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Atsushi ITO & Yusaku TIBA

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# CURVES IN QUADRIC AND CUBIC SURFACES WHOSE COMPLEMENTS ARE KOBAYASHI HYPERBOLICALLY IMBEDDED

by Atsushi ITO & Yusaku TIBA (\*)

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ABSTRACT. — We construct smooth irreducible curves of the lowest possible degrees in quadric and cubic surfaces whose complements are Kobayashi hyperbolically imbedded into those surfaces. Moreover we characterize line bundles on quadric and cubic surfaces such that the complete linear systems of the line bundles have a smooth irreducible curve whose complement is Kobayashi hyperbolically imbedded into those surfaces.

RÉSUMÉ. — Nous construisons des courbes complexes lisses de degré minimal possible dans des surfaces quadriques et cubiques à complémentaire hyperboliquement plongé, au sens de Kobayashi. De plus, nous caractérisons les fibrés en droites sur de telles surfaces dont les systèmes linéaires associés possèdent des courbes lisses à complémentaire hyperboliquement plongé.

## 1. Introduction and main result

Kobayashi [13], [14] introduced the Kobayashi hyperbolicity and proposed the following famous conjecture.

CONJECTURE 1.1 (Kobayashi conjecture). — *Let  $X$  be a general hypersurface of degree  $d$  in  $\mathbb{P}^n(\mathbb{C})$ .*

- (i) *If  $d \geq 2n - 1$ ,  $X$  is Kobayashi hyperbolic for  $n \geq 3$ .*

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- (ii) If  $d \geq 2n + 1$ ,  $\mathbb{P}^n(\mathbb{C}) \setminus X$  is Kobayashi hyperbolically imbedded into  $\mathbb{P}^n(\mathbb{C})$ .

A lot of researchers have studied the Kobayashi conjecture. For example, Păun [21] showed that a (very) general smooth surface of degree  $d \geq 18$  in  $\mathbb{P}^3(\mathbb{C})$  is Kobayashi hyperbolic improving the results of [17] and [4]. Rousseau [22] showed that  $\mathbb{P}^2(\mathbb{C}) \setminus X$  is Kobayashi hyperbolically imbedded into  $\mathbb{P}^2(\mathbb{C})$  if  $X$  is a very general curve of degree  $d \geq 14$  improving the results of [23] and [8]. It seems that there exist some difficulties to prove the Kobayashi conjecture with optimal degrees even in the case of low dimensions. We note that according to Fujimoto [10] and Green [11] the complement of a union of  $d \geq 2n + 1$  hyperplanes in  $\mathbb{P}^n(\mathbb{C})$  with simple normal crossings is Kobayashi hyperbolically imbedded.

On the other hand, concrete examples of Kobayashi hyperbolic hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  are known. Masuda and Noguchi [16] constructed algebraic families of hyperbolic hypersurfaces in  $\mathbb{P}^n(\mathbb{C})$  for large degrees. Duval [7] constructed smooth Kobayashi hyperbolic hypersurfaces of degree six in  $\mathbb{P}^3(\mathbb{C})$ . Shiffman and Zaidenberg [25] constructed smooth Kobayashi hyperbolic hypersurfaces of degree  $d$  in  $\mathbb{P}^3(\mathbb{C})$  for  $d \geq 8$ . Recently, Ciliberto and Zaidenberg [2] suggested a new construction of such hypersurfaces in  $\mathbb{P}^3(\mathbb{C})$  of degree six and seven. A smooth Kobayashi hyperbolic hypersurface of degree five in  $\mathbb{P}^3(\mathbb{C})$  is not known yet. Using a method of successive deformations, Zaidenberg [26] showed that, for each degree  $d \geq 5$ , there exists a smooth irreducible curve of degree  $d$  in  $\mathbb{P}^2(\mathbb{C})$  such that the complement of the curve is Kobayashi hyperbolically imbedded into  $\mathbb{P}^2(\mathbb{C})$ . In this paper, applying the same method and results of [9], [11] on degeneracy for entire curves, we construct smooth irreducible curves of the lowest possible degrees in quadric and cubic surfaces whose complements are Kobayashi hyperbolically imbedded into those surfaces.

**THEOREM 1.2.** — *Let  $Q = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ . Let  $L = \mathcal{O}_{\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})}(m, n)$  be a line bundle on  $Q$ . There exists a smooth irreducible curve  $X$  in the linear system  $|L|$  such that  $Q \setminus X$  is Kobayashi hyperbolically imbedded into  $Q$  if and only if  $m$  and  $n$  are larger than or equal to four.*

*Remark 1.3.* — We cannot remove the irreducibility of  $X$ . For example,  $(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) \times (\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})$  is Kobayashi hyperbolically imbedded into  $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ .

