ANNALES

DE

L'INSTITUT FOURIER

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Tome 65, nº 6 (2015), p. 2601-2612.

<http://aif.cedram.org/item?id=AIF_2015__65_6_2601_0>

Ann. Inst. Fourier, Grenoble
65, 6 (2015) 2601-2612

CHARACTER VARIETIES OF VIRTUALLY NILPOTENT KÄHLER GROUPS AND G–HIGGS BUNDLES

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ABSTRACT. — Let G be a connected complex reductive affine algebraic group, and let $K \subset G$ be a maximal compact subgroup. Let X be a compact connected Kähler manifold whose fundamental group Γ is virtually nilpotent. We prove that the character variety $\mathrm{Hom}(\Gamma,\, G)/\!\!/G$ admits a natural strong deformation retraction to the subset $\mathrm{Hom}(\Gamma,\, K)/K \subset \mathrm{Hom}(\Gamma,\, G)/\!\!/G$. The natural action of $\mathbb{C}^*$ on the moduli space of G–Higgs bundles over X extends to an action of $\mathbb{C}$. This produces the above mentioned deformation retraction.

RÉSUMÉ. — Soit G un groupe algébrique affine réductif complexe connexe, et soit $K \subset G$ un sous-groupe compact maximal. Soit X une variété Kählerienne compacte connexe dont le groupe fondamental Γ est virtuellement nilpotent. Nous montrons que la variété de caractères $\mathrm{Hom}(\Gamma,\, G)/\!\!/G$ admet une rétraction par déformation forte naturelle sur le sous-ensemble $\mathrm{Hom}(\Gamma,\, K)/K \subset \mathrm{Hom}(\Gamma,\, G)/\!\!/G$. L'action naturelle de $\mathbb{C}^*$ sur l'espace des modules de G-fibrés de Higgs sur X s'étend à une action de $\mathbb{C}$. Ceci produit la rétraction par déformation mentionnée ci-dessus.

1. Introduction

Let G be a complex reductive affine algebraic group, and let Γ be a finitely presentable group. Let $\mathcal{R}_\Gamma(G) := \mathrm{Hom}(\Gamma, G)/\!\!/G$ be the geometric invariant theoretic (GIT) quotient, of the space of all homomorphisms from Γ to G, for the conjugation action of G; it is known as the G–character variety of Γ. These moduli spaces $\mathcal{R}_\Gamma(G)$ play important roles in hyperbolic geometry

Keywords: Kähler group, character variety, G–Higgs bundle, virtually nilpotent group.
Math. classification: 20G20, 14J60.
(*) We thank the referee for helpful comments. The first author is supported by the J. C. Bose fellowship. The second author is partially supported by FCT (Portugal) through the projects EXCL/MAT-GEO/0222/2012, PTDC/MAT/120411/2010 and PTDC/MAT-GEO/0675/2012.

[7], the theory of bundles and connections [20], knot theory and quantum field theories [14] (see also the references in these papers).

Some particularly relevant cases of Γ include, for instance, the fundamental group of a compact connected Kähler manifold. These are called Kähler groups. If Γ is the fundamental group of a compact connected Kähler manifold X, then the corresponding character variety $\mathcal{R}_\Gamma(G)$ can be identified with a certain moduli space of G–Higgs bundles over X [15, 20, 6]; this identification is continuous but not holomorphic. Let K be a maximal compact subgroup of G. The above identification between $\mathcal{R}_{\pi_1(X,x_0)}(G) = \mathcal{R}_\Gamma(G)$ and a moduli space of G–Higgs bundles on X is an extension of the identification between $\mathrm{Hom}(\Gamma, K)/K$ and the moduli space of semistable principal G–bundles on X with vanishing characteristic classes of positive degrees [19], [9], [21], [17].

