdot s_{i_1}$ for any $\iota \in \mathcal{K}(n)$, and thus (d, v, s) satisfies (4.7) for $\check{\tau}$. Therefore, $\overline{f}$ is also a tentative realization of $\check{\tau}$. Moreover, it follows from the relation $S_\tau(t) = S_{\check{\tau}}(t)$ that $\lambda(\tau) = \lambda(\check{\tau})$.

Since $\sum_{\iota \in \mathcal{K}(n)} \kappa(\iota)$ is finite, we can assume that $\# P_2^1(\tau) = 0$ by repeating this argument. In particular, an integer $\overline{k} \geqslant 0$ with $\alpha^{\overline{k}}_{\iota,\overline{\iota}} \in P_2(\tau)$ for given $\iota, \overline{\iota} \in \mathcal{K}(n)$ is at most unique, since if $\alpha^{k'}_{\iota,\overline{\iota}}, \alpha^{k''}_{\iota,\overline{\iota}} \in P_2(\tau)$ for some $k' < k''$, then $\alpha^{\overline{k}}_{\iota,\iota} \in P_2^1(\tau)$ with $\overline{k} = k'' - k' > 0$.

Step 2. — Next we assume that $\alpha^k_{\iota',\iota''} \in P^2_2(\tau)$, which also belongs to $\Gamma^1_2(\tau)$ and thus satisfies $m = \theta_{i',i''}(k) > 0$. Then a new orbit data $\check\tau = (n, \check\sigma, \check\kappa)$ is defined by

$$\check\sigma(\iota) := \begin{cases} \sigma(\iota'') & (\iota = \iota') \\ \sigma(\iota') & (\iota = \iota'') \\ \sigma(\iota) & (\text{otherwise}), \end{cases} \qquad \check\mu(\iota) := \begin{cases} \mu(\iota') + m' & (\iota = \iota') \\ \mu(\iota'') - m' & (\iota = \iota'') \\ \mu(\iota) & (\text{otherwise}). \end{cases} \tag{5.3}$$

Since $p^m_{\iota'} \approx p^-_{\iota''}$, one has $p^{\check\mu(\iota')}_{\iota'} = p^{\mu(\iota')+m'}_{\iota'} = p^{\mu(\iota'')+m}_{\iota'} \approx p^{\mu(\iota'')}_{\iota''} \approx p^+_{\sigma(\iota'')} = p^+_{\check\sigma(\iota')}$ and $p^{\check\mu(\iota'')}_{\iota''} = p^{\mu(\iota'')-m'}_{\iota''} = p^{\mu(\iota')-m}_{\iota''} \approx p^{\mu(\iota')}_{\iota'} \approx p^+_{\sigma(\iota')} = p^+_{\check\sigma(\iota'')}$, which yield $d^{\check\kappa(\iota')-1} \cdot v_{\iota'} = u_{\check\sigma(\iota')}$ and $d^{\check\kappa(\iota'')-1} \cdot v_{\iota''} = u_{\check\sigma(\iota'')}$. For $\iota \neq \iota', \iota''$, the equation $d^{\kappa(\iota)-1} \cdot v_\iota = u_{\sigma(\iota)}$ leads to $d^{\check\kappa(\iota)-1} \cdot v_\iota = u_{\check\sigma(\iota)}$. This shows that $v_{\iota_{\check\sigma}} = d^{\check\kappa(\iota)} \cdot v_\iota + (d-1) \cdot s_{i_{\check\sigma}}$ for any $\iota \in \mathcal{K}(n)$, where $\check\sigma(\iota) = \iota_{\check\sigma} = (i_{\check\sigma}, j_{\check\sigma})$, and (d, v, s) satisfies (4.7) for $\check\tau$, which means that $\overline{f}$ is also a tentative realization of $\check\tau$ and that $\lambda(\tau) = \lambda(\check\tau)$.

Now, we will show that $\#P^2_2(\check\tau) < \#P^2_2(\tau)$. Indeed, for $\iota, \bar\iota \notin \{\iota', \iota''\}$, it follows that $\alpha^{k'}_{\iota,\bar\iota} \in P^2_2(\check\tau)$ if and only if $\alpha^{k'}_{\iota,\bar\iota} \in P^2_2(\tau)$. Hence, if one assumes that $\alpha^{k'}_{\iota,\bar\iota} \in P^2_2(\check\tau)$ and $\alpha^{k'}_{\iota,\bar\iota} \notin P^2_2(\tau)$, then $\iota, \bar\iota$ must satisfy $\{\iota, \bar\iota\} \cap \{\iota', \iota''\} \neq \emptyset$. When $\bar\iota = \iota'$ and $\iota \neq \iota''$, the relation $\check\mu(\iota) < \check\mu(\iota') + \theta_{i,i'}(k')$ yields $\mu(\iota) < \mu(\iota'') + \theta_{i,i''}(k'')$, and the relation $\mu(\iota) > \mu(\iota') + \theta_{i,i'}(k')$ yields $\check\mu(\iota) > \check\mu(\iota') + \theta_{i,i''}(k'')$, where $k'' = k + k'$. This means that $\alpha^{k''}_{\iota,\iota''} \notin P^2_2(\check\tau)$ and $\alpha^{k''}_{\iota,\iota''} \in P^2_2(\tau)$. On the other hand, when $\bar\iota = \iota'$ and $\iota = \iota''$, the relation $\alpha^{k'}_{\iota'',\iota'} \in P^2_2(\check\tau)$ gives $d^{k'} \cdot v_{\iota''} = v_{\iota'}$ with $0 < \theta_{i'',i'}(k') \leqslant \check\mu(\iota'')$. Combining it with $d^k \cdot v_{\iota'} = v_{\iota''}$, one has the equation $d^{k''} \cdot v_{\iota'} = v_{\iota'}$ for $k'' = k + k'$. Since $\mu(\iota') = \mu(\iota'') - m' + m = \check\mu(\iota'') + m \geqslant \theta_{i'',i'}(k') + m = \theta_{i',i'}(k'') > 0$, we have $\alpha^{k''}_{\iota',\iota'} \in P^1_2(\tau)$, which contradicts the assumption that $\#P^1_2(\tau) = 0$. In a similar manner, if $\alpha^{k'}_{\iota,\bar\iota} \in P^2_2(\check\tau)$ and $\alpha^{k'}_{\iota,\bar\iota} \notin P^2_2(\tau)$, then it can be seen that $\alpha^{k''}_{\bar\iota',\bar\iota''} \notin P^2_2(\check\tau)$ and $\alpha^{k''}_{\bar\iota',\bar\iota''} \in P^2_2(\tau)$, where $(\bar\iota', \bar\iota'', k'') = (\iota', \bar\iota, k + k')$ when $\iota = \iota''$, $(\bar\iota', \bar\iota'', k'') = (\iota, \iota', |k - k'|)$ when $\bar\iota = \iota''$, and $(\bar\iota', \bar\iota'', k'') = (\iota'', \bar\iota, |k - k'|)$ when $\iota = \iota'$. Finally, it follows from $\check\mu(\iota') > \check\mu(\iota'') + \theta_{i',i''}(k)$ that $\alpha^k_{\iota',\iota''} \notin P^2_2(\check\tau)$. Since $\alpha^k_{\iota',\iota''} \in P^2_2(\tau)$, these observations show that $\#P^2_2(\check\tau) < \#P^2_2(\tau)$.