Let  $S$  be a smooth cubic surface in  $\mathbb{P}^3(\mathbb{C})$ . A common point of three lines in  $S$  is called an Eckardt point. A generic cubic surface does not have any

such point, and cubic surfaces with an Eckardt point form a hypersurface in the moduli space (see Chapter 9 of [6]).

**THEOREM 1.4.** — *Let  $S$  be a smooth cubic surface in  $\mathbb{P}^3(\mathbb{C})$  without any Eckardt point. Let  $L$  be a holomorphic line bundle on  $S$ . There exists a smooth irreducible curve  $X$  in  $|L|$  such that  $S \setminus X$  is Kobayashi hyperbolically imbedded into  $S$  if and only if the intersection number  $L \cdot l$  is larger than or equal to three for any line  $l$  in  $S$ .*

*Remark 1.5.* — It is known that there exist 27 lines in  $S$ , where a line is a smooth rational curve of degree one in  $\mathbb{P}^3(\mathbb{C})$  (see [15], [1]).

*Remark 1.6.* — Let  $M$  be a compact complex manifold and let  $L$  be a line bundle on  $M$ . Let  $D \in |L|$  be a smooth divisor on  $M$  whose complement is Kobayashi hyperbolically imbedded into  $M$ . Then there exists a family of divisors in  $|L|$  whose complements are Kobayashi hyperbolically imbedded into  $M$ . More precisely, there exists a small neighborhood  $U$  of  $D$  in  $|L|$  in the sense of classical topology such that the complement of any element of  $U$  is Kobayashi hyperbolically imbedded into  $M$  (see [26]).

We note that any smooth quadric surface in  $\mathbb{P}^3(\mathbb{C})$  is isomorphic to  $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$  imbedded by  $\mathcal{O}_{\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})}(1, 1)$ . Let  $\mathcal{O}_Q(1) = \mathcal{O}_{\mathbb{P}^3(\mathbb{C})}(1)|_Q$  (resp.  $\mathcal{O}_S(1) = \mathcal{O}_{\mathbb{P}^3(\mathbb{C})}(1)|_S$ ). By Theorem 1.2 and Theorem 1.4, there exists a smooth irreducible curve  $Y$  in  $|\mathcal{O}_Q(d)|$  (resp.  $|\mathcal{O}_S(d)|$ ) such that  $Q \setminus Y$  (resp.  $S \setminus Y$ ) is Kobayashi hyperbolically imbedded into  $Q$  (resp.  $S$ ) if and only if  $d \geq 4$  (resp.  $d \geq 3$ ). The above  $Y$  for  $d = 4$  (resp.  $d = 3$ ) is a smooth irreducible curve of the lowest possible degree whose complement is Kobayashi hyperbolically imbedded into  $Q$  (resp.  $S$ ). To be precise,  $C \cdot \mathcal{O}_Q(1) \geq Y \cdot \mathcal{O}_Q(1) = 8$  (resp.  $C \cdot \mathcal{O}_S(1) \geq Y \cdot \mathcal{O}_S(1) = 9$ ) for any smooth irreducible curve  $C$  in  $Q$  (resp.  $S$ ) whose complement is Kobayashi hyperbolically imbedded into  $Q$  (resp.  $S$ ). More strongly,  $L - \mathcal{O}_Q(4)$  (resp.  $L - \mathcal{O}_S(3)$ ) is nef for any line bundle  $L$  on  $Q$  (resp.  $S$ ) which satisfies the condition in Theorem 1.2 (resp. Theorem 1.4). The nefness of  $L - \mathcal{O}_Q(4)$  is clear by Theorem 1.2. For the nefness of  $L - \mathcal{O}_S(3)$ , see Lemma 3.2 of Section 3.

The plan of this paper is as follows. Since the proof of Theorem 1.2 is similar to and simpler than that of Theorem 1.4, we first prove Theorem 1.4. In Section 2, we prove the necessity in Theorem 1.4 and prove that there exists a reducible curve of degree nine in  $S$  such that the complement of the curve is Kobayashi hyperbolically imbedded into  $S$ . In Section 3, we deform the above reducible curve to a smooth irreducible curve preserving

the Kobayashi hyperbolic imbedding as in [26] and so we complete the proof of Theorem 1.4. In Section 4, we prove Theorem 1.2.

