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ON MINIMAL SINGULAR METRICS OF CERTAIN CLASS OF LINE BUNDLES WHOSE SECTION RING IS NOT FINITELY GENERATED

by Takayuki KOIKE (*)

Abstract. — Wea are interested in the regularity of a minimal singular metric of a line bundle. One main conclusion of our general result in this paper is the existence of smooth Hermitian metrics with semi-positive curvatures on the so-called Zariski's example of a line bundle defined over the blow-up of $\mathbb{P}^2$ at twelve points. This is an example of a line bundle which is nef, big, not semi-ample, and whose section ring is not finitely generated. We generalize this result to the higher dimensional case when the stable base locus of a line bundle is a smooth hypersurface with a holomorphic tubular neighborhood.

Résumé. — On s'intéresse à la régularité d'une métrique singulière minimale d'un fibré en droites. Une des conséquences principales du résultat général de cet article est l'existence des métriques Hermitiennes lisses à courbure semi-positive sur X. Ici, X denote l'exemple de Zariski d'un fibré en droites défini sur l'éclatement du plan projectif en douze points. C'est un exemple de fibré en droites qui est nef, gros et non semi-ample et dont l'anneau des sections n'est pas de type fini. Nous généralisons ce résultat au cas de la dimension supérieure lorsque le lieu de base stable d'un fibré en droites est une hypersurface lisse avec un voisinage tubulaire holomorphe.

1. Introduction

Our interest is a regularity of a minimal singular metric of a line bundle. One main conclusion of our general result in this paper is the existence of smooth Hermitian metrics with semi-positive curvatures on the so-called Zariski's example ([8, 2.3.A]).

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Theorem 1.1 (Example 4.3). — *Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve, $\pi\colon X \to \mathbb{P}^2$ the blowing-up at general twelve points $p_1, p_2, \dots, p_{12} \in C$, H the pulled back divisor of a line in $\mathbb{P}^2$, and let D be the strict transform of C. Then the line bundle $L = \mathcal{O}_X(H + D)$ is semi-positive (i.e. L admits a smooth Hermitian metric with semi-positive curvature).*

This L is nef and big, however has a pathological property that $D \subset \mathrm{Bs}\,|L^{\otimes m}|$ holds for all $m \geqslant 1$, $|L^{\otimes m} \otimes \mathcal{O}_X(-D)|$ is globally generated for all $m \geqslant 1$, and that the section ring $\bigoplus_{m\geqslant 0} H^0(X, L^{\otimes m})$ of L is not finitely generated. When the twelve points $p_1, p_2, \dots, p_{12} \in C$ is special, the line bundle L is semi-ample and thus it is semi-positive. Minimal singular metrics of a line bundle L are metrics of L with the mildest singularities among singular Hermitian metrics of L whose local weights are plurisubharmonic. Minimal singular metrics have been introduced in [5, 1.4] as a (weak) analytic analogue of the Zariski decomposition, and always exist when L is pseudo-effective ([5, 1.5]). The main theorem is as follows.

Theorem 1.2. — *Let X be a smooth projective variety, D a smooth hypersurface of X, L a pseudo-effective line bundle over X, and let $h_{\min}$ be a minimal singular metric of L. Assume that $L \otimes \mathcal{O}_X(-D)$ is semi-positive, $\mathcal{O}_X(-D)|_D$ is ample, $\mathcal{O}_D(-K_D - D|_D)$ is nef and big, and that D has a holomorphic tubular neighborhood (i.e. an open neighborhood in X which is biholomorphic to an open neighborhood of the zero section in the normal bundle $N_{D/X}$). Then $h_{\min}|_D \not\equiv \infty$ holds if and only if $L|_D$ is pseudo-effective, moreover in this case $h_{\min}|_D$ is a minimal singular metric of $L|_D$.*

One of the typical cases of the situations in Theorem 1.2 is when X is a surface and the self-intersection number (D^2) is (sufficiently) negative. It is followed by a special case of Grauert's theorem [7, Satz 7]: A smooth compact complex curve D with genus g embedded in a complex surface X has a holomorphic tubular neighborhood if $(D^2) < \min\{0, 4 - 4g\}$ holds. Thus, we can apply our main theorem to Zariski's example to obtain Theorem 1.1, and we also can show the existence of a smooth Hermitian metric with semi-positive curvature for the same type examples introduced by Mumford ([8, 2.3.A], or Example 4.3 here). When $(L \otimes \mathcal{O}_X(-D))|_D$ is ample, we can write down more concretely a minimal singular metric of L around D by using equilibrium metrics, which are special minimal singular metrics, of $\mathbb{R}$-line bundles $(L \otimes \mathcal{O}_X(-tD))|_D$ for $0 \leqslant t \leqslant 1$ (see Theorem 2.2 and Remark 3.3).

Another application of Theorem 1.2 we can expect is a concrete description of minimal singular metrics of a pseudo-effective line bundle which is not big. It is because, it follows from next Theorem 1.3, which is a version of Theorem 1.2, that we can apply Bergman kernel construction argument even when L is not big but merely pseudo-effective. In more detail, the line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ in Theorem 1.3 is big if we chose A as an ample line bundle. Thus we can use Bergman kernel construction argument for this line bundle and we can study minimal singular metrics of L itself by restricting argument.

THEOREM 1.3. — *Let X be a smooth projective variety, A a semi-ample line bundle on X, and let L be a pseudo-effective line bundle on X. Then the restriction of a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ on $\mathbb{P}(A\oplus L)$ to the divisor $\mathbb{P}(L)\subset\mathbb{P}(A\oplus L)$ corresponding to the projection $A\oplus L\to L$ gives a minimal singular metric of L on X via the natural identification $(\mathbb{P}(L),\mathcal{O}_{\mathbb{P}(L)}(1))\cong(X,L)$ (see Remark 2.3).*

We can prove Theorem 1.3 directly by constructing an appropriate singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ from minimal singular metrics of A and L. In the proof of Theorem 1.2, we use the assumption on the existence of holomorphic tubular neighborhoods to reduce the situation in Theorem 1.2 to that in Theorem 1.3. Since L in Theorem 1.2 admits a singular Hermitian metric which is smooth on $X\setminus D$ and may be singular along D, all we have to do is to modify this metric around D. We will replace this metric on the tubular neighborhood of D by the metric constructed in the situation of Theorem 1.3.