In investigations of the topology of $\mathcal{R}_\Gamma(G)$ there are some notable situations where the analogous orbit space $\mathcal{R}_\Gamma(K) := \mathrm{Hom}(\Gamma, K)/K$ is a strong deformation retract of $\mathcal{R}_\Gamma(G)$. This happens when Γ is a free group [10] or a free abelian group [12, 5, 18] or a nilpotent group [3]. It should be mentioned that such a deformation retraction is not to be expected for arbitrary finitely presented groups Γ, not even for general Kähler groups. For example, this fails for surface groups [4].

In this article, we consider the case where Γ is a virtually nilpotent Kähler group. This means that Γ is the fundamental group of a compact connected Kähler manifold and it has a finite index subgroup which is nilpotent. Since any finite group is the fundamental group of a complex projective manifold [1, p. 6, Example 1.11], this class of groups include all finite groups.

Our approach uses non-abelian Hodge theory. When applied to a general compact connected Kähler manifold X, the non-abelian Hodge theory provides a natural correspondence between the flat principal G–bundles over X and a certain class of G–Higgs bundles on X. If X is a smooth complex projective variety, then this correspondence sends the flat principal G–bundles on X to the semistable G–Higgs bundles on X with vanishing characteristic classes of positive degrees. More generally, if X is a compact connected Kähler manifold, then the same correspondence remains valid once semistability is replaced by pseudostability.

Let X be a compact connected Kähler manifold such that its fundamental group $\Gamma := \pi_1(X, x_0)$ is virtually nilpotent. Take any homomorphism $\rho : \Gamma \longrightarrow G$. Let (F_G, θ) be the pseudostable G–Higgs bundle on X associated to ρ. (If X is a complex projective manifold, then (F_G, θ) is semistable.) We prove that the underlying principal G–bundle F_G is pseudostable (see Proposition 3.2); in [13], a similar result is proved for Higgs G–bundles on elliptic curves. In view of the earlier mentioned identification between $\mathcal{R}_\Gamma(G)$ and a moduli space of G–Higgs bundles, this produces a multiplication action of $\mathbb{C}$ on $\mathcal{R}_\Gamma(G)$ using the multiplication of Higgs fields by the scalars. The action of $1 \in \mathbb{C}$ is the identity map of $\mathcal{R}_\Gamma(G)$, and the action of 0 is a retraction of $\mathcal{R}_\Gamma(G)$ to $\mathrm{Hom}(\Gamma, K)/K$; this property of the action of 0 is deduced from the earlier mentioned identification between $\mathrm{Hom}(\Gamma, K)/K$ and the moduli space of pseudostable principal G–bundles on X with vanishing characteristic classes of positive degrees. Also, the action of every element of $\mathbb{C}$ fixes the subset $\mathrm{Hom}(\Gamma, K)/K$ pointwise. Therefore, this action of $\mathbb{C}$ on $\mathcal{R}_\Gamma(G)$ produces a strong deformation retraction of $\mathcal{R}_\Gamma(G)$ to $\mathrm{Hom}(\Gamma, K)/K$ (see Theorem 4.2).

2. Kähler groups and Flat G–bundles

Let G be a linear algebraic group defined over $\mathbb{C}$. A Borel subgroup of G is a maximal Zariski closed connected solvable subgroup of G. Any two Borel subgroups of G are conjugate [16, p. 134, § 21.3, Theorem]. Let Γ be a finitely presentable group.

2.1. Homomorphisms of virtually nilpotent Kähler groups

Recall that Γ is called *virtually solvable* (respectively, *virtually nilpotent*) if there is a finite index subgroup

$$\Gamma_1 \subset \Gamma$$

such that Γ_1 is a solvable (respectively, nilpotent) group.

Lemma 2.1. — *Let Γ be a virtually solvable group. Then, for any homomorphism $\rho : \Gamma \longrightarrow G$, there is a finite index subgroup*

$$\Gamma_0 \subset \Gamma\,,$$

and also a Borel subgroup $B \subset G$, such that $\rho(\Gamma_0) \subset B$.

