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THE LATTICE POINT COUNTING PROBLEM ON THE HEISENBERG GROUPS

by Rahul GARG, Amos NEVO & Krystal TAYLOR (*)

Abstract. — We consider radial and Heisenberg-homogeneous norms on the Heisenberg groups given by $N_{\alpha,A}((z,t)) = (|z|^\alpha + A\,|t|^{\alpha/2})^{1/\alpha}$, for $\alpha \geqslant 2$ and $A > 0$. This natural family includes the canonical Cygan-Korányi norm, corresponding to $\alpha = 4$. We study the lattice points counting problem on the Heisenberg groups, namely establish an error estimate for the number of points that the lattice of integral points has in a ball of large radius R. The exponent we establish for the error in the case $\alpha = 2$ is the best possible, in all dimensions.

Résumé. — Nous considérons les normes radiales et Heisenberg-homogènes sur les groupes de Heisenberg données par $N_{\alpha,A}((z,t)) = \left(|z|^\alpha + A\,|t|^{\alpha/2}\right)^{1/\alpha}$, pour $\alpha \geqslant 2$ et $A > 0$. Cette famille naturelle inclut la norme canonique de Cygan-Korányi, qui correspond à $\alpha = 4$. Nous étudions le problème de dénombrement des points d'un réseau dans les groupes de Heisenberg, et nous établissons un terme d'erreur sur le nombre d'éléments du réseau des points entiers dans une boule de grand rayon R. L'exposant utilisé pour le terme d'erreur dans le cas $\alpha = 2$ est optimal, en toute dimension.

1. Introduction, notation, and statement of results

1.1. Euclidean and non-Euclidean lattice point counting problem

The classical lattice point counting problem in Euclidean space considers a fixed compact convex set $B \subset \mathbb{R}^n$ with $0 \in B$ an interior point, and aims to establish an asymptotic of the form

$$(1.1) \quad |\mathbb{Z}^n \cap tB| \;=\; t^n \operatorname{vol}(B) + O_{\theta'}\left(t^{n-\theta'}\right) \;=\; \operatorname{vol}(tB) + O_{\kappa'}\left((\operatorname{vol}(tB))^{\kappa'}\right)$$

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with $\theta' > 0$ as large as possible (or $\kappa' < 1$ as small as possible) for large parameter t. We let θ denote the supremum of θ' that are admissible in (1.1) (and κ the infimum of admissible κ').

This problem has a long history, and arises naturally in many applications. Among those, we mention just the following two.

- The problem of obtaining the asymptotics of the Laplace eigenvalue counting function for the torus $\mathbb{R}^n/L$, where L is a lattice, is equivalent to the lattice point counting problem in the ellipsoid associated with L. Thus here the error estimate in the lattice point counting problem amounts to estimating the error in Weyl's law for the corresponding torus.
- When B is given as the level set of a positive homogeneous form with integral coefficients and degree, for example $x_1^k+\dots+x_n^k$, the lattice point counting problem is equivalent to the fundamental number-theoretic problem of bounding the error term in the average number of representations of positive integers by the form.

Let us note that three of the motivating themes in the development of this subject have been :

(1) Obtaining error estimates which are as sharp as possible in the case of Euclidean balls $tB^n \subset \mathbb{R}^n$. Here the best possible value of θ has been obtained for $n \geqslant 4$, and it is $\theta = 2$ (namely $\kappa = \frac{n-2}{n}$). The conjectured value for $\mathbb{R}^2$ is $\theta = 3/2$ (namely $\kappa = 1/4$), and for $\mathbb{R}^3$ it is $\theta = 2$ (namely $\kappa = 1/3$). We refer to [17] and [14] for detailed information on the historical development and current best results.

(2) Obtaining error estimates for Euclidean dilates of a general smooth compact convex body in $\mathbb{R}^n$ whose boundary has everywhere non-vanishing Gaussian curvature. Starting with [13], [12], many different results have been obtained, including, for example, for ellipsoids and other bodies of revolution. For more information we refer to [14] and the references therein, including [2].

(3) Obtaining error estimates for certain special bodies whose boundary surface contains points with vanishing Gaussian curvature. These include the unit balls of ℓ^p-norms on $\mathbb{R}^n$ and some generalizations, and the effect of vanishing curvature on the error estimates have been investigated extensively in e.g. [18], [19], [20], [21], [22], [23], [25], [26], [27], [28], [29] and [30].

It is natural to consider the following considerably more general set-up. Let $\mathsf{G} \subset \mathsf{GL}_n$ denote any connected linear algebraic group defined over $\mathbb{Q}$, such that the integral points $\Gamma = \mathsf{G}(\mathbb{Z})$ form a lattice subgroup in the group $G = \mathsf{G}(\mathbb{R}) \subset GL_n(\mathbb{R})$ of real points. For interesting gauge functions, for example a natural left-invariant distance dist on G, a natural problem is to establish an asymptotic of the form

$$|\Gamma \cap B_t| = m_G(B_t) + O\left((m_G(B_t))^\kappa\right) \tag{1.2}$$

with $B_t = \{g \in G : \mathrm{dist}(g, e) < t\}$, and m_G Haar measure on G, normalized so that the measure of a fundamental domain of Γ in G has measure 1.

When the group in question is a (non-compact) semi-simple algebraic group, a general solution to the problem of estimating κ has been developed in [10]. For simple groups of real rank at least two, this estimate is the best currently available. However, it should be noted that the best possible κ has never been established even in a single example, for any left-invariant distance on any (non-compact) simple Lie group.

Our purpose in the present paper is to investigate aspects of the lattice point counting problem (1.2) on the Heisenberg groups. Let us first note that unlike the Euclidean case, this problem is completely different from the eigenvalue counting problem for the natural Laplacian on compact Heisenberg homogeneous spaces G/Γ. The latter problem has been studied in detail, see [15] and the references therein. We note further that there has been considerable recent interest in geometric group theory in specific lattice point counting results in the Heisenberg groups. These pertain to Carnot-Carathéodory distances arising from word metrics on the lattice subgroup, and we refer to [1], [7], and the references therein for more on this topic. These counting problems are completely different from those we will consider in the present paper.

The lattice point counting problem on the Heisenberg groups that we will consider is that of counting in balls defined by natural radial Heisenberg-homogeneous gauge functions, and as far as the authors are aware, no prior results have been established regarding this problem. Let us now turn to describe our set-up, notation and results.

1.2. The Heisenberg group

The Heisenberg group, denoted H_d, has several equivalent descriptions which we will use below. One is given by

$$\mathsf{H}_d \;=\; \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \;=\; \left\{(x, y, t) \,:\, x, y \in \mathbb{R}^d, t \in \mathbb{R}\right\}$$

with multiplication given by

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \langle x, y' \rangle),$$

$\langle x, y'\rangle$ being the standard inner product on $\mathbb{R}^d$.
An equivalent formulation is given by the isomorphic group

$$\mathsf{H}_d \;=\; \mathbb{C}^d \times \mathbb{R} \;=\; \left\{(z,t) \,:\, x+iy = z \in \mathbb{C}^d, t \in \mathbb{R}\right\}$$

with multiplication

$$(z,t)\cdot(z',t') = (z+z',\, t+t' + 2\,\mathrm{Im}(z\cdot \bar{z}'))$$

so that the multiplication can also be described by the symplectic form :

$$(x,y,t)(x',y',t') = (x+x',\, y+y',\, t+t' - 2(\langle x,y'\rangle - \langle x',y\rangle))\,.$$

The *Heisenberg dilations* are defined by $(z,t) \mapsto \phi_a(z,t) = (az, a^2 t)$ for any given $a \in \mathbb{R}_+$, and constitute a group of automorphisms of $\mathsf{H_d}$. Another important group of automorphisms of $\mathsf{H_d}$ is the unitary group $U_d(\mathbb{C})$, whose action is given by $(z,t) \mapsto (Uz, t)$, for $U \in U_d(\mathbb{C})$.

1.2.1. Heisenberg-homogeneous radial norms

The action of the dilation group gives rise to a natural notion of homogeneity on the Heisenberg group, where $f : \mathsf{H}_d \to \mathbb{C}$ is homogeneous of degree μ if it satisfies $f(az, a^2t) = a^\mu f(z,t)$. The action of $U_d(\mathbb{C})$ gives rise to a natural notion of radiality on the Heisenberg group, where a function $f : \mathsf{H}_d \to \mathbb{C}$ is radial if it satisfies $f(Uz,t) = f(z,t)$, namely is invariant under the $U_d(\mathbb{C})$-action.

One of the most natural family of gauge functions on the Heisenberg group, the family we shall call Heisenberg norms, is given by, for $\alpha, A > 0$:

$$N_{\alpha,A}((z,t)) = \left(|z|^\alpha + A\,|t|^{\alpha/2}\right)^{1/\alpha}.$$

Clearly, Heisenberg norms $N_{\alpha,A}$ are radial and homogeneous of degree 1. For notational simplicity, at some places we will focus our attention on the family $N_\alpha = N_{\alpha,1}$, but the family $N_{\alpha,A}$ satisfies the same properties, and below we will point out briefly at the appropriate places that the arguments we use need only non-essential modifications. However, naturally the estimates involved are locally but not globally uniform in A.

An interesting special case arises when $\alpha = 2$, a gauge that was considered by a number of authors. For the balls associated with the Heisenberg norm N_2, we shall obtain the best possible result on the error estimate in the lattice point counting problem.

1.2.2. The Cygan-Korányi Heisenberg-norm

The most natural gauge function on the Heisenberg group arises when $\alpha = 4$, namely $N_{4,A}((z,t)) = \left(|z|^4 + A\,|t|^2\right)^{1/4}$. This norm was considered by Cygan [6], [5] and Korányi [16], and is often referred to as the Korányi norm, or Cygan-Korányi norm (see [24]), which is the designation we shall adopt. To gauge its full significance the reader should consult [4, §2, §3], [24] and [31], and here let us just mention the following. View the Heisenberg group as part of the Iwasawa AN group of the simple Lie group $SU(d,1)$, and embed it as a subset of the boundary of the complex hyperbolic space in the usual way. The Cygan-Korányi norm $N_{4,A}$, for suitable A depending on the structure constant associated with the symplectic form defining the bracket in the Lie algebra, can then be characterized uniquely in geometric terms, and appears in the explicit expression defining the following canonical geometric objects :

(1) the conformal inversion on $N \setminus \{e\}$,
(2) the Radon-Nikodym derivative of the conformal inversion acting on N,
(3) the Busemann cocycle and the density of the Patterson-Sullivan measure on the boundary of complex hyperbolic space,
(4) the cross ratio on N.

In addition to the above, the Cygan-Korányi norm has attracted a lot of attention in the context of the harmonic analysis on the Heisenberg group. For example, it appears in the expression defining the fundamental solution of a natural sublaplacian on the Heisenberg group and in other natural kernels, see [31] and [3] for an exposition and [9] and the references therein for recent results.

1.3. Statement of the main result

Let us recall that the Haar measure on H_d can be identified with the Lebesgue measure on $\mathbb{R}^{2d+1}$. We denote by $|B|$ the Haar measure (= Euclidean volume) of a set $B \subset \mathsf{H}_d$, and recall that it scales under dilations according to the homogeneous dimension, not the Euclidean dimension. In particular, if $B_R^{\alpha,A} = \{(z,t) \in \mathsf{H}_d : |z|^\alpha + A\,|t|^{\alpha/2} \leqslant R^\alpha\}$ is the $N_{\alpha,A}$-ball of radius R in H_d, which is also the Heisenberg dilate of the unit ball namely $\phi_R(B_1^{\alpha,A})$, then $\left|B_R^{\alpha,A}\right| = R^{2d+2}\left|B_1^{\alpha,A}\right|$.

Notation. — We let $\#(A)$ denote the size of a finite set $A \subset \mathbb{R}^{2d+1}$. We write $f \lesssim g$ if there exists a constant C such that $f \leqslant Cg$, and $f \sim g$ if $f \lesssim g$, $g \lesssim f$.