The organization of the paper is as follows. In §2, we treat the case when X has a suitable $\mathbb{P}^1$-bundle structure and L is the relative hyperplane bundle. In §3, we prove Theorem 1.3 and Theorem 1.2. Finally we give some examples in §4.

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2. The $\mathbb{P}^1$-bundle case

In this section, we treat the case when X has a suitable $\mathbb{P}^1$-bundle structure and L is the relative hyperplane bundle. Here we give a minimal singular metric of L concretely by using equilibrium metrics of $\mathbb{R}$-line bundles of the base space of X. First we define the equilibrium metrics for smooth Hermitian metrics on pseudo-effective line bundles.

Definition 2.1. — *Let X be a smooth projective variety, L a pseudo-effective line bundle over X, and let $h = e^{-\varphi}$ be a smooth Hermitian metric on L. We denote by h_e the* equilibrium metric, *whose local weight function φ_e is defined by*

$$\varphi_e = \varphi + \sup{}^*\{\psi\colon X \to \mathbb{R} \cup \{-\infty\} \mid \psi \text{ is a } \varphi\text{-psh function}, \psi \leqslant 0\},$$

where $\sup^*$ *stands for the upper semi-continuous regularization of the supremum.*

Equilibrium metrics are minimal singular metrics ([5, 1.5]). Using this notion, we prove the following theorem.

Theorem 2.2. — *Let X be a smooth projective variety, A an ample line bundle on X, and let L be a pseudo-effective line bundle on X. Let $h_L = e^{-\varphi_L}$ be a smooth Hermitian metric of L and let $h_A = e^{-\varphi_A}$ be a smooth Hermitian metric of A satisfying $dd^c\varphi_A > 0$. Fix a local coordinate system by $(z, x) \mapsto [zs_A^*(x) + s_L^*(x)] \in \mathbb{P}(A \oplus L)$, where s_A^* and s_L^* are local trivializations of A^{-1} and L^{-1}, respectively. Then the metric of the relative hyperplane line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ on $\mathbb{P}(A\oplus L)$ defined by the local weights*

$$\widetilde{\varphi}(z, x) = \log \max_{t\in[0,1]} |z|^{2t} e^{(t\varphi_A+(1-t)\varphi_L)_e(x)}$$

is a minimal singular metric, where $(t\varphi_A + (1-t)\varphi_L)_e$ is the local weight of the equilibrium metric associated to $h_A^t h_L^{1-t}$, which is a smooth Hermitian metric of the "$\mathbb{R}$-line bundle $A^{\otimes t} \otimes L^{\otimes(1-t)}$".

We denote by $\widetilde{X}$ the variety $\mathbb{P}(A \oplus L)$, by $\pi\colon \widetilde{X} \to X$ the canonical projection mapping, and by $\widetilde{L}$ the relative plane line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ on $\widetilde{X}$.

Remark 2.3. — Let us denote by X' the subset $\mathbb{P}(L)$ of $\widetilde{X}$, and X'' the subset $\mathbb{P}(A)$. Then $\mathcal{O}_{\widetilde{X}}(X') = \widetilde{L} \otimes \pi^*A^{-1}$ and $\widetilde{L}|_{X'} = \pi^*L|_{X'}$ hold as equalities of line bundles on $\widetilde{X}$ and X', respectively. Therefore we can regard the restriction of a metric of $\widetilde{L}$ to X' as a metric of L, and by regarding $X' \subset \widetilde{X}$ as $\mathbb{P}(\mathcal{O}_X) \subset \mathbb{P}(\mathcal{O}_X \oplus (L^{-1} \otimes A))$, we can identify $\widetilde{X} \setminus X''$ and X' with the total space of the normal bundle $N_{X'/\widetilde{X}}$, which is isomorphic to the bundle $L \otimes A^{-1}$ via π, and its zero-section.

We also remark here that $\widetilde{L}$ is big if A is ample (see [8, 2.3.2]).

From now on, we prove Theorem 2.2. Here we denote by U the domain of definition of s_A^*, s_L^* and x. We also use the smooth Hermitian metric $\widetilde{h}_\infty = e^{-\widetilde{\varphi}_\infty}$ of $\widetilde{L}$, whose local weight is defined as $\widetilde{\varphi}_\infty(z, x) =$

$\log\left(|z|^2 e^{\varphi_A(x)} + e^{\varphi_L(x)}\right)$. To prove Theorem 2.2, it is sufficient to show the following two propositions.

Proposition 2.4 (Plurisubharmonicity of $\widetilde{\varphi}$). — *The function*

$$\widetilde{\varphi}(z,x) = \log \max_{t\in[0,1]} |z|^{2t} e^{(t\varphi_A+(1-t)\varphi_L)_e(x)}$$

is plurisubharmonic and $\{e^{-\widetilde{\varphi}}\}$ glue up to define a singular Hermitian metric of $\widetilde{L}$.

Proposition 2.5 (Minimal singularity of $\widetilde{\varphi}$). — *There is a constant C such that $(\widetilde{\varphi}_\infty)_e \leqslant \widetilde{\varphi} + C$ holds.*

2.1. Proof of Proposition 2.4

Since $\log|z|^{2t}e^{(t\varphi_A+(1-t)\varphi_L)_e(x)}$ is plurisubharmonic and a local weight of a singular Hermitian metric of $\widetilde{L}$ for each $t \in [0,1]$, it is sufficient to show that $\widetilde{\varphi}$ is upper semi-continuous, and it follows immediately from Lemma 2.7.

