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# EIGENVALUE ASYMPTOTICS FOR SCHRÖDINGER OPERATORS ON SPARSE GRAPHS

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ABSTRACT. — We consider Schrödinger operators on sparse graphs. The geometric definition of sparseness turn out to be equivalent to a functional inequality for the Laplacian. In consequence, sparseness has in turn strong spectral and functional analytic consequences. Specifically, one consequence is that it allows to completely describe the form domain. Moreover, as another consequence it leads to a characterization for discreteness of the spectrum. In this case we determine the first order of the corresponding eigenvalue asymptotics.

RÉSUMÉ. — Nous considérons des opérateurs de Schrödinger agissant sur des graphes éparses. Le fait d'être épars est équivalent à une inégalité fonctionnelle pour le Laplacien. En particulier il y a des conséquences spectrales fortes pour le Laplacien quand le graphe est épars : caractérisation de son domaine de forme et de l'absence du spectre essentiel. Dans ce dernier cas, nous calculons l'asymptotique des valeurs propres.

## 1. Introduction

The spectral theory of discrete Laplacians on finite or infinite graphs has drawn a lot of attention for decades. One important aspect is to understand the relations between the geometry of the graph and the spectrum of the Laplacian. Often a particular focus lies on the study of the bottom of the spectrum and the eigenvalues below the essential spectrum.

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Certainly the most well-known estimates for the bottom of the spectrum of Laplacians on infinite graphs are so called isoperimetric estimates or Cheeger inequalities. Starting with [7] in the case of infinite graphs, these inequalities were intensively studied and resulted in huge body of literature, where we here mention only [2, 4, 8, 6, 11, 25, 26, 17, 15, 32]. In certain more specific geometric situations the bottom of the spectrum might be estimated in terms of curvature, see [3, 13, 14, 17, 18, 21, 23, 31]. There are various other more recent approaches such as Hardy inequalities in [12] and summability criteria involving the boundary and volume of balls in [20].

In this work we focus on sparse graphs to study discreteness of spectrum and eigenvalue asymptotics. In a moral sense, the term sparse means that there are not ‘too many’ edges, however, throughout the years various different definitions were investigated. We mention here [10, 24] as seminal works which are closely related to our definitions. As it is impossible to give a complete discussion of the development, we refer to some selected more recent works such as [1, 5, 22, 26] and references therein which also illustrates the great variety of possible definitions. Here, we discuss three notions of sparseness that result in a hierarchy of very general classes of graphs.

Let us highlight the work of Mohar [27], where large eigenvalues of the adjacency matrix on finite graphs are studied. Although our situation of infinite graphs with unbounded geometry requires fundamentally different techniques – functional analytic rather than combinatorial – in spirit our work is certainly closely related.

The techniques used in this paper owe on the one hand to considerations of isoperimetric estimates as well as a scheme developed in [12] for the special case of trees. In particular, we show that a notion of sparseness is a geometric characterization for an inequality of the type

$$(1 - a) \deg - k \leq \Delta \leq (1 + a) \deg + k$$

for some  $a \in (0, 1)$ ,  $k \geq 0$ , which holds in the form sense (precise definitions and details will be given below). The moral of this inequality is that the asymptotic behavior of the Laplacian  $\Delta$  is controlled by the vertex degree function  $\deg$  (the smaller  $a$  the better the control).

Furthermore, such an inequality has very strong consequences which follow from well-known functional analytic principles. These consequences include an explicit description of the form domain, characterization for discreteness of spectrum and eigenvalue asymptotics.

Let us set up the framework. Here, a graph  $\mathcal{G}$  is a pair  $(\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  denotes a countable set of vertices and  $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow \{0, 1\}$  is a symmetric

function with zero diagonal determining the edges. We say two vertices  $x$  and  $y$  are *adjacent* or *neighbors* whenever  $\mathcal{E}(x, y) = \mathcal{E}(y, x) = 1$ . In this case, we write  $x \sim y$  and we call  $(x, y)$  and  $(y, x)$  the (*directed*) edges connecting  $x$  and  $y$ . We assume that  $\mathcal{G}$  is *locally finite* that is each vertex has only finitely many neighbors. For any finite set  $\mathcal{W} \subseteq \mathcal{V}$ , the *induced subgraph*  $\mathcal{G}_{\mathcal{W}} := (\mathcal{W}, \mathcal{E}_{\mathcal{W}})$  is defined by setting  $\mathcal{E}_{\mathcal{W}} := \mathcal{E}|_{\mathcal{W} \times \mathcal{W}}$ , i.e., an edge is contained in  $\mathcal{G}_{\mathcal{W}}$  if and only if both of its vertices are in  $\mathcal{W}$ .

We consider the complex Hilbert space

$$\ell^2(\mathcal{V}) := \left\{ \varphi : \mathcal{V} \rightarrow \mathbb{C} \text{ such that } \sum_{x \in \mathcal{V}} |\varphi(x)|^2 < \infty \right\},$$

endowed with the scalar product  $\langle \varphi, \psi \rangle := \sum_{x \in \mathcal{V}} \overline{\varphi(x)}\psi(x)$ , for all functions  $\varphi, \psi \in \ell^2(\mathcal{V})$ .

For a function  $g : \mathcal{V} \rightarrow \mathbb{C}$ , we denote the operator of multiplication by  $g$  on  $\ell^2(\mathcal{V})$  given by  $\varphi \mapsto g\varphi$  and domain  $\mathcal{D}(g) := \{\varphi \in \ell^2(\mathcal{V}) \mid g\varphi \in \ell^2(\mathcal{V})\}$  with slight abuse of notation also by  $g$ .

Let  $q : \mathcal{V} \rightarrow [0, \infty)$ . We consider the Schrödinger operator  $\Delta + q$  defined as

$$\begin{aligned} \mathcal{D}(\Delta + q) &:= \left\{ \varphi \in \ell^2(\mathcal{V}) \mid \left( v \mapsto \sum_{w \sim v} (\varphi(v) - \varphi(w)) + q(v)\varphi(v) \right) \in \ell^2(\mathcal{V}) \right\} \\ (\Delta + q)\varphi(v) &:= \sum_{w \sim v} (\varphi(v) - \varphi(w)) + q(v)\varphi(v). \end{aligned}$$

The operator is non-negative and selfadjoint as it is essentially selfadjoint on  $\mathcal{C}_c(\mathcal{V})$ , the set of finitely supported functions  $\mathcal{V} \rightarrow \mathbb{R}$ , (confer [32, Theorem 1.3.1], [19, Theorem 6]). In Section 2 we will allow for potentials whose negative part is form bounded with bound strictly less than one. Moreover, in Section 4 we consider also magnetic Schrödinger operators.

As mentioned above sparse graphs have already been introduced in various contexts with varying definitions. In this article we also treat various natural generalizations of the concept. In this introduction we stick to an intermediate situation.

DEFINITION 1.1. — A graph  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  is called  $k$ -sparse if for any finite set  $\mathcal{W} \subseteq \mathcal{V}$  the induced subgraph  $\mathcal{G}_{\mathcal{W}} := (\mathcal{W}, \mathcal{E}_{\mathcal{W}})$  satisfies

$$2|\mathcal{E}_{\mathcal{W}}| \leq k|\mathcal{W}|,$$

where  $|A|$  denotes the cardinality of a finite set  $A$  and we define

$$|\mathcal{E}_{\mathcal{W}}| := \frac{1}{2} |\{(x, y) \in \mathcal{W} \times \mathcal{W} \mid \mathcal{E}_{\mathcal{W}}(x, y) = 1\}|,$$

that is we count the non-oriented edges in  $\mathcal{G}_{\mathcal{W}}$ .

Examples of sparse graphs are planar graphs and, in particular, trees. We refer to Section 6 for more examples.

For a function  $g : \mathcal{V} \rightarrow \mathbb{R}$  and a finite set  $\mathcal{W} \subseteq \mathcal{V}$ , we denote

$$g(\mathcal{W}) := \sum_{x \in \mathcal{W}} g(x).$$

Moreover, we define

$$\begin{aligned} \liminf_{|x| \rightarrow \infty} g(x) &:= \sup_{\mathcal{W} \subset \mathcal{V}} \inf_{\text{finite } x \in \mathcal{V} \setminus \mathcal{W}} g(x), \\ \limsup_{|x| \rightarrow \infty} g(x) &:= \inf_{\mathcal{W} \subset \mathcal{V}} \sup_{\text{finite } x \in \mathcal{V} \setminus \mathcal{W}} g(x). \end{aligned}$$

For two selfadjoint operators  $T_1, T_2$  on a Hilbert space and a subspace  $\mathcal{D}_0 \subseteq \mathcal{D}(T_1) \cap \mathcal{D}(T_2)$ , we write  $T_1 \leq T_2$  on  $\mathcal{D}_0$  if  $\langle T_1 \varphi, \varphi \rangle \leq \langle T_2 \varphi, \varphi \rangle$  for all  $\varphi \in \mathcal{D}_0$ . Moreover, for a selfadjoint semi-bounded operator  $T$  on a Hilbert space, we denote the eigenvalues below the essential spectrum by  $\lambda_n(T)$ ,  $n \geq 0$ , enumerated with increasing order counted with multiplicity.

The next theorem is a special case of the more general Theorem 2.3 in Section 2. It illustrates our results in the case of sparse graphs introduced above and includes the case of trees, [12, Theorem 1.1], as a special case. While the proof in [12] uses a Hardy inequality, we rely on some new ideas which have their roots in isoperimetric techniques. The proof is given in Section 2.3.

**THEOREM 1.2.** — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a  $k$ -sparse graph and a potential  $q : \mathcal{V} \rightarrow [0, \infty)$ . Then, we have the following:*

(a) *For all  $0 < \varepsilon \leq 1$ ,*

$$(1 - \varepsilon)(\text{deg} + q) - \frac{k}{2} \left( \frac{1}{\varepsilon} - \varepsilon \right) \leq \Delta + q \leq (1 + \varepsilon)(\text{deg} + q) + \frac{k}{2} \left( \frac{1}{\varepsilon} - \varepsilon \right),$$

*on  $\mathcal{C}_c(\mathcal{V})$ .*

(b)  $\mathcal{D}((\Delta + q)^{1/2}) = \mathcal{D}((\text{deg} + q)^{1/2})$ .

(c) *The operator  $\Delta + q$  has purely discrete spectrum if and only if*

$$\liminf_{|x| \rightarrow \infty} (\text{deg} + q)(x) = \infty.$$

*In this case, we obtain*

$$\liminf_{\lambda \rightarrow \infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\text{deg} + q)} = 1.$$

As a corollary, we obtain following estimate for the bottom and the top of the (essential) spectrum.