Keeping the notation introduced above, we now state our main result on upper bounds on the error in the lattice point counting problem.

Theorem 1.1. — *The error term in the lattice point counting problem in $B_R^{\alpha,A}$ is estimated as follows.*

(1) *For $d \geqslant 1$ and $\alpha = 2$,*

$$\left|\#\left(\mathbb{Z}^{2d+1} \cap B_R^{2,A}\right) - \left|B_R^{2,A}\right|\right| \lesssim R^{2d}.$$

Furthermore, this result is the best possible.

(2) *For $d = 1$ and $\alpha > 2$,*

$$\left|\#\left(\mathbb{Z}^{3} \cap B_R^{\alpha,A}\right) - \left|B_R^{\alpha,A}\right|\right| \lesssim R^{2+\max\{0,\,\delta(\alpha)\}}\log(R).$$

Here $\delta(\alpha) = \frac{1-\frac{4}{\alpha}}{\frac{3}{2}-\frac{2}{\alpha}}$, *so that in particular* $\max\{0,\,\delta(\alpha)\} = 0$ *for* $2 < \alpha \leqslant 4$.

(3) *For $d \geqslant 2$ and $\alpha > 0$,*

$$\left|\#\left(\mathbb{Z}^{2d+1} \cap B_R^{\alpha,A}\right) - \left|B_R^{\alpha,A}\right|\right| \lesssim \begin{cases} R^4(\log(R))^{2/3} & ;\ \textit{for } d = 2 \\ R^{2d} & ;\ \textit{for } d \geqslant 3. \end{cases}$$

We remark, that in the case of H_1, when α is sufficiently large it is possible to improve the error estimate $R^{2+\delta(\alpha)}$ given above. We will explain this further in § 5.

1.3.1. On the method of proof

Let us first remark that since the Heisenberg group H_d is parametrized by Euclidean space $\mathbb{R}^{2d+1}$, and the lattice Γ of integral points is parametrized by $\mathbb{Z}^{2d+1}$, the problem we consider can also be viewed as counting elements in the Euclidean lattice $\mathbb{Z}^{2d+1}$ contained in the family of increasing bodies $B_R^{\alpha,A} \subset R^{2d+1}$ as $R \to \infty$. Nevertheless, no Euclidean counting result of lattice points in dilates of convex bodies is directly relevant to our problem, since the Heisenberg dilations used to expand the given body $B_1^{\alpha,A}$ are materially different than the Euclidean dilations.

Our method of bounding the error term in the lattice point counting problem in $B_R^{\alpha,A} \subset \mathsf{H}_d$ uses a blend of Euclidean and Heisenberg notions. In §2, we will estimate the *Euclidean Fourier transform* of the characteristic function of $B_1^{\alpha,A}$. In §3, we will dominate the lattice point count in $B_R^{\alpha,A}$

from the above and below by the *Euclidean convolution* $\chi_{B_R^{\alpha,A}} * \rho_\epsilon$, where ρ_ϵ is a bump function. A key new point here is that ρ_ϵ is defined using *Heisenberg dilations*, rather than the Euclidean ones. We will then apply the *Euclidean Poisson summation* formula to $\chi_{B_R^{\alpha,A}} * \rho_\epsilon$, and estimate the resulting product expression using the spectral decay estimates established in §2. We will argue separately in several different regions, whose structure reflects the fact that ρ_ϵ was defined using *Heisenberg dilations*. In §4, we will compare our upper bound on the error term with the upper bound that can be obtained from slicing - namely by viewing our lattice point problem in each hyperplane (z, t) with t fixed separately. In the hyperplane we apply the known results regarding the classical Euclidean sum-of-squares problem in $\mathbb{R}^{2d}$.

Let us remark that estimating the decay of the Euclidean Fourier transform of $\chi_{B_1^{\alpha,A}}$ is crucial to our argument, but we have not found in the literature any result that applies directly to this problem. Indeed, as we shall see, it turns out that the bodies $B_1^{\alpha,A}, \alpha \geqslant 2$, are in fact Euclidean-convex bodies of revolution, but their surfaces have points of vanishing curvature, and this renders the elaborate results on the Fourier transform decay for bodies with surfaces of non-vanishing Gaussian curvature irrelevant. In fact, the surface of each $B_1^{\alpha,A}, \alpha > 2$, has the property that the curvature vanishes to maximal order at some points, namely the Hessian is zero at those points. Our spectral decay estimates are direct and are based on the observation that the condition of radiality reduces the problem to estimating oscillatory integrals in the variables $(|z|, |t|)$. We analyze the latter using ideas developed in estimating oscillatory integrals on plane curves initiated in [32], using van-der-Corput classical results. The estimates we obtain exceed, in our particular situation, those that can be deduced from the current standard estimates for the decay of the Fourier transform of a general Euclidean convex body whose surface has points of zero Gaussian curvature to maximal order.

The closest point of comparison to our spectral decay result for $B_R^{\alpha,A}$ would seem to be spectral decay results for special Euclidean bodies such as ℓ^p-balls and other related bodies, which were considered in e.g. [27], [28], [18], [19], [21], [22]. As we shall show in §6, our method, when applied to the Euclidean lattice point counting problem in some of these bodies, actually yields the same (main) error estimate obtained for some of them in the references cited. We should note however that the analysis in these references is much more elaborate and produces a secondary summand in the asymptotic expansion.

2. Harmonic analysis of Heisenberg norm balls

2.1. Properties of Heisenberg norm balls

Let us begin by establishing three properties of the norm balls $B_1^{\alpha,A} = \{(z,t) \in \mathsf{H}_d : N_{\alpha,A}((z,t)) \leqslant 1\}$ that are the subject of our discussion. Essential use will be made of the third property later on.

2.1.1. Euclidean convexity of the $N_{\alpha,A}$-norm balls

Proposition 2.1. — *In every Heisenberg group H_d, $d \geqslant 1$, the unit balls $B_1^{\alpha,A}$ are Euclidean-convex if and only if $\alpha \geqslant 2$.*

Proof. — Let $\alpha \geqslant 2$, fix $(z,t),(w,s) \in B_1^{\alpha,A}$ and $0 < \lambda < 1$, and write

$$\begin{aligned}(N_{\alpha,A}(\lambda(z,t) + (1-\lambda)(w,s)))^{\alpha} \\ &= (N_{\alpha,A}((\lambda z + (1-\lambda)w, \lambda t + (1-\lambda)s)))^{\alpha} \\ &= |\lambda z + (1-\lambda)w|^{\alpha} + A\,|\lambda t + (1-\lambda)s|^{\alpha/2} \\ &\leqslant (\lambda\,|z| + (1-\lambda)\,|w|)^{\alpha} + A\,(\lambda\,|t| + (1-\lambda)\,|s|)^{\alpha/2}.\end{aligned}$$

Using the convexity of $x \mapsto x^{\alpha}$ and $x \mapsto x^{\alpha/2}$ for $\alpha \geqslant 2$, the latter expression is bounded by

$$\lambda\,|z|^{\alpha} + (1-\lambda)\,|w|^{\alpha} + A\lambda\,|t|^{\alpha/2} + A(1-\lambda)\,|s|^{\alpha/2}$$

which equals

$$\lambda\,(N_{\alpha,A}((z,t)))^{\alpha} + (1-\lambda)\,(N_{\alpha,A}((w,s)))^{\alpha}.$$

Thus if $N_{\alpha,A}((z,t)) \leqslant 1$ and $N_{\alpha,A}((w,s)) \leqslant 1$ then also $N_{\alpha,A}(\lambda(z,t) + (1-\lambda)(w,s)) \leqslant 1$, and the unit $N_{\alpha,A}$-norm ball is Euclidean-convex.

To see the non-convexity of $B_1^{\alpha,A}$ in the case $\alpha < 2$, consider the set $D_{\alpha,A} = \{(a,b) \in \mathbb{R}^2 : a,b \geqslant 0 \text{ and } a^{\alpha} + A\,b^{\alpha/2} \leqslant 1\}$. It can be isometrically identified with the intersection of the unit ball $B_1^{\alpha,A}$ with the set $U = \{(z,t) : z = a(1,0,...,0) \text{ and } a,t \geqslant 0\}$, and U is clearly a Euclidean-convex set. If the unit ball $B_1^{\alpha,A}$ was Euclidean-convex as well, then $D_{\alpha,A}$ being the intersection of two Euclidean-convex sets would also be a Euclidean-convex subset of $\mathbb{R}^2$. But, for $\alpha < 2$ this is not the case as can easily be verified directly. Thus the unit balls for the $N_{\alpha,A}$-norm are convex if and only if $\alpha \geqslant 2$. □

2.1.2. Vanishing of principal and Gaussian curvatures for $N_{\alpha,A}$-norm balls

Considering the curvature of the surface bounding the body $B_1^{\alpha,A}$ for $d \geqslant 1$ and $\alpha > 2$, we note the following.

The north and south poles. The Gaussian curvature of the surface of $B_1^{\alpha,A} \subset \mathbb{R}^{2d+1}$ vanishes at both the points of intersection of $B_1^{\alpha,A}$ and the t-axis, namely the north and south poles. In fact all of the $2d$ principal curvatures vanish at these two points, so that the Hessian of the defining equation at these points is the zero matrix. In view of the symmetry of the surface, it is enough to compute the principal curvatures at the point $t = -1$. The surface near the point $t = -1$ (after translation) is given by

$$t \;=\; \varphi(X) \;=\; \varphi(X_1, \ldots, X_{2d}) \;=\; A^{-2/\alpha}\left(1 - (1 - |X|^\alpha)^{2/\alpha}\right),$$

with $\varphi(\vec{0}) = 0 = \nabla\varphi(\vec{0})$. Differentiating directly, the Hessian matrix $H = \left(\frac{\partial^2\varphi}{\partial X_i \partial X_j}\right)$ obtained at the origin is the zero matrix, so that the principal curvatures at the origin all vanish, being the eigenvalues of the Hessian.

The equator. On the equator of the surface, namely the intersection of $B_1^{\alpha,A}$ with the hyperplane $t = 0$, the Gaussian curvature vanishes as well, but here only one principal curvature vanishes. Indeed, isolating the first variable X_1, the surface near the point $X_1 = 1$ (after translation) is given by

$$\begin{aligned} X_1 &= \psi(X_2, \ldots, X_{2d}, t) \\ &= 1 - \left(\left(1 - A\,|t|^{\alpha/2}\right)^{2/\alpha} - \left(X_2^2 + \cdots + X_{2d}^2\right)\right)^{1/2}, \end{aligned}$$

with $\psi(\vec{0}) = 0 = \nabla\psi(\vec{0})$. Differentiating directly, the Hessian matrix at origin is in fact diagonal, and only one diagonal entry is 0, namely ψ_{tt}. All other diagonal entries, i.e. $\psi_{X_j X_j}$ are non-zero. Thus, the equator forms a curve where the Gaussian curvature vanishes to the first order.

2.1.3. Euclidean subadditivity of the $N_{\alpha,A}$-norm balls

Our analysis below will involve Euclidean convolution with a special family ρ_ϵ of bump functions, defined as follows. Let us fix a bump function $\rho : \mathbb{R}^{2d} \times \mathbb{R} \to \mathbb{R}$, which is a smooth non-negative function with support contained in the unit ball $B_1^{\alpha,A}$, such that $\rho(0) > 0$ and $\int_{B_1^{\alpha,A}} \rho(z,t)\,dz\,dt = 1$. We then consider the family of functions defined by the normalized

Heisenberg dilates of ρ, namely

$$\rho_\epsilon(z,t) \;=\; \frac{1}{\epsilon^{2d+2}}\rho\left(\frac{z}{\epsilon},\frac{t}{\epsilon^2}\right) \;=\; \frac{1}{\epsilon^{2d+2}}\rho\circ\phi_{1/\epsilon}(z,t).$$

Clearly, ρ_ϵ is supported in the ball $B_\epsilon^{\alpha,A}$, and of course, $\int_{\mathbb{R}^{2d+1}}\rho_\epsilon(z,t)\,dz\,dt = 1$ for all $\epsilon>0$.