Let us denote by ψ_t the function

$$\sup{}^*\{\psi\colon X \to \mathbb{R}\cup\{-\infty\} \mid \psi \text{ is a } (t\varphi_A+(1-t)\varphi_L)\text{-psh function}, \ \psi \leqslant 0\}.$$

For proving Lemma 2.7, we need the following lemma.

Lemma 2.6.

(1) *The sequence $\{\frac{\psi_t}{1-t}\}_{t\in[0,1]}$ is monotonically increasing with respect to t.*

(2) *For all $t \in [0,1)$, $\lim_{s\downarrow t} \frac{\psi_s}{1-s} = \frac{\psi_t}{1-t}$ holds.*

Proof.

(1) Let $t \leqslant s$ be elements of $[0,1]$. Since

$$s\varphi_A + (1-s)\varphi_L + \frac{1-s}{1-t}\psi_t = \frac{1-s}{1-t}(t\varphi_A+(1-t)\varphi_L)_e + \frac{s-t}{1-t}\varphi_A$$

holds, $\frac{1-s}{1-t}\psi_t$ is a $(s\varphi_A+(1-s)\varphi_L)$-psh function. As $\frac{1-s}{1-t}\psi_t \leqslant 0$, $\frac{1-s}{1-t}\psi_t \leqslant \psi_s$ holds.

(2) According to (1), it is sufficient to show that $\lim_{s\downarrow t}\frac{\psi_s}{1-s} \leqslant \frac{\psi_t}{1-t}$ holds. Since the sequence $\frac{s}{1-s}\varphi_A + \varphi_L + \frac{\psi_s}{1-s}$ $(= \frac{1}{1-s}(s\varphi_A+(1-s)\varphi_L)_e)$ is monotonically increasing psh functions, the limit $\frac{t}{1-t}\varphi_A + \varphi_L + \lim_{s\downarrow t}\frac{\psi_s}{1-s}$ is also psh. As $\lim_{s\downarrow t}\frac{\psi_s}{1-s} \leqslant 0$, $(1-t)\lim_{s\downarrow t}\frac{\psi_s}{1-s} \leqslant \psi_t$ holds. $\square$

Lemma 2.7. — *The function* $F\colon \mathbb{C}\times U\times[0,1]\to\mathbb{R}\cup\{-\infty\}$ *defined by* $F(z,x,t)=(t\varphi_A+(1-t)\varphi_L)_e(x)+t\log|z|^2$ *is upper semi-continuous.*

Proof. — Let us set the function $H\colon U\times[0,1]\to\mathbb{R}\cup\{-\infty\}$ as $H(x,t)=\frac{\psi_t(x)}{1-t}$. Since $F(z,x,t)$ is a sum of upper semi-continuous functions and $(1-t)H(x,t)$, it is sufficient to show that H is upper semi-continuous. Let us fix an element $(x_0,t_0)\in U\times[0,1)$ and sufficiently small positive number ε. Then, by Lemma 2.6 (1),

$$\limsup_{(x,t)\mapsto(x_0,t_0)} H(x,t)=\lim_{r\downarrow 0}\sup_{\substack{|x-x_0|<r\\|t-t_0|<r}} H(x,t)\leqslant \lim_{r\downarrow 0}\sup_{|x-x_0|<r} H(x,t_0+\varepsilon)$$

holds. As $H(-,t_0+\varepsilon)=\frac{\psi_{t_0+\varepsilon}}{1-(t_0+\varepsilon)}$ is upper semi-continuous, we obtain an inequality $\limsup_{(x,t)\mapsto(x_0,t_0)} H(x,t)\leqslant H(x_0,t_0+\varepsilon)$. By Lemma 2.6 (2), we can show that the equality $\limsup_{(x,t)\mapsto(x_0,t_0)} H(x,t)\leqslant H(x_0,t_0)$ holds. We can also show this inequality when $t_0=1$ by the same argument, and this shows the lemma. □

2.2. Proof of Proposition 2.5

Next we prove Proposition 2.5. Let us fix a (sufficiently positive) Kähler metric ω of X and define

$$\widetilde{\omega}=\pi^*\omega+dd^c\log(|z|^2e^{\varphi_A}+e^{\varphi_L})-\frac{|z|^2e^{\varphi_A}\pi^*dd^c\varphi_A+e^{\varphi_L}\pi^*dd^c\varphi_L}{|z|^2e^{\varphi_A}+e^{\varphi_L}},$$

where $dd^c=\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}$. This $\widetilde{\omega}$ defines a global smooth $(1,1)$-form on $\widetilde{X}$, since $dd^c\log(|z|^2e^{\varphi_A}+e^{\varphi_L})$ is the curvature form of the smooth Hermitian metric of $\widetilde{L}$ associated to the Finsler metric of $A\oplus L$ induced from h_A and h_L, and both of the coefficients $|z|^2e^{\varphi_A}/(|z|^2e^{\varphi_A}+e^{\varphi_L})$ of $\pi^*dd^c\varphi_A$ and $e^{\varphi_L}/(|z|^2e^{\varphi_A}+e^{\varphi_L})$ of $\pi^*dd^c\varphi_L$ glue up to define $\mathbb{R}$-valued functions defined on whole $\widetilde{X}$. It is because, the values $|z|^2e^{\varphi_A}$, e^{φ_L}, and $|z|^2e^{\varphi_A}+e^{\varphi_L}$ can be regarded as the norms of the points $(z,0)$, $(0,1)$, and $(z,1)$ of a fiber of the vector bundle $A^{-1}\oplus L^{-1}$, respectively, computed by using the metric induced from h_A and h_L. Thus ratios of these values define genuine functions on the whole of $\widetilde{X}$.