Proof. — Since Γ is virtually solvable, there is a solvable subgroup $\Gamma_1 \subset \Gamma$ of finite index. Let H denote the Zariski closure of the image $\rho(\Gamma_1)$. In particular, H is an algebraic subgroup of G. Moreover, H is a solvable subgroup of G because Γ_1 is solvable. Let

$$H_0 \subset H$$

be the connected component of H containing the identity element. Since H_0 is a connected solvable subgroup of G, there is a Borel subgroup $B \subset G$ with

$$H_0 \subset B\,.$$

The group H has only finitely many connected components because it is algebraic. This implies that

$$\Gamma' := H_0 \bigcap \rho(\Gamma_1) \subset \rho(\Gamma_1)$$

is a finite index subgroup of $\rho(\Gamma_1)$. Indeed, the index of Γ' in $\rho(\Gamma_1)$ coincides with the number of connected components of H.

Now define

$$\Gamma_0 := \rho^{-1}(\Gamma') \bigcap \Gamma_1 = \rho^{-1}(H_0) \bigcap \Gamma_1 \subset \Gamma_1.$$

The index of the subgroup $\Gamma_0 \subset \Gamma_1$ coincides with the index of the subgroup $\Gamma' \subset \rho(\Gamma_1)$. In particular, Γ_0 is a subgroup of Γ_1 of finite index. Since Γ_1 is a subgroup of Γ of finite index, we now conclude that Γ_0 is a finite index subgroup of Γ. We also have $\rho(\Gamma_0) \subset H_0 \subset B$, so the proof is complete. □

By a *Kähler group* we mean a finitely presentable group isomorphic to the fundamental group $\pi_1(X, x_0)$ of some compact connected Kähler manifold X, where $x_0 \in X$ is a base point.

Remark 2.2. — Suppose that Γ is a virtually solvable Kähler group, and $\Gamma_1 \subset \Gamma$ is a solvable subgroup of finite index. Then Γ_1 is a solvable Kähler group, because finite index subgroups of Kähler groups are also Kähler groups. By a recent result of Delzant (see [8]), Γ_1 is virtually nilpotent. So there is a subgroup $\Gamma_2 \subset \Gamma_1$ of finite index such that Γ_2 is nilpotent. Therefore, Γ itself is virtually nilpotent.

In view of Remark 2.2, while considering virtually solvable Kähler groups, we can restrict ourselves to virtually nilpotent Kähler groups.

The following proposition is immediate from Lemma 2.1.

Proposition 2.3. — *Let X be a compact connected Kähler manifold with a virtually nilpotent fundamental group $\pi_1(X, x_0)$. Let $\rho : \pi_1(X, x_0) \longrightarrow G$ be a homomorphism. Then there is a finite index subgroup*

$$\Gamma_0 \subset \pi_1(X, x_0)$$

and a Borel subgroup $B \subset G$, such that $\rho(\Gamma_0) \subset B$.

2.2. Flat principal G–bundles and G–Higgs bundles

Let G be a connected linear algebraic group defined over $\mathbb{C}$. Let X be a compact connected Kähler manifold equipped with a Kähler form ω. There is an equivalence between the category of pseudostable Higgs G–bundles on X with vanishing characteristic classes of positive degrees and the category of flat principal G–bundles on X [6, p. 20, Theorem 1.1].

Let $p : Y \longrightarrow X$ be a finite étale covering with Y connected. So $(Y, p^*\omega)$ is a compact connected Kähler manifold.

Lemma 2.4. — *Let (E_G, ∇) be a flat principal G–bundle on X, and let (F_G, θ) be the pseudostable Higgs G–bundle over X associated to (E_G, ∇). Then the pullback $(p^*F_G, p^*\theta)$ is isomorphic to the pseudostable Higgs G–bundle over Y associated to the flat principal G–bundle $(p^*E_G, p^*\nabla)$.*

Proof. — First assume that G is reductive. We recall that a flat G–connection on X is called irreducible if it does not admit a reduction of structure group to a proper parabolic subgroup of G. A flat G–connection is called completely reducible if it admits a reduction of structure group to a Levi subgroup of a parabolic subgroup of G such that the reduction is irreducible.