We can now state the following :

Proposition 2.2. — *For every $d\geqslant 1$, and $\alpha\geqslant 1$,*

(1) the $N_{\alpha,A}$-norms are subadditive on $\mathbb{R}^{2d+1}$ with respect to Euclidean addition, namely :

$$N_{\alpha,A}((z,t)+(w,s)) \leqslant N_{\alpha,A}((z,t)) + N_{\alpha,A}((w,s)).$$

(2) The balls $B_R^{\alpha,A}$ satisfy the following two-sided inequality with respect to Euclidean convolution : $\chi_{B^{\alpha,A}_{R-\epsilon}} * \rho_\epsilon \leqslant \chi_{B^{\alpha,A}_{R}} \leqslant \chi_{B^{\alpha,A}_{R+\epsilon}} * \rho_\epsilon$.

Proof. — To see that the Heisenberg norms are subadditive with respect to Euclidean addition, fix any $(z,t),(w,s)\in\mathbb{R}^{2d+1}=\mathsf{H}_d$. Then

$$\begin{aligned} N_{\alpha,A}((z,t)+(w,s)) &= \left(|z+w|^\alpha + A\,|t+s|^{\alpha/2}\right)^{1/\alpha} \\ &\leqslant \left((|z|+|w|)^\alpha + A\left((|t|+|s|)^{1/2}\right)^\alpha\right)^{1/\alpha} \\ &\leqslant \left((|z|+|w|)^\alpha + \left(A^{1/\alpha}\,|t|^{1/2} + A^{1/\alpha}\,|s|^{1/2}\right)^\alpha\right)^{1/\alpha} \\ &\leqslant \left(|z|^\alpha + A\,|t|^{\alpha/2}\right)^{1/\alpha} + \left(|w|^\alpha + A\,|s|^{\alpha/2}\right)^{1/\alpha} \\ &= N_{\alpha,A}((z,t)) + N_{\alpha,A}((w,s))\,, \end{aligned}$$

where we have used $(a+b)^{1/2}\leqslant a^{1/2}+b^{1/2}$ for $a,b\geqslant 0$, and following that we have used the triangle inequality in $l_\alpha(\mathbb{R}^2)$ applied to the vectors $\left(|z|\,,A^{1/\alpha}\,|t|^{1/2}\right)$ and $\left(|w|\,,A^{1/\alpha}\,|s|^{1/2}\right)$.

To prove the second inequality of part (2), note first that if $(z,t)\notin B_R^{\alpha,A}$, then the inequality holds trivially. Taking $(z,t)\in B_R^{\alpha,A}$, we have

$$\begin{aligned} \chi_{B^{\alpha,A}_{R+\epsilon}} * \rho_\epsilon(z,t) &= \int_{B^{\alpha,A}_{R+\epsilon}} \rho_\epsilon(z-w,t-s)\,dw\,ds \\ &= \int_{(z,t)-B^{\alpha,A}_{R+\epsilon}} \rho_\epsilon(w,s)\,dw\,ds \\ &\geqslant \int_{B^{\alpha,A}_{\epsilon}} \rho_\epsilon(w,s)\,dw\,ds\,, \end{aligned}$$

where the last step holds true because of the non-negativity of ρ_ϵ and the Euclidean subadditivity of the $N_{\alpha,A}$-norms, which implies that for $(z,t) \in B_R^{\alpha,A}$, the set $(z,t) - B_{R+\epsilon}^{\alpha,A}$ contains $B_\epsilon^{\alpha,A}$. Thus,

$$\chi_{B_{R+\epsilon}^{\alpha,A}} * \rho_\epsilon(z,t) \;\geqslant\; \int_{B_\epsilon^{\alpha,A}} \rho_\epsilon(w,s)\,dw\,ds \;=\; 1 \;=\; \chi_{B_R^{\alpha,A}}(z,t).$$

Finally, note that for any $(z,t) \in B_R^{\alpha,A}$, the first inequality follows from the fact that $\chi_{B_{R-\epsilon}^{\alpha,A}} * \rho_\epsilon \leqslant \int \rho_\epsilon = 1$, whereas for $(z,t) \notin B_R^{\alpha,A}$,

$$\begin{aligned}\chi_{B_{R-\epsilon}^{\alpha,A}} * \rho_\epsilon(z,t) &= \int_{(z,t)-B_{R-\epsilon}^{\alpha,A}} \rho_\epsilon(w,s)\,dw\,ds \\ &\leqslant \int_{\mathbb{R}^{2d+1}\setminus B_\epsilon^{\alpha,A}} \rho_\epsilon(w,s)\,dw\,ds \\ &= 0.\end{aligned}$$

This completes the proof of Proposition 2.2. □

2.2. Decay of Fourier transforms of Heisenberg norm balls

We continue to view the Heisenberg-homogeneous norm balls

$$B_1^{\alpha,A} = \left\{(w,s) \in \mathbb{R}^{2d} \times \mathbb{R} : |w|^\alpha + A\,|s|^{\alpha/2} \leqslant 1\right\},$$

as subsets of $\mathbb{R}^{2d+1}$, and in the present section we will give estimates on the rate of decay of the Euclidean Fourier transform of $\chi_{B_1^{\alpha,A}}$. These decay estimates will be applied in the next section to the problem of finding the error term in the lattice point counting problem in the sets $B_R^{\alpha,A} = \phi_R(B_1^{\alpha,A})$, where $\phi_R(z,t) = (Rz, R^2t)$ is the Heisenberg dilation.

Notation. — We write B_1^α and B_R^α to denote $B_1^{\alpha,1}$ and $B_R^{\alpha,1}$ respectively.

The set $\left\{(w,s) : w \in \mathbb{R}^{2d}, s \in \mathbb{R}\right\}$ parametrizes the unitary characters of $\mathbb{R}^{2d+1}$. Let $\widehat{f}$ denote the Euclidean Fourier transform of a function f, in the form

$$\widehat{f}(w,s) = \int_{\mathbb{R}^{2d+1}} e^{-2\pi i(\langle z,w\rangle + ts)} f(z,t)\,dz\,dt\,.$$

We first record here the following useful identity :

$$\widehat{\chi_{B_1^{\alpha,A}}}(w,s) = A^{-2/\alpha}\,\widehat{\chi_{B_1^\alpha}}\left(w, A^{-2/\alpha}s\right) \tag{2.1}$$

which relates the Euclidean Fourier transform of $\chi_{B_1^{\alpha,A}}$ with that of $\chi_{B_1^\alpha}$. We divide the parameter space into three subsets and argue separately in each. The domains of our consideration are given by

(1) $w = \vec{0}$, namely the s-axis,
(2) the hyperplane $s = 0$,
(3) the set $|w| \geqslant 1$ and $|s| \geqslant A^{2/\alpha}$.

As we shall see below, this decomposition is natural in the context of the Heisenberg group and is dictated by the decomposition of $\mathbb{R}^{2d+1}$ to eigenspaces of the Heisenberg dilation. Furthermore, since after applying the Euclidean Poisson summation formula we will be interested in summing the Fourier transform over the points in the integer (dual) lattice, estimates on the sets listed above will suffice.

2.2.1. Decay of the Fourier transform along s-axis

Lemma 2.3. — *For $\alpha > 0, |s| \geqslant A^{2/\alpha}$,*

$$\left|\widehat{\chi_{B_1^{\alpha,A}}}(\vec{0}, s)\right| \lesssim |s|^{-\left(1+\min\left\{\frac{2d}{\alpha}, \frac{\alpha}{2}\right\}\right)} .$$

However, for a positive integer $\alpha \in 4\mathbb{N}$ we have a better estimate

$$\left|\widehat{\chi_{B_1^{\alpha,A}}}(\vec{0}, s)\right| \lesssim |s|^{-\left(1+\frac{2d}{\alpha}\right)} .$$

Proof. — In view of identity (2.1), it suffices to establish the lemma for $A = 1$. Now,

$$\begin{aligned}
\widehat{\chi_{B_1^\alpha}}(\vec{0}, s) &= \int_{B_1^\alpha} e^{-2\pi its}\, dz\, dt \\
&= \int_{-1}^{1} e^{-2\pi its} \left(\int_{|z| \leqslant \left(1-|t|^{\alpha/2}\right)^{1/\alpha}} dz\right) dt \\
&= C_d \int_{-1}^{1} e^{-2\pi its} \left(1 - |t|^{\alpha/2}\right)^{2d/\alpha} dt\,.
\end{aligned}$$

Before beginning the proof, let us first note that for $\alpha = 4$, the integral above is, up to a constant, the Fourier transform of the Euclidean unit ball in $\mathbb{R}^{d+1}$ evaluated at $(\vec{0}, s)$; in this case, the decay is well-known and agrees with the claim. Proceeding with the analysis of a general $\alpha > 0$ and B_1^α, we consider the following two cases :

Case I. — When $\alpha = 4k$ for some positive integer k, then we write $\frac{2d}{\alpha} = \frac{d}{2k} = m + \mu$ for $m \in \mathbb{N}$, $0 < \mu \leqslant 1$ and apply integration by parts

$m+1$-times to get

$$
\begin{aligned}
\widehat{\chi_{B_1^\alpha}}(\vec{0},s) &= \frac{C_{k,d}}{s^{m+1}} \int_{-1}^{1} e^{-2\pi its} \left(1-t^{2k}\right)^{\mu-1} P(t)\, dt \\
&= \frac{C_{k,d}}{s^{m+1}} \int_{0}^{1} e^{-2\pi its} (1-t)^{\mu-1} \left\{ \left(\sum_{j=0}^{2k-1} t^j \right)^{\mu-1} P(t) \right\} dt \\
&\quad + \frac{C_{k,d}}{s^{m+1}} \int_{-1}^{0} e^{-2\pi its} (1+t)^{\mu-1} \left\{ \left(\sum_{j=0}^{2k-1} (-t)^j \right)^{\mu-1} P(t) \right\} dt
\end{aligned}
$$

for some polynomial P. We now use the standard estimates of oscillatory integrals stated below, which apply to integrals having singularity at one of the end points. In case $\mu = 1$, the integrals have no singularity and we integrate by parts once more. Thus, we have shown that the above integrals decay at least at the rate of $|s|^{-\mu}$ which proves the claimed estimate in the case of an integer $\alpha \in 4\mathbb{N}$.

Case II. — In general, we consider the following integral for $\beta, \gamma > 0$:

$$
I_{\beta,\gamma}(s) = \int_0^1 \cos(ts) \left(1-t^\beta\right)^\gamma \, dt\, .
$$

Let $q \geqslant 1$ be the integer for which $-1 < \min\{\beta - q, \gamma - q\} \leqslant 0$. Applying integration by parts q-times we get

$$
\begin{aligned}
I_{\beta,\gamma}(s) =& \frac{1}{s^q} \int_0^1 F(ts) t^{\beta-q} \left(1-t^\beta\right)^{\gamma-q} \left\{ \sum_{j=0}^{q-1} C_{q,j} (t^\beta)^j \right\} dt \\
=& \frac{1}{s^q} \int_{\frac{1}{2}}^1 F(ts) (1-t)^{\gamma-q} \left\{ \left(\frac{1-t^\beta}{1-t} \right)^{\gamma-q} \sum_{j=1}^{q} C_{q,j} t^{j\beta-q} \right\} dt \\
&+ \frac{1}{s^q} \sum_{j=1}^{q} C_{q,j} \int_0^{\frac{1}{2}} F(ts) t^{j\beta-q} \left\{ \left(1-t^\beta\right)^{\gamma-q} \right\} dt.
\end{aligned}
$$

Here $F(s) = \cos(s)$ if q is even, otherwise $F(s) = \sin(s)$. An argument similar to that of Case I then implies that the above integrals in the expression of $I_{\beta,\gamma}(s)$ decay at least at the rate of $|s|^{-(1+\min\{\beta-q,\, \gamma-q\})}$. Substituting the appropriate values of β and γ completes the proof of Lemma 2.3. $\square$

2.2.2. Estimates of oscillatory integrals with singularities at one endpoint.

It is well known that one can use the method of stationary phase to prove the following estimates (for full details regarding even more refined asymptotic estimates, see e.g. [8], Sec 2.8). For any $0 < \lambda \leqslant 1$, and $-\infty < a < b < \infty$, we have

$$\left| \int_a^b e^{-ist}(t-a)^{\lambda-1}\, dt \right| \leqslant C_\lambda \, |s|^{-\lambda}. \tag{2.2}$$

Here the constant C_λ does not depend on a and b. From this one can easily deduce that for any differentiable function g on $[a,b]$ such that g' is integrable on $[a,b]$

$$\left| \int_a^b e^{-ist}(t-a)^{\lambda-1} g(t)\, dt \right| \leqslant C_\lambda \left(|g(b)| + \int_a^b |g'(t)|\, dt \right) |s|^{-\lambda}. \tag{2.3}$$

In fact, the integral in (2.3) can be written as $\int_a^b F'(t)g(t)\, dt$ with

$$F(t) = \int_a^t e^{-isr}(r-a)^{\lambda-1}\, dr\,; \qquad a < t < b.$$

The estimate of (2.3) then follows by doing integration by parts and using (2.2).