Lemma 2.8. — *The form* $\widetilde{\omega}$ *and the measures* $dV_\omega=\frac{\omega^n}{n!}$ *of* X *and* $dV_{\widetilde{\omega}}=\frac{\widetilde{\omega}^{n+1}}{(n+1)!}$ *of* $\widetilde{X}$ *satisfy the following properties when* ω *is sufficiently positive.*

(1) $\widetilde{\omega}$ *is a smooth strictly positive* $(1,1)$*-form on* $\widetilde{X}$*.*
(2) *For all* $x\in X$, $\int_{z\in\pi^{-1}(x)}\widetilde{\omega}|_{\pi^{-1}(x)}=1$ *holds.*

(3) *For all $\mathbb{R}$-valued measurable function F on $\widetilde{X}$, the equation*

$$\int_{(z,x)\in\widetilde{X}} F(z,x)\, dV_{\widetilde{\omega}} = \int_{x\in X}\left(\int_{z\in\pi^{-1}(x)} F(z,x)\, dV_{\widetilde{\omega}}|_{\pi^{-1}(x)}\right) dV_\omega$$

holds.

(4) *Moreover, when F depends only on x and $|z|$, an equation*

$$\int_{(z,x)\in\widetilde{X}} F(z,x)\, dV_{\widetilde{\omega}} = \int_{x\in X}\left(\int_0^\infty \frac{2rG(r,x)e^{\varphi_A(x)+\varphi_L(x)}}{(r^2e^{\varphi_A(x)}+e^{\varphi_L(x)})^2}\, dr\right) dV_\omega$$

holds, where G is the function such that $G(|z|,x) = F(z,x)$ holds.

Proof. — By straightforward computations, we can obtain the formula

$$\widetilde{\omega} = \pi^*\omega + C(\eta\wedge\overline{\eta} + dz\wedge\overline{\eta} + \eta\wedge d\overline{z} + dz\wedge d\overline{z}),$$

where $C = \frac{\sqrt{-1}}{2\pi}\frac{e^{\varphi_A+\varphi_L}}{(|z|^2e^{\varphi_A}+e^{\varphi_L})^2}$ and $\eta = z\partial(\varphi_A - \varphi_L)$. From this formula, it is shown that $\widetilde{\omega}^{n+1} = (n+1)Cdz\wedge d\overline{z}\wedge(\pi^*\omega)^n$ holds, which shows the lemma. □

We also use the following lemma, which can be proved by straightforward computations.

Lemma 2.9.

$$-\log\int_{z\in\pi^{-1}(x)} |z|^{2t}e^{-\widetilde{\varphi}_\infty(z,x)}\,\widetilde{\omega}|_{\pi^{-1}(x)} = t\varphi_A(x) + (1-t)\varphi_L(x) - \log\frac{\Gamma(1+t)\Gamma(2-t)}{4}$$

holds for all $t\in[0,1]$ and $x\in X$, where Γ stands for the Gamma function.

The following lemma can be shown by using the approximation theorem [3, 13.21].

Lemma 2.10. — *Let Y be a smooth projective variety, dV_Y a smooth volume form of Y, M a pseudo-effective line bundle over Y, and let $h_M = e^{-\psi_\infty}$ be a smooth Hermitian metric of M. Fix points $y_0, y_1, \dots y_N \in Y$ and local coordinates systems around each y_j such that $\bigcup_{j=0}^N\{y \mid |y-y_j| < \frac{1}{\sqrt{\pi}}\} = Y$ holds. Let $h_{M,1} = e^{-\psi_1}$ and $h_{M,2} = e^{-\psi_2}$ be singular Hermitian metrics given by*

$$\psi_1 = \psi_\infty + \sup{}^*\left\{\frac{1}{m}\log|f|^2_{h_M^m}\;\middle|\; m\in\mathbb{N}, f\in H^0(Y,mM), \log|f|^2_{h_M^m}\leqslant 0\right\},$$

$$\psi_2 = \psi_\infty + \sup{}^*\left\{\frac{1}{m}\log|f|^2_{h_M^m}\;\middle|\; m\in\mathbb{N}, f\in H^0(Y,mM), \int_Y |f|^2_{h_M^m}dV_Y\leqslant 1\right\}.$$

Let $C' = C'_1 + C'_2$ *with* $C'_1 = \max_j \left(\max_{|y-y_j|\leqslant\frac{2}{\sqrt{\pi}}} \psi_\infty(y) - \min_{|y-y_j|\leqslant\frac{2}{\sqrt{\pi}}} \psi_\infty(y)\right)$ *and* $C'_2 = \log\max_j \max_{|y-y_j|\leqslant\frac{2}{\sqrt{\pi}}} \frac{d\lambda}{dV_Y}$*, where* $d\lambda$ *is the Euclidean measure. Then an inequality* $\psi_2 - C' \leqslant \psi_1 \leqslant (\psi_\infty)_e$ *holds. Moreover, if* M *is big, then an inequality* $\psi_2 - C' \leqslant \psi_1 \leqslant (\psi_\infty)_e \leqslant \psi_2$ *holds.*

Proof of Proposition 2.5.