Let (E', ∇') be a completely reducible flat principal G–bundle on X. Suppose that

$$h = E_K \subset E'$$

is a harmonic metric on (E', ∇'). Then clearly $p^*h = p^*E_K \subset p^*E'$ is a harmonic metric on the flat principal G–bundle $(p^*E', p^*\nabla')$ on Y.

On the other hand, if h_1 is a Hermitian structure on a polystable Higgs G–bundle (F', θ') on X that satisfies the Yang-Mills-Higgs equation, then the pulled back Hermitian structure p^*h_1 on $(p^*F', p^*\theta')$ also satisfies the Yang-Mills-Higgs equation. From this it follows immediately that the correspondence in [20, p. 36, Lemma 3.5] is compatible with taking finite étale coverings.

The correspondence in [6, p. 20, Theorem 1.1] for general G is constructed from the correspondence in [20, p. 36, Lemma 3.5]. Therefore, it is also compatible with taking finite étale coverings. In particular, the pseudostable Higgs G–bundle on Y associated to the flat principal G–bundle $(p^*E_G, p^*\nabla)$ coincides with the pullback $(p^*F_G, p^*\theta)$, where (F_G, θ) as before is the pseudostable Higgs G–bundle over X associated to (E_G, ∇). $\square$

3. Semistability of holomorphic G–bundles underlying Flat G–bundles

From now on, G will be assumed to be connected and reductive.

We start, for simplicity, with the projective case. So, let X denote a connected smooth complex projective variety such that $\pi_1(X, x_0)$ is virtually nilpotent. Note that for any finite index subgroup of $\Gamma_0 \subset \pi_1(X, x_0)$, the covering of X associated to Γ_0 is also a connected smooth complex projective variety.

To define (semi)stability of bundles on X, we need to fix a polarization on X (first Chern class of an ample line bundle) in order to compute the degree of torsionfree coherent sheaves on X. However, for bundles on X with vanishing characteristic classes of positive degrees, the notion of (semi)stability is independent of the choice of polarization. Since we are solely dealing with bundles with vanishing characteristic classes of positive degrees, we will not refer to a particular choice of polarization.

Proposition 3.1. — *Let X be a connected smooth complex projective variety such that $\pi_1(X, x_0)$ is virtually nilpotent. Let (F_G, θ) be a semistable G–Higgs bundle on X whose characteristic classes of positive degrees vanish. Then the holomorphic principal G–bundle F_G is semistable.*

Proof. — Let (E_G, ∇) be the flat principal G–bundle over X corresponding to the given semistable G–Higgs bundle (F_G, θ). Suppose that (E_G, ∇) is given by the homomorphism (its monodromy representation)

$$\rho : \pi_1(X, x_0) \longrightarrow G. \tag{3.1}$$

Since $\pi_1(X, x_0)$ is virtually nilpotent, from Proposition 2.3 we know that there is a finite index subgroup

$$\Gamma_0 \subset \pi_1(X, x_0)$$

and a Borel subgroup $B \subset G$, such that

$$\rho(\Gamma_0) \subset B. \tag{3.2}$$

Let

$$p : Y \longrightarrow X \tag{3.3}$$

be the finite étale covering corresponding to the subgroup Γ_0 in (3.2). We note that Y is a connected smooth complex projective variety.

Let $y_0 \in p^{-1}(x_0) \subset Y$ be the base point of the covering Y. Consider the homomorphism

$$\rho' : \pi_1(Y, y_0) = \Gamma_0 \xrightarrow{\rho|_{\Gamma_0}} B$$

(see (3.2)). Let $(E_B\,,\nabla^B)$ be the flat principal B–bundle on Y associated to ρ'. Let $(F_B\,,\theta_B)$ be the semistable B–Higgs on Y corresponding to $(E_B\,,\nabla^B)$. It should be clarified that from [6, p. 26, Proposition 2.4] we know that a Higgs principal bundle on X with vanishing characteristic classes of positive degrees is semistable if and only if it is pseudostable.