If $\alpha^{k'}_{\iota,\iota} \notin P^1_2(\check\tau)$, which means in fact that $\iota = \iota'$, then we further take a new orbit data so that $\#P^1_2(\check\tau) = 0$. Hence, this step yields the orbit data $\check\tau$ satisfying either $\#P^2_2(\check\tau) < \#P^2_2(\tau)$ and $\sum_{\iota\in\mathcal{K}(n)} \check\kappa(\iota) = \sum_{\iota\in\mathcal{K}(n)} \kappa(\iota)$, or $\sum_{\iota\in\mathcal{K}(n)} \check\kappa(\iota) < \sum_{\iota\in\mathcal{K}(n)} \kappa(\iota)$. So we can also assume that $\#P^1_2(\tau) = \#P^2_2(\tau) = 0$ by repeating this argument.

Step 3. — Finally, suppose that $\alpha^k_{\iota',\iota''} \in P^3_2(\tau)$, which can be chosen so that $h = h(\iota',\iota'') \geqslant 0$ is minimal among all elements in $P^3_2(\tau)$, where $1 \leqslant h \leqslant \infty$ is determined by the relations $\mu(\sigma^\ell(\iota')) = \mu(\sigma^\ell(\iota''))$ for $1 \leqslant \ell \leqslant h-1$ and $\mu(\sigma^h(\iota')) < \mu(\sigma^h(\iota''))$. Then we define a new orbit data $\check{\tau} = (n, \check{\sigma}, \kappa)$ as in (5.3) with $m' = 0$. It can be checked that $\overline{f}$ is also a tentative realization of $\check{\tau}$ and $\lambda(\tau) = \lambda(\check{\tau})$, and that $\#P^1_2(\check{\tau}) = \#P^2_2(\check{\tau}) = 0$ as μ is invariant in this procedure.

Now we claim that $\mu(\check{\sigma}^\ell(\iota')) = \mu(\sigma^\ell(\iota''))$ and $\mu(\check{\sigma}^\ell(\iota'')) = \mu(\sigma^\ell(\iota'))$ for any $1 \leqslant \ell \leqslant h$. In particular, $\alpha^k_{\iota',\iota''}$ does not belong to $P^3_2(\check{\tau})$ since $\mu(\check{\sigma}^\ell(\iota')) = \mu(\check{\sigma}^\ell(\iota''))$ for $1 \leqslant \ell < h$ and $\mu(\check{\sigma}^h(\iota')) > \mu(\check{\sigma}^h(\iota''))$. Indeed, when $s', s'' \geqslant 1$ are the minimal integers such that $\sigma^{s'}(\iota') \in \{\iota', \iota''\}$ and $\sigma^{s''}(\iota'') \in \{\iota', \iota''\}$, one has $\check{\sigma}^\ell(\iota'') = \sigma^\ell(\iota')$ or $\mu(\check{\sigma}^\ell(\iota'')) = \mu(\sigma^\ell(\iota'))$ for $1 \leqslant \ell \leqslant s'$ and $\check{\sigma}^\ell(\iota') = \sigma^\ell(\iota'')$ or $\mu(\check{\sigma}^\ell(\iota')) = \mu(\sigma^\ell(\iota''))$ for $1 \leqslant \ell \leqslant s''$. Moreover, assuming that $s' < \ell' \leqslant s''$ and that the claim is verified for $\ell \leqslant \ell'-1$, we fix $k' \geqslant 1$ with $1 \leqslant \ell'' := \ell' - k' \cdot s' \leqslant s'$. If $\check{\sigma}^{s'}(\iota'') = \sigma^{s'}(\iota') = \iota'$, then it follows that $\sigma^{\ell'}(\iota') = \sigma^{\ell''}(\iota')$ and $\mu(\check{\sigma}^{\ell'}(\iota'')) = \mu(\check{\sigma}^{\ell'-s'}(\iota')) = \mu(\check{\sigma}^{\ell'-s'}(\iota'')) = \cdots = \mu(\check{\sigma}^{\ell''}(\iota'))$, which show that $\mu(\check{\sigma}^{\ell'}(\iota'')) = \mu(\sigma^{\ell'}(\iota'))$. The case $\check{\sigma}^{s'}(\iota'') = \sigma^{s'}(\iota') = \iota''$ can be treated in the same manner. Furthermore, when $s'' < \ell' \leqslant s'$ or $s', s'' < \ell'$, the claim also can be verified for $\ell = \ell'$ in a similar way under the assumption that the claim is already verified for $\ell \leqslant \ell' - 1$.

We also claim that $\#P^3_2(\check{\tau};\ell) = 0$ for $1 \leqslant \ell < h$ and $\#P^3_2(\check{\tau};h) < \#P^3_2(\tau;h)$, where $P^3_2(\tau;\ell) := \{\alpha^{k'}_{\iota,\bar{\iota}} \in P^3_2(\tau) \,|\, h(\iota,\bar{\iota}) = \ell\}$. Indeed, since $\mu(\sigma^\ell(\iota)) = \mu(\check{\sigma}^\ell(\iota))$ for any $\iota \in \mathcal{K}(n)$ and $\ell < h$ by the above claim, it follows from the assumption $\#P^3_2(\tau;\ell) = 0$ that $\#P^3_2(\check{\tau};\ell) = 0$ for $1 \leqslant \ell < h$. Moreover, if $\alpha^{k'}_{\iota,\bar{\iota}} \in P^3_2(\check{\tau};h)$ and $\alpha^{k'}_{\iota,\bar{\iota}} \notin P^3_2(\tau;h)$, then $\iota, \bar{\iota}$ should satisfy $\{\iota,\bar{\iota}\} \cap \{\iota',\iota''\} \neq \emptyset$. A straightforward calculation shows that they further satisfy either $\iota = \iota''$ or $\bar{\iota} = \iota'$, and that $\alpha^{k''}_{\iota',\bar{\iota}} \in P^3_2(\tau;h)$ does not belong to $P^3_2(\check{\tau};h)$ when $\iota = \iota''$, and $\alpha^{k''}_{\iota,\iota''} \in P^3_2(\tau;h)$ does not belong to $P^3_2(\check{\tau};h)$ when $\bar{\iota} = \iota'$, where $k'' = k' + k$. Finally, since $\alpha^k_{\iota',\iota''} \in P^3_2(\tau;h)$ and $\alpha^k_{\iota',\iota''} \notin P^3_2(\check{\tau};h)$, we can show that $\#P^3_2(\check{\tau};h) < \#P^3_2(\tau;h)$.