## 2. A line arrangement on $S$ with Kobayashi hyperbolically imbedded complement

Let  $S \subset \mathbb{P}^3(\mathbb{C})$  be a smooth cubic surface. Then  $S$  is isomorphic to the blow-up of  $\mathbb{P}^2(\mathbb{C})$  at six points (see [15], [1]). Let  $\pi : S \rightarrow \mathbb{P}^2(\mathbb{C})$  be the blow-up at six points  $p_1, \dots, p_6 \in \mathbb{P}^2(\mathbb{C})$ . Let  $E_i$  ( $1 \leq i \leq 6$ ) be the exceptional divisor over  $p_i$ , let  $L_{ij} \subset S$  ( $1 \leq i < j \leq 6$ ) be the strict transform of the line in  $\mathbb{P}^2(\mathbb{C})$  through  $p_i$  and  $p_j$  under  $\pi$ , and let  $C_i$  ( $1 \leq i \leq 6$ ) be the strict transform of the conic in  $\mathbb{P}^2(\mathbb{C})$  through five points  $p_j$  ( $j \neq i$ ) under  $\pi$ . Such conic does exist and is unique since  $S$  is a smooth cubic and so the six points  $p_i$  in  $\mathbb{P}^2(\mathbb{C})$  are in general position. It is known that  $E_i$  ( $1 \leq i \leq 6$ ),  $L_{ij}$  ( $1 \leq i < j \leq 6$ ), and  $C_i$  ( $1 \leq i \leq 6$ ) are all the lines on  $S$  (see [15], [1]).

The necessity in Theorem 1.4 is easily shown as follows.

*Proof of the necessity in Theorem 1.4.* — Assume that there exists a line  $l$  on  $S$  such that  $L \cdot l \leq 2$ . Let  $X \in |L|$  be a smooth irreducible curve in  $S$ . If  $X = l$ ,  $X$  meets another line  $l'$  in at most one point. Hence, there exists a non-constant holomorphic map from  $\mathbb{C}$  to  $l' \setminus X|_{l'} \subset S \setminus X$  and  $S \setminus X$  is not Kobayashi hyperbolically imbedded into  $S$ . If  $X \neq l$ ,  $X \cap l$  contains at most two points. Hence there exists a non-constant holomorphic map from  $\mathbb{C}$  to  $l \setminus X|_l$ . This implies that  $S \setminus X$  is not Kobayashi hyperbolically imbedded into  $S$ .  $\square$

We mention that there exists a very ample line bundle  $L$  on  $S$  with any high degree such that  $|L|$  does not contain any smooth irreducible curve whose complement is Kobayashi hyperbolically imbedded into  $S$ . For example, let  $L = \pi^* \mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(k) + \mathcal{O}_S(1)$ ,  $k \geq 1$  and let  $X \in |L|$  be a smooth irreducible curve. Then  $E_i \setminus X = \mathbb{C}$  since  $X$  intersects  $E_i$  at only one point. Therefore  $S \setminus X$  is not Kobayashi hyperbolically imbedded into  $S$  and the degree  $L \cdot \mathcal{O}_S(1) = 3k + 3$ .

To prove the sufficiency in Theorem 1.4, we show the following proposition. In the first version of this paper, we use results of [18], [19], and [3] to prove Proposition 2.1. The following simplified proof, where we use results of [9], [11] instead, is due to the referee.

PROPOSITION 2.1. — *Let  $S$  be a smooth cubic surface without any Eckardt point. Let  $D$  be the union of nine lines on  $S$  as follows:*

$$D = E_4 + E_5 + E_6 + L_{45} + L_{46} + L_{56} + C_4 + C_5 + C_6.$$

Then  $S \setminus D$  is Kobayashi hyperbolically imbedded into  $S$ .

We define  $L_{ji} = L_{ij}$  for  $1 \leq i < j \leq 6$ . Since  $L_{ij}$  is linearly equivalent to  $\pi^* \mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(1) - E_i - E_j$  and  $C_i$  is linearly equivalent to  $\pi^* \mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(2) - \sum_{j \neq i} E_j$ ,  $E_i + L_{ij} + C_j$  ( $4 \leq i, j \leq 6, i \neq j$ ) is linearly equivalent to  $\mathcal{O}_S(1) = \pi^* \mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(3) - \sum_{i=1}^6 E_i$  (see [15], [1]). Hence there exists a hyperplane  $H_{ij}$  in  $\mathbb{P}^3(\mathbb{C})$  such that  $H_{ij}|_S = E_i + L_{ij} + C_j$ . Note that

$$2D = \sum_{4 \leq i \neq j \leq 6} H_{ij}|_S$$

as divisors on  $S$ . First we show the following lemma.