2.2.3. Decay of the Fourier transform on the hyperplane s=0

We begin by re-writing $\widehat{\chi_{B_1^\alpha}}(w,s)$ as follows, and we use this expression in the next two lemmas.

$$\begin{aligned}
\widehat{\chi_{B_1^\alpha}}(w,s) &= \int_{B_1^\alpha} e^{-2\pi i(\langle z,w\rangle + ts)}\, dz\, dt \\
&= \int_{|z|\leqslant 1} e^{-2\pi i\langle z,w\rangle} \left(\int_{|t|\leqslant (1-|z|^\alpha)^{2/\alpha}} e^{-2\pi its}\, dt \right) dz \\
&= \int_0^1 \left(\int_{S^{2d-1}} e^{-2\pi i r\theta\cdot w} d\theta \right) \left(\int_{|t|\leqslant (1-r^\alpha)^{2/\alpha}} e^{-2\pi its}\, dt \right) r^{2d-1}\, dr \\
&= \int_0^1 \left(\int_{S^{2d-1}} e^{-2\pi i r\theta\cdot w} d\theta \right) \frac{2\sin\left(2\pi s(1-r^\alpha)^{2/\alpha}\right)}{2\pi s} r^{2d-1}\, dr \\
&= \int_0^1 \widehat{\sigma}(rw) \frac{2\sin\left(2\pi s(1-r^\alpha)^{2/\alpha}\right)}{2\pi s} r^{2d-1}\, dr\,,
\end{aligned}$$

where σ denotes the surface measure on the $(2d-1)-$dimensional sphere. Notice that by the rotation invariance of σ, and hence of $\widehat{\sigma}$,

$$\begin{aligned}\widehat{\sigma}(rw) &= \int_{S^{2d-1}} e^{-2\pi i r|w|\theta\cdot e_1}\,d\theta\\ &= c_d\int_0^\pi e^{-2\pi i r|w|\cos(\phi)}(\sin(\phi))^{2d-2}\,d\phi\\ &= c_d\int_{-1}^1 e^{-2\pi i r|w|u}(1-u^2)^{\frac{2d-3}{2}}\,du.\end{aligned}$$

By definition of the Bessel function J_{d-1}, the last integral equals to a constant multiple of $(2\pi r\,|w|)^{-(d-1)}J_{d-1}(2\pi r\,|w|)$.

We now state the estimate along the hyperplane.

Lemma 2.4. — *For any $\alpha>2$ we have for $|w|\geqslant 1$,*

$$\left|\widehat{\chi_{B_1^{\alpha,A}}}(w,0)\right| \lesssim \begin{cases}|w|^{-2d} & ;\ \textit{if } \frac{2}{\alpha}>d-\frac12\\ |w|^{-\left(d+\frac12+\frac{2}{\alpha}\right)} & ;\ \textit{if } \frac{2}{\alpha}\leqslant d-\frac12\,.\end{cases}$$

Moreover, for $\alpha=2$, one has the following identity

$$\widehat{\chi_{B_1^{2,A}}}(w,0) = C_{d,A}\,\frac{J_{d+1}(2\pi\,|w|)}{|w|^{d+1}}\,.$$

Proof. — In view of (2.1), it again suffices to prove the lemma for $A=1$. Putting $s=0$ in the expression derived for $\widehat{\chi_{B_1^\alpha}}(w,s)$ in the beginning of this section, we have

$$\widehat{\chi_{B_1^\alpha}}(w,0) = \frac{C_d}{|w|^{d-1}}\int_0^1 J_{d-1}(2\pi r\,|w|)\,(1-r^\alpha)^{2/\alpha}\,r^d\,dr.$$

Recall first that the Bessel functions satisfy following identity (see e.g [11] Appendix B.3, p. 427) :

$$\int_0^1 J_\mu(rs)(1-r^2)^\nu r^{\mu+1}\,dr = C_\nu\frac{J_{\mu+\nu+1}(s)}{s^{\nu+1}}\,,$$

for any $\mu>-1/2$, $\nu>-1$. From this identity we get in the case $\alpha=2$,

$$\widehat{\chi_{B_1^2}}(w,0) = C_d\frac{J_{d+1}(2\pi\,|w|)}{|w|^{d+1}}\,.$$

In general, for any $\alpha>2$, we are left to study the bound (as $1\leqslant\xi\to\infty$) for

$$I(\xi) = \int_0^1 J_{d-1}(\xi r)\,(1-r^\alpha)^{2/\alpha}\,r^d\,dr.$$

The proof will be completed once we show that

$$|I(\xi)| \lesssim \begin{cases} \xi^{-(d+1)} & ;\ \text{if } \frac{2}{\alpha} > d - \frac{1}{2} \\ \xi^{-(\frac{3}{2}+\frac{2}{\alpha})} & ;\ \text{if } \frac{2}{\alpha} \leqslant d - \frac{1}{2}. \end{cases}$$

In order to prove the above estimate, we utilize the asymptotic expansion of J_{d-1} which is valid when $\xi r \geqslant 1$. Therefore, we divide the integration over $r \in [0,1]$ to two parts : on the interval $[0,\delta]$ and on the interval $[\delta,1]$, where $\delta \geqslant \xi^{-1}$ is fixed and will be chosen momentarily. Since J_{d-1} is a bounded function, we clearly have

$$\left| \int_0^\delta J_{d-1}(\xi r)\,(1-r^\alpha)^{2/\alpha}\, r^d\, dr \right| \lesssim \int_0^\delta r^d dr \lesssim \delta^{d+1}\,.$$

To estimate the second interval, we recall the following asymptotic expression for $J_{d-1}(\xi r)$, valid when $\xi r \geqslant 1$ [31, p. 356] :

$$\begin{aligned} J_{d-1}(\xi r) \sim \left(\frac{\pi \xi r}{2}\right)^{-1/2} \cos\left(\xi r - \frac{(d-1)\pi}{2} - \frac{\pi}{4}\right) \sum_{j=0}^{\infty} a_j\, (\xi r)^{-2j} \\ + \left(\frac{\pi \xi r}{2}\right)^{-1/2} \sin\left(\xi r - \frac{(d-1)\pi}{2} - \frac{\pi}{4}\right) \sum_{j=0}^{\infty} b_j\, (\xi r)^{-2j-1}\,. \end{aligned}$$

with the implied constant independent of ξr.

We make use of the first two terms in the above asymptotic expression. The estimate in question then becomes equivalent to

$$\int_\delta^1 \left[e^{i\xi r} \left(\sum_{k=0}^{1} c_j (\xi r)^{-\left(k+\frac{1}{2}\right)} \right) + O\left((\xi r)^{-5/2} \right) \right] (1-r^\alpha)^{2/\alpha} r^d\, dr\,.$$

We estimate the O-term as follows :

$$\left| \xi^{-5/2} \int_\delta^1 r^{d-5/2}\, dr \right| = \xi^{-5/2} \left| \frac{1-\delta^{d-3/2}}{d-3/2} \right| \leqslant 2 \begin{cases} \xi^{-5/2}\delta^{-1/2} & ;\ \text{if } d=1 \\ \xi^{-5/2} & ;\ \text{if } d>1. \end{cases}$$

The main term can be written as the difference of two integrals over the intervals of $[0,1]$ and $[0,\delta]$. The integral over the interval $[0,\delta]$ is bounded by

$$\sum_{k=0}^{1} |c_j|\, \xi^{-\left(k+\frac{1}{2}\right)} \int_0^\delta r^{d-k-\frac{1}{2}}\, dr = \sum_{k=0}^{1} \frac{|c_j|}{d-k+\frac{1}{2}} \xi^{-\left(k+\frac{1}{2}\right)} \delta^{d-k+\frac{1}{2}}\,.$$

Finally we proceed with the main term over the interval $[0,1]$:

$$\sum_{k=0}^{1} c_j \xi^{-\left(k+\frac{1}{2}\right)} \int_0^1 e^{i\xi r} r^{d-k-\frac{1}{2}} (1-r^\alpha)^{2/\alpha}\, dr\,.$$

Now an argument similar to that of Case II in the proof of Lemma 2.3 can be given to prove that

$$\left|\int_0^1 e^{i\xi r} r^{d-k-\frac{1}{2}}(1-r^\alpha)^{2/\alpha}\,dr\right| \lesssim \xi^{-\left(1+\min\left\{d-k-\frac{1}{2},\,\frac{2}{\alpha}\right\}\right)}.$$

Collecting all the bounds, we see that $I(\xi)=\int_0^1 J_{d-1}(\xi r)\,(1-r^\alpha)^{2/\alpha}\,r^d dr$ is dominated by a finite sum of terms of the form

$$\delta^{d+1},\ \xi^{-\frac{5}{2}}\delta^{-\frac{1}{2}},\ \xi^{-\left(k+\frac{1}{2}\right)}\delta^{d-k+\frac{1}{2}},\ \xi^{-\left(k+\frac{3}{2}\right)}\xi^{-\min\left\{d-k-\frac{1}{2},\,\frac{2}{\alpha}\right\}}$$

for $k=0,1$. Choosing $\delta=\xi^{-1}$, we verify that

$$|I(\xi)| \lesssim \begin{cases}\xi^{-(d+1)} & ;\ \text{if } \frac{2}{\alpha}>d-\frac{1}{2}\\ \xi^{-\left(\frac{3}{2}+\frac{2}{\alpha}\right)} & ;\ \text{if } \frac{2}{\alpha}\leqslant d-\frac{1}{2}.\end{cases}$$

This completes the of proof Lemma 2.4. □

2.2.4. Completion of the Fourier transform decay estimates

The third case we consider is establishing decay when $|w|\geqslant 1$ and $|s|\geqslant A^{2/\alpha}$.