COROLLARY 1.3. — Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a  $k$ -sparse graph and a potential  $q : \mathcal{V} \rightarrow [0, \infty)$ . Define

$$d := \inf_{x \in \mathcal{V}} (\deg + q)(x) \text{ and } D := \sup_{x \in \mathcal{V}} (\deg + q)(x)$$

Assume  $d < k \leq D < +\infty$ . Then,

$$d - 2\sqrt{\frac{k}{2} \left( d - \frac{k}{2} \right)} \leq \inf \sigma(\Delta + q) \leq \sup \sigma(\Delta + q) \leq D - 2\sqrt{\frac{k}{2} \left( D - \frac{k}{2} \right)}.$$

Define

$$d_{\text{ess}} := \liminf_{|x| \rightarrow \infty} (\deg + q)(x) \text{ and } D_{\text{ess}} := \limsup_{|x| \rightarrow \infty} (\deg + q)(x).$$

Assume  $d_{\text{ess}} < k \leq D_{\text{ess}} < +\infty$ . Then,

$$\begin{aligned} d_{\text{ess}} - 2\sqrt{\frac{k}{2} \left( d_{\text{ess}} - \frac{k}{2} \right)} &\leq \inf \sigma_{\text{ess}}(\Delta + q) \\ &\leq \sup \sigma_{\text{ess}}(\Delta + q) \leq D_{\text{ess}} - 2\sqrt{\frac{k}{2} \left( D_{\text{ess}} - \frac{k}{2} \right)}. \end{aligned}$$

*Proof of Corollary 1.3.* — Set  $\varepsilon = \min \left( \sqrt{\frac{k}{2d-k}}, 1 \right)$  in the part (a) of Theorem 1.2. □

*Remark 1.4.* — Since trees are 2-sparse, see Lemma 6.2, the bounds in Corollary 1.3 are optimal for the bottom and the top of the (essential) spectrum in the case of regular trees.

The paper is structured as follows. In the next section an extension of the notion of sparseness is introduced which is shown to be equivalent to a functional inequality and equality of the form domains of  $\Delta$  and  $\deg$ . In Section 3 we consider almost sparse graphs for which we obtain precise eigenvalue asymptotics. Furthermore, in Section 4 we shortly discuss magnetic Schrödinger operators. Our notion of sparseness has very explicit but non-trivial connections to isoperimetric inequalities which are made precise in Section 5. Finally, in Section 6 we discuss some examples.

## 2. A geometric characterization of the form domain

In this section we characterize equality of the form domains of  $\Delta + q$  and  $\deg + q$  by a geometric property. This geometric property is a generalization of the notion of sparseness from the introduction. Before we come to this definition, we introduce the class of potentials that is treated in this paper.

Let  $\alpha > 0$ . We say a potential  $q : \mathcal{V} \rightarrow \mathbb{R}$  is in the class  $\mathcal{K}_\alpha$  if there is  $C_\alpha \geq 0$  such that

$$q_- \leq \alpha(\Delta + q_+) + C_\alpha,$$

where  $q_\pm := \max(\pm q, 0)$ . For  $\alpha \in (0, 1)$ , we define the operator  $\Delta + q$  via the form sum of the operators  $\Delta + q_+$  and  $-q_-$  (i.e., by the KLMN Theorem, see e.g., [28, Theorem X.17]). Note that  $\Delta + q$  is bounded from below and

$$\mathcal{D}(|\Delta + q|^{\frac{1}{2}}) = \mathcal{D}((\Delta + q_+)^{\frac{1}{2}}) = \mathcal{D}(\Delta^{\frac{1}{2}}) \cap \mathcal{D}(q_+^{\frac{1}{2}}),$$

where  $|\Delta + q|$  is defined by the spectral theorem. The last equality follows from [16, Theorem 5.6]. in the sense of functions and forms.

Another important class are the potentials

$$\mathcal{K}_{0+} := \bigcap_{\alpha \in (0,1)} \mathcal{K}_\alpha.$$

In our context of sparseness, we can characterize the class  $\mathcal{K}_{0+}$  to be the potentials whose negative part  $q_-$  is morally  $o(\deg + q_+)$ , see Corollary 2.10. Let us mention that if  $q_-$  is in the Kato class with respect to  $\Delta + q_+$ , i.e., if we have  $\limsup_{t \rightarrow 0+} \|e^{-t(\Delta + q_+)} q_-\|_\infty = 0$ , then  $q := q_+ - q_- \in \mathcal{K}_{0+}$  by [29, Theorem 3.1].

Next, we come to an extension of the notion of sparseness. For a set  $\mathcal{W} \subseteq \mathcal{V}$ , let the boundary  $\partial\mathcal{W}$  of  $\mathcal{W}$  be the set of edges emanating from  $\mathcal{W}$

$$\partial\mathcal{W} := \{(x, y) \in \mathcal{W} \times \mathcal{V} \setminus \mathcal{W} \mid x \sim y\}.$$

DEFINITION 2.1. — Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and  $q : \mathcal{V} \rightarrow \mathbb{R}$ . For given  $a \geq 0$  and  $k \geq 0$ , we say that  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse if for any finite set  $\mathcal{W} \subseteq \mathcal{V}$  the induced subgraph  $\mathcal{G}_\mathcal{W} := (\mathcal{W}, \mathcal{E}_\mathcal{W})$  satisfies

$$2|\mathcal{E}_\mathcal{W}| \leq k|\mathcal{W}| + a(|\partial\mathcal{W}| + q_+(\mathcal{W})).$$

Remark 2.2.

- (a) Observe that the definition depends only on  $q_+$ . The negative part of  $q$  will be taken in account through the hypothesis  $\mathcal{K}_\alpha$  or  $\mathcal{K}_{0+}$  in our theorems.
- (b) If  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse, then  $(\mathcal{G}, q')$  is  $(a, k)$ -sparse for every  $q' \geq q$ .
- (c) As mentioned above there is a great variety of definitions which were so far predominantly established for (families of) finite graphs. For example it is asked that  $|\mathcal{E}| = C|\mathcal{V}|$  in [10],  $|\mathcal{E}_\mathcal{W}| \leq k|\mathcal{W}| + l$  in [24, 22],  $|\mathcal{E}| \in O(|\mathcal{V}|)$  in [1] and  $\deg(\mathcal{W}) \leq k|\mathcal{W}|$  in [27].

We now characterize the equality of the form domains in geometric terms.

**THEOREM 2.3.** — Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and  $q \in \mathcal{K}_\alpha$ ,  $\alpha \in (0, 1)$ . The following assertions are equivalent:

(i) There are  $a, k \geq 0$  such that  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse.

(ii) There are  $\tilde{a} \in (0, 1)$  and  $\tilde{k} \geq 0$  such that on  $\mathcal{C}_c(\mathcal{V})$

$$(1 - \tilde{a})(\text{deg} + q) - \tilde{k} \leq \Delta + q \leq (1 + \tilde{a})(\text{deg} + q) + \tilde{k}.$$

(iii) There are  $\tilde{a} \in (0, 1)$  and  $\tilde{k} \geq 0$  such that on  $\mathcal{C}_c(\mathcal{V})$

$$(1 - \tilde{a})(\text{deg} + q) - \tilde{k} \leq \Delta + q.$$

(iv)  $\mathcal{D}(|\Delta + q|^{1/2}) = \mathcal{D}(|\text{deg} + q|^{1/2})$ .

Furthermore,  $\Delta + q$  has purely discrete spectrum if and only if

$$\liminf_{|x| \rightarrow \infty} (\text{deg} + q)(x) = \infty.$$

In this case, we obtain

$$1 - \tilde{a} \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\text{deg} + q)} \leq \limsup_{n \rightarrow \infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\text{deg} + q)} \leq 1 + \tilde{a}.$$

Before we come to the proof of Theorem 2.3, we summarize the relation between the sparseness parameters  $(a, k)$  and the constants  $(\tilde{a}, \tilde{k})$  in the inequality in Theorem 2.3 (ii).

*Remark 2.4.* — Roughly speaking  $a$  tends to  $\infty$  as  $\tilde{a}$  tends to  $1^-$  and  $a$  tends to  $0^+$  as  $\tilde{a}$  tends to  $0^+$  and vice-versa. More precisely, by Lemma 2.6 we obtain that for given  $\tilde{a}$  and  $\tilde{k}$  the values of  $a$  and  $k$  can be chosen to be

$$a = \frac{\tilde{a}}{1 - \tilde{a}} \quad \text{and} \quad k = \frac{\tilde{k}}{1 - \tilde{a}}.$$

Reciprocally, given  $a, k \geq 0$  and  $q : \mathcal{V} \rightarrow [0, \infty)$ , Lemma 2.8 distinguishes the case where the graph is sparse  $a = 0$  and  $a > 0$ . For  $a = 0$ , we may choose  $\tilde{a} \in (0, 1)$  arbitrary and

$$\tilde{k} = \frac{k}{2} \left( \frac{1}{\tilde{a}} - \tilde{a} \right).$$

For an  $(a, k)$ -sparse graph with  $a > 0$ , the precise constants are found below in Lemma 2.8. Here, we discuss the asymptotics. For  $a \rightarrow 0^+$ , we obtain

$$\tilde{a} \simeq \sqrt{2a} \quad \text{and} \quad \tilde{k} \simeq \frac{k}{2a},$$

and, for  $a \rightarrow \infty$ , we obtain

$$\tilde{a} \simeq 1 - \frac{3}{8a^2} \quad \text{and} \quad \tilde{k} \simeq \frac{3k}{4a}.$$

In the case  $q \in \mathcal{K}_\alpha$ , the constants  $\tilde{a}, \tilde{k}$  from the case  $q \geq 0$  have to be replaced by constants whose formula can be explicitly read from Lemma A.3.



For  $\alpha \rightarrow 0^+$ , the constant replacing  $\tilde{a}$  tends to  $\tilde{a}$  while the asymptotics of the constant replacing  $\tilde{k}$  depend also on the behavior of  $C_\alpha$  from the assumption  $q_- \leq \alpha(\Delta + q) + C_\alpha$ .

*Remark 2.5.*

(a) Observe that in the context of Theorem 2.3 statement (iv) is equivalent to

$$(iv') \quad \mathcal{D}(|\Delta + q|^{1/2}) = \mathcal{D}((\deg + q_+)^{1/2}).$$

Indeed, (ii) implies the corresponding inequality for  $q = q_+$ . Thus, as  $q \in \mathcal{K}_\alpha$ ,

$$\mathcal{D}(|\Delta + q|^{\frac{1}{2}}) = \mathcal{D}((\Delta + q_+)^{\frac{1}{2}}) = \mathcal{D}((\deg + q_+)^{\frac{1}{2}}).$$

(b) The definition of the class  $\mathcal{K}_{0+}$  is rather abstract. Indeed, Theorem 2.3 yields a very concrete characterization of these potentials, see Corollary 2.10 below.

(c) Theorem 2.3 characterizes equality of the form domains. Another natural question is under which circumstances the operator domains agree. For a discussion on this matter we refer to [12, Section 4.1].

The rest of this section is devoted to the proof of the results which is divided into four parts. The following three lemmas essentially show the equivalences (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii) for  $q \geq 0$  providing the explicit dependence of  $(a, k)$  on  $(\tilde{a}, \tilde{k})$  and vice versa. The third part uses general functional analytic principles collected in the appendix to treat general  $q$  and show the spectral consequences.

The first lemma shows (iii)  $\Rightarrow$  (i).