(1) We fix points $x_0, x_1, \dots x_N \in X$ and local coordinates systems around each x_j such that $\bigcup_{j=0}^N \{x \mid |x - x_j| < \frac{1}{\sqrt{\pi}}\} = X$ holds. Let us denote by $\varphi_{\infty,t}$ the weight of the "smooth Hermitian metric" $t\varphi_A + (1-t)\varphi_L - \log\frac{\Gamma(1+t)\Gamma(2-t)}{4}$ of $tA + (1-t)L$. We let $C = C_1 + C_2 + \log 2$ with

$$\begin{aligned} C_1 &= \max_j \left(\max_{\substack{|x-x_j|\leqslant\frac{2}{\sqrt{\pi}}\\ t\in[0,1]}} \widetilde{\varphi}_{\infty,t}(x) - \min_{\substack{|x-x_j|\leqslant\frac{2}{\sqrt{\pi}}\\ t\in[0,1]}} \widetilde{\varphi}_{\infty,t}(x) \right), \\ C_2 &= \log\max_j \max_{|x-x_j|\leqslant\frac{2}{\sqrt{\pi}}} \frac{d\lambda}{dV_\omega}. \end{aligned}$$

Since $(\varphi_{\infty,t})_e = (t\varphi_A + (1-t)\varphi_L)_e - \log\frac{\Gamma(1+t)\Gamma(2-t)}{4}$ holds, it is sufficient to show that

$$(\widetilde{\varphi}_\infty)_e \leqslant \log\max_{t\in[0,1]} |z|^{2t} e^{(\varphi_{\infty,t})_e(x)} + C$$

holds. According to the last part of Lemma 2.10, this is reduced to show that for each $F \in H^0(\widetilde{X}, m\widetilde{L})$ such that $\int_{\widetilde{X}} |F|^2 e^{-m\widetilde{\varphi}_\infty}\, dV_{\widetilde{\omega}} \leqslant 1$, an inequality

$$\frac{1}{m}\log|F|^2 \leqslant \log\max_{t\in[0,1]} |z|^{2t} e^{(\varphi_{\infty,t})_e(x)} + C$$

holds.

(2) We show the last inequality. The holomorphic section $F(z,x)$ can be expanded as $F(z,x) = \sum_{\ell=0}^m z^\ell f_\ell(x)$ with $f_\ell \in H^0(X, \ell A + (m-\ell)L)$. We first show that an inequality

$$(*) \qquad \int_{\widetilde{X}} |z^\ell f_\ell|^2 e^{-m\widetilde{\varphi}_\infty}\, dV_{\widetilde{\omega}} \leqslant 1$$

holds for $\ell = 1, 2, \dots, m$. For proving $(*)$, we use an inequality

$$|f_\ell(x)|^2 = \left|\frac{1}{\ell!}\frac{\partial^\ell}{\partial z^\ell} F(0,x)\right|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{|F(re^{\sqrt{-1}\theta}, x)|^2}{r^{2\ell}}\, d\theta$$

for each positive number r. We denote by $\widetilde{\varphi}_\infty(r,x)$ the function such that $\widetilde{\varphi}_\infty(|z|,x)=\widetilde{\varphi}_\infty(z,x)$ holds. By multiplying the metric terms and integrating these with r, we obtain the following inequality.

$$\begin{aligned}
&\int_0^\infty \frac{2r(r^{2\ell}|f_\ell(x)|^2 e^{-m\widetilde{\varphi}_\infty(r,x)})e^{\varphi_A(x)+\varphi_L(x)}}{(r^2e^{\varphi_A(x)}+e^{\varphi_L(x)})^2}\,dr\\
&\leqslant\frac{1}{2\pi}\int_0^\infty\left(\int_0^{2\pi}\frac{2r(|F(re^{\sqrt{-1}\theta},x)|^2e^{-m\widetilde{\varphi}_\infty(r,x)})e^{\varphi_A(x)+\varphi_L(x)}}{(r^2e^{\varphi_A(x)}+e^{\varphi_L(x)})^2}\,d\theta\right)dr\\
&=\int_{z\in\pi^{-1}(x)}|F(z,x)|^2e^{-m\widetilde{\varphi}(z,x)}\,dV_{\widetilde{\omega}}|_{\pi^{-1}(x)}.
\end{aligned}$$

This inequality and Lemma 2.8 (2), (3), (4) implies the inequality $(*)$.

Then, by Lemma 2.8 (3),

$$\begin{aligned}
1&\geqslant\int_{(z,x)\in\widetilde{X}}|z^\ell f_\ell(x)|^2e^{-m\widetilde{\varphi}_\infty(z,x)}\,dV_{\widetilde{\omega}}\\
&=\int_{x\in X}|f_\ell(x)|^2\left(\int_{z\in\pi^{-1}(x)}|z^\ell|^2e^{-m\widetilde{\varphi}_\infty(z,x)}\,dV_{\widetilde{\omega}}|_{\pi^{-1}(x)}\right)dV_\omega\\
&=\int_{x\in X}|f_\ell(x)|^2\left(\left(\int_{z\in\pi^{-1}(x)}\left(|z|^{2\frac{\ell}{m}}e^{-\widetilde{\varphi}_\infty(z,x)}\right)^m dV_{\widetilde{\omega}}|_{\pi^{-1}(x)}\right)^{\frac{1}{m}}\right.\\
&\qquad\qquad\left.\cdot\left(\int_{z\in\pi^{-1}(x)}1^{\frac{m}{m-1}}\,dV_{\widetilde{\omega}}|_{\pi^{-1}(x)}\right)^{\frac{m-1}{m}}\right)^m dV_\omega\\
&\geqslant\int_{x\in X}|f_\ell(x)|^2\left(\int_{z\in\pi^{-1}(x)}|z|^{2\frac{\ell}{m}}e^{-\widetilde{\varphi}_\infty(z,x)}\right)^m dV_\omega
\end{aligned}$$

holds (Here we used Hölder's inequality). Therefore, by Lemma 2.9, $\int_{x\in X}|f_\ell(x)|^2e^{-m\varphi_{\infty,t}(x)}\,dV_\omega\leqslant 1$ holds for $t=\frac{\ell}{m}$. Then by Lemma 2.10, we obtain an inequality $\frac{1}{m}\log|f_\ell|^2\leqslant(\varphi_{\infty,t})_e+C_1+C_2$.