Therefore, by repeating this argument, we can conclude that $\#P^1_2(\check{\tau}) = \#P^2_2(\check{\tau}) = \#P^3_2(\check{\tau}) = 0$ and establish the proposition. □

Proposition 5.9. — *Let τ be an orbit data satisfying conditions* (1) *and* (2) *in Theorem 1.6. Then we have*

$$\overline{\Gamma}_2(\tau) \cap P(\tau) = \{\alpha^0_{\iota,\iota'} \,|\, i_m = i'_m,\ \kappa(\sigma^m(\iota)) = \kappa(\sigma^m(\iota')),\ m \geqslant 0\}.$$

In addition, if τ satisfies condition (3) *in Theorem 1.6, then it also satisfies condition* (5.1).

The proof of this proposition is given in Section 6. We are now in a position to establish the main theorems. To this end, we make the definition of the set $\Gamma(\tau)$.

DEFINITION 5.10. — *The finite subset $\Gamma(\tau)$ of the root system Φ_N is defined by*

$$\Gamma(\tau) := \Gamma_1(\tau) \cup \Gamma_2(\tau) \subset \Phi_N.$$

THEOREM 5.11. — *Let τ be an orbit data with $\lambda(\tau) > 1$ and d be a root of $S_\tau(t) = 0$. Then, τ satisfies the condition*

$$\Gamma(\tau) \cap P(\tau) = \emptyset, \tag{5.4}$$

if and only if there is a realization $\overline{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ of τ such that $\delta(\overline{f}) = d$. The realization $\overline{f} \in \mathcal{Q}(C)^n$ of τ with $\delta(\overline{f}) = d$ is uniquely determined. Moreover, the blowup $\pi_\tau : X_\tau \to \mathbb{P}^2$ of $N = \sum_{\iota \in \mathcal{K}(n)} \kappa(\iota)$ points $\{p_\iota^m \,|\, \iota = (i,j) \in \mathcal{K}(n), m = \theta_{i,0}(k), 1 \leqslant k \leqslant \kappa(\iota)\}$ on C^* lifts $f = f_n \circ \cdots \circ f_1$ to the automorphism $F_\tau : X_\tau \to X_\tau$. Finally, (π_τ, F_τ) realizes w_τ and F_τ has positive entropy $h_{\mathrm{top}}(F_\tau) = \log \lambda(\tau) > 0$.

This theorem follows from Propositions 3.10, 4.13 and 5.7.

Remark 5.12. — Lemmas 4.12 and 5.1 provide another realizability condition instead of (5.4). Namely, for an orbit data τ with $\lambda(\tau) > 1$, let d be a root of $S_\tau(t) = 0$ and $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ be a solution of (4.7) as in Corollary 4.8. Then τ satisfies condition (5.4) if and only if

(1) $s_\ell \neq 0$ for any $1 \leqslant \ell \leqslant n$, and
(2) $d^k \cdot v_\iota \neq v_{\iota'}$ for any (k, ι, ι') with $\alpha^k_{\iota,\iota'} \in \Gamma_2(\tau)$.

Proofs of Theorems 1.4–1.6. — Theorem 1.4 is an immediate consequence of Theorem 5.11, since α is a periodic root of w_τ with period some $k \geqslant 1$ if and only if α is a periodic root of w_τ with period ℓ_{w_τ} (see Lemma 4.11). Moreover, Theorem 1.5 follows from Propositions 4.14 and 5.8, and Theorem 1.6 follows from Propositions 4.15 and 5.9. $\square$

Proof of Theorem 1.1. — For any value $\lambda \neq 1 \in \Lambda$, Theorem 1.5 and Proposition 3.12 show that there is an orbit data τ such that $\lambda = \lambda(\tau)$ and τ satisfies the realizability condition (1.8). In particular, the automorphism F_τ mentioned in Theorem 1.4 has entropy $h_{\mathrm{top}}(F_\tau) = \log \lambda > 0$. Note that when $\lambda = 1 \in \Lambda$, the automorphism $\mathrm{id}_{\mathbb{P}^2} : \mathbb{P}^2 \to \mathbb{P}^2$ satisfies $\lambda(\mathrm{id}^*_{\mathbb{P}^2}) = \lambda = 1$ and $h_{\mathrm{top}}(\mathrm{id}_{\mathbb{P}^2}) = 0$. On the other hand, it follows from Proposition 3.3 that the entropy of any automorphism $F : X \to X$ is given by $h_{\mathrm{top}}(F) = \log \lambda$ for some $\lambda \in \Lambda$. Therefore, Theorem 1.1 is proved. $\square$

Example 5.13. — We consider the orbit data $\tau = (n, \sigma, \kappa)$ given by $n = 2$, $\sigma = \mathrm{id}$, $\kappa(1, \ell) = 3$ and $\kappa(2, \ell) = 4$ for any $\ell = 1, 2, 3$. Then τ satisfies the assumptions in Theorem 1.6, and thus w_τ is realized by a pair (π_τ, F_τ), where $\pi_\tau : X_\tau \to \mathbb{P}^2$ is a blowup of 21 points. We can check that equations (4.7) admit a solution (d, v, s) with $d = \lambda(\tau) \approx 3.87454251$, which is a root of $t^6 - 4t^5 + t^4 - 2t^3 + t^2 - 4t + 1 = 0$, $v = (v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}) \approx (1.749, 1.749, 1.749, 0.233, 0.233, 0.233)$ and $s = (s_1, s_2) \approx (-100, -52.274)$. Therefore, the entropy of F_τ is given by $h_{\mathrm{top}}(F_\tau) = \log \lambda(\tau) \approx 1.35442759$. Moreover, for any $\iota \neq \iota'$ with $i = i' \in \{1, 2\}$, the equality $v_\iota = v_{\iota'}$ shows that the element w_τ admits a periodic root $\alpha^0_{\iota,\iota'}$, which is not contained in $\Gamma(\tau)$. Therefore, the automorphism F_τ does not appear in the paper of McMullen [11]. On the other hand, for any data $\hat{\tau} = (1, \hat{\sigma}, \hat{\kappa})$, let $F_{\hat{\tau}} : X_{\hat{\tau}} \to X_{\hat{\tau}}$ be an automorphism that Diller in [6] constructs from a single quadratic map preserving a cuspidal cubic. We claim that F_τ is not topologically conjugate to $F_{\hat{\tau}}^m$ for any $m \geqslant 1$. Indeed, assume the contrary that F_τ is topologically conjugate to $F_{\hat{\tau}}^m$ for some data $\hat{\tau}$ and $m \geqslant 1$. Then the topological conjugacy yields $\lambda(\tau) = \lambda(\hat{\tau})^m$. Moreover, since X_τ is obtained by blowing up 21 points, so is $X_{\hat{\tau}}$, which means that $\sum_{\ell=1}^{3} \hat{\kappa}(1, \ell) = 21$ and thus there are 190 possibilities for $\hat{\kappa}$. As $\hat{\sigma}$ has 6 possibilities, $\hat{\tau}$ has 1140 possibilities. However, with the help of a computer, it is seen that there are no data $\hat{\tau}$ and $m \geqslant 1$ satisfying these conditions. Our claim is proved.