LEMMA 2.6. — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and  $q : \mathcal{V} \rightarrow \mathbb{R}$ . If there are  $\tilde{a} \in (0, 1)$  and  $\tilde{k} \geq 0$  such that for all  $f$  in  $C_c(\mathcal{V})$ ,*

$$(1 - \tilde{a})\langle f, (\deg + q)f \rangle - \tilde{k}\|f\|^2 \leq \langle f, \Delta f + qf \rangle,$$

*then  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse with*

$$a = \frac{\tilde{a}}{1 - \tilde{a}} \quad \text{and} \quad k = \frac{\tilde{k}}{1 - \tilde{a}}.$$

*Remark 2.7.* — We stress that we suppose solely that  $q : \mathcal{V} \rightarrow \mathbb{R}$  and work with  $\Delta|_{C_c(\mathcal{V})} + q|_{C_c(\mathcal{V})}$ . We do not specify any self-adjoint extension of the latter.

*Proof.* — Let  $f \in C_c(\mathcal{V})$ . By adding  $q_-$  to the assumed inequality we obtain immediately

$$(1 - \tilde{a})\langle f, (\deg + q_+)f \rangle - \tilde{k}\|f\|^2 \leq \langle f, \Delta f + q_+f \rangle.$$

Let  $\mathcal{W} \subseteq \mathcal{V}$  be a finite set and denote by  $\mathbf{1}_{\mathcal{W}}$  the characteristic function of the set  $\mathcal{W}$ . We recall the basic equalities

$$\deg(\mathcal{W}) = 2|\mathcal{E}_{\mathcal{W}}| + |\partial\mathcal{W}| \quad \text{and} \quad \langle \mathbf{1}_{\mathcal{W}}, \Delta \mathbf{1}_{\mathcal{W}} \rangle = |\partial\mathcal{W}|.$$

Therefore, applying the asserted inequality with  $f = \mathbf{1}_{\mathcal{W}}$ , we obtain

$$2|\mathcal{E}_{\mathcal{W}}| \leq \frac{\tilde{k}}{1-\tilde{a}}|W| + \frac{\tilde{a}}{1-\tilde{a}}(|\partial W| + q_+(\mathcal{W})).$$

This proves the statement. □

The second lemma gives (i)⇒(ii) for  $q \geq 0$ . The case of general  $q$  is later taken care of by Lemma A.3.

LEMMA 2.8. — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and  $q : \mathcal{V} \rightarrow [0, \infty)$ . If there are  $a, k \geq 0$  such that  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse, then*

$$(1 - \tilde{a})(\deg + q) - \tilde{k} \leq \Delta + q \leq (1 + \tilde{a})(\deg + q) + \tilde{k} \quad \text{on } \mathcal{C}_c(\mathcal{V})$$

where if  $(\mathcal{G}, q)$  is sparse, i.e.,  $a = 0$ , we may choose  $\tilde{a} \in (0, 1)$  arbitrary and

$$\tilde{k} = \frac{k}{2} \left( \frac{1}{\tilde{a}} - \tilde{a} \right).$$

In the other case, i.e.  $a > 0$ , we may choose

$$\tilde{a} = \frac{\sqrt{\min\left(\frac{1}{4}, a^2\right) + 2a + a^2}}{(1 + a)}$$

and  $\tilde{k} = \max\left(\frac{\max\left(\frac{3}{2}, \frac{1}{a} - a\right)k}{2(1 + a)}, 2k(1 - \tilde{a})\right).$

*Proof.* — Let  $f \in \mathcal{C}_c(\mathcal{V})$  be complex valued. Assume first that

$$(2.1) \quad \langle f, (\deg + q)f \rangle < k\|f\|^2.$$

In this case, remembering  $\Delta \leq 2 \deg$ , we choose  $\tilde{a} \in (0, 1)$  arbitrary and  $\tilde{k}$  such that

$$\tilde{k} \geq 2(1 - \tilde{a})k.$$

So, assume  $\langle f, (\deg + q)f \rangle \geq k\|f\|^2$ . Using an area and a co-area formula (cf. [15, Theorem 12 and Theorem 13]) with

$$\Omega_t := \{x \in \mathcal{V} \mid |f(x)|^2 > t\},$$

in the first step,  $\text{deg}(\Omega_t) = 2|\mathcal{E}(\Omega_t)| + |\partial\Omega_t|$  and the sparseness in the third step, we obtain

$$\begin{aligned} & \langle f, (\text{deg} + q)f \rangle - k\|f\|^2 \\ &= \int_0^\infty \left( \text{deg}(\Omega_t) + q(\Omega_t) - k|\Omega_t| \right) dt \\ &= \int_0^\infty \left( 2|\mathcal{E}\Omega_t| + |\partial\Omega_t| + q(\Omega_t) - k|\Omega_t| \right) dt \\ &\leq (1+a) \int_0^\infty \left( |\partial\Omega_t| + q(\Omega_t) \right) dt \\ &= \frac{(1+a)}{2} \sum_{x,y,x\sim y} \left| |f(x)|^2 - |f(y)|^2 \right| + (1+a) \sum_x q(x)|f(x)|^2 \\ &\leq \frac{(1+a)}{2} \sum_{x,y,x\sim y} \left| (f(x) - f(y))(\overline{f(x)} + \overline{f(y)}) \right| + (1+a) \sum_x q(x)|f(x)|^2 \\ &\leq \frac{(1+a)}{2} \left( \sum_{x,y,x\sim y} |f(x) - f(y)|^2 + 2 \sum_x q(x)|f(x)|^2 \right)^{1/2} \\ &\quad \times \left( \sum_{x,y,x\sim y} |f(x) + f(y)|^2 + 2 \sum_x q(x)|f(x)|^2 \right)^{1/2} \\ &= (1+a) \langle f, (\Delta + q)f \rangle^{\frac{1}{2}} \left( 2 \langle f, (\text{deg} + q)f \rangle - \langle f, (\Delta + q)f \rangle \right)^{\frac{1}{2}}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last inequality and basic algebraic manipulation in the last equality. Since the left hand side is non-negative by the assumption  $\langle f, (\text{deg} + q)f \rangle \geq k\|f\|^2$ , we can take square roots on both sides. To shorten notation, we assume for the rest of the proof  $q \equiv 0$  since the proof with  $q \neq 0$  is completely analogous.

Reordering the terms yields

$$(1+a)^2 \langle f, \Delta f \rangle^2 - 2(1+a)^2 \langle f, \text{deg} f \rangle \langle f, \Delta f \rangle + (\langle f, (\text{deg} - k)f \rangle)^2 \leq 0.$$

Resolving the quadratic expression above gives

$$\langle f, \text{deg} f \rangle - \sqrt{\delta} \leq \langle f, \Delta f \rangle \leq \langle f, \text{deg} f \rangle + \sqrt{\delta},$$

with

$$\delta := \langle f, \text{deg} f \rangle^2 - (1+a)^{-2} (\langle f, (\text{deg} - k)f \rangle)^2.$$

Using  $4\xi\zeta \leq (\xi + \zeta)^2$ ,  $\xi, \zeta \geq 0$ , for all  $0 < \lambda < 1$ , we estimate  $\delta$  as follows

$$\begin{aligned} (1 + a)^2\delta &= (2a + a^2)\langle f, \deg f \rangle^2 + k\|f\|^2\langle f, (2 \deg - k)f \rangle \\ &\leq (2a + a^2)\langle f, \deg f \rangle^2 + \left( \lambda\langle f, \deg f \rangle + \frac{k}{2} \left( \frac{1}{\lambda} - \lambda \right) \|f\|^2 \right)^2 \\ &\leq \left( \sqrt{\lambda^2 + 2a + a^2}\langle f, \deg f \rangle + \frac{k}{2} \left( \frac{1}{\lambda} - \lambda \right) \|f\|^2 \right)^2. \end{aligned}$$

If  $a = 0$ , i.e., the  $k$ -sparse case, then we take  $\lambda = \tilde{a}$  to get

$$\delta \leq k\|f\|^2\langle f, 2 \deg f \rangle \leq \left( \tilde{a}\langle f, \deg f \rangle + \frac{k}{2} \left( \frac{1}{\tilde{a}} - \tilde{a} \right) \|f\|^2 \right)^2.$$

As  $k/2\tilde{a} \geq 2(1 - \tilde{a})k$ , this proves the desired inequality with  $\tilde{k} = k/2\tilde{a}$ .

If  $a > 0$ , we take  $\lambda = \min\left(\frac{1}{2}, a\right)$  to get

$$\begin{aligned} (1 + a)^2\delta &= \left( \sqrt{\min\left(\frac{1}{4}, a^2\right) + 2a + a^2} \langle f, \deg f \rangle \right. \\ &\quad \left. + \frac{k}{2} \max\left(\frac{3}{2}, \left(\frac{1}{a} - a\right)\right) \|f\|^2 \right)^2. \end{aligned}$$

Keeping in mind the restriction  $\tilde{k} \geq 2(1 - \tilde{a})k$  for (2.1), this gives the statement with the choice of  $(\tilde{a}, \tilde{k})$  in the statement of the lemma.  $\square$

The two lemmas above are sufficient to prove Theorem 2.3 for the case  $q \geq 0$ . An application of Lemma A.3 turns the lower bound of Lemma 2.8 for  $q \geq 0$  into a corresponding lower bound for general  $q$ . This straightforward argument does not work for the upper bound. However, the following surprising lemma shows that such a lower bound by  $\deg$  automatically implies the corresponding upper bound. There is a deeper reason for this fact which shows up in the context of magnetic Schrödinger operators. We present the non-magnetic version of the statement here for the sake of being self-contained in this section. For the more conceptual and more general magnetic version, we refer to Lemma 4.4.