Thus

$$\begin{aligned}\frac{1}{m}\log|F(z,x)|^2 &\leqslant \frac{1}{m}\log\sum_{l=0}^{m}|z^\ell f_\ell(x)|^2\\ &\leqslant \frac{1}{m}\log\left((m+1)\max_{\ell}|z^\ell f_\ell(x)|^2\right)\\ &= \frac{1}{m}\log(m+1)+\log\max_{0\leqslant\ell\leqslant m}|z|^{2\frac{\ell}{m}}|f_\ell(x)|^{\frac{2}{m}}\\ &\leqslant \log 2+\log\max_{t\in[0,1]}|z|^{2t}e^{(\varphi_{\infty,t})_e(x)+C_1+C_2}\\ &= \log\max_{t\in[0,1]}|z|^{2t}e^{(\varphi_{\infty,t})_e(x)}+C\end{aligned}$$

holds. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 and Theorem 1.3. We first prove Theorem 1.3.

Proof of Theorem 1.3. — As Theorem 1.3 immediately follows from Theorem 2.2 when A is ample, we show the theorem when A is merely semi-ample (Although we can not obtain a concrete description of a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ as in Theorem 2.2 in this case, we can show the theorem). Let $\widetilde{h}_{\min}$ be a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$, h_A be a smooth Hermitian metric of A with semi-positive curvature, and h_L be a minimal singular metric of L. Let us consider the singular Hermitian metric $\widetilde{h}$ with local weight function $\log(|z|^2e^{\varphi_A(x)}+e^{\varphi_L(x)})$ where (z,x) is the coordinates just as in Theorem 2.2 and φ_A,φ_L is the local weight of h_A,h_L, respectively. Since $\widetilde{h}_{\min}$ is less singular than $\widetilde{h}$ and $\widetilde{h}|_{\mathbb{P}(L)}=h_L$ holds, there exists a positive constant C such that

$$\widetilde{h}_{\min}|_{\mathbb{P}(L)}\leqslant C\cdot\widetilde{h}|_{\mathbb{P}(L)}=C\cdot h_L$$

holds, which shows that the metric $\widetilde{h}_{\min}|_{\mathbb{P}(L)}$ has minimal singularity. □

Proof of Theorem 1.2. — Let X be a smooth projective variety, D a 1-codimensional smooth subvariety of X, and let L be a pseudo-effective line bundle over X. We assume that $A=L\otimes\mathcal{O}_X(-D)$ is semi-positive and that there is an open neighborhood U of $D\subset X$ biholomorphic to an open neighborhood U' of the zero section of the normal bundle $N_{D/X}$. Here we may assume that $U'=\{\xi\in N_{D/X}\mid|\xi|_{h_{X/D}}<\varepsilon_0\}$ for some smooth

Hermitian metric $h_{X/D}$ with negative curvature of $N_{D/X}$ and a positive number ε_0.

Since $L|_D$ has no singular Hermitian metric of with psh local weights (which is not identically equal to $-\infty$) when $L|_D$ is not pseudo-effective, all we have to do is showing the existence of a singular Hermitian metric of L with psh local weights which is an extension of a minimal singular metric of $L|_D$ assuming $L|_D$ is pseudo-effective but not big. We set X' as the total space $\pi\colon \mathbb{P}(L|_D \oplus A|_D) \to D$ and L' as the relative hyperplane bundle $\mathcal{O}_{\mathbb{P}(L|_D \oplus A|_D)}(1)$. Let us fix a minimal singular metric $h_{L'} = e^{-\varphi_{L'}}$ of L'. We set V' as the subset $\{\xi \in N_{D/X} \mid |\xi|_{h_{X/D}} < \frac{\varepsilon_0}{2}\}$. By Remark 2.3, we can regard U' and V' be neighborhoods of $D' = \mathbb{P}(L|_D) \subset X'$. From the assumption, the natural biholomorphic mapping $\pi|_{D'}\colon D' \to D$ extends to a biholomorphic mapping $f\colon U' \to U$. We denote by V the set $f(V') \subset U$.

By Proposition 3.1 (2) below, there exists a line bundle F on U' which admits a flat structure and $f^*(L|_U) \cong L'|_{U'} \otimes F$ holds. We fix a flat metric $h_F = e^{-\varphi_F}$ of F. By choosing appropriate local trivialization, we may assume $\varphi_F \equiv 0$. Thus we can regard $(f^{-1})^*\varphi_{L'}$ as the local weight function of the singular Hermitian metric $(f^{-1})^*h_{L'}h_F$ of $L|_U$. To show the theorem, according to Theorem 1.3, it is sufficient to construct a singular Hermitian metric $e^{-\varphi_L}$ of L with $dd^c\varphi_L \geqslant 0$ and $\varphi_L|_V = (f^{-1})^*\varphi_{L'}|_{V'}$ holds. Let $h_A = e^{-\varphi_A}$ be a smooth Hermitian metric of A with $dd^c\varphi_A \geqslant 0$ and let $f_D \in H^0(X, \mathcal{O}_X(D))$ be a section which vanishes only on D. Without loss of generality, we may assume $\varphi_A \geqslant 0, (f^{-1})^*\varphi_{L'} \leqslant -1$ holds on each fixed open set $W_j (j = 1, 2, \ldots, N)$ covering the whole U, and $\log|f_D|^2 \geqslant -1$ holds on each intersection $W_j \cap (\overline{U} \setminus V)$. We define φ_L as the function $\max\{\varphi_A + \log|f_D|^2, (f^{-1})^*\varphi_{L'}\}$ on each $W_j \cap U$. Since $\varphi_L = \varphi_A + \log|f_D|^2$ holds on each intersection $W_j \cap (\overline{U} \setminus V)$, $e^{-\varphi_L}$ on U and $e^{-(\varphi_A + \log|f_D|^2)}$ on $X \setminus V$ glue up to define a new singular Hermitian metric of L, which proves the theorem. □

Proposition 3.1. — *(1) (a version of Rossi's theorem) The natural map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/I_{D'}^n)$ is injective for some $n \geqslant 1$, where $I_{D'}$ the defining ideal sheaf of $D \subset U$.*

(2) There is a line bundle E on D' such that $c_1(E) = 0$ and $f^*(L|_U) \cong (L' \otimes \pi^*E)|_{U'}$ hold.