6. Proof of Realizability with Estimates

As is seen in Section 5, Propositions 4.15 and 5.9 prove Theorem 1.6, or the realizability of orbit data. In this section, we establish these propositions by applying some estimates mentioned below. Let $\bar{c}_{\iota,k}(d)$ and $c_{i,j}(d)$ be polynomials of d defined by

$$v_\iota(d) = \sum_{k=1}^{n} \bar{c}_{\iota,k}(d) \cdot s_k, \tag{6.1}$$

$$v_i(d) := v_{i,1}(d) + v_{i,2}(d) + v_{i,3}(d) = -\sum_{j=1}^{n} c_{i,j}(d) \cdot s_j, \tag{6.2}$$

where $v_\iota(d)$ is given in (4.6), and let $\mathcal{A}_n(d, x)$ be an $n \times n$ matrix having the (i, j)-th entry:

$$\mathcal{A}_n(d, x)_{i,j} = \begin{cases} d - 2 + x_{i,i} & (i = j) \\ -1 + x_{i,j} & (i > j) \\ -d + x_{i,j} & (i < j) \end{cases}$$

with $x = (x_1, \dots, x_n) = (x_{ij}) \in M_n(\mathbb{R})$. Then equations (4.7) yield

$$\mathcal{A}_\tau(d)\, s = 0, \qquad s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, \tag{6.3}$$

where $\mathcal{A}_\tau(d) := \mathcal{A}_n(d, c(d))$ with $c(d) := (c_{i,j}(d))$. Finally, let $\chi_\tau(d)$ be the determinant $|\mathcal{A}_\tau(d)|$ of the matrix $\mathcal{A}_\tau(d)$.

Lemma 6.1. — *Assume that d is not a root of unity. Then, d is a root of $\chi_\tau(t) = 0$ if and only if d is a root of $S_\tau(t) = 0$.*

Proof. — If d is a root of $\chi_\tau(t) = 0$, then there is a solution $s \neq 0$ of (6.3). Thus, (d, v, s) satisfies (4.7), where $v = (v_\iota)$ is given in (4.6). This means that d is a root of $S_\tau(t) = 0$ (see Corollary 4.8). Conversely, if d is a root of $S_\tau(t) = 0$, then there is a unique solution $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ of (4.7). Moreover, s is a solution of (6.3) and thus d is a root of $\chi_\tau(t) = 0$. □

Now fix an orbit data τ satisfying conditions (1) and (2) in Theorem 1.6.

Lemma 6.2. — *If $d > 1$, $k \in \{1, \dots, n\}$ and $\iota \in \mathcal{K}(n)$, then we have*

$$-\frac{1}{d^2 + d + 1} \leqslant \overline{c}_{\iota,k}(d) \leqslant 0.$$

Proof. — In view of equation (4.6), $\overline{c}_{\iota,k}(d)$ may be expressed as either $\overline{c}_{\iota,k}(d) = 0$, or $\overline{c}_{\iota,k}(d) = -(d-1) \cdot d^{\eta_1}/(d^\eta - 1)$ with $\eta_1 + 3 \leqslant \eta$, or $\overline{c}_{\iota,k}(d) = -(d-1) \cdot (d^{\eta_1} + d^{\eta_2})/(d^\eta - 1)$ with $\eta_1 + 3 \leqslant \eta_2$ and $\eta_2 + 3 \leqslant \eta$, or $\overline{c}_{\iota,k}(d) = -(d-1) \cdot (d^{\eta_1} + d^{\eta_2} + d^{\eta_3})/(d^\eta - 1)$ with $\eta_1 + 3 \leqslant \eta_2$, $\eta_2 + 3 \leqslant \eta_3$ and $\eta_3 + 3 \leqslant \eta$, since $\#\{m \,|\, 1 \leqslant m \leqslant |\iota|,\, i_m = k\} \leqslant \#\{(k,1), (k,2), (k,3)\} = 3$. We only consider the case $\overline{c}_{\iota,k}(d) = -(d-1) \cdot (d^{\eta_1} + d^{\eta_2})/(d^\eta - 1)$ as the remaining cases can be treated in the same manner. Since $d > 1$, the inequality $\overline{c}_{\iota,k}(d) < 0$ is trivial. Moreover, one has

$$\begin{aligned}
\frac{\overline{c}_{\iota,k}(d)}{d-1} = -\frac{d^{\eta_1} + d^{\eta_2}}{d^\eta - 1} &\geqslant -\frac{d^{\eta_2 - 3} + d^{\eta_2}}{d^\eta - 1} \\
&\geqslant -\frac{d^{\eta - 6} + d^{\eta - 3}}{d^\eta - 1} = -(1 + \frac{1}{d^\eta - 1})(d^{-6} + d^{-3}) \\
&\geqslant -\frac{1}{d^3 - 1}.
\end{aligned}$$

Thus the lemma is established. □

Since $c_{i,j}(d) = -\sum_{\ell=1}^{3} \overline{c}_{(i,\ell),j}(d)$ from (6.2), the above lemma leads to the inequality

$$0 \leqslant c_{i,j}(d) \leqslant h(d), \qquad h(d) := \frac{3}{1 + d + d^2}.$$

Note that for any $d \geqslant 2$ and $0 \leqslant x_{i,j} \leqslant h(d)$, each diagonal entry $\mathcal{A}_n(d,x)_{i,i}$ of $\mathcal{A}_n(d,x)$ is positive and each non-diagonal entry $\mathcal{A}_n(d,x)_{i,j}$ with $i \neq j$ is negative. Let $\overline{\mathcal{A}}_n(d,x)_{i,j}$ be the (i,j)-cofactor of the matrix $\mathcal{A}_n(d,x)$. Then, the relation $|\mathcal{A}_n(d,x)| = \sum_{i=1}^n \overline{\mathcal{A}}_n(d,x)_{i,j} \cdot \mathcal{A}_n(d,x)_{i,j}$ holds for any $j = 1, \ldots, n$, where $|\mathcal{A}_n(d,x)|$ is the determinant of the matrix $\mathcal{A}_n(d,x)$.