LEMMA 2.5. — *For any $\alpha\geqslant 2$ and $|w|\geqslant 1$ and $|s|\geqslant A^{2/\alpha}$,*

$$\left|\widehat{\chi_{B_1^{\alpha,A}}}(w,s)\right| \lesssim |w|^{-d}\,|s|^{-1}. \tag{2.4}$$

Moreover, for $\alpha=2$, we have the better decay estimate

$$\left|\widehat{\chi_{B_1^{2,A}}}(w,s)\right| \lesssim |w|^{-\left(d-\frac{1}{2}\right)}\,|s|^{-1}\,|(w,s)|^{-1/2}. \tag{2.5}$$

Proof. — As observed in the earlier lemmas, here also it suffices to prove the estimates for $A=1$, thanks to (2.1). As mentioned in the beginning of § 2.2.3, we have

$$\widehat{\chi_{B_1^\alpha}}(w,s)=c_d\,|w|^{-(d-1)}\,s^{-1}\int_0^1 J_{d-1}(2\pi r\,|w|)\sin\left(2\pi s(1-r^\alpha)^{2/\alpha}\right)r^d\,dr.$$

We divide the integration over $r\in[0,1]$ to two parts : on the interval $[0,\delta]$ and on the interval $[\delta,1]$. $J_{d-1}(\cdot)$ and $\sin(\cdot)$ being bounded functions on the interval, we clearly have

$$\left|\int_0^\delta J_{d-1}(2\pi r\,|w|)\sin\left(2\pi s(1-r^\alpha)^{2/\alpha}\right)r^d\,dr\right| \lesssim \int_0^\delta r^d\,dr=\frac{\delta^{d+1}}{d+1}.$$

To estimate the integral on the interval $[\delta,1]$, we once again make use of the asymptotic expression for the Bessel function $J_{d-1}(\cdot)$ as we did in the proof of Lemma 2.4. This time we make use only of the first term. This integral can be written as the difference of two integrals over the intervals of $[0,1]$ and $[0,\delta]$. The integral over the interval $[0,\delta]$ is bounded by

$$|w|^{-1/2}\,\delta^{d+1/2}.$$

The integral over $[\delta,1]$ is then bounded by

$$\left(|w|^{-1/2}\int_0^1 e^{2\pi i|w|r}\sin\left(2\pi s(1-r^\alpha)^{2/\alpha}\right)r^{d-\frac12}\,dr\right)+|w|^{-1/2}\,\delta^{d+1/2}.$$

Now it suffices to estimate (from here onwards we drop the constant 2π for the sake of convenience)

$$|w|^{-1/2}\int_0^1 e^{i|w|r}e^{is(1-r^\alpha)^{2/\alpha}}r^{d-\frac12}\,dr\,,$$

in both cases when s is positive or negative. The above integral is the same as

$$|w|^{-1/2}\int_0^1 \exp\left(i\,|w|\,\phi_{|w|,s}(r)\right)r^{d-\frac12}\,dr\,, \tag{2.6}$$

where the phase function is given by $\phi_{|w|,s}(r)=r+\frac{s}{|w|}(1-r^\alpha)^{2/\alpha}$.

We now use the van-der-Corput lemma to estimate this integral. Note that the derivative of the phase function is given by

$$\phi'_{|w|,s}(r)=1-\frac{2s}{|w|}r^{\alpha-1}(1-r^\alpha)^{\frac{2}{\alpha}-1}.$$

Clearly, $\phi'_{|w|,s}(r)\geqslant 1$ for all $r\in[0,1]$ when s is negative. The difficulty arises when s is positive. To handle this case, notice first that if $\alpha\geqslant 2$, then $r^{\alpha-1}(1-r^\alpha)^{\frac{2}{\alpha}-1}$ is a monotonically increasing function for $r\in[0,1)$, as follows from the fact that its derivative is strictly positive in $(0,1)$. Assume for now that $\alpha>2$, and we will take care of the case when $\alpha=2$ separately. When $\alpha>2$ the limit of the latter function as $r\to 1^-$ is $+\infty$, and therefore there exists unique point $r_0\equiv r_0(|w|,s)\in(0,1)$ at which $r_0^{\alpha-1}(1-r_0^\alpha)^{\frac{2}{\alpha}-1}=\frac{|w|}{4s}$. Then for $r\in[0,r_0]$, the first derivative is bounded from below by

$$\left|\phi'_{|w|,s}(r)\right|\geqslant\frac12\,.$$

In the complementary range $[r_0,1)$, we will now show that $\left|\phi''_{|w|,s}(r)\right|$ is also bounded from below by $1/2$. Indeed, for $\alpha>2$,

$$\phi''_{|w|,s}(r)=\frac{-2s}{|w|}r^{\alpha-2}(1-r^\alpha)^{\frac{2}{\alpha}-2}\left((\alpha-1)-r^\alpha\right).$$

Notice that $\frac{(\alpha-1)-r^\alpha}{1-r^\alpha} \geqslant 1$, and therefore for $r \in [r_0, 1)$, it is easy to verify that

$$\left|\phi''_{|w|,s}(r)\right| \geqslant \frac{2s}{|w|} r^{\alpha-1}(1-r^\alpha)^{\frac{2}{\alpha}-1} \geqslant \frac{1}{2}.$$

In summary,

$$\begin{aligned} \left|\phi'_{|w|,s}(r)\right| &\geqslant \frac{1}{2} \text{ for } r \in [0, r_0], \\ \left|\phi''_{|w|,s}(r)\right| &\geqslant \frac{1}{2} \text{ for } r \in [r_0, 1). \end{aligned}$$

So far we have seen that one can divide the interval $[0,1]$ into exactly two parts such that on one part $\left|\phi'_{|w|,s}\right|$ is bounded below by $1/2$, while on the other part $\left|\phi''_{|w|,s}\right|$ is bounded below by $1/2$. In addition, $\phi'_{|w|,s}$ is monotone on $(0,1)$, since $\phi''_{|w|,s}$ has a constant sign there. We conclude that on each of such subintervals one can apply van-der Corput lemma (in the form stated e.g. in [31, p. 332-334], which is valid as long as there is a fixed positive lower bound for absolute values of all applicable derivatives), so that :

$$|w|^{-1/2} \left| \int_0^1 \exp\left(i\,|w|\,\phi_{|w|,s}(r)\right) r^{d-\frac{1}{2}}\, dr \right| \lesssim |w|^{-1/2}\,|w|^{-1/2} = |w|^{-1}.$$

As to the case $\alpha = 2$, let now $|w|\,\phi_{|w|,s}(r) = \zeta(r) = |w|\,r + s(1-r^2)$. Again we should consider just the case where s is positive and $s \geqslant 1$, and then observe that if $\frac{|w|}{4s} \geqslant 1$, then $\zeta'(r) \geqslant \frac{|w|}{2} \geqslant \frac{2}{\sqrt{17}}\,|(w,s)|$ for all $r \in [0,1]$ (here $|(w,s)|$ is the Euclidean norm). On the other hand, if $0 < \frac{|w|}{4s} \leqslant 1$, then $\zeta''(r) = 2s \geqslant \frac{2}{\sqrt{17}}\,|(w,s)|$ for all $r \in [0,1]$. We now apply van-der Corput lemma on the interval $[0,1]$ and we have

$$|w|^{-1/2} \left| \int_0^1 e^{i\zeta(r)} r^{d-\frac{1}{2}}\, dr \right| \lesssim |w|^{-1/2}\,|(w,s)|^{-1/2}.$$

Finally, comparing all the bounds leads to the choice of $\delta = |w|^{-1}$ and this completes the proof of Lemma 2.5. □

3. Counting lattice point result in Heisenberg norm balls

In the present section we will prove the following result :

Theorem 3.1. — *For every $d \geqslant 1$, and $\alpha \geqslant 2$*

$$\left|\#\left(\mathbb{Z}^{2d+1} \cap B_R^{\alpha,A}\right) - \left|B_R^{\alpha,A}\right|\right| \lesssim \begin{cases} R^{2d} & ; \text{ for } \alpha = 2 \\ R^{2d}\log(R) & ; \text{ for } 2 < \alpha \leqslant 4 \\ R^{2d+\delta} & ; \text{ for } \alpha > 4\,, \end{cases}$$

where $\delta = \delta(\alpha) = \frac{2\left(\frac{1}{2}-\frac{2}{\alpha}\right)}{d+\frac{1}{2}-\frac{2}{\alpha}}$.

We recall that the volume of $B_R^{\alpha,A}$ is given by $\left|B_R^{\alpha,A}\right| = R^{2d+2}\left|B_1^{\alpha,A}\right|$. Our proof will utilize the spectral estimates of the (Euclidean) Fourier transform of the unit ball $B_1^{\alpha,A}$ that were proved in the previous section, via the (Euclidean) Poisson summation formula, to which we now turn.

3.1. Euclidean Poisson summation

As noted in §2.1, we fix a bump function $\rho : \mathbb{R}^{2d} \times \mathbb{R} \to \mathbb{R}$, which is a smooth non-negative function with support contained in the unit ball $B_1^{\alpha,A}$, such that $\rho(0) > 0$ and $\int_{B_1^{\alpha,A}} \rho(z,t)\,dz\,dt = 1$. We then consider the family of functions defined by the Heisenberg dilates of ρ, namely $\rho_\epsilon(z,t) = \frac{1}{\epsilon^{2d+2}}\rho(\frac{z}{\epsilon}, \frac{t}{\epsilon^2})$.

Notation. — From here onward, to simplify the notation we write B_R to denote $B_R^{\alpha,A}$.

We begin by obtaining estimates for $\#\left(\mathbb{Z}^{2d+1} \cap B_R\right)$ from above and below using Euclidean convolution. First, observe that by Proposition 2.2

$$\begin{aligned} \#\left(\mathbb{Z}^{2d+1} \cap B_R\right) &= \sum_{k \in \mathbb{Z}^{2d+1}} \chi_{B_R}(k) \\ &\leqslant \sum_{k \in \mathbb{Z}^{2d+1}} \chi_{B_{R+\epsilon}} * \rho_\epsilon(k), \end{aligned}$$

where ρ_ϵ was defined above, and the convolution is Euclidean. Similarly,

$$\#\left(\mathbb{Z}^{2d+1} \cap B_R\right) \geqslant \sum_{k \in \mathbb{Z}^{2d+1}} \chi_{B_{R-\epsilon}} * \rho_\epsilon(k).$$

We now use the Euclidean Poisson summation formula, so that the upper bound becomes :

$$\begin{aligned} \#\left(\mathbb{Z}^{2d+1} \cap B_R\right) &\leqslant \sum_{k \in \mathbb{Z}^{2d+1}} \widehat{\left(\chi_{B_{R+\epsilon}} * \rho_\epsilon\right)}(k) \\ &= \sum_{k \in \mathbb{Z}^{2d+1} \setminus \{\vec{0}\}} \widehat{\chi_{B_{R+\epsilon}}}(k)\widehat{\rho_\epsilon}(k) + \left|B_{R+\epsilon}\right|\,. \end{aligned}$$

Since ρ is a compactly supported smooth bump function, the decay of the Fourier transform of $\rho_\epsilon(z,t) = \epsilon^{-2d-2}\rho(z/\epsilon, t/\epsilon^2)$ for any $N \in \mathbb{N}$ and any (w,s) is bounded by

$$(3.1) \qquad |\widehat{\rho_\epsilon}(w,s)| = \left|\widehat{\rho}(\epsilon w, \epsilon^2 s)\right| \leqslant C_N \left(1 + \left|(\epsilon w, \epsilon^2 s)\right|\right)^{-N} .$$

Continuing with the estimate, we have

$$\begin{aligned} \#\left(\mathbb{Z}^{2d+1} \cap B_R\right) - |B_R| &\leqslant \sum_{k\in\mathbb{Z}^{2d+1}\setminus\{\vec{0}\}} \widehat{\chi_{B_{R+\epsilon}}}(k)\widehat{\rho_\epsilon}(k) + |B_{R+\epsilon}| - |B_R| \\ &= \sum_{k\in\mathbb{Z}^{2d+1}\setminus\{\vec{0}\}} \widehat{\chi_{B_{R+\epsilon}}}(k)\widehat{\rho_\epsilon}(k) + O\left(R^{2d+1}\epsilon\right). \end{aligned}$$

Similarly,

$$\#\left(\mathbb{Z}^{2d+1} \cap B_R\right) - |B_R| \geqslant - \left|\sum_{k\in\mathbb{Z}^{2d+1}\setminus\{\vec{0}\}} \widehat{\chi_{B_{R-\epsilon}}}(k)\widehat{\rho_\epsilon}(k)\right| - O\left(R^{2d+1}\epsilon\right).$$