LEMMA 2.2. — *Let  $S$  be a smooth cubic surface without any Eckardt point. Then the six hyperplanes  $H_{ij}$  ( $4 \leq i, j \leq 6, i \neq j$ ) are in general position in  $\mathbb{P}^3(\mathbb{C})$ .*

*Proof.* — We denote  $H_{ij}$  ( $4 \leq i, j \leq 6, i \neq j$ ) by  $H_1, \dots, H_6$ . Assume that there exists a point  $p$  in  $H_1 \cap H_2 \cap H_3 \cap H_4$ . Since each line in  $H_1|_S$  is contained in the unique hyperplane in  $\{H_i\}_{2 \leq i \leq 6}$ , there exist three hyperplanes in  $\{H_i\}_{2 \leq i \leq 6}$  such that  $H_1 \cap H_i \subset S$ . Hence we may assume that  $l := H_1 \cap H_2 \subset S$ . Then  $p \in l \subset S$ . Because  $p \in H_3|_S$ , there exists a line  $l' \subset H_3|_S$  such that  $p \in l'$ . We have that  $l \neq l'$  since each line in  $D$  is contained in only two hyperplanes in  $\{H_i\}_{1 \leq i \leq 6}$ . By the same argument, there exists a line  $l'' \subset H_4|_S$  which contains  $p$  and  $l \neq l''$ . Because of our assumption on absence of an Eckardt point on  $S$ , we have that  $l' = l''$ . Note that  $H_1$  or  $H_2$  contains  $l'$  since there are only four lines in  $D$  which intersect  $l$  and these lines are all contained in  $H_1|_S \cup H_2|_S$ . Hence  $l'$  is contained in three hyperplanes in  $\{H_i\}_{1 \leq i \leq 4}$  and this is a contradiction.  $\square$

*Proof of Proposition 2.1.* — Let  $D_i$  ( $1 \leq i \leq 9$ ) be irreducible components of  $D$ . Hence  $D_i$  is equal to  $E_k$  or  $L_{kl}$  or  $C_l$ . Assume that  $S \setminus D$  is not Kobayashi hyperbolically imbedded into  $S$ . Then there exists a partition of indices  $I \cup J = \{1, 2, \dots, 9\}$  and a non-constant holomorphic map from  $\mathbb{C}$  to  $\bigcap_{i \in I} D_i \setminus \bigcup_{j \in J} D_j$  (see Theorem (1.8.3) of [20]). Here  $I$  may be the empty set and we define  $\bigcap_{i \in I} D_i = S$  in that case. Since each  $D_i$  is isomorphic to  $\mathbb{P}^1(\mathbb{C})$  and intersects  $\bigcup_{j \neq i} D_j$  at four points, there exists no non-constant holomorphic map from  $\mathbb{C}$  to  $D_i \setminus \bigcup_{j \neq i} D_j$  by the little Picard Theorem. Therefore  $I$  must be empty and there exists a non-constant holomorphic

map

$$f : \mathbb{C} \rightarrow S \setminus D \subset \mathbb{P}^3(\mathbb{C}) \setminus \bigcup_{4 \leq i \neq j \leq 6} H_{ij}.$$

Because of Lemma 2.2,  $H_{ij}$  ( $4 \leq i \neq j \leq 6$ ) are in general position. According to Theorem B of [9] or Theorem 1 of [11], it follows that  $f(\mathbb{C})$  is contained in a line, say,  $l$  on  $S$ . By the little Picard Theorem and Lemma 2.2, there exist two points  $A, B$  ( $A \neq B$ ) in  $S$  such that  $l \cap D = \{A, B\}$ . On the other hand,  $D \cdot l = 3$  since  $D$  is linearly equivalent to  $\mathcal{O}_S(3)$ . Thus  $l$  contains a node of  $D$ . Hence three lines in  $S$  has a common point  $A$  or  $B$  and that point is an Eckardt point of  $S$ , contrary to our assumption.  $\square$

### 3. Deformation of curves

We show the sufficiency in Theorem 1.4. From *First Step* to *Fourth Step*, we deform the nine lines in Proposition 2.1 to three smooth elliptic curves keeping the Kobayashi hyperbolic imbedding.