LEMMA 6.3. — *For any $n \geqslant 2$, the following inequalities hold:*

$$\begin{cases} \overline{\mathcal{A}}_n(d,x)_{i,j} > 0 & (d > 2^n - 1,\ 0 \leqslant x_{i,j} \leqslant h(d)) \\ |\mathcal{A}_n(d,x)| > 0 & (d > 2^n,\ 0 \leqslant x_{i,j} \leqslant h(d)) \\ |\mathcal{A}_n(2^n - 1, x)| < 0 & (0 \leqslant x_{i,j} \leqslant h(d)). \end{cases}$$

Proof. — We prove the inequalities by induction on n. For $n = 2$, the first inequality holds since

$$\overline{\mathcal{A}}_2(d,x)_{i,j} = \begin{cases} -\mathcal{A}_2(d,x)_{j,i} > 0 & (i \neq j) \\ \mathcal{A}_2(d,x)_{i+1,i+1} > 0 & (i = j \in \mathbb{Z}/2\mathbb{Z}). \end{cases}$$

As $h(d) < \frac{3}{13}$ when $d > 3$, the remaining inequalities are consequences of the estimates

$$\begin{cases} |\mathcal{A}_2(d,x)| \ = \ (d-2+x_{1,1})(d-2+x_{2,2}) - (1-x_{2,1})(d-x_{1,2}) \\ \qquad\qquad > \ 2^2 - 1 \cdot 4 \ = \ 0 \\ |\mathcal{A}_2(3,x)| = (1+x_{1,1})(1+x_{2,2}) - (1-x_{2,1})(3-x_{1,2}) \\ \qquad\qquad < \ (1+\frac{3}{13})^2 - (1-\frac{3}{13})(3-\frac{3}{13}) \ < \ 0. \end{cases}$$

Therefore, the lemma is proved when $n = 2$. Assume that the inequalities hold when $n = l-1$. A straightforward calculation shows that $\overline{\mathcal{A}}_{i,j} := \overline{\mathcal{A}}_l(d,x)_{i,j}$ can be expressed as

$$\overline{\mathcal{A}}_{i,j} = \begin{cases} -\sum_{k=1}^{i-1} \overline{\mathcal{A}}_{l-1}(d,x^i)_{k,j-1} \cdot \mathcal{A}_l(d,x)_{k,i} \\ \qquad - \sum_{k=i+1}^{l} \overline{\mathcal{A}}_{l-1}(d,x^i)_{k-1,j-1} \cdot \mathcal{A}_l(d,x)_{k,i} & (i < j) \\ -\sum_{k=1}^{i-1} \overline{\mathcal{A}}_{l-1}(d,x^i)_{k,j} \cdot \mathcal{A}_l(d,x)_{k,i} \\ \qquad - \sum_{k=i+1}^{l} \overline{\mathcal{A}}_{l-1}(d,x^i)_{k-1,j} \cdot \mathcal{A}_l(d,x)_{k,i} & (i > j) \\ |\mathcal{A}_{l-1}(d,x^i)| & (i = j), \end{cases}$$

where x^i is the $(l-1,l-1)$-matrix obtained from x by removing the i-th row and column vectors. Hence, the first assertion follows from the induction hypothesis. Moreover, as $|\mathcal{A}_l(d,x)| = \sum_{i=1}^{l} \overline{\mathcal{A}}_l(d,x)_{i,j} \cdot \mathcal{A}_l(d,x)_{i,j}$, the bounds

$$\begin{aligned}|\mathcal{A}_l(d,(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_l)|\\ &\leqslant |\mathcal{A}_l(d,x)|\\ &\leqslant |\mathcal{A}_l(d,(x_1,\dots,x_{j-1},\overline{h}(d),x_{j+1},\dots,x_l)|\end{aligned}$$

hold for any j, where $\overline{h}(d)$ is the column vector having each component equal to $h(d)$. Thus, we have

$$\begin{aligned}|\mathcal{A}_l(d,x)| &\geqslant |\mathcal{A}_l(d,(0,\dots,0))|\\ &= (d-1)^{l-1}(d-2^l) > 0 \qquad (d > 2^l),\\ |\mathcal{A}_l(2^l-1,x)| &\leqslant |\mathcal{A}_l(2^l-1,(\overline{h}(2^l-1),\dots,\overline{h}(2^l-1)))|\\ &= -\frac{(2^l-2)^{l+1}}{2^{2l}-2^l+1} < 0,\end{aligned}$$

which show that the assertions are verified when $n = l$. Therefore, the induction is complete, and the lemma is established. ☐

Let $\delta > 1$ be the root of $S_\tau(t) = 0$ in $|t| > 1$ and $s \neq 0$ be the solution of equation (6.3) with $d = \delta$. Then (δ, v, s) satisfies equations (4.7), where $v = (v_\iota)$ is given in (4.6) with $d = \delta$. We notice that δ, which is also a root of $\chi_\tau(d) = |\mathcal{A}_\tau(d)| = 0$, satisfies $2^n - 1 < \delta < 2^n$, since $\chi_\tau(2^n-1) < 0$ and $\chi_\tau(2^n) > 0$ from Lemma 6.3.

Lemma 6.4. — *For any $1 \leqslant i \leqslant n-1$, the ratio s_{i+1}/s_i satisfies*

$$z_1(n) < \frac{s_{i+1}}{s_i} < z_2(n),$$

where

$$z_1(n) := \frac{2^{n-1}(2^n+2)}{2^{2n}+2^{n+1}+6}, \quad z_2(n) := \frac{2^{2n-1}+2^n+3}{2^{2n}+2^{n+1}+3}.$$

Proof. — For each $k_1, k_2 \geqslant 0$ with $k_1+k_2 \leqslant n-2$, let $\mathcal{A}_n^{k_1,k_2}(\delta)$ be the $n \times n$ matrix defined inductively as follows. First, put $\mathcal{A}_n^{0,0}(\delta) := \mathcal{A}_\tau(\delta)$. Next, let $\mathcal{A}_n^{k_1,0}(\delta)$ be the matrix obtained from $\mathcal{A}_n^{k_1-1,0}(\delta)$ by replacing the i-th row of $\mathcal{A}_n^{k_1-1,0}(\delta)$ with the sum of the i-th row and the k_1-th row multiplied by $-\mathcal{A}_n^{k_1-1,0}(\delta)_{i,k_1}/\mathcal{A}_n^{k_1-1,0}(\delta)_{k_1,k_1}$, where i runs from k_1+1 to n. Finally, let $\mathcal{A}_n^{k_1,k_2}(\delta)$ be the matrix obtained from $\mathcal{A}_n^{k_1,k_2-1}(\delta)$ by replacing the i-th row of $\mathcal{A}_n^{k_1,k_2-1}(\delta)$ with the sum of the i-th row and the $(n-k_2+1)$-th row multiplied by $-\mathcal{A}_n^{k_1,k_2-1}(\delta)_{i,n-k_2+1}/\mathcal{A}_n^{k_1,k_2-1}(\delta)_{n-k_2+1,n-k_2+1}$, where