Combining these observations, we conclude that $\left|\#\left(\mathbb{Z}^{2d+1} \cap B_R\right) - |B_R|\right|$ is bounded from above by

$$\left|\sum_{k\in\mathbb{Z}^{2d+1}\setminus\{\vec{0}\}} \widehat{\chi_{B_{R+\epsilon}}}(k)\widehat{\rho_\epsilon}(k)\right| + \left|\sum_{\sum_{k\in\mathbb{Z}^{2d+1}\setminus\{\vec{0}\}}} \widehat{\chi_{B_{R-\epsilon}}}(k)\widehat{\rho_\epsilon}(k)\right| + O\left(R^{2d+1}\epsilon\right).$$

3.2. Estimates

Let us write for $k \in \mathbb{Z}^{2d+1}$, $k = (k', k'')$ where $k' \in \mathbb{Z}^{2d}$ and $k'' \in \mathbb{Z}$. Our task is now to bound

$$(3.2) \qquad R^{2d+2} \sum_{k\in\mathbb{Z}^{2d+1}\setminus\{\vec{0}\}} \left|\widehat{\chi_{B_1}}(Rk', R^2k'')\right| \left|\widehat{\rho}(\epsilon k', \epsilon^2 k'')\right| + O\left(R^{2d+1}\epsilon\right).$$

We will break the sum further to three regions to utilize our three separate spectral decay estimates as follows :

$$
\begin{aligned}
R^{2d+2}\sum \left|\widehat{\chi_{B_1}}(Rk', R^2k'')\right| \left|\widehat{\rho}(\epsilon k', \epsilon^2 k'')\right| \\
&= R^{2d+2}\left(\sum_{k''\neq 0}\left|\widehat{\chi_{B_1}}\left(\vec{0}, R^2k''\right)\right| \left|\widehat{\rho}(\vec{0}, \epsilon^2 k'')\right|\right) \\
&\quad + R^{2d+2}\left(\sum_{k'\neq \vec{0}}\left|\widehat{\chi_{B_1}}\left(Rk', 0\right)\right| \left|\widehat{\rho}(\epsilon k', 0)\right|\right) \\
&\quad + R^{2d+2}\left(\sum_{k'\neq \vec{0},\, k''\neq 0}\left|\widehat{\chi_{B_1}}\left(Rk', R^2k''\right)\right| \left|\widehat{\rho}(\epsilon k', \epsilon^2 k'')\right|\right) \\
&:= I + II + III.
\end{aligned}
$$

For I, we use Lemma 2.3 and decay of $\widehat{\rho}$ provided by equation (3.1) (for any N),

$$
\begin{aligned}
I &= R^{2d+2}\sum_{k''\neq 0}\left|\widehat{\chi_{B_1}}\left(\vec{0}, R^2k''\right)\right| \left|\widehat{\rho}(\vec{0}, \epsilon^2 k'')\right| \\
&\lesssim R^{2d-2\min\{\frac{\alpha}{2},\frac{2d}{\alpha}\}}\sum_{k''\neq 0}|k''|^{-\left(1+\min\{\frac{\alpha}{2},\frac{2d}{\alpha}\}\right)}\left(1+\epsilon^2|k''|\right)^{-N} \\
&\lesssim R^{2d-2\min\{\frac{\alpha}{2},\frac{2d}{\alpha}\}}.
\end{aligned}
$$

In order to take advantage of the decay of $\widehat{\rho}$ (as stated in equation (3.1)) in estimating II and III, we break the sum in $k=(k',k'')$ further into four pieces :

— *sum 1* : $|k'|\leqslant \epsilon^{-1}$, $|k''|\leqslant \epsilon^{-2}$, and $k\neq \vec{0}$;
— *sum 2* : $|k'|\geqslant \epsilon^{-1}$, $|k''|\leqslant \epsilon^{-2}$, and $k\neq \vec{0}$;
— *sum 3* : $|k'|\leqslant \epsilon^{-1}$, $|k''|\geqslant \epsilon^{-2}$, and $k\neq \vec{0}$;
— *sum 4* : $|k'|\geqslant \epsilon^{-1}$, $|k''|\geqslant \epsilon^{-2}$, and $k\neq \vec{0}$.

At various steps in the remaining computations of this section, we use the decay estimate of equation (3.1) with N large enough to ensure the convergence of series' involved. For II, we use Lemma 2.4. We consider the case when $d-\frac{1}{2}\geqslant \frac{2}{\alpha}$ and $d-\frac{1}{2}\leqslant \frac{2}{\alpha}$ separately. In either case, we need only consider II over the set of $k\in\mathbb{Z}^{2d+1}$ satisfying the conditions listed in *sum 1* and *sum 2* with last coordinate equal to zero.

When $d-\frac{1}{2}\geqslant\frac{2}{\alpha}$,

$$\begin{aligned}II &= R^{2d+2}\sum_{k'\neq\vec{0}}\left|\widehat{\chi_{B_1}}\,(Rk',0)\right|\left|\widehat{\rho}(\epsilon k',0)\right|\\ &\lesssim R^{2d+2}\sum_{k'\neq\vec{0}}\left|Rk'\right|^{-(d+\frac{1}{2}+\frac{2}{\alpha})}\left(1+\epsilon\left|k'\right|\right)^{-N}.\end{aligned}$$

We first sum over the conditions listed in *sum 1* to get

$$\begin{aligned}II_{sum1} &\lesssim R^{2d+2}\sum_{1\leqslant|k'|\leqslant\epsilon^{-1}}\left|Rk'\right|^{-(d+\frac{1}{2}+\frac{2}{\alpha})}\left(1+\epsilon\left|k'\right|\right)^{-N}.\\ &\lesssim R^{d+\frac{3}{2}-\frac{2}{\alpha}}\left(1/\epsilon\right)^{d-\frac{1}{2}-\frac{2}{\alpha}}\ .\end{aligned}$$

Next, summing over the conditions listed in *sum 2*, we get

$$\begin{aligned}II_{sum2} &\lesssim R^{2d+2}\sum_{|k'|\geqslant\epsilon^{-1}}\left|Rk'\right|^{-(d+\frac{1}{2}+\frac{2}{\alpha})}\left|\epsilon k'\right|^{-N}.\\ &\lesssim R^{d+\frac{3}{2}-\frac{2}{\alpha}}\left(1/\epsilon\right)^{d-\frac{1}{2}-\frac{2}{\alpha}}\ .\end{aligned}$$

We conclude that, for $d-\frac{1}{2}\geqslant\frac{2}{\alpha}$, $II\lesssim R^{d+\frac{3}{2}-\frac{2}{\alpha}}\left(\frac{1}{\epsilon}\right)^{d-\frac{1}{2}-\frac{2}{\alpha}}$.
When $d-\frac{1}{2}\leqslant\frac{2}{\alpha}$, which is only the case for $d=1$ since $\alpha\geqslant 2$,

$$\begin{aligned}II &= R^{4}\sum_{k'\neq\vec{0}}\left|\widehat{\chi_{B_1}}\,(Rk',0)\right|\left|\widehat{\rho}(\epsilon k',0)\right|\\ &\lesssim R^{4}\sum_{k'\neq\vec{0}}\left|Rk'\right|^{-2}\left(1+\epsilon\left|k'\right|\right)^{-N}.\end{aligned}$$

Once again, we first sum over the conditions listed in *sum 1* to get

$$\begin{aligned}II_{sum1} &\lesssim R^{4}\sum_{1\leqslant|k'|\leqslant\epsilon^{-1}}\left|Rk'\right|^{-2}\\ &\sim R^{2}\log\left(1/\epsilon\right).\end{aligned}$$

And, summing over the conditions listed in *sum 2*, we get

$$\begin{aligned}II_{sum2} &\lesssim R^{4}\sum_{|k'|\geqslant\epsilon^{-1}}\left|Rk'\right|^{-2}\left|\epsilon k'\right|^{-N}\\ &\sim R^{2}\log\left(1/\epsilon\right).\end{aligned}$$

We conclude that, for $d-\frac{1}{2}\leqslant\frac{2}{\alpha}$, $II\lesssim R^2\log\left(1/\epsilon\right)$. We note for future use that for $d=1$ and $\alpha=2$ we have $II\lesssim R^{3/2}$, as follows from the fact that the decay of $\left|\widehat{\chi_{B_1^{2,A}}}\,(Rk',0)\right|$ is $R^{-5/2}$, as stated in Lemma 2.4.

For III, we use Lemma 2.5 and (3.1). We begin by considering the case when $\alpha > 2$. We have

$$\begin{aligned} III &= R^{2d+2} \sum_{k' \neq \vec{0},\, k'' \neq 0} \left|\widehat{\chi_{B_1}}(Rk', R^2 k'')\right| \left|\widehat{\rho}(\epsilon k', \epsilon^2 k'')\right| \\ &\lesssim R^d \sum_{k' \neq \vec{0},\, k'' \neq 0} |k'|^{-d} |k''|^{-1} \left(1 + \left|(\epsilon k', \epsilon^2 k'')\right|\right)^{-N}. \end{aligned}$$

Let us first consider III with conditions listed in *sum 1*.

$$\begin{aligned} III_{sum1} &\leqslant R^d \sum_{sum1} |k'|^{-d} |k''|^{-1} \\ &\sim R^d (1/\epsilon)^d \log(1/\epsilon). \end{aligned}$$

Next, we consider III with conditions listed in *sum 2*.

$$\begin{aligned} III_{sum2} &\lesssim R^d \sum_{sum2} |k'|^{-d} |k''|^{-1} |\epsilon k'|^{-N} \\ &\lesssim R^d (1/\epsilon)^d \log(1/\epsilon). \end{aligned}$$

Now, we consider III with conditions listed in *sum 3*.

$$\begin{aligned} III_{sum3} &\lesssim R^d \sum_{sum3} |k'|^{-d} |k''|^{-1} \left|\epsilon^2 k''\right|^{-N} \\ &\lesssim R^d (1/\epsilon)^d. \end{aligned}$$

Finally, we consider III with conditions listed in *sum 4*.

$$\begin{aligned} III_{sum4} &\lesssim R^d \sum_{sum4} |k'|^{-d} |k''|^{-1} |\epsilon k'|^{-N/2} \left|\epsilon^2 k''\right|^{-N/2} \\ &\lesssim R^d (1/\epsilon)^d. \end{aligned}$$

We conclude that for $\alpha > 2$, $III \lesssim R^d (1/\epsilon)^d \log\left(\frac{1}{\epsilon}\right)$.

In the case $\alpha = 2$ we have $III \lesssim R^{d-1/2} (1/\epsilon)^{d+1/2}$ which follows from the previous calculation using the decay estimate $\left|\widehat{\chi_{B_1}}\left(Rk', R^2 k''\right)\right| \lesssim |Rk'|^{-(d-1/2)} \left|R^2 k''\right|^{-3/2}$ stated in Lemma 2.5.

We conclude that (3.2) is bounded for $\alpha > 2$ by

$$\begin{aligned} R^{2d-\min\left\{\frac{\alpha}{2},\, \frac{2d}{\alpha}\right\}} + R^{d+2-\min\left\{d,\, \frac{1}{2}+\frac{2}{\alpha}\right\}} (1/\epsilon)^{d-\min\left\{d, \frac{1}{2}+\frac{2}{\alpha}\right\}} \log(1/\epsilon) \\ + R^d (1/\epsilon)^d \log(1/\epsilon) + O\left(R^{2d+1}\epsilon\right), \end{aligned}$$

and for $\alpha = 2$ the upper bound is $R^{d-\frac{1}{2}} (1/\epsilon)^{d+\frac{1}{2}}$, as mentioned earlier in the proof, and appears without the logarithmic term.

Next, we find the value of ϵ which minimizes the sum above. This is accomplished by graphing each of the four quantities in the sum above as functions of ϵ in order to find the value of epsilon which minimizes the maximum over the four quantities. For $2 \leqslant \alpha \leqslant 4$, we set $\epsilon = \frac{1}{R}$ and conclude that (3.2) is bounded by R^{2d} when $\alpha = 2$, and $R^{2d}\log(R)$ when $\alpha > 2$.