Note that $(p^*E_G\,,p^*\nabla)$ is identified with the flat principal G–bundle on Y obtained by extending the structure group of the flat principal B–bundle $(E_B\,,\nabla^B)$ using the inclusion of B in G. The correspondence in [6, p. 20, Theorem 1.1] is compatible with extensions of structure group. Therefore, using Proposition 2.4 we know that the pullback $(p^*F_G\,,p^*\theta)$ is identified with the G–Higgs bundle obtained by extending the structure group of the Higgs B–bundle $(F_B\,,\theta_B)$ using the inclusion of B in G.

For any holomorphic character

$$\chi\,:\,B\,\longrightarrow\,\mathbb{C}^*$$

of B, we have

$$c_1(F_B\times^\chi\mathbb{C})\,=\,0\,, \tag{3.4}$$

where $F_B\times^\chi\mathbb{C}$ is the holomorphic line bundle on Y associated to the principal B–bundle F_B for the character χ [6, p. 20, Theorem 1.1].

Let $\mathfrak{g}$ denote the Lie algebra of G. We will consider $\mathfrak{g}$ as a B–module using the adjoint action. Since B is solvable, there is a filtration of B–modules

$$0\,=\,V_0\,\subset\,V_1\,\subset\,V_2\,\subset\,\cdots\,\subset\,V_{d-1}\,\subset\,V_d\,=\,\mathfrak{g}\,, \tag{3.5}$$

where $d\,=\,\dim_{\mathbb{C}}\mathfrak{g}$, such that each successive quotient V_i/V_{i-1}, $1\,\leqslant\,i\,\leqslant\,d$, is a B–module of dimension one.

Let

$$W\,:=\,F_B\times^B\mathfrak{g}\,\longrightarrow\,Y$$

be the holomorphic vector bundle on Y associated to the principal B–bundle F_B for the above B–module $\mathfrak{g}$. We note that this holomorphic vector bundle W is identified with the adjoint vector bundle

$$(p^*F_G)\times^G\mathfrak{g}\,=\,\mathrm{ad}(p^*F_G)\,=\,p^*\mathrm{ad}(F_G)$$

for the principal G–bundle p^*F_G, because p^*F_G the the extension of structure group of F_B constructed using the inclusion of B in G. So, we write

$$W\,=\,\mathrm{ad}(p^*F_G)\,=\,p^*\mathrm{ad}(F_G)\,. \tag{3.6}$$

For $0\,\leqslant\,i\,\leqslant\,d$, let

$$W_i\,:=\,F_B\times^B V_i\,\longrightarrow\,Y$$

be the holomorphic vector bundle associated to the principal B–bundle F_B for the B–module V_i in (3.5). The filtration of B–modules in (3.5) produces a filtration of W by holomorphic vector subbundles

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{d-1} \subset W_d = W = p^*\mathrm{ad}(F_G)\,, \tag{3.7}$$

where $\mathrm{rank}(W_i) = i$ (see (3.6)).

For any $1 \leqslant i \leqslant d$, the line bundle W_i/W_{i-1} on Y coincides with the one associated to the principal B–bundle F_B for the B–module V_i/V_{i-1}. Therefore, from (3.4) we conclude that

$$c_1(W_i/W_{i-1}) = 0 \tag{3.8}$$

for all $1 \leqslant i \leqslant d$.

From (3.7) and (3.8) we conclude that the vector bundle $p^*\mathrm{ad}(F_G)$ is semistable. This implies that $\mathrm{ad}(F_G)$ is semistable. Indeed, if a subsheaf $V' \subset \mathrm{ad}(F_G)$ contradicts the semistability of $\mathrm{ad}(F_G)$, then the pullback p^*V' contradicts the semistability of $p^*\mathrm{ad}(F_G)$. Since $\mathrm{ad}(F_G)$ is semistable, we conclude that the principal G–bundle F_G is semistable [2, p. 214, Proposition 2.10]. □

Let M be a compact connected Kähler manifold equipped with a Kähler form. A Higgs vector bundle $(E\,,\theta)$ over M is called *pseudostable* if E admits a filtration by holomorphic subbundles