i runs from k_1+1 to $n-k_2$. Therefore, each entry of $\mathcal{A}_n^{k_1,k_2}(\delta)$ may be expressed as

$$\mathcal{A}_n^{k_1,k_2}(\delta)_{i,j} = \begin{cases} \delta_{i,j} + \xi_{i,j}^{i-1} & (i \leqslant k_1 \text{ and } i \leqslant j) \\ \delta_{i,j} + \xi_{i,j}^{k_1+k_2} & (k_1+1 \leqslant i, j \leqslant n-k_2) \\ \delta_{i,j} + \xi_{i,j}^{k_1+n-i} & (n-k_2+1 \leqslant i \text{ and } k_1+1 \leqslant j \leqslant i) \\ 0 & (\text{otherwise}), \end{cases}$$

where

$$\delta_{i,j} = \begin{cases} \delta - 2 & (i=j) \\ -1 & (i>j) \\ -\delta & (i<j), \end{cases}$$

and $\xi_{i,j}^k$ is given inductively by

$$\xi_{i,j}^0 = c_{i,j}(\delta), \quad \xi_{i,j}^{k+1} = \begin{cases} \xi_{i,j}^k - \dfrac{(1-\xi_{i,k}^k)(\delta - \xi_{k,j}^k)}{\delta - 2 + \xi_{k,k}^k} & (k < k_1) \\ \xi_{i,j}^k - \dfrac{(\delta - \xi_{i,n-k+k_1}^k)(1-\xi_{n-k+k_1,j}^k)}{\delta - 2 + \xi_{n-k+k_1,n-k+k_1}^k} & (k \geqslant k_1). \end{cases}$$

Moreover, it is seen that $\xi_{i,j}^k$ satisfies the estimates

$$-\frac{(2^k-1)\delta}{\delta-2^k} \leqslant \xi_{i,j}^k \leqslant -\overline{\xi}_k, \qquad \overline{\xi}_k := \frac{(2^k-1)\delta - (2^k\delta-1)h(\delta)}{(\delta-2^k)+(2^k-1)h(\delta)}.$$

Note that s satisfies $\mathcal{A}_n^{k_1,k_2}(\delta)\, s = 0$ for any $k_1, k_2 \geqslant 0$. In particular, one has $\mathcal{A}_n^{i-1,n-i-1}(\delta)\, s = 0$, the i-th and $(i+1)$-th components of which are given by

$$\begin{cases} (\delta - 2 + \xi_{i,i}^{n-2})\, s_i + (-\delta + \xi_{i,i+1}^{n-2})\, s_{i+1} = 0 \\ (-1 + \xi_{i+1,i}^{n-2})\, s_i + (\delta - 2 + \xi_{i+1,i+1}^{n-2})\, s_{i+1} = 0. \end{cases}$$

Therefore, we have

$$\frac{s_{i+1}}{s_i} = \frac{\delta - 2 + \xi_{i,i}^{n-2}}{\delta - \xi_{i,i+1}^{n-2}} < \frac{\delta - 2 - \overline{\xi}_{n-2}}{\delta + \overline{\xi}_{n-2}} = \frac{2\delta^2 - (2^n-4)\delta - (2^{n+1}-6)}{2(\delta^2+2\delta+3)},$$

the righthand side of which is monotone increasing with respect to δ, and thus is less than $z_2(n)$ since $\delta < 2^n$. In a similar manner, we have

$$\frac{s_{i+1}}{s_i} = \frac{1 - \xi_{i+1,i}^{n-2}}{\delta - 2 + \xi_{i+1,i+1}^{n-2}} > \frac{1 + \overline{\xi}_{n-2}}{\delta - 2 - \overline{\xi}_{n-2}} = \frac{2^{n-2}(1-h(\delta))}{\delta - 2^{n-1} + (2^{n-1}-1)h(\delta)} > z_1(n).$$

Thus, the lemma is established. □

We remark that the functions $z_1(n)$ and $z_2(n)$ satisfy
$$0 < z_1(n) < \frac{1}{2} < z_2(n) < 1.$$

Proof of Proposition 4.15. — Recall that δ satisfies $2^n - 1 < \delta < 2^n$ from Lemma 6.3. Moreover, it follows from Lemma 6.4 that $s_\ell \neq 0$ for any ℓ. Thus Lemma 4.12 yields $\Gamma_1(\tau) \cap P(\tau) = \emptyset$. □

Next we prove Proposition 5.9.

Lemma 6.5. — *For any $n \geqslant 2$, we have the following two inequalities:*

(1) $g_1(n) < 0$, *where* $g_1(n) := \dfrac{1}{\delta^3 - 1} + 1 - \delta \cdot z_1(n)^{n-1}$,

(2) $g_2(n) > 0$, *where* $g_2(n) := z_1(n)^{n-2} - z_2(n)^{n-1} - \dfrac{1}{\delta^3 - 1}$.

Proof. — First, we claim that the following inequality holds:
$$z_1(n)^{n-1} > \frac{1}{2^{n-1}} - (n-1)\Big(\frac{1}{2^{3n-4}} + \frac{1}{2^{4n-3}}\Big). \tag{6.4}$$
Indeed, since
$$\Big(1 - \frac{1}{2^{n-1}} - \frac{1}{2^{2n-1}}\Big)\Big(1 + \frac{1}{2^{n-1}} + \frac{6}{2^{2n}}\Big) = 1 - \frac{1}{2^{4n-2}}(2^{n+2} + 3) \leqslant 1,$$
one has
$$\begin{aligned} z_1(n) &\geqslant \frac{1}{2}\Big(1 + \frac{1}{2^{n-1}}\Big)\Big(1 - \frac{1}{2^{n-1}} - \frac{1}{2^{2n-1}}\Big) = \frac{1}{2}\Big\{1 - \big(\frac{3}{2^{2n-1}} + \frac{1}{2^{3n-2}}\big)\Big\} \\ &> \frac{1}{2}\Big\{1 - \big(\frac{1}{2^{2n-3}} + \frac{1}{2^{3n-2}}\big)\Big\}. \end{aligned}$$
Therefore, the claim holds from the Bernoulli inequality, namely, $(1+x)^n \geqslant 1 + nx$ for any $x \geqslant -1$. By using inequality (6.4), we prove the two inequalities in the lemma.