For $\alpha > 4$, we set $\epsilon = R^{\delta-1}$, where $\delta = \frac{2(\frac{1}{2}-\frac{2}{\alpha})}{d+\frac{1}{2}-\frac{2}{\alpha}}$, and conclude that (3.2) is bounded by $R^{2d+\delta}\log(R)$.

This completes the proof of Theorem 3.1.

4. Hyperplane slicing arguments

We now turn to the standard method of counting the lattice points in a given body by slicing it with hyperplanes, and counting the lattice points in each hyperplane separately, and establish what this method yields in the case under consideration. For notational simplicity, we write the arguments only for $A = 1$, though the same estimates hold for general case with only non-essential modifications.

4.1. Error bounds in higher dimensions

Slicing by hyperplane produces the following obvious identity :

$$\begin{aligned}\#\left(\mathbb{Z}^{2d+1} \cap B_R^\alpha\right) &= \#\left(\left\{(k',k'') \in \mathbb{Z}^{2d} \times \mathbb{Z} \,:\, |k'|^\alpha + |k''|^{\alpha/2} \leqslant R^\alpha\right\}\right) \\ &= \sum_{|k''| \leqslant R^2} \#\left(\left\{k' \in \mathbb{Z}^{2d} \,:\, |k'| \leqslant \left(R^\alpha - |k''|^{\alpha/2}\right)^{1/\alpha}\right\}\right) \\ &= \sum_{|k''| \leqslant R^2} G_{2d}\left(\left(R^\alpha - |k''|^{\alpha/2}\right)^{1/\alpha}\right),\end{aligned}$$

where $G_{2d}(T)$ denotes $\#\left(\left\{k' \in \mathbb{Z}^{2d} : |k'| \leqslant T\right\}\right)$, the standard lattice point counting function in Euclidean balls of radius T in $\mathbb{R}^{2d}$. Let us write, with A_{2d} denoting the volume of the unit ball in $\mathbb{R}^{2d}$:

$$\left|G_{2d}(T) - A_{2d}T^{2d}\right| \lesssim T^{\theta_1(2d)}\left(\log(2+T)\right)^{\theta_2(2d)}.$$

From the known error estimates on counting lattice points in Euclidean balls in $\mathbb{R}^{2d}$ (see [17] for a full discussion), we have $\theta_1(2d) = 2d-2$ for all $d \geqslant 2$, $\theta_1(2)$ is conjectured to be $\frac{1}{2}+\epsilon$ for any $\epsilon > 0$ (the conjectured error in the Gauss circle problem), $\theta_2(4) = 2/3$ while $\theta_2(2d) = 0$ for all $d \geqslant 3$. Thus,

$$\left|\#\left(\mathbb{Z}^{2d+1} \cap B_R^\alpha\right) - |B_R^\alpha|\right| \lesssim |E_1 - |B_R^\alpha|| + E_2\,,$$

where,

$$E_1 = A_{2d} \sum_{|k''|\leqslant R^2} \left(R^\alpha - |k''|^{\alpha/2}\right)^{2d/\alpha},$$

$$E_2 = \sum_{|k''|\leqslant R^2} \left(R^\alpha - |k''|^{\alpha/2}\right)^{\theta_1(2d)/\alpha} \left(\log\left(2+\left(R^\alpha - |k''|^{\alpha/2}\right)^{1/\alpha}\right)\right)^{\theta_2(2d)}.$$

We will first show that the main term given by the volume satisfies for $d \geqslant 1$:

$$|E_1 - |B_R^\alpha|| \lesssim R^{2d-\min\left\{2,\frac{4d}{\alpha}\right\}}.$$

For this, first notice that

$$|B_R^\alpha| = \int_{-R^2}^{R^2}\left(\int_{|z|\leqslant\left(R^\alpha-|t|^{\alpha/2}\right)^{1/\alpha}} dz\right) dt = A_{2d}\int_{-R^2}^{R^2}(R^\alpha - |t|^{\alpha/2})^{2d/\alpha}\, dt,$$

and therefore $E_1 - |B_R^\alpha|$ equals (up to a constant multiple) to

$$\begin{aligned}\sum_{|k''|\leqslant R^2} \left(R^\alpha - |k''|^{\alpha/2}\right)^{2d/\alpha} &- \int_{-R^2}^{R^2}\left(R^\alpha - |t|^{\alpha/2}\right)^{2d/\alpha} dt \\ &= R^{2d}\sum_{-R^2\leqslant k''\leqslant R^2} g\left(k''/R^2\right) - R^{2d}\int_{-R^2}^{R^2} g\left(t/R^2\right)\, dt\,,\end{aligned}$$

where $g(t) = \left(1-|t|^{\alpha/2}\right)^{2d/\alpha}$ on $[-1,1]$.

Now we apply the Euler-MacLaurin formula (as in [21, § 3], see also [17, p.20-23]) which says that

$$\begin{aligned}\sum_{-R^2\leqslant k''\leqslant R^2} g\left(k''/R^2\right) - \int_{-R^2}^{R^2} g\left(t/R^2\right)\, dt &= \int_{-R^2}^{R^2}\psi(t)\frac{d}{dt}\left(g\left(t/R^2\right)\right)\, dt \\ &= 2\int_0^1 \psi\left(R^2 t\right) g'(t)\, dt\,,\end{aligned}$$

with $\psi(t) = t - [t] - \frac{1}{2}$. Making use of the Fourier series expansion of ψ, namely $\psi(t) = -\pi^{-1}\sum_{j=1}^\infty j^{-1}\sin(2\pi j t)$, the above estimate equals to

$$\frac{2d}{\pi}\sum_{j=1}^\infty \frac{1}{j}\int_0^1 t^{\frac{\alpha}{2}-1}\left(1-t^{\alpha/2}\right)^{\frac{2d}{\alpha}-1}\sin\left(2\pi j R^2 t\right)\, dt.$$

To estimate each integral of the above sum, we perform one integration by parts if $\frac{2d}{\alpha} \geqslant 1$, otherwise we use the estimates for oscillatory integrals with singularities at an endpoint as stated in § 2.2.2, to conclude that

$$\left|\int_0^1 t^{\frac{\alpha}{2}-1}\left(1-t^{\alpha/2}\right)^{\frac{2d}{\alpha}-1}\sin\left(2\pi jR^2t\right)\,dt\right|\lesssim\left(R^2j\right)^{-\min\left\{1,\frac{2d}{\alpha}\right\}}.$$

Collecting all these estimates, we get the claimed bound on $|E_1-|B_R^\alpha||$.

Now we perform the following calculation to estimate E_2 : When $d\geqslant 2$, we get

$$E_2\sim R^{2+\theta_1(2d)}\left(\log(R)\right)^{\theta_2(2d)}. \tag{4.1}$$

Combining the above estimates we see that

$$\left|\#\left(\mathbb{Z}^{2d+1}\cap B_R^\alpha\right)-|B_R^\alpha|\right|\lesssim\begin{cases}R^4(\log(R))^{2/3} & ;\text{ for } d=2\\ R^{2d} & ;\text{ for } d\geqslant 3.\end{cases} \tag{4.2}$$

4.2. Error bounds in the first Heisenebrg group H_1

In the case $d=1$ of the first Heisenebrg group H_1, we can perform the same calculation as in the previous section, but must decide on the value we take for $\theta_1(2)$. Let us note that even if we assume the best possible conjectured error estimate in the Gauss circle problem, namely $\theta_1(2)=\frac{1}{2}+\epsilon$, then (4.1) implies

$$|E_2|=O_\epsilon\left(R^{\frac{5}{2}+\epsilon}\right).$$

Therefore,

$$\left|\#\left(\mathbb{Z}^3\cap B_R^\alpha\right)-|B_R^\alpha|\right|=O_\epsilon\left(R^{\frac{5}{2}+\epsilon}\right) \tag{4.3}$$

for every $\epsilon>0$. However, as stated in Theorem 3.1, the estimate we obtain for example for $2\leqslant\alpha\leqslant 4$ is at most $R^2\log(R)$, which is better, see below for further discussion.

5. Proof of the main theorem

We will now put everything together and prove Theorem 1.1.

5.1. The best possible estimate for $\alpha = 2$ in all dimensions

The fact that

$$\left|\#\left(\mathbb{Z}^{2d+1}\cap B_R^{2,A}\right) - \left|B_R^{2,A}\right|\right| = O(R^{2d})$$

follows from the corresponding case in Theorem 3.1. The matching lower bound for the error, namely the fact that

$$\left|\#\left(\mathbb{Z}^{2d+1}\cap B_R^{2,A}\right) - \left|B_R^{2,A}\right|\right| = \Omega(R^{2d})$$

follows from the following elementary argument. Assume to the contrary that

$$\left|\#\left(\mathbb{Z}^{2d+1}\cap B_R^{2,A}\right) - \left|B_R^{2,A}\right|\right| = o(R^{2d}).$$

Then for any $M \in \mathbb{N}$, using the fact that we are counting integer points

$$\begin{aligned}
0 &= \#\left(\mathbb{Z}^{2d+1}\cap B^{2,A}_{\sqrt{M+\frac{1}{2}}}\right) - \#\left(\mathbb{Z}^{2d+1}\cap B^{2,A}_{\sqrt{M}}\right)\\
&= \left|B^{2,A}_{\sqrt{M+\frac{1}{2}}}\right| - \left|B^{2,A}_{\sqrt{M}}\right| + o(M^d)\\
&= C\,M^d + o(M^d),
\end{aligned}$$

which is a contradiction.

5.2. The first Heisenebrg group H_1

5.2.1. The case $\alpha > 2$.

Theorem 3.1 implies that

$$\left|\#\left(\mathbb{Z}^{3}\cap B_R^{\alpha,A}\right) - \left|B_R^{\alpha,A}\right|\right| \leqslant \begin{cases} R^2\log(R) & ;\ \text{for } 2<\alpha\leqslant 4\\ R^{2+\delta} & ;\ \text{for } \alpha > 4\,, \end{cases}$$

where $\delta = \delta(\alpha) = \frac{2(\frac{1}{2}-\frac{2}{\alpha})}{\frac{3}{2}-\frac{2}{\alpha}}$.

Note that the bound on the error term for the Cygan-Korányi norm ($\alpha = 4$) for example is the same (up to a log factor) as in the case of $\alpha = 2$ which was best possible. However, we do not have any meaningful Ω-result here, and the upper bound on the error term can most likely be improved in this case.

5.2.2. Slicing and the error bound for large α.

As soon as $\alpha > 4$, we have $\delta(\alpha) > 0$ and so the quality of the error term obtained in Theorem 3.1 declines as $\alpha \to \infty$. As we saw at the end of the previous section, in the case of H_1 the slicing argument produces different results, depending on the quality of the error estimate in the classical Gauss circle problem. Using the best possible conjectured estimate in the latter problem produces an estimate of the error term in $\left|\#\left(\mathbb{Z}^3 \cap B_R^{\alpha,A}\right) - \left|B_R^{\alpha,A}\right|\right|$ which is $R^{\frac{5}{2}+\epsilon}$, so that it is inferior to the one stated in Theorem 3.1, namely $R^2 \log(R)$ when $2 < \alpha \leqslant 4$. However, for sufficiently large α, the error estimate obtained by slicing, namely $R^{2+\theta_1(2)}$ is superior to the one mentioned in Theorem 1.1, namely $R^{2+\delta(\alpha)}$. Indeed, $\lim_{\alpha\to\infty} \delta(\alpha) = 2/3$, so for any value of $\theta_1(2)$ which is less than $2/3$, there exists α_θ such that for $\alpha > \alpha_\theta$ the estimate produced by slicing is better than that stated in Theorem 3.1. For example, when we choose the conjectured best possible value $\theta_1(2) = 1/2$, we have $\alpha_{1/2} = 12$. In particular, for $2 \leqslant \alpha \leqslant 12$, the estimate stated in Theorem 3.1 is superior to the one produced by slicing.