*First Step.* — We first deform  $E_4$  and  $L_{45}$  to an irreducible conic in  $S$  keeping other lines. Recall that  $H_{ij}$  is the hyperplane in  $\mathbb{P}^3(\mathbb{C})$  such that  $H_{ij}|_S = E_i + L_{ij} + C_j$ . Let  $H_t$  ( $t \in \mathbb{C}, |t| < 1$ ) be a deformation family of hyperplanes in  $\mathbb{P}^3(\mathbb{C})$  such that  $C_5 \subset H_t$  for all  $t$  and  $H_0 = H_{45}$ . Since there exists the only finite set of lines in  $S$ ,  $H_t|_S$  is a union of  $C_5$  and an irreducible conic for small  $t \neq 0$ . We show that there exists a small  $\delta > 0$  such that  $S \setminus (H_t \cup H_{46} \cup H_{56})|_S$  is Kobayashi hyperbolically imbedded into  $S$  for  $|t| < \delta$ . Otherwise, there exists a sequence  $\{t_\nu\}_{\nu \in \mathbb{N}}$  such that  $t_\nu \rightarrow 0, t_\nu \neq 0$  and  $S \setminus (H_{t_\nu} \cup H_{46} \cup H_{56})|_S$  is not Kobayashi hyperbolically imbedded into  $S$ . Then there exist holomorphic maps  $f_\nu$  from  $\Delta(\nu) = \{z \in \mathbb{C} : |z| < \nu\}$  to  $S \setminus (H_{t_\nu} \cup H_{46} \cup H_{56})|_S$  such that  $f_\nu$  converges to a non-constant holomorphic map  $f : \mathbb{C} \rightarrow S$  uniformly on compact subsets (see Section 2 of [26]). Let  $\mathcal{D}$  be the hypersurface in  $S \times \Delta(1)$  such that  $\mathcal{D}|_{S \times \{t\}} = (H_t \cup H_{46} \cup H_{56})|_S$ . Then we may consider that  $f_\nu$  (resp.  $f$ ) is a holomorphic map from  $\mathbb{C}$  to  $S \times \{t_\nu\} \setminus \mathcal{D}|_{S \times \{t_\nu\}}$  (resp.  $S \times \{0\} \setminus \mathcal{D}|_{S \times \{0\}}$ ). Let  $\mathcal{C}$  be an irreducible component of  $\mathcal{D}$ . By Hurwitz' theorem,  $f(\mathbb{C}) \subset \mathcal{C}|_{S \times \{0\}}$  or  $f(\mathbb{C}) \cap \mathcal{C}|_{S \times \{0\}} = \emptyset$ . Because there exists no non-constant holomorphic map from  $\mathbb{C}$  to  $S \setminus D$ ,  $f(\mathbb{C})$  is contained in an irreducible component of  $H_{45}|_S$  or  $H_{46}|_S$  or  $H_{56}|_S$ . Since any irreducible component of  $D$  intersects other components of  $D$  at four points,  $f(\mathbb{C})$  must be contained in  $E_4$  or  $L_{45}$ . Without loss of generality, we may assume that  $f(\mathbb{C}) \subset E_4$ . Since  $E_4$  intersects the divisor  $D - (E_4 + L_{45})$  at three

points, it follows that  $f$  is constant by the little Picard Theorem. This is a contradiction and we have that  $S \setminus (H_{t_0} \cup H_{46} \cup H_{56})|_S$  is Kobayashi hyperbolically imbedded into  $S$  for small  $t_0$ .

*Second Step.* — Next we deform  $H_{t_0}|_S$  to an irreducible curve with a node in  $S$ . Since two components of  $H_{t_0}|_S$  intersect at two points, there exist two nodes in  $H_{t_0}|_S$ . Let  $p$  be one of those nodes. Then  $H_{t_0}$  is the tangent plane of  $S$  at  $p$ . Let  $\epsilon : \mathbb{P}^3(\mathbb{C}) \times S \rightarrow S$  be the second projection and let  $\mathcal{H}$  be the divisor on  $\mathbb{P}^3(\mathbb{C}) \times S$  such that  $\mathcal{H}_q = \mathcal{H}|_{\mathbb{P}^3(\mathbb{C}) \times \{q\}}$  ( $q \in S$ ) is the tangent plane of  $S$  at  $q$  in  $\mathbb{P}^3(\mathbb{C})$ .

LEMMA 3.1. — *Let  $U$  be a sufficiently small neighborhood of  $p$  in  $S$ . Then the divisor  $\mathcal{H}|_{S \times S}$  on  $S \times S$  consists of two smooth transversally intersecting irreducible components in  $U \times U$ .*

*Proof.* — Let  $(x, y)$  be a local coordinate system on  $U$  such that  $x(p) = y(p) = 0$  and  $H_{t_0}|_S$  is equal to the divisor defined by  $xy$  in  $U$ . Let  $\pi_i : S \times S \rightarrow S$  be the  $i$ -th projection. We will still write  $x = \pi_1^*x, y = \pi_1^*y$  by abuse of notation. Let  $u = \pi_2^*x, v = \pi_2^*y$ . There exists a holomorphic function  $f(x, y, u, v)$  on  $U \times U$  which defines the divisor  $\mathcal{H}|_{U \times U}$ . Then we write

$$\begin{aligned} f(x, y, u, v) &= f_{1,0}(u, v)(x - u) + f_{0,1}(u, v)(y - v) + f_{2,0}(u, v)(x - u)^2 \\ &\quad + f_{1,1}(u, v)(x - u)(y - v) + f_{0,2}(u, v)(y - v)^2 \\ &\quad + O((|x - u| + |y - v|)^3). \end{aligned}$$

Let  $q = (a, b)$  be any point in  $U$ . The divisor  $\mathcal{H}|_{S \times \{q\}}$  is defined by  $f(x, y, a, b)$  on  $U$ . Because of the singularity of  $\mathcal{H}|_{S \times \{q\}}$  at  $q$ , we have  $f_{1,0}(a, b) = f_{0,1}(a, b) = 0$ . Since  $(a, b)$  is any point in  $U$ , we have

$$\begin{aligned} f(x, y, u, v) &= f_{2,0}(u, v)(x - u)^2 + f_{1,1}(u, v)(x - u)(y - v) \\ &\quad + f_{0,2}(u, v)(y - v)^2 + O((|x - u| + |y - v|)^3). \end{aligned}$$