(3) The groups $\operatorname{Pic}(U)$ *and* $\operatorname{Pic}(D)$ *are isomorphic.*

Proof of Proposition 3.1.

(1) We intrinsically use Rossi's theorem [10, Theorem 3]. Here we remark that, from the assumption that $\mathcal{O}_X(-D)|_D$ is ample, U' is a strongly

pseudoconvex domain. Thus, from Rossi's theorem, it turns out that there exists an ideal sheaf $J \subset \mathcal{O}_{U'}$ satisfying the condition that

(i) $V(J) \subset D' \cup \{p_1, p_2, \cdots .p_l\}$ for some finitely many points $p_1, p_2, \cdots, p_l \in U' \setminus D'$, where $V(J) \subset U'$ stands for the zero set of the ideal sheaf J, and that

(ii) the natural map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/J)$ is injective.

Here we remark that $H^1(U', \mathcal{O}_{U'}/J) = H^1(D', \mathcal{O}_{U'}/J)$ holds. It is because the condition (i) and the fact that the first sheaf cohomology vanishes on the zero-dimensional sets $p_1, p_2, \cdots, p_l$.

Let us denote by $I_{D'}$ the defining ideal sheaf of D', by I_{p_j} the defining ideal sheaf of p_j for $1 \leqslant j \leqslant l$, and by $\hat{J}$ the ideal sheaf $I_{p_1} I_{p_2} \cdots I_{p_l} I_{D'}$. By Hilbert's Nullstellensatz, there exists an integer n such that $\hat{J}^n \subset J$ holds. Thus the natural map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/J)$ is decomposed into the composition of two natural maps $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/\hat{J}^n)$ and $H^1(U', \mathcal{O}_{U'}/\hat{J}^n) \to H^1(U', \mathcal{O}_{U'}/J)$. From the condition (ii), it turns out that the map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/\hat{J}^n)$ is also injective, and since $H^1(U', \mathcal{O}_{U'}/\hat{J}^n) = H^1(D', \mathcal{O}_{U'}/\hat{J}^n) = H^1(D', \mathcal{O}_{U'}/I_{D'}^n)$ holds, this proves the first assertion.

(2) The projection $\pi\colon U' \to D$ and the injection $i\colon D' \to U'$ induce the maps $\pi^*\colon H^1(D', \mathcal{O}_{D'}) \to H^1(U', \mathcal{O}_{U'})$ and $i^*\colon H^1(U', \mathcal{O}_{U'}) \to H^1(D', \mathcal{O}_{D'})$, respectively. Since $\pi \circ i = \mathrm{id}_{D'}$, π^* is injective.

$$
\begin{tikzcd}
H^1(U', \mathcal{O}_{U'}) \arrow[r, "\alpha"] & H^1(U', \mathcal{O}^*_{U'}) \arrow[r, "\delta"] & H^2(U', \mathbb{Z}) \\
H^1(D', \mathcal{O}_{D'}) \arrow[r, "\beta"] \arrow[u, "\pi^*"] \arrow[ur, phantom, "\circlearrowleft"] & H^1(D', \mathcal{O}^*_{D'}) \arrow[u, "\pi^*"'] &
\end{tikzcd}
$$

We first check that $f^*(L|_U) \otimes L'|_{U'}^{-1}$ is topologically trivial line bundle. Indeed, $(f^*(L|_U) \otimes L'|_{U'}^{-1})|_{D'}$ is the trivial bundle and $i \circ \pi$ is homotopic to $\mathrm{id}_{U'}$. Thus we conclude that $\delta(f^*(L|_U) \otimes L'|_{U'}^{-1}) = 0$ and we can take an element $\xi \in H^1(U', \mathcal{O}_{U'})$ satisfying $\alpha(\xi) = f^*(L|_U) \otimes L'|_{U'}^{-1}$. When ξ lies in the image of π^*, we can take an element $\eta \in H^1(D', \mathcal{O}_{D'})$ such that $\pi^*(\eta) = \xi$ holds. In this case, $f^*(L|_U) \otimes L'|_{U'}^{-1} = \pi^*\beta(\eta)$ holds and since $\beta(\eta)$ is a flat line bundle, $f^*(L|_U) \otimes L'|_{U'}^{-1}$ is also a flat line bundle.

Thus all we have to do is showing that the inequality $\dim H^1(U', \mathcal{O}_{U'}) \leqslant \dim H^1(D', \mathcal{O}_{D'})$ holds. Let us consider the short exact sequence $0 \to I_{D'}^l / I_{D'}^{l+1} \to \mathcal{O}_{U'}/I_{D'}^{l+1} \to \mathcal{O}_{U'}/I_{D'}^l \to 0$ for $l \geqslant 1$. Then it follows that the natural map $H^1(U', \mathcal{O}_{U'}/I_{D'}^{l+1}) \to H^1(U', \mathcal{O}_{U'}/I_{D'}^l)$ is injective. It is because $H^1(U', I_{D'}^l/I_{D'}^{l+1}) = H^1(U', I_{D'}^l \otimes (\mathcal{O}_{U'}/I_{D'})) =$

$H^1(D', \mathcal{O}_{D'}(-lD'|_{D'}))$ vanishes for each $l \geqslant 1$, since $\mathcal{O}_{D'}(-K_{D'} - lD'|_{D'}) = \mathcal{O}_{D'}(-K_{D'} - D'|_{D'}) \otimes \mathcal{O}_{D'}(-(l-1)D'|_{D'})$ is nef and big from the assumption. From this combined with the injection in Lemma 3.1 (1), it holds that the natural map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/I_{D'}) = H^1(D', \mathcal{O}_{D'})$ is injective, and thus we obtain the inequality $\dim H^1(U', \mathcal{O}_{U'}) \leqslant \dim H^1(D', \mathcal{O}_{D'})$.