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = E$$

such that

(1) $\theta(F_i) \subset F_i \otimes \Omega^1_M$ for all $i \in [1\,,n]$,
(2) for each integer $i \in [1\,,n]$, the Higgs vector bundle defined by the quotient F_i/F_{i-1} equipped with the Higgs field induced by θ is stable, and
(3) $\mathrm{degree}(F_1)/\mathrm{rank}(F_1) = \cdots = \mathrm{degree}(F_n)/\mathrm{rank}(F_n)$, where the degree is defined using the Kähler form on M.

A pseudostable Higgs vector bundle is semistable (see [6]). A holomorphic vector bundle E on M is called pseudostable if the Higgs vector bundle $(E\,,0)$ is pseudostable. A G–Higgs bundle $(E_G\,,\theta)$ on M is called *pseudostable* if the adjoint vector bundle $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ equipped with the Higgs field induced by θ is pseudostable. A holomorphic principal G–bundle E_G on M is called pseudostable if the G–Higgs bundle $(E_G\,,0)$ is pseudostable.

Proposition 3.2. — *Let X be a compact connected Kähler manifold such that $\pi_1(X,x_0)$ is virtually nilpotent. Let $(F_G\,,\theta)$ be a pseudostable*

G–Higgs bundle on X with zero characteristic classes of positive degrees. Then the holomorphic principal G–bundle F_G is pseudostable.

Proof. — The proof of Proposition 3.2 is very similar to the proof of Proposition 3.1. Since $(F_G\,,\theta)$ is pseudostable with vanishing characteristic classes of positive degrees, it corresponds to a flat G–bundle on X [6, p. 20, Theorem 1.1]. Let $(E_G\,,\nabla)$ be the flat G–bundle corresponding to $(F_G\,,\theta)$. Construct ρ as in (3.1). Define Γ_0 as in (3.2), and consider p as in (3.3). Now the proof proceeds exactly as the proof of Proposition 3.1 does. The only point to note is that the adjoint vector bundle $\mathrm{ad}(F_G)$, which we get at the end, is pseudostable. But this means that F_G is pseudostable (see the above definition). □

4. Deformation retraction of character varieties

As before, X is a compact connected Kähler manifold such that $\Gamma := \pi_1(X, x_0)$ is virtually nilpotent. Since Γ is a finitely presented group, and G is an affine algebraic group, the representation space

$$\widetilde{\mathcal{R}}_\Gamma(G) := \mathrm{Hom}(\Gamma,\, G)$$

is an affine algebraic scheme over $\mathbb{C}$. The reductive group G acts on $\widetilde{\mathcal{R}}_\Gamma(G)$ via the conjugation action of G on itself. The geometric invariant theoretic quotient

$$\mathcal{R}_\Gamma(G) := \widetilde{\mathcal{R}}_\Gamma(G)/\!\!/G \tag{4.1}$$

is also an affine algebraic scheme over $\mathbb{C}$.

Fix a maximal compact subgroup

$$K \subset G\,.$$

Let

$$\mathcal{R}_\Gamma(K) := \mathrm{Hom}(\Gamma,\, K)/K \tag{4.2}$$

be the space of all equivalence classes of homomorphisms from $\Gamma = \pi_1(X, x_0)$ to K. The inclusion of K in G produces an inclusion

$$\mathcal{R}_\Gamma(K) \hookrightarrow \mathcal{R}_\Gamma(G)\,, \tag{4.3}$$

where $\mathcal{R}_\Gamma(G)$ and $\mathcal{R}_\Gamma(K)$ are constructed in (4.1) and (4.2) respectively (see [11, Proposition 4.5], [11, Theorem 4.3]).