Since  $H_{t_0}|_S$  is defined by  $xy$  on  $U$ , we may assume  $f(x, y, 0, 0) = xy$ . In particular, we have that  $f_{2,0}(0, 0) = f_{0,2}(0, 0) = 0$  and  $f_{1,1}(0, 0) = 1$ . Hence  $f_{1,1}(u, v)^2 - 4f_{2,0}(u, v)f_{0,2}(u, v) \neq 0$  on the small open set  $U$ , and we take the branch of the function  $\sqrt{f_{1,1}(u, v)^2 - 4f_{2,0}(u, v)f_{0,2}(u, v)}$  satisfying

$$-\frac{\pi}{2} < \arg \left( \sqrt{f_{1,1}(u, v)^2 - 4f_{2,0}(u, v)f_{0,2}(u, v)} \right) < \frac{\pi}{2}.$$



It follows that

$$\begin{aligned}
 & f(x, y, u, v) \\
 &= \left( x' + \frac{2f_{0,2}(u, v)}{f_{1,1}(u, v) + \sqrt{f_{1,1}(u, v)^2 - 4f_{2,0}(u, v)f_{0,2}(u, v)}} y' \right) \\
 &\quad \times \left( f_{2,0}(u, v)x' + \frac{1}{2} \left( f_{1,1}(u, v) + \sqrt{f_{1,1}(u, v)^2 - 4f_{2,0}(u, v)f_{0,2}(u, v)} \right) y' \right) \\
 &\quad + O((|x'| + |y'|)^3),
 \end{aligned}$$

where  $x' = x - u, y' = y - v$ . We let

$$\begin{aligned}
 x'' &= x' + \frac{2f_{0,2}(u, v)}{f_{1,1}(u, v) + \sqrt{f_{1,1}(u, v)^2 - 4f_{2,0}(u, v)f_{0,2}(u, v)}} y', \\
 y'' &= f_{2,0}(u, v)x' + \frac{1}{2} \left( f_{1,1}(u, v) + \sqrt{f_{1,1}(u, v)^2 - 4f_{2,0}(u, v)f_{0,2}(u, v)} \right) y'.
 \end{aligned}$$

We have that  $(x'', y'', u, v)$  is a local coordinate system on  $U \times U$  and  $f = x''y'' + O((|x''| + |y''|)^3)$ . It is easy to see that  $f = \tilde{x}\tilde{y}$  for suitable  $\tilde{x} = x'' + O((|x''| + |y''|)^2)$  and  $\tilde{y} = y'' + O((|x''| + |y''|)^2)$  by an argument similar to that of Example 5.6.3 in Chapter 1 of [12]. Since  $(\tilde{x}, \tilde{y}, u, v)$  is a local coordinate system on  $U \times U$ , this completes the proof of the lemma. □

Now let us show that  $S \setminus (\mathcal{H}_q \cup H_{46} \cup H_{56})|_S$  is Kobayashi hyperbolically imbedded into  $S$  if  $q$  is sufficiently close to  $p$ . Otherwise, there exists a non-constant holomorphic map  $f : \mathbb{C} \rightarrow S$  such that  $f(\mathbb{C}) \subset \mathcal{H}_p|_S \setminus (H_{46} \cup H_{56})|_S$  by the same argument as that used on the *First Step*. We note that  $\mathcal{H}_p = H_{t_0}$ . Because of Hurwitz' theorem and Lemma 3.1, we have that  $f(\mathbb{C})$  does not contain the point  $p$  (see Lemma-Definition 3.2 of [26]). Each component of  $\mathcal{H}_p|_S$  intersects  $H_{46}|_S \cup H_{56}|_S$  in at least two points. Hence  $f(\mathbb{C})$  is contained in the complement of three points in a rational curve and  $f$  is a constant map by the little Picard Theorem. This is a contradiction, and so  $S \setminus (\mathcal{H}_q \cup H_{46} \cup H_{56})|_S$  is Kobayashi hyperbolically imbedded into  $S$  if  $q$  is sufficiently close to  $p$ .