In order to prove assertion (1), we consider the function of n:
$$\check{g}_1(n) := \frac{1}{(2^n - 1)^3 - 1} + 1 - (2^n - 1) \cdot z_1(n)^{n-1}.$$
Then the inequality $g_1(n) < \check{g}_1(n)$ holds since $\delta > 2^n - 1$. Moreover, as $\check{g}_1(2) < 0$, one has $g_1(2) < 0$. On the other hand, when $n \geqslant 3$, inequality (6.4) yields
$$\begin{aligned} \check{g}_1(n) &< \frac{(2^n - 1)^3}{(2^n - 1)^3 - 1} - \frac{2^n - 1}{2^{n-1}}\Big\{1 - (n-1)\big(\frac{1}{2^{2n-3}} + \frac{1}{2^{3n-2}}\big)\Big\} \\ &< \frac{2^n - 1}{2^{n-1}}\Big(-1 + \frac{2^{n-1}(2^n - 1)^2}{(2^n - 1)^3 - 1} + \frac{n-1}{2^{2n-3}} + \frac{n-1}{2^{3n-2}}\Big). \end{aligned}$$

Since the terms $\frac{2^{n-1}(2^n-1)^2}{(2^n-1)^3-1}$, $\frac{n-1}{2^{2n-3}}$ and $\frac{n-1}{2^{3n-2}}$ are monotone decreasing with respect to n, the function $-1+\frac{2^{n-1}(2^n-1)^2}{(2^n-1)^3-1}+\frac{n-1}{2^{2n-3}}+\frac{n-1}{2^{3n-2}}$ is maximized when $n=3$, which is negative. Therefore, we have $\check{g}_1(n)<0$, and thus $g_1(n)<0$.

Finally, in order to prove assertion (2), we consider the function of n:

$$\check{g}_2(n) := z_1(n)^{n-2} - z_2(n)^{n-1} - \frac{1}{(2^n-1)^3-1}.$$

Then the inequality $g_2(n) > \check{g}_2(n)$ holds since $\delta > 2^n-1$. Moreover, as $\check{g}_2(2), \check{g}_2(3) > 0$, one has $g_2(2), g_2(3) > 0$. On the other hand, when $n \geqslant 4$, $\check{g}_2(n)$ can be estimated as

$$\begin{aligned}
\check{g}_2(n) &= z_1(n)^{n-2}\big(1-z_1(n)\big) - \big(z_2(n)^{n-1} - z_1(n)^{n-1}\big) - \frac{1}{(2^n-1)^3-1}\\
&\geqslant z_1(n)^{n-2}\big(1-z_1(n)\big) - (n-1)\big(z_2(n)-z_1(n)\big)z_2(n)^{n-2}\\
&\qquad\qquad - \frac{1}{(2^n-1)^3-1},
\end{aligned}$$

where the last inequality follows from the general inequality $x^n - y^n \leqslant n(x-y)x^{n-1}$ for any $x \geqslant y \geqslant 0$. Since $z_2(n)-z_1(n) = \frac{9}{2}\frac{2^{2n}+2^{n+1}+4}{(2^{2n}+2^{n+1}+3)(2^{2n}+2^{n+1}+6)} < \frac{9}{2}\frac{1}{2^{2n}+2^{n+1}+3} < \frac{9}{8}\frac{1}{2^{2(n-1)}}$, and $z_2(n) = \frac{1}{2}+\frac{3}{2}\frac{1}{2^{2n}+2^{n+1}+3}$ is monotone decreasing with respect to n, and thus is less than $\frac{13}{24}$, we have

$$(n-1)\big(z_2(n)-z_1(n)\big)z_2(n)^{n-2} < (n-1)\frac{9}{8}\Big(\frac{13}{24}\Big)^{n-2}\frac{1}{2^{2(n-1)}} < \frac{1}{2^{2(n-1)}},$$

where we use the fact that the function $(n-1)\frac{9}{8}\left(\frac{13}{24}\right)^{n-2}$ is monotone decreasing and is less than 1. Moreover, as $1-z_1(n) > z_1(n)$, one has

$$\begin{aligned}
\check{g}_2(n) &> z_1(n)^{n-1} - \frac{1}{2^{2(n-1)}} - \frac{1}{(2^n-1)^3-1}\\
&> \frac{1}{2^{n-1}}\Big\{1-(n-1)\Big(\frac{1}{2^{2n-3}}+\frac{1}{2^{3n-2}}\Big)\Big\} - \frac{1}{2^{2(n-1)}} - \frac{1}{(2^n-1)^3-1}\\
&= \frac{1}{2^{n-1}}\Big(1-\frac{n-1}{2^{2n-3}}-\frac{n-1}{2^{3n-2}}-\frac{1}{2^{n-1}}-\frac{2^{n-1}}{(2^n-1)^3-1}\Big).
\end{aligned}$$

Since the terms $\frac{n-1}{2^{2n-3}}$, $\frac{n-1}{2^{3n-2}}$, $\frac{1}{2^{n-1}}$ and $\frac{2^{n-1}}{(2^n-1)^3-1}$ are monotone decreasing with respect to n, the function $1-\frac{n-1}{2^{2n-3}}-\frac{n-1}{2^{3n-2}}-\frac{1}{2^{n-1}}-\frac{2^{n-1}}{(2^n-1)^3-1}$ is minimized when $n=4$, which is positive. Therefore, we have $\check{g}_2(n)>0$ and thus $g_2(n)>0$, and so the proof is complete. □

Proposition 6.6. — *Assume that* $v_{\iota'}(\delta) = \delta^k \cdot v_\iota(\delta)$. *Then we have* $k = 0$, $i'_m = i_m$ *and* $\kappa(\sigma^m(\iota')) = \kappa(\sigma^m(\iota))$ *for any* $m \geqslant 0$.

Proof. — Viewing $v_{\iota'}(\delta)/(\delta-1)$ and $\delta^k \cdot v_\iota(\delta)/(\delta-1)$ as functions of δ (see (4.6))), we expand them into Taylor series around infinity:

$$\frac{v_{\iota'}(\delta)}{\delta-1} = -s_{i'_1} \cdot \delta^{-\varepsilon_1(\iota')} - \cdots - s_{i'_{|\iota'|}} \cdot \delta^{-\varepsilon_{|\iota'|}(\iota')} - s_{i'_{|\iota'|+1}} \cdot \delta^{-\varepsilon_{|\iota'|+1}(\iota')} - \cdots ,$$

$$\frac{\delta^k \cdot v_\iota(\delta)}{\delta-1} = -s_{i_1} \cdot \delta^{-\varepsilon_1(\iota)+k} - \cdots - s_{i_{|\iota|}} \cdot \delta^{-\varepsilon_{|\iota|}(\iota)+k} - \cdots .$$