5.3. The case of H_d, $d \geqslant 2$

For the higher-dimensional Heisenberg groups, the method of slicing carried out in § 4.1 is surprisingly effective and produces the following bound on the error term

$$\left|\#\left(\mathbb{Z}^{2d+1} \cap B_R^{\alpha,A}\right) - \left|B_R^{\alpha,A}\right|\right| \leqslant \begin{cases} R^4(\log(R))^{\frac{2}{3}} & ;\ \text{for } d = 2 \\ R^{2d} & ;\ \text{for } d \geqslant 3\,. \end{cases}$$

For $2 < \alpha \leqslant 4$ this is the same bound (up to a log factor) that was obtained in Theorem 3.1. However, note that in the slicing argument we utilized the best possible result for the classical lattice point counting problem in Euclidean balls in dimensions $2d \geqslant 4$. This is a highly non-elementary result, and it is interesting to note that Theorem 3.1 produces the bound $R^{2d}\log(R)$ for $2 < \alpha \leqslant 4$ just using van-der-Corput lemma and Poisson summation.

Another interesting comment is that convexity of $B_R^{\alpha,A}$ is irrelevant to the slicing argument, and in fact the error estimates stated in the beginning of this subsection are valid for all $\alpha > 0$, but $B_1^{\alpha,A}$ is convex if and only if $\alpha \geqslant 2$.

This concludes the proof of Theorem 1.1.

6. Comparison with some Euclidean lattice point counting results

In this section we consider the problem of applying an analogue of our method to the problem of lattice point counting in the *Euclidean dilates* of the compact, convex 3-dimensional Euclidean bodies

$$D_1^\alpha = \{(z,t) \in \mathbb{R}^2 \times \mathbb{R} \,:\, \widetilde{N}_\alpha((z,t)) \leqslant 1\}.$$

where $\widetilde{N}_\alpha((z,t)) = \left(|z|^\alpha + |t|^\alpha\right)^{1/\alpha}$. Namely we will estimate

$$\left|\#\left(\mathbb{Z}^3 \cap D_R^\alpha\right) - |D_R^\alpha|\right|$$

where $D_R^\alpha = \{(Rz, Rt) \,:\, (z,t) \in D_1^\alpha\}$ so that $|D_R^\alpha| = R^3\,|D_1^\alpha|$.

This problem was studied in great detail in [21] (see also the references therein) where the authors performed very fine analysis obtaining sharp asymptotic results. Our goal in the present section is to demonstrate that our method of utilizing direct spectral decay estimates via Poisson's summation formula also gives the right *first order* error estimate (up to a log factor) for $\alpha \geqslant 4$, equal to the one produced in [21]. Our method cannot reproduce the much finer results regarding the secondary main terms obtained in [21]. The reason we include this analysis here is in order to establish some point of comparison with which to gauge the quality of the error estimate stated in Theorem 1.1. As noted in the introduction, the authors are not aware of any previous result on the lattice point counting problem on the Heisenberg groups which could serve as a basis for such a comparison.

Our main result in the Euclidean setting is as follows.

Theorem 6.1. — *For $\alpha > 2$,*

$$\left|\#\left(\mathbb{Z}^3 \cap D_R^\alpha\right) - |D_R^\alpha|\right| = O\left(\left(R^{\frac{3}{2}} + R^{2-\frac{2}{\alpha}}\right)\log(R)\right).$$

In the next section, we state spectral decay estimates analogous to those of § 2, which we will use to prove Theorem 6.1 in § 6.2.

6.1. Spectral decay estimates

We begin with a remark that though for simplicity we have stated Theorem 6.1 for $d = 1$, our result extends to higher dimensions, namely $\mathbb{R}^{2d+1}$ as well. We restrict our attention only to the case $\alpha > 2$. The estimates in this section are stated for $d \geqslant 1$ and $\alpha > 2$. We start with the analogs of Lemmas 2.3, 2.4 and 2.5. Each of the three lemmas stated in this section is proved using the same techniques as above. Therefore, rather than repeat each of the proofs above, we include the statement of the key estimates with some brief comments regarding the proof.

Lemma 6.2. — *For $\alpha > 2$, $|s| \geqslant 1$,*

$$\left|\widehat{\chi_{D_1^\alpha}}(\vec{0}, s)\right| \lesssim |s|^{-\left(1+\min\left\{\frac{2d}{\alpha}, \alpha\right\}\right)}.$$

However, for a positive integer $\alpha \in 2\mathbb{N}$, we have a better estimate

$$\left|\widehat{\chi_{D_1^\alpha}}(\vec{0}, s)\right| \lesssim |s|^{-\left(1+\frac{2d}{\alpha}\right)}.$$

For the purpose of proving Theorem 6.1 where $d = 1$, let us first note that $\min\left\{\frac{2}{\alpha}, \alpha\right\} = \frac{2}{\alpha}$. The proof of Lemma 6.2 is similar to that of Lemma 2.3, and so we do not repeat it here.

Lemma 6.3. — *For $\alpha > 2$, $|w| \geqslant 1$,*

$$\left|\widehat{\chi_{D_1^\alpha}}(w, 0)\right| \lesssim \begin{cases} |w|^{-2d} & ;\ \textit{if } \frac{1}{\alpha} > d - \frac{1}{2} \\ |w|^{-\left(d+\frac{1}{2}+\frac{1}{\alpha}\right)} & ;\ \textit{if } \frac{1}{\alpha} \leqslant d - \frac{1}{2}. \end{cases}$$

For the purpose of proving Theorem 6.1 where $d = 1$, we will only need to use the second estimate.

Proof. — The conclusion of the first item agrees with the first item in Lemma 2.4, and the proof follows similarly. In the case that $\frac{1}{\alpha} \leqslant d - \frac{1}{2}$, we follow the proof of Lemma 2.4 to see that for $k = 0, 1$:

$$\left|\int_0^1 e^{i\xi r} r^{d-k-\frac{1}{2}} (1 - r^\alpha)^{1/\alpha}\, dr\right| \lesssim \xi^{-\left(1+\min\left\{\alpha, \frac{1}{\alpha}, d-k-\frac{1}{2}\right\}\right)}.$$

Collecting all the bounds, we see that $I(\xi) = \int_0^1 J_{d-1}(\xi r)\, (1 - r^\alpha)^{2/\alpha}\, r^d\, dr$ is dominated by a finite sum of terms of the form

$$\delta^{d+1},\ \xi^{-\frac{5}{2}}\delta^{-\frac{1}{2}},\ \xi^{-\left(k+\frac{1}{2}\right)}\delta^{d-k+\frac{1}{2}},\ \xi^{-\left(k+\frac{3}{2}\right)}\xi^{-\min\left\{\alpha, \frac{1}{\alpha}, d-k-\frac{1}{2}\right\}}$$

where $k = 0, 1$. Choosing $\delta = \xi^{-1}$, we verify that

$$|I(\xi)| \lesssim \xi^{-\left(\frac{3}{2}+\frac{1}{\alpha}\right)};\ \text{ whenever } \frac{1}{\alpha} \leqslant d - \frac{1}{2}.$$

□

Lemma 6.4. — *For $\alpha > 2$ we have for $|w| \geqslant 1$ and $|s| \geqslant 1$,*

$$\left|\widehat{\chi_{D_1^\alpha}}(w, s)\right| \lesssim |w|^{-d}\, |s|^{-1}. \tag{6.1}$$

Proof. — Considering the proof of Lemma 2.5, we see that it suffices to estimate

$$|w|^{-1/2} \int_0^1 e^{i|w|r} e^{is(1-r^\alpha)^{1/\alpha}} r^{d-\frac{1}{2}}\, dr\,.$$

The above integral is same as

$$|w|^{-1/2}\int_0^1 \exp\left(i\,|w|\,\phi_{|w|,s}(r)\right) r^{d-\frac{1}{2}}\,dr\,,$$

where $\phi_{|w|,s}(r) = r + \frac{s}{|w|}(1-r^\alpha)^{1/\alpha}$. Now

$$\begin{aligned}\phi'_{|w|,s}(r) &= 1 - \frac{s}{|w|} r^{\alpha-1}(1-r^\alpha)^{\frac{1}{\alpha}-1}\,,\\ \phi''_{|w|,s}(r) &= -(\alpha-1)\frac{s}{|w|} r^{\alpha-2}(1-r^\alpha)^{\frac{1}{\alpha}-2}\,.\end{aligned}$$

Notice once again that $\phi'_{|w|,s}$ is monotone on $(0,1)$, since $\phi''_{|w|,s}$ has a constant sign there. Clearly, $\phi'_{|w|,s}(r) \geqslant 1$ for all $r \in [0,1]$ when s is negative. The difficulty arises only when s is positive.

Since $r^{\alpha-1}(1-r^\alpha)^{\frac{1}{\alpha}-1}$ is a strictly increasing function mapping $[0,1)$ onto $[0,\infty)$, there exists unique point $r_0 \equiv r_0(|w|\,,s) \in (0,1)$ at which $r_0^{\alpha-1}(1-r_0^\alpha)^{\frac{1}{\alpha}-1} = \frac{|w|}{2s}$ and

$$\begin{aligned}\left|\phi'_{|w|,s}(r)\right| &\geqslant \frac{1}{2} \text{ for all } r \in [0,r_0],\\ \left|\phi''_{|w|,s}(r)\right| &\geqslant \frac{\alpha-1}{2} \text{ for all } r \in [r_0,1).\end{aligned}$$

The rest of the proof is analogous to that of Lemma 2.5. □

6.2. Proof of Theorem 6.1

We fix a bump function $\rho : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, which is a smooth non-negative function with support contained in the unit ball D_1^α, such that $\rho(0) > 0$ and $\int_{D_1^\alpha} \rho(z,t)\,dz\,dt = 1$. We then consider the family of functions defined by the Euclidean dilates of ρ, namely $\rho_\epsilon(z,t) = \epsilon^{-3}\rho(\frac{z}{\epsilon},\frac{t}{\epsilon})$.

In order to prove Theorem 6.1, we will follow the proof of the Theorem 3.1. Rather than repeat the proof above, we include key estimates where the two arguments differ. As in § 3.2, the proof reduces to estimating :

$$R^3 \sum_{k\in\mathbb{Z}^3\setminus\{\vec{0}\}} \left|\widehat{\chi_{D_1^\alpha}}(Rk',Rk'')\right| |\widehat{\rho}(\epsilon k',\epsilon k'')| + O\left(R^2\epsilon\right). \tag{6.2}$$

In order to utilize the spectral decay estimates of Lemmas 6.2, 6.3 and 6.4, we break the sum in $k = (k',k'') \in \mathbb{Z}^2 \times \mathbb{Z}$ into following four pieces :

- *sum 1* : $|k'| \leqslant \epsilon^{-1}$, $|k''| \leqslant \epsilon^{-1}$, and $k \neq \vec{0}$;
- *sum 2* : $|k'| \geqslant \epsilon^{-1}$, $|k''| \leqslant \epsilon^{-1}$, and $k \neq \vec{0}$;

— *sum 3* : $|k'| \leqslant \epsilon^{-1}$, $|k''| \geqslant \epsilon^{-1}$, and $k \neq \vec{0}$;
— *sum 4* : $|k'| \geqslant \epsilon^{-1}$, $|k''| \geqslant \epsilon^{-1}$, and $k \neq \vec{0}$.

Continuing the analysis analogous to that of § 3.2, we conclude that (6.2) is bounded by a constant times :

$$R^{2-\frac{2}{\alpha}} + R^{\frac{3}{2}-\frac{1}{\alpha}}\,(1/\epsilon)^{\frac{1}{2}-\frac{1}{\alpha}} + (R/\epsilon)\log(R) + O\left(R^2\epsilon\right). \tag{6.3}$$

When $2 < \alpha \leqslant 4$, we choose $\epsilon = R^{-1/2}$ to see that the expression in (6.3) is bounded by

$$R^{3/2}\log(R).$$

When $\alpha > 4$, we choose $\epsilon = R^{\frac{2}{\alpha}-1}$ to see that (6.3) is bounded by

$$R^{2-\frac{2}{\alpha}}\log(R).$$

This completes the proof of Theorem 6.1.

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