*Third Step.* — Let  $q_0$  be a point in  $S$  such that  $q_0$  is sufficiently close to  $p$  and  $\mathcal{H}_{q_0}|_S$  is irreducible. We deform the nodal rational curve  $\mathcal{H}_{q_0}|_S$  to a smooth elliptic curve in  $S$ . Let  $E'$  be a smooth irreducible elliptic curve in  $S$  which is sufficiently close to  $\mathcal{H}_{q_0}|_S$  in  $|\mathcal{O}_S(1)|$ . Since  $\mathcal{H}_{q_0}|_S$  is a small deformation of  $H_{45}|_S$ ,  $\mathcal{H}_{q_0}|_S$  intersects  $H_{46}|_S \cup H_{56}|_S$  at six points and  $S \setminus (E' \cup H_{46}|_S \cup H_{56}|_S)$  is Kobayashi hyperbolically imbedded into  $S$  by the same argument as that used on the *First Step*.

*Fourth Step.* — Deforming  $H_{46}|_S$  and  $H_{56}|_S$  in a similar way as  $H_{45}|_S$ , there exist smooth irreducible elliptic curves  $E'', E'''$  in  $|\mathcal{O}_S(1)|$  such that  $S \setminus (E' \cup E'' \cup E''')$  is Kobayashi hyperbolically imbedded into  $S$ .

We need the following lemma.

LEMMA 3.2. — *Let  $L$  be a line bundle on  $S$  such that  $L \cdot l \geq 3$  for any line  $l$  on  $S$ . Then  $L - \mathcal{O}_S(3)$  is nef and, moreover, spanned by its global sections.*

*Proof.* — Let  $L' = L - \mathcal{O}_S(3)$ . Then  $L' \cdot l \geq 0$  for any line  $l$  on  $S$  and this implies that  $L'$  is spanned by its global sections (see [5]).  $\square$

Let  $L$  be a line bundle as that of Lemma 3.2. Recall that  $D$  is linearly equivalent to  $\mathcal{O}_S(3)$ . By Lemma 3.2,  $L - D$  is spanned by its global sections. We take a general curve  $D'$  in  $|L - D|$ . We have that  $S \setminus (E' \cup E'' \cup E''' \cup D')$  is Kobayashi hyperbolically imbedded into  $S$  and  $E' + E'' + E''' + D'$  is linearly equivalent to  $L$ .

In the remaining steps, we deform  $E' \cup E'' \cup E''' \cup D'$  to a smooth irreducible curve in  $S$  keeping the Kobayashi hyperbolic imbedding.

*Fifth Step.* — Let  $G$  be a smooth irreducible curve in  $S$  which is a small deformation of  $E' \cup E''$ . Since both elliptic curves  $E'$  and  $E''$  intersect  $E''' \cup D'$  in at least three points respectively,  $S \setminus (G \cup E''' \cup D')$  is Kobayashi hyperbolically imbedded into  $S$  by the same argument as that used on the *First Step*. Let  $G'$  be a smooth irreducible curve in  $S$  which is a small deformation of  $E''' \cup D'$ . Here we let  $G' = E'''$  if  $D' = 0$ . Note that  $D'$  is an irreducible smooth curve which is not a line if  $D' \neq 0$ . We have that  $E'''$  intersects  $G$  at six points and  $D'$  also intersects  $G$  in at least four points if  $D' \neq 0$ . Hence  $S \setminus (G \cup G')$  is Kobayashi hyperbolically imbedded into  $S$  by the same argument as that used on the *First Step*.

*Sixth Step.* — We deform  $G \cup G'$  to an irreducible curve with a node in  $S$ . Take a point  $s \in G \cap G'$ . It follows that  $G$  intersects  $G'$  transversally at  $s$ . Let  $\mu : \tilde{S} \rightarrow S$  be the blow-up at  $s$  and let  $Z$  be the exceptional divisor of  $\mu$ . The strict transform  $\tilde{G}$  of  $G + G'$  under  $\mu$  is linearly equivalent to  $\mu^*L - 2Z$ , and  $\mu^*L - 2Z$  is a very ample line bundle on  $\tilde{S}$ . We can take a smooth irreducible divisor  $\tilde{G}'$  in  $|\tilde{G}|$  which is sufficiently close to  $\tilde{G}$  and intersects  $Z$  transversally at two points. The image  $\mu(\tilde{G}')$  is an irreducible curve in  $S$  with a node at  $s$  and  $S \setminus \mu(\tilde{G}')$  is Kobayashi hyperbolically imbedded into  $S$  by the same argument as that used on the *Second Step* (note that the genera of  $G$  and  $G'$  are bigger than or equal to one. Hence there exists no non-constant holomorphic map from  $\mathbb{C}$  to  $G \setminus \{s\}$  or  $G' \setminus \{s\}$ ).

*Seventh Step.* — Finally we deform  $\mu(\tilde{G}')$  to a smooth irreducible curve in  $S$ . Since the genus of the normalization of  $\mu(\tilde{G}')$  is larger than or equal to two, there exists no non-constant holomorphic map from  $\mathbb{C}$  to  $\mu(\tilde{G}')$ . Hence the complement of a small deformation of  $\mu(\tilde{G}')$  in  $S$  is Kobayashi hyperbolically imbedded into  $S$  by the same argument as that used on the *First Step*. This completes the proof of Theorem 1.4.