LEMMA 2.9 (Upside-Down-Lemma – non-magnetic version). — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and  $q : \mathcal{V} \rightarrow \mathbb{R}$ . Assume there are  $\tilde{a} \in (0, 1)$ ,  $\tilde{k} \geq 0$  such that for all  $f \in \mathcal{C}_c(\mathcal{V})$ ,*

$$(1 - \tilde{a})\langle f, (\deg + q)f \rangle - \tilde{k}\|f\|^2 \leq \langle f, \Delta f + qf \rangle,$$

then, for all  $f \in \mathcal{C}_c(\mathcal{V})$ , we also have

$$\langle f, \Delta f + qf \rangle \leq (1 + \tilde{a})\langle f, (\deg + q)f \rangle + \tilde{k}\|f\|^2.$$

*Proof.* — By a direct calculation we find for  $f \in \mathcal{C}_c(\mathcal{V})$

$$\begin{aligned} \langle f, (2 \operatorname{deg} - \Delta)f \rangle &= \frac{1}{2} \sum_{x,y \in \mathcal{V}, x \sim y} (2|f(x)|^2 + 2|f(y)|^2) - |f(x) - f(y)|^2 \\ &= \frac{1}{2} \sum_{x,y,x \sim y} |f(x) + f(y)|^2 \geq \frac{1}{2} \sum_{x,y,x \sim y} ||f(x)| - |f(y)||^2 \\ &= \langle |f|, \Delta|f| \rangle. \end{aligned}$$

Adding  $q$  to the inequality and using the assumption gives after reordering

$$\begin{aligned} \langle f, (\Delta + q)f \rangle - 2\langle f, (\operatorname{deg} + q)f \rangle &\leq -\langle |f|, (\Delta + q)|f| \rangle \\ &\leq -(1 - \tilde{a})\langle |f|, (\operatorname{deg} + q)|f| \rangle + \tilde{k}\langle |f|, |f| \rangle \\ &= -(1 - \tilde{a})\langle f, (\operatorname{deg} + q)f \rangle + \tilde{k}\langle f, f \rangle \end{aligned}$$

which yields the assertion. □

*Proof of Theorem 2.3.* — First, the implication (i) $\Rightarrow$ (iii) follows from Lemma 2.8 applied with  $q_+$  and then from Lemma A.3 with  $q$ . Second, the implication (iii) $\Rightarrow$ (ii) follows from the Upside-Down-Lemma above. Furthermore, (ii) $\Rightarrow$ (i) is implied by Lemma 2.6. The equivalence (ii) $\Leftrightarrow$ (iv) follows from an application of the Closed Graph Theorem, Theorem A.1. Finally, the statements about discreteness of spectrum and eigenvalue asymptotics follow from Theorem A.2. □

*Proof of Theorem 1.2.* — (a) follows from Lemma 2.8. The other statements follow directly from Theorem 2.3. □

As a corollary we can now determine the potentials in the class  $\mathcal{K}_{0+}$  explicitly and give necessary and sufficient criteria for potentials being in  $\mathcal{K}_\alpha$ ,  $\alpha \in (0, 1)$ .

**COROLLARY 2.10.** — *Let  $(\mathcal{G}, q)$  be an  $(a, k)$ -sparse graph for  $a, k \geq 0$ .*

(a) *The potential  $q$  is in  $\mathcal{K}_{0+}$  if and only if for all  $\alpha \in (0, 1)$  there is  $\kappa_\alpha \geq 0$  such that*

$$q_- \leq \alpha(\operatorname{deg} + q_+) + \kappa_\alpha.$$

(b) *Let  $\alpha \in (0, 1)$  and  $\tilde{a} = \sqrt{\min(1/4, a^2) + 2a + a^2} / (1 + a)$  (as given by Lemma 2.8). If there is  $\kappa_\alpha \geq 0$  such that  $q_- \leq \alpha(\operatorname{deg} + q_+) + \kappa_\alpha$ , then  $q \in \mathcal{K}_{\alpha/(1-\tilde{a})}$ . On the other, hand if  $q \in \mathcal{K}_\alpha$ , then there is  $\kappa_\alpha \geq 0$  such that  $q_- \leq \alpha(1 + \tilde{a})(\operatorname{deg} + q_+) + \kappa_\alpha$ .*

*Proof.* — Using the assumption  $q_- \leq \alpha(\text{deg} + q_-) + \kappa_\alpha$  and the lower bound of Theorem 2.3 (ii), we infer

$$q_- \leq \alpha(\text{deg} + q_+) + \kappa_\alpha \leq \frac{\alpha}{(1 - \tilde{a})}(\Delta + q_+) + \frac{\alpha}{(1 - \tilde{a})}\tilde{k} + \kappa_\alpha.$$

Conversely,  $q \in \mathcal{X}_\alpha$  and the upper bound of Theorem 2.3 (ii) yields

$$q_- \leq \alpha(\Delta + q_+) + C_\alpha \leq \alpha(1 + \tilde{a})(\text{deg} + q_+) + \alpha\tilde{k} + C_\alpha.$$

Hence, (a) follows. For (b), notice that  $\tilde{a} = \sqrt{\min(1/4, a^2) + 2a + a^2}/(1+a)$  by Lemma 2.8. □

### 3. Almost-sparseness and asymptotic of eigenvalues

In this section we prove better estimates on the eigenvalue asymptotics in a more specific situation. Looking at the inequality in Theorem 2.3 (ii) it seems desirable to have  $\tilde{a} = 0$ . As this is impossible when the degree is unbounded, we consider a sequence of  $\tilde{a}$  that tends to 0. Keeping in mind Remark 2.4, this leads naturally to the following definition.

**DEFINITION 3.1.** — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and  $q : \mathcal{V} \rightarrow \mathbb{R}$ . We say  $(\mathcal{G}, q)$  is almost sparse if for all  $\varepsilon > 0$  there is  $k_\varepsilon \geq 0$  such that  $(\mathcal{G}, q)$  is  $(\varepsilon, k_\varepsilon)$ -sparse, i.e., for any finite set  $\mathcal{W} \subseteq \mathcal{V}$  the induced subgraph  $\mathcal{G}_\mathcal{W} := (\mathcal{W}, \mathcal{E}_\mathcal{W})$  satisfies*

$$2|\mathcal{E}_\mathcal{W}| \leq k_\varepsilon|\mathcal{W}| + \varepsilon(|\partial\mathcal{W}| + q_+(\mathcal{W})).$$

**Remark 3.2.** — (a) Every sparse graph  $\mathcal{G}$  is almost sparse.  
 (b) For an almost sparse graph  $(\mathcal{G}, q)$ , every graph  $(\mathcal{G}, q')$  with  $q' \geq q$  is almost sparse.

The main result of this section shows how the first order of the eigenvalue asymptotics in the case of discrete spectrum can be determined for almost sparse graphs.

**THEOREM 3.3.** — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and  $q \in \mathcal{X}_{0+}$ . The following assertions are equivalent:*

- (i)  $(\mathcal{G}, q)$  is almost sparse.
- (ii) For every  $\varepsilon > 0$  there are  $k_\varepsilon \geq 0$  such that on  $\mathcal{C}_c(\mathcal{V})$ 

$$(1 - \varepsilon)(\text{deg} + q) - k_\varepsilon \leq \Delta + q \leq (1 + \varepsilon)(\text{deg} + q) + k_\varepsilon.$$
- (iii) For every  $\varepsilon > 0$  there are  $k_\varepsilon \geq 0$  such that on  $\mathcal{C}_c(\mathcal{V})$ 

$$(1 - \varepsilon)(\text{deg} + q) - k_\varepsilon \leq \Delta + q.$$

Moreover,  $\mathcal{D}((\Delta + q)^{1/2}) = \mathcal{D}((\deg + q)^{1/2})$  and the operator  $\Delta + q$  has purely discrete spectrum if and only if  $\liminf_{|x| \rightarrow \infty} (\deg + q)(x) = \infty$ . In this case, we have

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\deg + q)} = 1.$$

*Proof.* — The statement is a direct application of Theorem 2.3 if one keeps track of the constants given explicitly by Lemma 2.6, Lemma 2.8 and Lemma A.3. □

### 4. Magnetic Laplacians

In this section, we consider magnetic Schrödinger operators. Clearly, every lower bound can be deduced from Kato’s inequality. However, for the eigenvalue asymptotics we also need to prove an upper bound.

We fix a phase

$$\theta : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}/2\pi\mathbb{Z} \text{ such that } \theta(x, y) = -\theta(y, x).$$

For a potential  $q : \mathcal{V} \rightarrow [0, \infty)$ , we consider the magnetic Schrödinger operator  $\Delta_\theta + q$  defined on

$$\mathcal{D}(\Delta_\theta + q) := \left\{ \varphi \in \ell^2(\mathcal{V}) \mid \left( v \mapsto \sum_{x \sim y} (\varphi(x) - e^{i\theta(x,y)}\varphi(y)) + q(x)\varphi(x) \right) \in \ell^2(\mathcal{V}) \right\}$$

by

$$(\Delta_\theta + q)\varphi(x) := \sum_{x \sim y} (\varphi(x) - e^{i\theta(x,y)}\varphi(y)) + q(x)\varphi(x).$$

A computation for  $\varphi \in \mathcal{C}_c(\mathcal{V})$  gives

$$\langle \varphi, (\Delta_\theta + q)\varphi \rangle = \frac{1}{2} \sum_{x,y,x \sim y} \left| \varphi(x) - e^{i\theta(x,y)}\varphi(y) \right|^2 + \sum_x q(x)|\varphi(x)|^2.$$

The operator is non-negative and selfadjoint as it is essentially selfadjoint on  $\mathcal{C}_c(\mathcal{V})$  (confer e.g. [12]). For  $\alpha > 0$ , let  $\mathcal{H}_\alpha^\theta$  be the class of real-valued potentials  $q$  such that  $q_- \leq \alpha(\Delta_\theta + q_+) + C_\alpha$  for some  $C_\alpha \geq 0$ . Denote

$$\mathcal{H}_{0+}^\theta = \bigcap_{\alpha \in (0,1)} \mathcal{H}_\alpha^\theta.$$

Again, for  $\alpha \in (0, 1)$  and  $q \in \mathcal{H}_\alpha^\theta$ , we define  $\Delta_\theta + q$  to be the form sum of  $\Delta_\theta + q_+$  and  $-q_-$ .

We present our result for magnetic Schrödinger operators which has one implication from the equivalences of Theorem 2.3 and Theorem 3.3.

**THEOREM 4.1.** — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph,  $\theta$  be a phase and  $q \in \mathcal{K}_{0+}^\theta$  be a potential. Assume  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse for some  $a, k \geq 0$ . Then, we have the following:*

(a) *There are  $\tilde{a} \in (0, 1)$ ,  $k \geq 0$  such that on  $\mathcal{C}_c(\mathcal{V})$*

$$(1 - \tilde{a})(\deg + q) - k \leq \Delta_\theta + q \leq (1 + \tilde{a})(\deg + q) + k.$$

(b)  $\mathcal{D}(|\Delta_\theta + q|^{1/2}) = \mathcal{D}(|\deg + q|^{1/2})$ .

(c) *The operator  $\Delta_\theta + q$  has purely discrete spectrum if and only if*

$$\liminf_{|x| \rightarrow \infty} (\deg + q)(x) = \infty.$$

*In this case, if  $(\mathcal{G}, q)$  is additionally almost sparse, then*

$$\liminf_{\lambda \rightarrow \infty} \frac{\lambda_n(\Delta_\theta + q)}{\lambda_n(\deg + q)} = 1.$$

**Remark 4.2.**

(a) The constants  $\tilde{a}$  and  $\tilde{k}$  can be chosen to be the same as the ones we obtained in the proof of Theorem 2.3, i.e., these constants are explicitly given combining Lemma 2.8 and Lemma A.3.

(b) The statements of the theorem remain true for  $q \in \mathcal{K}_{0+}$  since  $\mathcal{K}_{0+} \subseteq \mathcal{K}_{0+}^\theta$  holds by Kato's inequality below. Furthermore, by Lemma 4.5, the statements (a), (b) and the characterization of discrete spectrum can be extended to  $\mathcal{K}_\alpha$  for certain small  $\alpha > 0$ .