In view of these expressions, the coefficient of δ^{-l} is either $-s_\bullet$ or 0. Now assume the contrary that $v_{\iota'}(\delta)/(\delta-1)$ and $\delta^k \cdot v_\iota(\delta)/(\delta-1)$ have different coefficients. Let l_1 and l_2 be the minimal integers such that $v_{\iota'}(\delta)/(\delta-1)$ and $\delta^k \cdot v_\iota(\delta)/(\delta-1)$ have the coefficient $-s_{m_1}$ of δ^{-l_1} and the coefficient $-s_{m_2}$ of δ^{-l_2} which are different from the coefficient of δ^{-l_1} in $\delta^k \cdot v_\iota(\delta)/(\delta-1)$ and the coefficient of δ^{-l_2} in $v_{\iota'}(\delta)/(\delta-1)$ for some $1 \leqslant m_1 \leqslant n$ and $1 \leqslant m_2 \leqslant n$ respectively. Note that $s_1 > s_2 > \cdots > s_n$ and $\varepsilon_{m+1}(\iota'') - \varepsilon_m(\iota'') \geqslant 3$ for any $m \geqslant 1$ and $\iota'' \in \mathcal{K}(n)$. Thus, $v_{\iota'}(\delta)/(\delta-1) - \delta^k \cdot v_\iota(\delta)/(\delta-1) = 0$ satisfies the estimates

$$\begin{aligned} s_{m_1}\delta^{-l_1} - s_{m_2}\delta^{-l_2} - s_1\frac{\delta^{-l_2}}{\delta^3-1} &< \frac{v_{\iota'}(\delta)}{\delta-1} - \frac{\delta^k \cdot v_\iota(\delta)}{\delta-1} \\ &< s_{m_1}\delta^{-l_1} - s_{m_2}\delta^{-l_2} + s_1\frac{\delta^{-l_1}}{\delta^3-1}. \end{aligned}$$

If $l_1 > l_2$, then it follows that

$$\begin{aligned} 0 &< s_{m_1}\delta^{-l_1} - s_{m_2}\delta^{-l_2} + s_1\frac{\delta^{-l_1}}{\delta^3-1} \\ &< s_1\delta^{-l_1} - s_1 z_1(n)^{n-1}\delta^{-l_1+1} + s_1\frac{\delta^{-l_1}}{\delta^3-1} \\ &< s_1\delta^{-l_1} g_1(n), \end{aligned}$$

which contradicts Lemma 6.5. On the other hand, if $l_1 = l_2$ and $m_1 > m_2$, then we have

$$\begin{aligned} 0 &< s_{m_1}\delta^{-l_1} - s_{m_2}\delta^{-l_1} + s_1\frac{\delta^{-l_1}}{\delta^3-1} \\ &< s_1 z_2(n)^{m_1-1}\delta^{-l_1} - s_1 z_1(n)^{m_1-2}\delta^{-l_1} + s_1\frac{\delta^{-l_1}}{\delta^3-1} \\ &< -s_1\delta^{-l_1} g_2(n), \end{aligned}$$

where the last inequality is a consequence of the fact that $z_2(n)^{m_1-1} - z_1(n)^{m_1-2} = -z_2(n)^{m_1-2}\left((\frac{z_1(n)}{z_2(n)})^{m_1-2} - z_2(n)\right)$ is monotone increasing with respect to m_1 since $0 < z_2(n), \frac{z_1(n)}{z_2(n)} < 1$ and $(\frac{z_1(n)}{z_2(n)})^{m_1-2} - z_2(n) > \frac{g_2(n)}{z_2(n)^{m_1-2}} > 0$. This contradicts Lemma 6.5. In a similar manner, if $l_1 < l_2$, then it follows that

$$\begin{aligned}0 &> s_{m_1}\delta^{-l_1} - s_{m_2}\delta^{-l_2} - s_1\frac{\delta^{-l_2}}{\delta^3-1}\\ &> s_1 z_1(n)^{n-1}\delta^{-l_2+1} - s_1\delta^{-l_2} - s_1\frac{\delta^{-l_2}}{\delta^3-1}\\ &> -s_1\delta^{-l_2}g_1(n),\end{aligned}$$

which is a contradiction. On the other hand, if $l_1 = l_2$ and $m_1 < m_2$, then we have

$$\begin{aligned}0 &> s_{m_1}\delta^{-l_2} - s_{m_2}\delta^{-l_2} - s_1\frac{\delta^{-l_2}}{\delta^3-1}\\ &> s_1 z_1(n)^{m_2-2}\delta^{-l_2} - s_1 z_2(n)^{m_2-1}\delta^{-l_2} - s_1\frac{\delta^{-l_2}}{\delta^3-1}\\ &> s_1\delta^{-l_2}g_2(n),\end{aligned}$$

which is also a contradiction. Thus, $v_{\iota'}(\delta)/(\delta-1)$ and $\delta^k \cdot v_\iota(\delta)/(\delta-1)$ have the same coefficients. In particular, we have $i'_m = i_m$ and $\varepsilon_m(\iota') = \varepsilon_m(\iota) - k$ for any $m \geqslant 1$. The relations $\varepsilon_m(\iota') = \varepsilon_m(\iota) - k$ yield $\kappa(\iota') = \varepsilon_1(\iota') = \varepsilon_1(\iota) - k = \kappa(\iota) - k$ and $\kappa(\sigma^m(\iota')) = \kappa(\sigma^m(\iota))$ for any $m \geqslant 1$. Now, $L \geqslant 1$ is chosen so that $\sigma^L = \mathrm{id}$. Then we have $i' = i'_L = i_L = i$ and $\kappa(\iota') = \kappa(\sigma^L(\iota')) = \kappa(\sigma^L(\iota)) = \kappa(\iota)$, which shows that $k = 0$. Therefore, the proposition is established. □

Proof of Proposition 5.9. — From Proposition 6.6, if the relation $\delta^k \cdot v_{\overline{\iota}}(\delta) = v_{\overline{\iota}'}(\delta)$ holds, then one has $k = 0$, $i_m = i'_m$ and $\kappa(\sigma^m(\iota)) = \kappa(\sigma^m(\iota'))$ for any $m \geqslant 0$. Conversely, it is easily seen that if $i_m = i'_m$ and $\kappa(\sigma^m(\iota)) = \kappa(\sigma^m(\iota'))$ for any $m \geqslant 0$ then $v_\iota(\delta) = v_{\iota'}(\delta)$ holds. In particular, it follows from Lemma 5.1 that $\overline{\Gamma}_2(\tau) \cap P(\tau) = \{\alpha^0_{\iota,\iota'} \,|\, i_m = i'_m, \kappa(\sigma^m(\iota)) = \kappa(\sigma^m(\iota')), m \geqslant 0\}$ and hence $(\overline{\Gamma}_2(\tau) \cap P(\tau)) \cap \Gamma^1_2(\tau) = \emptyset$. Moreover, if τ satisfies condition (3) in Theorem 1.6, then any element $\alpha^0_{\iota,\iota'} \in \overline{\Gamma}_2(\tau) \cap P(\tau)$ does not belong to $\Gamma^2_2(\tau)$, which shows that $\Gamma_2(\tau) \cap P(\tau) = \emptyset$. Therefore we establish Proposition 5.9. □

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Takato UEHARA
Saga University
Department of Mathematics
1 Honjo-machi, Saga, 840-8502 (Japan)
tuehara@cc.saga-u.ac.jp