#### 4. The case of quadric surfaces

*Proof of the sufficiency in Theorem 1.2.* — Let  $Q = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ . Let  $[X_0 : X_1]$  and  $[Y_0 : Y_1]$  be the homogeneous coordinates on the first and second factors of  $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ . Let  $H_1, \dots, H_4$  be divisors on  $Q$  defined by  $X_0 = 0$ ,  $X_1 = 0$ ,  $Y_0 = 0$  and  $Y_1 = 0$ , respectively. Let  $D$  be a general divisor in  $|\mathcal{O}_Q(m-2, n-2)|$  ( $m, n \geq 4$ ). Then  $\sum_{i=1}^4 H_i + D$  is linearly equivalent to  $L = \mathcal{O}_Q(m, n)$ . It follows that

$$Q \setminus \left( \bigcup_{i=1}^4 H_i \cup D \right)$$

is Kobayashi hyperbolically imbedded into  $Q$  (see Theorem 1 of [24]). Let  $C$  be a smooth irreducible curve in  $Q$  which is sufficiently close to  $H_1 + H_3$  in  $|\mathcal{O}_Q(1, 1)|$ . We have that  $H_i$  ( $i = 1, 3$ ) intersects  $H_2 + H_4 + D$  in at least three points. Then  $Q \setminus (C \cup H_2 \cup H_4 \cup D)$  is Kobayashi hyperbolically imbedded into  $Q$  by the same argument as that used on the *First Step* of Section 3. Let  $C'$  be a smooth irreducible curve in  $Q$  which is sufficiently close to  $H_2 + H_4$  in  $|\mathcal{O}_Q(1, 1)|$ . As in the case of  $C$ , it follows that  $Q \setminus (C \cup C' \cup D)$  is Kobayashi hyperbolically imbedded into  $Q$ . Let  $D'$  be a smooth irreducible curve in  $Q$  which is sufficiently close to  $C + C'$  in  $|\mathcal{O}_Q(2, 2)|$ . Then both  $C$  and  $C'$  intersect  $D$  in at least four points respectively and  $Q \setminus (D \cup D')$  is Kobayashi hyperbolically imbedded into  $Q$  by the same argument as that used on the *First Step* of Section 3. Let  $p \in D \cap D'$ . We have that  $D$  intersects  $D'$  transversally at  $p$ . Let  $\mu : \tilde{Q} \rightarrow Q$  be the blowing up at  $p$  and let  $Z$  be the exceptional divisor of  $\mu$ . Let  $\tilde{D}$  be the strict transform of  $D + D'$  under  $\mu$ . We have that  $\tilde{D}$  is an element of  $|\mu^* \mathcal{O}_Q(m, n) - 2Z|$  and the line bundle  $\mu^* \mathcal{O}_Q(m, n) - 2Z$  is very ample. Note that genera of  $D$  and  $D'$  are at least one. By the same arguments as that used on the *Sixth Step* and the *Seventh Step* of Section 3, there exists a smooth irreducible curve in  $Q$  whose complement is Kobayashi hyperbolically imbedded into  $Q$ .  $\square$

*Proof of the necessity in Theorem 1.2.* — Let  $L = \mathcal{O}_Q(m, n)$ . Without loss of generality, we may assume that  $m \leq n$ . Assume that  $m \leq 3$ . Let

$X \in |L|$  be a smooth irreducible curve in  $Q$ . Let  $\pi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$  be the second projection.

If  $m \leq 2$ , a general fiber  $F$  of  $\pi$  intersects  $X$  in at most two points. Then there exists a non-constant holomorphic map from  $\mathbb{C}$  to  $F \setminus X$ , and  $Q \setminus X$  is not Kobayashi hyperbolically imbedded into  $Q$ .

If  $m = 3$ ,  $\pi|_X : X \rightarrow \mathbb{P}^1(\mathbb{C})$  is ramified because of the Riemann–Hurwitz formula. Let  $q \in \mathbb{P}^1(\mathbb{C})$  be a branch point of  $\pi|_X$ . The fiber  $\pi^{-1}(q)$  intersects  $X$  in at most two points. Hence there exists a non-constant holomorphic map from  $\mathbb{C}$  to  $\pi^{-1}(q) \setminus X$ . This implies that  $Q \setminus X$  is not Kobayashi hyperbolically imbedded into  $Q$ .  $\square$

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Atsushi ITO  
Department of Mathematics  
Kyoto University  
Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto (Japan)  
aito@math.kyoto-u.ac.jp

Yusaku TIBA  
Graduate School of Mathematical Sciences  
University of Tokyo  
Komaba, Meguro-ku, Tokyo (Japan)  
tiba@ms.u-tokyo.ac.jp