We will prove the theorem by applying Theorem 2.3 and Theorem 3.3. The considerations heavily rely on Kato's inequality and a conceptual version of the Upside-Down-Lemma, Lemma 2.9, which shows that a lower bound for  $\Delta + q$  implies an upper and lower bound on  $\Delta_\theta + q$ . Secondly, in Theorem 3.3 potentials in  $\mathcal{K}_{0+}$  are considered, while here we start with the class  $\mathcal{K}_{0+}^\theta$ . However, it can be seen that  $\mathcal{K}_{0+} = \mathcal{K}_{0+}^\theta$  in the case of  $(a, k)$ -sparse graph, see Lemma 4.5 below.

As mentioned above a key fact is Kato's inequality, see for instance [9, Lemma 2.1] or [16, Theorem 5.2.b].

**PROPOSITION 4.3** (Kato's inequality). — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph,  $\theta$  be a phase and  $q : \mathcal{V} \rightarrow \mathbb{R}$ . For all  $f \in \mathcal{C}_c(\mathcal{V})$ , we have*

$$\langle |f|, (\Delta|f| + q|f|) \rangle \leq \langle f, (\Delta_\theta f + qf) \rangle.$$



In particular, for all  $\alpha > 0$

$$\mathcal{K}_\alpha \subseteq \mathcal{K}_\alpha^\theta \text{ and } \mathcal{K}_{0+} \subseteq \mathcal{K}_{0+}^\theta.$$

*Proof.* — The proof of the inequality can be obtained by a direct calculation. The second statement is an immediate consequence.  $\square$

The next lemma is a rather surprising observation. It is the magnetic version of the Upside-Down-Lemma, Lemma 2.9.

LEMMA 4.4 (Upside-Down-Lemma – magnetic version). — Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph,  $\theta$  be a phase and  $q : \mathcal{V} \rightarrow \mathbb{R}$  be a potential. Assume that there are  $\tilde{a} \in (0, 1)$  and  $\tilde{k} \geq 0$  such that for all  $f \in \mathcal{C}_c(\mathcal{V})$ , we have

$$(1 - \tilde{a})\langle f, (\text{deg} + q)f \rangle - \tilde{k}\|f\|^2 \leq \langle f, \Delta f + qf \rangle$$

then for all  $f \in \mathcal{C}_c(\mathcal{V})$ , we also have

$$\begin{aligned} (1 - \tilde{a})\langle f, (\text{deg} + q)f \rangle - \tilde{k}\|f\|^2 &\leq \langle f, \Delta_\theta f + qf \rangle \\ &\leq (1 + \tilde{a})\langle f, (\text{deg} + q)f \rangle + \tilde{k}\|f\|^2. \end{aligned}$$

*Proof.* — The lower bound follows directly from Kato’s inequality and the lower bound from the assumption (as  $\langle f, (\text{deg} + q)f \rangle = \langle |f|, (\text{deg} + q)|f| \rangle$  for all  $f \in \mathcal{C}_c(\mathcal{V})$ ). Now, observe that for all  $\theta$

$$\Delta_\theta = 2 \text{deg} - \Delta_{\theta+\pi}.$$

So, the upper bound for  $\Delta_\theta + q$  follows from the lower bound of  $\Delta_{\theta+\pi} + q$  which we deduced from Kato’s inequality.  $\square$

The lemma above allows to relate the classes  $\mathcal{K}_\alpha$  and  $\mathcal{K}_\alpha^\theta$  for  $(a, k)$ -sparse graphs.

LEMMA 4.5. — For  $a, k \geq 0$  let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be an  $(a, k)$ -sparse graph,  $\theta$  be a phase and  $\alpha > 0$ . Then,

$$\mathcal{K}_\alpha^\theta \subseteq \mathcal{K}_{\alpha'}, \text{ for } \alpha' = \frac{1 + \tilde{a}}{1 - \tilde{a}} \alpha,$$

where  $\tilde{a}$  is given in Lemma 2.8. In particular,

$$\mathcal{K}_{0+}^\theta = \mathcal{K}_{0+}.$$

Moreover, if  $(\mathcal{G}, q)$  is almost-sparse, then

$$\mathcal{K}_\alpha^\theta \subseteq \mathcal{K}_{\alpha'}, \text{ for all } \alpha' > \alpha.$$

*Proof.* — Let  $q \in \mathcal{K}_\alpha^\theta$ . Applying Lemma 2.8, we get

$$\Delta + q_+ \geq (1 - \tilde{a})(\text{deg} + q_+) - \tilde{k}.$$

Now, by the virtue of the Upside-Down-Lemma, Lemma 4.4, we infer

$$\Delta_\theta + q_+ \leq (1 + \tilde{a})(\text{deg} + q_+) + \tilde{k} \leq \frac{1 + \tilde{a}}{1 - \tilde{a}}(\Delta + q_+) + \frac{2}{1 - \tilde{a}}\tilde{k}$$

which implies the first statement and  $\mathcal{K}_{0+}^\theta \subseteq \mathcal{K}_{0+}$ . The reverse inclusion  $\mathcal{K}_{0+}^\theta \supseteq \mathcal{K}_{0+}$  follows from Kato’s inequality, Lemma 4.3. For almost sparse graphs  $a$  can be chosen arbitrary small and accordingly  $\tilde{a}$  (from Lemma 2.8) becomes arbitrary small. Hence, the statement  $\mathcal{K}_\alpha^\theta \subseteq \mathcal{K}_{\alpha'}$ , for  $\alpha' > \alpha$  follows from the inequality above.  $\square$

*Proof of Theorem 4.1.* — Let  $q \in \mathcal{K}_{0+}^\theta$ . By Lemma 4.5,  $q \in \mathcal{K}_{0+}$ . Thus, (a) follows from Theorem 2.3 and Lemma 4.4. Using (a), statement (b) follows from an application of the Closed Graph Theorem, Theorem A.1 and statement (c) follows from an application of the Min Max Principle, Theorem A.2.  $\square$

*Remark 4.6.* — Instead of using Kato’s inequality one can also reproduce the proof of Lemma 2.8 using the following estimate

$$||f(x)|^2 - |f(y)|^2| \leq |(f(x) - e^{i\theta(x,y)}f(y))(\overline{f(x)} + e^{-i\theta(x,y)}\overline{f(y)})|.$$

So, we infer the key estimate:

$$\begin{aligned} \langle f, (\text{deg} + q)f \rangle - k\|f\|^2 &\leq (1 + a)\langle f, (\Delta_\theta + q)f \rangle^{\frac{1}{2}} \langle f, (\Delta_{\theta+\pi} + q)f \rangle^{\frac{1}{2}}, \\ &= (1 + a)\langle f, (\Delta_\theta + q)f \rangle^{\frac{1}{2}} (2\langle f, (\text{deg} + q)f \rangle - \langle f, (\Delta_\theta + q)f \rangle)^{\frac{1}{2}}. \end{aligned}$$

The rest of the proof is analogous.

It can be observed that unlike in Theorem 2.3 or Theorem 3.3 we do not have an equivalence in the theorem above. A reason for this seems to be that our definition of sparseness does not involve the magnetic potential. This direction shall be pursued in the future. Here, we restrict ourselves to some remarks on the perturbation theory in the context of Theorem 4.1 above.

*Remark 4.7.*

- (a) If the inequality Theorem 4.1 (a) holds for some  $\theta$ , then the inequality holds with the same constants for  $-\theta$  and  $\theta \pm \pi$ . This can be seen by the fact  $\Delta_{\theta+\pi} = 2 \text{deg} - \Delta$  and  $\langle f, \Delta_\theta f \rangle = \langle \bar{f}, \Delta_{-\theta} \bar{f} \rangle$  while  $\langle f, \text{deg} f \rangle = \langle \bar{f}, \text{deg} \bar{f} \rangle$  for  $f \in \mathcal{C}_c(\mathcal{V})$ .

- (b) The set of  $\theta$  such that Theorem 4.1 (a) holds true for some fixed  $\tilde{a}$  and  $\tilde{k}$  is closed in the product topology, i.e., with respect to pointwise convergence. This follows as  $\langle f, \Delta_{\theta_n} f \rangle \rightarrow \langle f, \Delta_{\theta} f \rangle$  if  $\theta_n \rightarrow \theta$ ,  $n \rightarrow \infty$ , for fixed  $f \in \mathcal{C}_c(\mathcal{V})$ .
- (c) For two phases  $\theta$  and  $\theta'$  let  $h(x) = \max_{y \sim x} |\theta(x, y) - \theta'(x, y)|$ . By a straight forward estimate  $\limsup_{|x| \rightarrow \infty} h(x) = 0$  implies that for every  $\varepsilon > 0$  there is  $C \geq 0$  such that

$$-\varepsilon \deg - C \leq \Delta_{\theta} - \Delta_{\theta'} \leq \varepsilon \deg + C$$

on  $\mathcal{C}_c(\mathcal{V})$ . We discuss three consequences of this inequality:

First, this inequality immediately yields that if  $\mathcal{D}(\Delta_{\theta}^{1/2}) = \mathcal{D}(\deg^{1/2})$  then  $\mathcal{D}(\Delta_{\theta'}^{1/2}) = \mathcal{D}(\deg^{1/2})$  (by the KLMN Theorem, see e.g., [28, Theorem X.17]) which in turn yields equality of the form domains of  $\Delta_{\theta}$  and  $\Delta_{\theta'}$ .

Secondly, combining this inequality with Theorem 3.3 we obtain the following: If  $\limsup_{|x| \rightarrow \infty} \max_{y \sim x} |\theta(x, y)| = 0$  and for every  $\varepsilon > 0$  there is  $k_{\varepsilon} \geq 0$  such that

$$(1 - \varepsilon) \deg - k_{\varepsilon} \leq \Delta_{\theta} \leq (1 + \varepsilon) \deg + k_{\varepsilon}$$

then the graph is almost sparse and in consequence the inequality in Theorem 4.1 (a) holds for any phase.

Thirdly, using the techniques in the proof of [12, Proposition 5.2] one shows that the essential spectra of  $\Delta_{\theta}$  and  $\Delta_{\theta'}$  coincide. With slightly more effort and the help of the Kuroda-Birman Theorem, [28, Theorem XI.9] one can show that if  $h \in \ell^1(\mathcal{V})$ , then even the absolutely continuous spectra of  $\Delta_{\theta}$  and  $\Delta_{\theta'}$  coincide.

### 5. Isoperimetric estimates and sparseness

In this section we relate the concept of sparseness with the concept of isoperimetric estimates. First, we present a result which should be viewed in the light of Theorem 2.3 as it points out in which sense isoperimetric estimates are stronger than our notions of sparseness. In the second subsection, we present a result related to Theorem 3.3. Finally, we present a concrete comparison of sparseness and isoperimetric estimates. As this section is of a more geometric flavor we restrict ourselves to the case of potentials  $q : \mathcal{V} \rightarrow [0, \infty)$ .