(3) By the same argument in the proof of Lemma 3.1 (2), it can be shown that the restriction map $\mathrm{Pic}\,(U') \to \mathrm{Pic}\,(D')$ is the inverse map of $\pi^* \colon \mathrm{Pic}\,(D') \to \mathrm{Pic}\,(U')$. □

Remark 3.2. — When $\mathcal{O}(K_D)$ is semi-negative, we can prove $H^1(U', \mathcal{O}_{U'}) \cong H^1(D', \mathcal{O}_{D'})$ more shortly. Let us consider the short exact sequence $0 \to I_{D'} \to \mathcal{O}_{U'} \to \mathcal{O}_{U'}/I_{D'} \to 0$ and the induced exact sequence $H^1(U', I_{D'}) \to H^1(U', \mathcal{O}_{U'}) \to H^1(D', \mathcal{O}_{D'}) \to H^2(U', I_{D'})$. By the assumption that $\mathcal{O}_D(-K_D) = \mathcal{O}_U(-K_U - D)|_D$ is semi-positive and by Ohsawa's theorem [9, 4.5], it follows that the cohomology group $H^p(U', I_{D'})$ vanishes for all $p > 0$. Thus $H^1(U', \mathcal{O}_{U'}) \cong H^1(D', \mathcal{O}_{D'})$ holds.

Remark 3.3. — In the above proof of Theorem 1.2, we compared the singular Hermitian metric of L with that of L' around the tubular neighborhoods of the divisors. By using this technique, it turns out to be clear that the metric $e^{-\varphi_L}$ we constructed above is a minimal singular metric. Moreover, $\varphi_{L'}$ in the above proof of Theorem 1.2 can be taken as in Theorem 2.2 when $A|_D$ is ample, and thus we can conclude that the minimal singular metric we constructed has just the same form as the metric in Theorem 2.2 around D (up to smooth harmonic function). This means that we here determined a minimal singular metric of L around D by only using equilibrium metrics of $tA|_D + (1-t)L|_D$ for $0 \leqslant t \leqslant 1$ in the above proof in this case.

When L in Theorem 1.2 satisfies that $L|_D$ is semi-positive, we can say that L is also semi-positive.

Corollary 3.4. — *Let X, D, L be those in Theorem 1.2. When $L|_D$ is semi-positive, L is also semi-positive.*

Proof. — We use notations in the proof of 1.2. By the proof of Theorem 1.3, it is clear that we can choose smooth $h_{L'}$ when $L|_D$ is semi-positive. We define φ_L as the function $M(\varphi_A + \log|f_D|^2, (f^{-1})^*\varphi_{L'})$ (instead of $\max\{\varphi_A + \log|f_D|^2, (f^{-1})^*\varphi_{L'}\}$) on each $W_j \cap U$, where M is a regularized max function (see [2, §5.E] for the definition). Then $\{e^{-\varphi_L}\}$ glues up to define a smooth Hermitian metric of L with semi-positive curvature. □

We here remark that the idea to use a regularized max function instead of the function “max" is pointed out by Prof. Shin-ichi Matsumura.

4. Some examples

4.1. Nef and big line bundles with no locally bounded minimal singular metrics

One can obtain the following corollary immediately from Theorem 1.3.

Corollary 4.1. — *Let X be a smooth projective variety, L a nef line bundle over X and let A be an ample line bundle over X. Then a minimal singular metric of L is locally bounded if and only if a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ over $\mathbb{P}(A \oplus L)$ is locally bounded.*

We remark that the line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ above is nef and big ([8, 2.3.2]).

Example 4.2. — Let (X, L) be these in Example 1.7 of [4], which are defined as the relative hyperplane bundle on $X = \mathbb{P}(E)$, where E is a vector bundle defined over an elliptic curve C given by the non-spitting extension $0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$. In this example, L is nef, not big, and possesses no locally-bounded minimal singular metric. Then we can conclude that the nef and big line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ defined on $\mathbb{P}(L\oplus A)$ for some ample line bundle A on X also has no locally-bounded minimal singular metric. We remark that the similar example is introduced in [1, 5.4], [6, 5.2].

4.2. Zariski's and Mumford's examples

We can apply Theorem 1.2 to Zariski's and Mumford's examples [8, 2.3.A].

Example 4.3. — Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve and let $p_1, p_2, \dots, p_{12} \in C$ be twelve general points. We define X as the blow up of $\mathbb{P}^2$ at these twelve points. We denote by H the pulled back divisor of X of a line in $\mathbb{P}^2$ and by D the strict transform of C. In this case, since $(D^2) = 9 - 12 = -3$ and the genus $g(D) = 1$, we can apply Grauert's theorem [7, Satz 7] (see §1 here) to see that $X, L = \mathcal{O}_X(H + D)$, and D satisfy the condition of Theorem 1.2. Moreover, for $L|_D$ is semi-positive, we can apply Corollary 3.4. Thus L is semi-positive.

There is a generalization of this Zariski's example pointed out by Mumford (see also [8, 2.3.1]). Let X be a smooth projective surface, A a very ample divisor on X, and let $D \subset X$ be a curve with $(D^2) < 0$ holds and the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is injective. We denote by a, b the positive number $(A.D), -(D^2)$, respectively. Then the line bundle $L = \mathcal{O}_X(bA+aD)$ is nef, big, satisfying $D \subset \mathrm{Bs}\,|L^{\otimes m}|$ for all $m \geqslant 1$, and there exists a positive integer p_0 such that $|L^{\otimes m} \otimes \mathcal{O}_X(-p_0 D)|$ is generated by global sections for all $m \geqslant 1$. These X, L, and D satisfy the condition of Corollary 3.4 also in this situation when D is smooth, b is sufficiently large, and $p_0 = 1$. Thus, such L is semi-positive, too.

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