Theorem 4.1. — *Let X be a connected smooth complex projective variety with a virtually nilpotent fundamental group $\Gamma = \pi_1(X, x_0)$. Let G be a reductive linear algebraic group. The character variety* $\mathrm{Hom}(\Gamma,\, G)/\!\!/G$ *admits a strong deformation retraction to the subset*

$$\mathrm{Hom}(\Gamma,\, K)/K \subset \mathrm{Hom}(\Gamma,\, G)/\!\!/G\,.$$

Proof. — We will construct a continuous map

$$\Phi\,:\,\mathbb{C}\times\mathcal{R}_\Gamma(G) \longrightarrow \mathcal{R}_\Gamma(G)\,. \tag{4.4}$$

Take any $\rho \in \mathcal{R}_\Gamma(G)$. Choose a homomorphism $\widetilde{\rho} \in \widetilde{\mathcal{R}}_\Gamma(G)$ (see (4.1)) that projects to ρ. Let $(E_G\,,\nabla)$ be the flat principal G–bundle on X associated to $\widetilde{\rho}$. Let $(F_G\,,\theta)$ be the semistable G–Higgs bundle on X associated to the flat principal G–bundle $(E_G\,,\nabla)$. From Proposition 3.1 we know that F_G is semistable. Therefore, $(F_G\,,\lambda\cdot\theta)$ is a semistable Higgs G–bundle for every $\lambda \in \mathbb{C}$. Hence $(F_G\,,\lambda\cdot\theta)$ corresponds to a flat principal G–bundle on X. Let $(E^\lambda_G\,,\nabla^\lambda)$ be the flat principal G–bundle on X corresponding to $(F_G\,,\lambda\cdot\theta)$. The map Φ in (4.4) sends the point $(\lambda\,,\rho) \in \mathbb{C}\times\mathcal{R}_\Gamma(G)$ to the monodromy representation of the flat connection $(E^\lambda_G\,,\nabla^\lambda)$. The bijection between $\mathcal{R}_\Gamma(G)$ and the moduli space of semistable G–Higgs bundles on X with vanishing characteristic classes of positive degrees is continuous. Also, the action of $\mathbb{C}$ on this moduli space of semistable G–Higgs bundles is continuous. Therefore, Φ is a continuous map.

Clearly, $\rho \longmapsto \Phi(1\,,\rho)$ is the identity map of $\mathcal{R}_\Gamma(G)$. We have $\Phi(\lambda\,,\rho) = \rho$ for every $\rho \in \mathcal{R}_\Gamma(K)$ and $\lambda \in \mathbb{C}$. Also

$$\rho \longmapsto \Phi(0\,,\rho)$$

is a retraction to the subset $\mathcal{R}_\Gamma(K)$ in (4.3). □

Theorem 4.2. — *Let X be a compact connected Kähler manifold with a virtually nilpotent fundamental group $\Gamma = \pi_1(X, x_0)$. Let G be a reductive linear algebraic group. Then the character variety* $\mathrm{Hom}(\Gamma,\, G)/\!\!/G$ *admits a strong deformation retraction to the subset*

$$\mathrm{Hom}(\Gamma,\, K)/K \subset \mathrm{Hom}(\Gamma,\, G)/\!\!/G\,.$$

In view of Proposition 3.2, the proof of Theorem 4.2 is identical to the proof of Theorem 4.1.

Remark 4.3. — In Theorem 4.1, the assumption that $\pi_1(X, x_0)$ is virtually nilpotent is only used in deducing the following: if $(E_G\,,\theta)$ is a semistable G–Higgs bundle on the complex projective manifold X such that all the characteristic classes of E_G of positive degree vanish, then the principal G–bundle E_G is semistable. Similarly, in Theorem 4.2, the assumption that $\pi_1(X, x_0)$ is virtually nilpotent is only used in deducing the following: if (E_G, θ) is a pseudostable G–Higgs bundle on the compact connected Kähler manifold X such that all the characteristic classes of E_G of positive degree vanish, then the principal G–bundle E_G is pseudostable. It is natural to ask which other complex projective varieties (or compact connected Kähler manifolds) satisfy this condition on G–Higgs bundles.

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Manuscrit reçu le 16 juillet 2014,
révisé le 24 janvier 2015,
accepté le 24 février 2015.

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