### 5.1. Isoperimetric estimates

Let  $\mathcal{U} \subseteq \mathcal{V}$  and define the Cheeger or isoperimetric constant of  $\mathcal{U}$  by

$$\alpha_{\mathcal{U}} := \inf_{\mathcal{W} \subseteq \mathcal{U} \text{ finite}} \frac{|\partial \mathcal{W}| + q(\mathcal{W})}{\deg(\mathcal{W}) + q(\mathcal{W})}.$$

In the case where  $\deg(\mathcal{W}) + q(\mathcal{W}) = 0$ , for instance when  $W$  is an isolated point, by convention the above quotient is set to be equal to 0. Note that  $\alpha_{\mathcal{U}} \in [0, 1)$ .

The following theorem illustrates in which sense positivity of the Cheeger constant is linked with  $(a, 0)$ -sparseness. We refer to Theorem 5.4 for precise constants.

**THEOREM 5.1.** — *Given  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  a graph and  $q : \mathcal{V} \rightarrow [0, \infty)$ . The following assertions are equivalent*

- (i)  $\alpha_{\mathcal{V}} > 0$ .
- (ii) There is  $\tilde{a} \in (0, 1)$

$$(1 - \tilde{a})(\deg + q) \leq \Delta + q \leq (1 + \tilde{a})(\deg + q).$$

- (iii) There is  $\tilde{a} \in (0, 1)$  such that

$$(1 - \tilde{a})(\deg + q) \leq \Delta + q.$$

The implication (iii) $\Rightarrow$ (i) is already found in [12, Proposition 3.4]. The implication (i) $\Rightarrow$ (ii) is a consequence from standard isoperimetric estimates which can be extracted from the proof of [15, Proposition 15].

**PROPOSITION 5.2** ([15]). — *Let  $\mathcal{G} := (\mathcal{E}, \mathcal{V})$  be a graph and a potential  $q : \mathcal{V} \rightarrow [0, \infty)$ . Then, for all  $\mathcal{U} \subseteq \mathcal{V}$  we have on  $\mathcal{C}_c(\mathcal{U})$ .*

$$(1 - \sqrt{1 - \alpha_{\mathcal{U}}^2})(\deg + q) \leq \Delta + q \leq (1 + \sqrt{1 - \alpha_{\mathcal{U}}^2})(\deg + q).$$

### 5.2. Isoperimetric estimates at infinity

Let the Cheeger constant at infinity be defined as

$$\alpha_{\infty} = \sup_{\mathcal{X} \subseteq \mathcal{V} \text{ finite}} \alpha_{\mathcal{V} \setminus \mathcal{X}}.$$

Clearly,  $0 \leq \alpha_{\mathcal{V}} \leq \alpha_{\mathcal{U}} \leq \alpha_{\infty} \leq 1$  for any  $\mathcal{U} \subseteq \mathcal{V}$ .

As a consequence of Proposition 5.2, we get the following theorem.

**THEOREM 5.3.** — *Let  $\mathcal{G} := (\mathcal{E}, \mathcal{V})$  be a graph and  $q : \mathcal{V} \rightarrow [0, \infty)$  be a potential. Assume  $\alpha_{\infty} > 0$ . Then, we have the following:*

(a) For every  $\varepsilon > 0$  there is  $k_\varepsilon \geq 0$  such that on  $\mathcal{C}_c(\mathcal{V})$

$$\begin{aligned} (1 - \varepsilon)(1 - \sqrt{1 - \alpha_\infty^2})(\deg + q) - k_\varepsilon & \\ & \leq \Delta + q \\ & \leq (1 + \varepsilon)(1 + \sqrt{1 - \alpha_\infty^2})(\deg + q) + k_\varepsilon. \end{aligned}$$

(b)  $\mathcal{D}((\Delta + q)^{1/2}) = \mathcal{D}((\deg + q)^{1/2})$ .

(c) The operator  $\Delta + q$  has purely discrete spectrum if and only if we have  $\liminf_{|x| \rightarrow \infty} (\deg + q)(x) = \infty$ . In this case, if additionally  $\alpha_\infty = 1$ , we get

$$\liminf_{\lambda \rightarrow \infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\deg + q)} = 1.$$

*Proof.* — Starting with (a), let  $\varepsilon > 0$  and  $\mathcal{K} \subseteq \mathcal{V}$  be finite and large enough such that

$$\begin{aligned} (1 - \varepsilon)(1 - \sqrt{1 - \alpha_\infty^2}) & \leq (1 - \sqrt{1 - \alpha_{\mathcal{V} \setminus \mathcal{K}}^2}) \\ (1 + \sqrt{1 - \alpha_{\mathcal{V} \setminus \mathcal{K}}^2}) & \leq (1 + \varepsilon)(1 + \sqrt{1 - \alpha_\infty^2}). \end{aligned}$$

From Proposition 5.2 we conclude on  $\mathcal{C}_c(\mathcal{V} \setminus \mathcal{K})$

$$\begin{aligned} (1 - \varepsilon)(1 - \sqrt{1 - \alpha_\infty^2})(\deg + q) & \leq (1 - \sqrt{1 - \alpha_{\mathcal{V} \setminus \mathcal{K}}^2})(\deg + q) \\ & \leq \Delta + q \leq (1 + \sqrt{1 - \alpha_{\mathcal{V} \setminus \mathcal{K}}^2})(\deg + q) \\ & \leq (1 + \varepsilon)(1 + \sqrt{1 - \alpha_\infty^2})(\deg + q) \end{aligned}$$

By local finiteness note that  $\mathbf{1}_{\mathcal{V} \setminus \mathcal{K}}(\Delta + q)\mathbf{1}_{\mathcal{V} \setminus \mathcal{K}}$  and  $\mathbf{1}_{\mathcal{V} \setminus \mathcal{K}}(\deg + q)\mathbf{1}_{\mathcal{V} \setminus \mathcal{K}}$  are bounded (indeed, finite rank) perturbations of  $\Delta + q$  and  $\deg + q$ . This gives rise to the constants  $k_\varepsilon$  and the inequality of (a) follows. Now, (b) is an immediate consequence of (a), and (c) follows by the Min-Max-Principle, Theorem A.2. □

### 5.3. Relating sparseness and isoperimetric estimates.

We now explain how the notions of sparseness and isoperimetric estimates are exactly related.

First, we consider classical isoperimetric estimates.

**THEOREM 5.4.** — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph,  $a, k \geq 0$ , and let  $q : \mathcal{V} \rightarrow [0, \infty)$  be a potential.*

- (a)  $\alpha_{\mathcal{V}} \geq \frac{1}{1+a}$  if and only if  $(\mathcal{G}, q)$  is  $(a, 0)$ -sparse.
- (b) If  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse, then

$$\alpha_{\mathcal{V}} \geq \frac{d-k}{d(1+a)},$$

where  $d := \inf_{x \in \mathcal{V}} (\deg + q)(x)$ . In particular,  $\alpha_{\mathcal{V}} > 0$  if  $d > k$ .

- (c) Suppose that  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse graph that is not  $(a, k')$ -sparse for all  $k' < k$ . Suppose also that there is  $d$  such that  $d = \deg(x) + q(x)$  for all  $x \in \mathcal{V}$ . Then

$$\alpha_{\mathcal{V}} = \frac{d-k}{d(1+a)}.$$

*Proof.* — Let  $\mathcal{W} \subset \mathcal{V}$  be a finite set. Recalling the identity  $\deg(\mathcal{W}) = 2|\mathcal{E}_{\mathcal{W}}| + |\partial\mathcal{W}|$  we observe that

$$\frac{1}{1+a} \leq \frac{|\partial\mathcal{W}| + q(\mathcal{W})}{(\deg + q)(\mathcal{W})}$$

is equivalent to

$$2|\mathcal{E}_{\mathcal{W}}| \leq a(|\partial\mathcal{W}| + q(\mathcal{W})).$$

This proves (a).

For (b), the definition of  $(a, k)$ -sparseness yields

$$\frac{|\partial\mathcal{W}| + q(\mathcal{W})}{(\deg + q)(\mathcal{W})} = 1 - \frac{2|\mathcal{E}_{\mathcal{W}}|}{(\deg + q)(\mathcal{W})} \geq 1 - a \frac{|\partial\mathcal{W}| + q(\mathcal{W})}{(\deg + q)(\mathcal{W})} - k \frac{|\mathcal{W}|}{(\deg + q)(\mathcal{W})}.$$

This gives immediately the statement.

For (c), the lower bound of  $\alpha_{\mathcal{V}}$  follows from (b). Since  $(\mathcal{G}, q)$  is not  $(a, k')$ -sparse, there is a finite  $\mathcal{W}_0 \subset \mathcal{V}$  such that

$$\frac{|\partial\mathcal{W}_0| + q(\mathcal{W}_0)}{(\deg + q)(\mathcal{W}_0)} < 1 - a \frac{|\partial\mathcal{W}_0| + q(\mathcal{W}_0)}{(\deg + q)(\mathcal{W}_0)} - k' \frac{|\mathcal{W}_0|}{(\deg + q)(\mathcal{W}_0)}.$$

Therefore,  $\alpha_{\mathcal{V}} < (d - k') / (d(1 + a))$ . □

We next address the relation between almost sparseness and isoperimetry and show two “almost equivalences”.

**THEOREM 5.5.** — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph and let  $q : \mathcal{V} \rightarrow [0, \infty)$  be a potential.*

- (a) If  $\alpha_{\infty} > 0$ , then  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse for some  $a > 0, k \geq 0$ . On the other hand, if  $(\mathcal{G}, q)$  is  $(a, k)$ -sparse for some  $a > 0, k \geq 0$  and

$$l := \liminf_{|x| \rightarrow \infty} (\deg + q)(x) > k,$$

then

$$\alpha_\infty \geq \frac{l - k}{l(a + 1)} > 0,$$

if  $l$  is finite and  $\alpha_\infty \geq 1/(1 + a)$  otherwise.

(b) If  $\alpha_\infty = 1$ , then  $(\mathcal{G}, q)$  is almost sparse. On the other hand, if  $(\mathcal{G}, q)$  is almost sparse and  $\liminf_{|x| \rightarrow \infty} (\deg + q)(x) = \infty$ , then  $\alpha_\infty = 1$ .

*Proof.* — The first implication of (a) follows from Theorem 5.3 (a) and Theorem 2.3 (ii) $\Rightarrow$ (i). For the opposite direction let  $\varepsilon > 0$  and  $\mathcal{K} \subseteq \mathcal{V}$  be finite such that  $\deg + q \geq l - \varepsilon$  on  $\mathcal{V} \setminus \mathcal{K}$ . Using the formula in the proof of Theorem 5.4 above, yields for  $\mathcal{W} \subseteq \mathcal{V} \setminus \mathcal{K}$

$$\begin{aligned} \frac{|\partial \mathcal{W}| + q(\mathcal{W})}{(\deg + q)(\mathcal{W})} &= 1 - \frac{2|\mathcal{E}_{\mathcal{W}}|}{(\deg + q)(\mathcal{W})} \geq 1 - \frac{k|\mathcal{W}| + a(|\partial \mathcal{W}| + q(\mathcal{W}))}{(\deg + q)(\mathcal{W})} \\ &\geq 1 - \frac{k}{(l - \varepsilon)} - \frac{a(|\partial \mathcal{W}| + q(\mathcal{W}))}{(\deg + q)(\mathcal{W})}. \end{aligned}$$

This proves (a).

The first implication of (b) follows from Theorem 5.3 (a) and Theorem 3.3 (ii) $\Rightarrow$ (i). The other implication follows from (a) using the definition of almost sparseness.  $\square$

*Remark 5.6.* — We point out that without the assumptions on  $(\deg + q)$  the converse implications do not hold. For example the Cayley graph of  $\mathbb{Z}$  is 2-sparse (cf. Lemma 6.2), but has  $\alpha_\infty = 0$ .

The previous theorems provides a slightly simplified proof of [17] which also appeared morally in somewhat different forms in [7, 31].

**COROLLARY 5.7.** — *Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a planar graph.*

- (a) *If for all vertices  $\deg \geq 7$ , then  $\alpha_\mathcal{V} > 0$ .*
- (b) *If for all vertices away from a finite set  $\deg \geq 7$ , then  $\alpha_\infty > 0$ .*

*Proof.* — Combine Theorem 5.4 and Theorem 5.5 with Lemma 6.2.  $\square$

## 6. Examples

### 6.1. Examples of sparse graphs

To start off, we exhibit two classes of sparse graphs. First we consider the case of graphs with bounded degree.

LEMMA 6.1. — Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph. Assume  $D := \sup_{x \in \mathcal{V}} \deg(x)$  is finite, then  $\mathcal{G}$  is  $D$ -sparse.

*Proof.* — Let  $\mathcal{W}$  be a finite subset of  $\mathcal{V}$ . Then,  $2|\mathcal{E}_{\mathcal{W}}| \leq \deg(\mathcal{W}) \leq D|\mathcal{W}|$ . □

We turn to graphs which admit a 2-cell embedding into  $S_g$ , where  $S_g$  denotes a compact orientable topological surface of genus  $g$ . (The surface  $S_g$  might be pictured as a sphere with  $g$  handles.) Admitting a 2-cell embedding means that the graphs can be embedded into  $S_g$  without self-intersection. By definition we say that a graph is planar when  $g = 0$ . Note that unlike other possible definitions of planarity, we do not impose any local compactness on the embedding.

LEMMA 6.2.

- (a) Trees are 2-sparse.
- (b) Planar graphs are 6-sparse.
- (c) Graphs admitting a 2-cell embedding into  $S_g$  with  $g \geq 1$  are  $4g + 2$ -sparse.

*Proof.* — Starting with (a), let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a tree and  $\mathcal{G}_{\mathcal{W}} := (\mathcal{W}, \mathcal{E}_{\mathcal{W}})$  be a finite induced subgraph of  $\mathcal{G}$ . Clearly  $|\mathcal{E}_{\mathcal{W}}| \leq |\mathcal{W}| - 1$ . Therefore, every tree is 2-sparse.

We treat the cases (b) and (c) simultaneously. Let  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  be a graph which is connected 2-cell embedded in  $S_g$  with  $g \geq 0$  (as remarked above planar graphs correspond to  $g = 0$ ). Let  $\mathcal{G}_{\mathcal{W}} := (\mathcal{W}, \mathcal{E}_{\mathcal{W}})$  be a finite induced subgraph of  $\mathcal{G}$  which, clearly, also admits a 2-cell embedding into  $S_g$ . The statement is clear for  $|\mathcal{W}| \leq 2$ . Assume  $|\mathcal{W}| \geq 3$ . Let  $\mathcal{F}_{\mathcal{W}}$  be the faces induced by  $\mathcal{G}_{\mathcal{W}} := (\mathcal{W}, \mathcal{E}_{\mathcal{W}})$  in  $S_g$ . Here, all faces (even the outer one) contain at least 3 edges, each edge belongs only to 2 faces, thus,

$$2|\mathcal{E}_{\mathcal{W}}| \geq 3|\mathcal{F}_{\mathcal{W}}|.$$

Euler’s formula,  $|\mathcal{W}| - |\mathcal{E}_{\mathcal{W}}| + |\mathcal{F}_{\mathcal{W}}| = 2 - 2g$ , gives then

$$2 - 2g + |\mathcal{E}_{\mathcal{W}}| = |\mathcal{W}| + |\mathcal{F}_{\mathcal{W}}| \leq |\mathcal{W}| + \frac{2}{3}|\mathcal{E}_{\mathcal{W}}|$$

that is

$$|\mathcal{E}_{\mathcal{W}}| \leq 3|\mathcal{W}| + 6(g - 1) \leq \max(2g + 1, 3)|\mathcal{W}|.$$

This concludes the proof. □

Next, we explain how to construct sparse graphs from existing sparse graphs.

LEMMA 6.3. — Let  $\mathcal{G}_1 := (\mathcal{V}_1, \mathcal{E}_1)$  and  $\mathcal{G}_2 := (\mathcal{V}_2, \mathcal{E}_2)$  be two graphs.



(a) Assume  $\mathcal{V}_1 = \mathcal{V}_2$ ,  $\mathcal{G}_1$  is  $k_1$ -sparse and  $\mathcal{G}_2$  is  $k_2$ -sparse. Then,  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  with  $\mathcal{E} := \max(\mathcal{E}_1, \mathcal{E}_2)$  is  $(k_1 + k_2)$ -sparse.

(b) Assume  $\mathcal{G}_1$  is  $k_1$ -sparse and  $\mathcal{G}_2$  is  $k_2$ -sparse. Then  $\mathcal{G}_1 \oplus \mathcal{G}_2 := (\mathcal{V}, \mathcal{E})$  with where  $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$  and

$$\mathcal{E}((x_1, x_2), (y_1, y_2)) := \delta_{\{x_1\}}(y_1) \cdot \mathcal{E}_2(x_2, y_2) + \delta_{\{x_2\}}(y_2) \cdot \mathcal{E}_1(x_1, y_1),$$

is  $(k_1 + k_2)$ -sparse.

(c) Assume  $\mathcal{V}_1 = \mathcal{V}_2$ ,  $\mathcal{G}_1$  is  $k$ -sparse and  $\mathcal{E}_2 \leq \mathcal{E}_1$ . Then,  $\mathcal{G}_2$  is  $k$ -sparse.

*Proof.* — For (a) let  $\mathcal{W} \subseteq \mathcal{V}$  be finite and note that  $|\mathcal{E}_{\mathcal{W}}| \leq |\mathcal{E}_{1, \mathcal{W}}| + |\mathcal{E}_{2, \mathcal{W}}|$ . For (b) let  $p_1, p_2$  the canonical projections from  $\mathcal{V}$  to  $\mathcal{V}_1$  and  $\mathcal{V}_2$ . For finite  $\mathcal{W} \subseteq \mathcal{V}$  we observe

$$|\mathcal{E}_{\mathcal{W}}| = |\mathcal{E}_{1, p_1(\mathcal{W})}| + |\mathcal{E}_{2, p_2(\mathcal{W})}| \leq k_1 |p_2(\mathcal{W})| + k_2 |p_1(\mathcal{W})| \leq (k_1 + k_2) |\mathcal{W}|.$$

For (c) and  $\mathcal{W} \subseteq \mathcal{V}$  finite, we have  $|\mathcal{E}_{2, \mathcal{W}}| \leq |\mathcal{E}_{1, \mathcal{W}}|$  which yields the statement. □

*Remark 6.4.*

- (a) We point out that there are bi-partite graphs which are not sparse. See for example [12, Proposition 4.11] or take an antitree, confer [20, Section 6], where the number of vertices in the spheres grows monotonously to  $\infty$ .
- (b) The last point of the lemma states that the  $k$ -sparseness is non-decreasing when we remove edges from the graph. This is not the case for the isoperimetric constant.

### 6.2. Examples of almost-sparse and $(a, k)$ -sparse graph

We construct a series of examples which are perturbations of a radial tree. They illustrate that sparseness, almost sparseness and  $(a, k)$ -sparseness are indeed different concepts.

Let  $\beta = (\beta_n), \gamma = (\gamma_n)$  be two sequences of natural numbers. Let  $\mathcal{T} = \mathcal{T}(\beta)$  with  $\mathcal{T} = (\mathcal{V}, \mathcal{E}^{\mathcal{T}})$  be a radial tree with root  $o$  and vertex degree  $\beta_n$  at the  $n$ -th sphere, that is every vertex which has natural graph distance  $n$  to  $o$  has  $(\beta_n - 1)$  forward neighbors. We denote the distance spheres by  $S_n$ . We let  $\mathcal{G}(\beta, \gamma)$  be the set of graphs  $\mathcal{G} := (\mathcal{V}, \mathcal{E}^{\mathcal{G}})$  that are super graphs of  $\mathcal{T}$  such that the induced subgraphs  $\mathcal{G}_{S_n}$  are  $\gamma_n$ -regular and  $\mathcal{E}^{\mathcal{G}}(x, y) = \mathcal{E}^{\mathcal{T}}(x, y)$  for  $x \in S_n, y \in S_m, m \neq n$ .

Observe that  $\mathcal{G}(\beta, \gamma)$  is non empty if and only if  $\gamma_n \prod_{j=0}^n (\beta_j - 1)$  is even and  $\gamma_n < |S_n| = \prod_{j=0}^n (\beta_j - 1)$  for all  $n \geq 0$ .

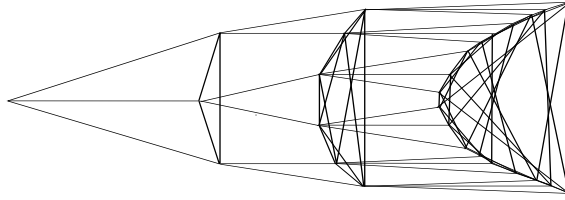


Figure 6.1.  $\mathcal{G}$  with  $\beta = (3, 3, 4, \dots)$  and  $\gamma = (0, 2, 4, 5, \dots)$ .

PROPOSITION 6.5. — Let  $\beta, \gamma \in \mathbb{N}_0^{\mathbb{N}}$ ,  $a = \limsup_{n \rightarrow \infty} \gamma_n / \beta_n$  and  $\mathcal{G} \in \mathcal{G}(\beta, \gamma)$ .

- (a) If  $a = 0$ , then  $\mathcal{G}$  is almost sparse. The graph  $\mathcal{G}$  is sparse if and only if  $\limsup_{n \rightarrow \infty} \gamma_n < \infty$ .
- (b) If  $a > 0$ , then  $\mathcal{G}$  is  $(a', k)$ -sparse for some  $k \geq 0$  if  $a' > a$ . Conversely, if  $\mathcal{G}$  is  $(a', k)$ -sparse for some  $k \geq 0$ , then  $a' \geq a$ .

Proof. — Let  $\varepsilon > 0$  and let  $N \geq 0$  be so large that

$$\gamma_n \leq (a + \varepsilon)\beta_n, \quad n \geq N.$$

Set  $C_\varepsilon := \sum_{n=0}^{N-1} \deg^{\mathcal{G}}(S_n)$ . Let  $\mathcal{W}$  be a non-empty finite subset of  $\mathcal{V}$ . We calculate

$$\begin{aligned} 2|\mathcal{E}_{\mathcal{W}}^{\mathcal{G}}| + |\partial^{\mathcal{G}} \mathcal{W}| &= \deg^{\mathcal{G}}(\mathcal{W}) \\ &= \deg^{\mathcal{T}}(\mathcal{W}) + \sum_{n \geq 0} |\mathcal{W} \cap S_n| \gamma_n \\ &\leq \deg^{\mathcal{T}}(\mathcal{W}) + (a + \varepsilon) \sum_{n \geq 0} |\mathcal{W} \cap S_n| \beta_n + \sum_{n=0}^{N-1} |\mathcal{W} \cap S_n| \gamma_n \\ &\leq (1 + a + \varepsilon) \deg^{\mathcal{T}}(\mathcal{W}) + C_\varepsilon |\mathcal{W}| \\ &= 2(1 + a + \varepsilon) |\mathcal{E}_{\mathcal{W}}^{\mathcal{T}}| + (1 + a + \varepsilon) |\partial^{\mathcal{T}} \mathcal{W}| + C_\varepsilon |\mathcal{W}| \\ &\leq (2(1 + a + \varepsilon) + C_\varepsilon) |\mathcal{W}| + (1 + a + \varepsilon) |\partial^{\mathcal{T}} \mathcal{W}|, \end{aligned}$$

where we used that trees are 2-sparse in the last inequality. Finally, since  $|\partial^{\mathcal{G}} \mathcal{W}| \geq |\partial^{\mathcal{T}} \mathcal{W}|$ , we conclude

$$2|\mathcal{E}_{\mathcal{W}}^{\mathcal{G}}| \leq (2(1 + a + \varepsilon) + C_\varepsilon) |\mathcal{W}| + (a + \varepsilon) |\partial^{\mathcal{G}} \mathcal{W}|.$$

This shows that the graph in (a) with  $a = 0$  is almost sparse and that the graph in (b) with  $a > 0$  is  $(a + \varepsilon, k_\varepsilon)$ -sparse for  $\varepsilon > 0$  and  $k_\varepsilon = 2(1 + a + \varepsilon) + C_\varepsilon$ . Moreover, for the other statement of (a) let  $k_0 = \limsup_{n \rightarrow \infty} \gamma_n$

and note that for  $\mathcal{G}_{S_n}$

$$2|\mathcal{E}_{S_n}| = \gamma_n|S_n|.$$

Hence, if  $k_0 = \infty$ , then  $\mathcal{G}$  is not sparse. On the other hand, if  $k_0 < \infty$ , then  $\mathcal{G}$  is  $(k_0 + 2)$ -sparse by Lemma 6.3 as  $\mathcal{T}$  is 2-sparse by Lemma 6.2. This finishes the proof of (a). Finally, assume that  $\mathcal{G}$  is  $(a', k)$ -sparse with  $k \geq 0$ . Then, for  $\mathcal{W} = S_n$

$$\gamma_n|S_n| = 2|\mathcal{E}_{S_n}| \leq k|S_n| + a'|\partial^{\mathcal{G}} S_n| = k|S_n| + a'\beta_n|S_n|$$

Dividing by  $\beta_n|S_n|$  and taking the limit yields  $a \leq a'$ . This proves (b).  $\square$

*Remark 6.6.* — In (a), we may suppose alternatively that we have the complete graph on  $S_n$  and the exponential growth:  $\lim_{n \rightarrow \infty} \frac{|S_n|}{|S_{n+1}|} = 0$ .

### Appendix A. Some general operator theory

We collect some consequences of standard results from functional analysis that are used in the paper. Let  $H$  be a Hilbert space with norm  $\|\cdot\|$ . For a quadratic form  $Q$ , denote the form norm by  $\|\cdot\|_Q := \sqrt{Q(\cdot) + \|\cdot\|^2}$ . The following is a direct consequence of the Closed Graph Theorem, (confer e.g. [30, Satz 4.7]).

**THEOREM A.1.** — *Let  $(Q_1, \mathcal{D}(Q_1))$  and  $(Q_2, \mathcal{D}(Q_2))$  be closed non-negative quadratic forms with a common form core  $\mathcal{D}_0$ . Then, the following are equivalent:*

- (i)  $\mathcal{D}(Q_1) \leq \mathcal{D}(Q_2)$ .
- (ii) *There are constants  $c_1 > 0$ ,  $c_2 \geq 0$  such that  $c_1Q_2 - c_2 \leq Q_1$  on  $\mathcal{D}_0$ .*

*Proof.* — If (ii) holds, then any  $\|\cdot\|_{Q_1}$ -Cauchy sequence is a  $\|\cdot\|_{Q_2}$ -Cauchy sequence. Thus, (ii) implies (i). On the other hand, consider the identity map  $j : (\mathcal{D}(Q_1), \|\cdot\|_{Q_1}) \rightarrow (\mathcal{D}(Q_2), \|\cdot\|_{Q_2})$ . The map  $j$  is closed as it is defined on the whole Hilbert space  $(\mathcal{D}(Q_1), \|\cdot\|_{Q_1})$  and, thus, bounded by the Closed Graph Theorem [28, Theorem III.12] which implies (i).  $\square$

For a selfadjoint operator  $T$  which is bounded from below, we denote the bottom of the spectrum by  $\lambda_0(T)$  and the bottom of the essential spectrum by  $\lambda_0^{\text{ess}}(T)$ . Let  $n(T) \in \mathbb{N}_0 \cup \{\infty\}$  be the dimension of the range of the spectral projection of  $(-\infty, \lambda_0^{\text{ess}}(T))$ . For  $\lambda_0(T) < \lambda_0^{\text{ess}}(T)$  we denote the eigenvalues below  $\lambda_0^{\text{ess}}(T)$  by  $\lambda_n(T)$ , for  $0 \leq n \leq n(T)$ , in increasing order counted with multiplicity.

**THEOREM A.2.** — *Let  $(Q_1, \mathcal{D}(Q_1))$  and  $(Q_2, \mathcal{D}(Q_2))$  be closed non-negative quadratic forms with a common form core  $\mathcal{D}_0$  and let  $T_1$  and  $T_2$  be the corresponding selfadjoint operators. Assume there are constants  $c_1 > 0, c_2 \in \mathbb{R}$  such that on  $\mathcal{D}_0$*

$$c_1 Q_2 - c_2 \leq Q_1.$$

*Then,  $c_1 \lambda_n(T_2) - c_2 \leq \lambda_n(T_1)$ , for  $0 \leq n \leq \min(n(T_1), n(T_2))$ . Moreover,  $c_1 \lambda_0^{\text{ess}}(T_2) - c_2 \leq \lambda_0^{\text{ess}}(T_1)$ , in particular,  $\sigma_{\text{ess}}(T_1) = \emptyset$  if  $\sigma_{\text{ess}}(T_2) = \emptyset$  and in this case*

$$c_1 \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n(T_1)}{\lambda_n(T_2)}.$$

*Proof.* — Letting

$$\mu_n(T) = \sup_{\varphi_1, \dots, \varphi_n \in H} \inf_{0 \neq \psi \in \{\varphi_1, \dots, \varphi_n\}^\perp \cap \mathcal{D}_0} \frac{\langle T\psi, \psi \rangle}{\langle \psi, \psi \rangle},$$

for a selfadjoint operator  $T$ , we know by the Min-Max-Principle [28, Chapter XIII.1]  $\mu_n(T) = \lambda_n(T)$  if  $\lambda_n(T) < \lambda_0^{\text{ess}}(T)$  and  $\mu_n(T) = \lambda_0^{\text{ess}}(T)$  otherwise,  $n \geq 0$ . Assume  $n \leq \min\{n(T_1), n(T_2)\}$  and let  $\varphi_0^{(j)}, \dots, \varphi_n^{(j)}$  be the eigenfunctions of  $T_j$  to  $\lambda_0(T_j), \dots, \lambda_n(T_j)$  we get

$$\begin{aligned} c_1 \lambda_n(T_2) - c_3 &= \inf_{0 \neq \psi \in \{\varphi_1^{(2)}, \dots, \varphi_n^{(2)}\}^\perp \cap \mathcal{D}_0} \left( c_1 \frac{\langle T_2 \psi, \psi \rangle}{\langle \psi, \psi \rangle} - c_3 \right) \\ &\leq \inf_{0 \neq \psi \in \{\varphi_1^{(2)}, \dots, \varphi_n^{(2)}\}^\perp \cap \mathcal{D}_0} \frac{\langle T_1 \psi, \psi \rangle}{\langle \psi, \psi \rangle} \leq \mu_n(T_1) = \lambda_n(T_1) \end{aligned}$$

This directly implies the first statement. By a similar argument the statement about the bottom of the essential spectrum follows, in particular,  $\lambda_0^{\text{ess}}(T_2) = \infty$  implies  $\lim_{n \rightarrow \infty} \mu_n(T_1) = \infty$  and, thus,  $\lambda_0^{\text{ess}}(T_1) = \infty$ . In this case  $\lambda_n(T_2) \rightarrow \infty, n \rightarrow \infty$ , which implies the final statement.  $\square$

Finally, we give a lemma which helps us to transform inequalities under form perturbations.

**LEMMA A.3.** — *Let  $(Q_1, \mathcal{D}(Q_1)), (Q_2, \mathcal{D}(Q_2))$  and  $(q, \mathcal{D}(q))$  be closed symmetric non-negative quadratic forms with a common form core  $\mathcal{D}_0$  such that there are  $\alpha \in (0, 1), C_\alpha \geq 0$  such that*

$$q \leq \alpha Q_1 + C$$

*on  $\mathcal{D}_0$ . If for  $a \in (0, 1)$  and  $k \geq 0$*

$$(1 - a)Q_2 - k \leq Q_1 \quad \text{on } \mathcal{D}_0,$$

then

$$\frac{(1-\alpha)(1-a)}{(1-\alpha(1-a))}(Q_2 - q) - \frac{(1-\alpha)k + aC_\alpha}{(1-\alpha(1-a))} \leq Q_1 - q, \quad \text{on } \mathcal{D}_0.$$

In particular, if  $a \rightarrow 0^+$ , then  $(1-\alpha)(1-a)/(1-\alpha(1-a)) \rightarrow 1^-$  and if  $\alpha \rightarrow 0^+$ , then  $(1-\alpha)(1-a)/(1-\alpha(1-a)) \rightarrow (1-a)$ .

*Proof.* — The assumption on  $q$  implies

$$q \leq \frac{\alpha}{(1-\alpha)}(Q_1 - q) + \frac{C_\alpha}{(1-\alpha)}.$$

We subtract  $(1-a)q$  on each side of the lower bound in  $(1-a)Q_2 - k \leq Q_1$ . Then, we get

$$(1-a)(Q_2 - q) - k \leq (Q_1 - q) + aq \leq \frac{1-\alpha(1-a)}{(1-\alpha)}(Q_1 - q) + \frac{aC_\alpha}{(1-\alpha)}$$

and, thus, the asserted inequality follows.  $\square$

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