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UNIQUE CONTINUATION FOR QUASIMODES ON SURFACES OF REVOLUTION: ROTATIONALLY INVARIANT NEIGHBOURHOODS

by Hans CHRISTIANSON (*)

ABSTRACT. — We introduce the definition of irreducible quasimodes, which are quasimodes with h-wavefront sets living on the smallest invariant sets in phase space. We prove a strong conditional unique continuation estimate for these quasimodes in rotationally invariant neighbourhoods on compact surfaces of revolution. The estimate states that irreducible Laplace quasimodes have L^2 mass bounded below by $C_\epsilon \lambda^{-1-\epsilon}$ for any $\epsilon > 0$ on any open rotationally invariant neighbourhood which meets the semiclassical wavefront set of the quasimode. For an analytic manifold, we conclude the same estimate with a lower bound of $C_\delta \lambda^{-1+\delta}$ for some fixed $\delta > 0$.

RÉSUMÉ. — Nous introduisons la définition de quasimodes irréductibles, qui sont des quasimodes du laplacien dont les h-front d'onde est localisé sur les ensembles invariants minimaux de l'espace des phases. Nous prouvons une estimation de prolongement unique conditionnelle pour ces quasimodes sur les ensembles invariants par rotation des surfaces compactes de révolution. L'estimée affirme que les quasimodes ont une norme L^2 minorée par $C_\epsilon \lambda^{-1-\epsilon}$ pour tout $\epsilon > 0$ et sur tout ensemble ouvert invariant par rotation qui intersecte le front d'onde semi-classique du quasimode. Si la surface est analytique, nous obtenons la même estimation minorée par $C_\delta \lambda^{-1+\delta}$ pour $\delta > 0$ fixe.

1. Introduction

We consider a compact periodic surface of revolution $X = \mathbb{S}^1_x \times \mathbb{S}^1_\theta$, equipped with a metric of the form

$$ds^2 = dx^2 + A^2(x) d\theta^2,$$

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where $A \in \mathcal{C}^\infty$ is a smooth function, $A \geqslant \epsilon > 0$. Our analysis is microlocal, so applies also to any compact surface of revolution with no boundary, and to certain surfaces of revolution with boundary under mild assumptions, however we will concentrate on the toral case for ease of exposition.

From such a metric, we get the volume form

$$d\text{Vol} = A(x)dxd\theta,$$

and the Laplace-Beltrami operator acting on 0-forms

$$\Delta f = (\partial_x^2 + A^{-2}\partial_\theta^2 + A^{-1}A'\partial_x)f.$$

We are concerned with quasimodes, which are the building blocks from which eigenfunctions are made, however we need to define the most basic kind of quasimodes, which we will call *irreducible quasimodes*. The very rough idea in this definition is that the quasimode cannot be decomposed as a sum of two or more nontrivial quasimodes. However, we will use a slightly weaker definition than this (see Section 2.3 for more on this distinction).

Quasimodes are known to live in phase space on sets which are invariant under the geodesic flow. For a two dimensional surface of revolution, the classical flow is completely integrable. Quasimodes can live on invariant tori, but they can also live on degenerate sets of higher codimension (for example, a periodic geodesic). This paper is primarily concerned with understanding how quasimodes are concentrated as invariant sets degenerate from tori into lower dimensional sets. Our definition must distinguish between these different types of sets. To make our definition, we recall first that the geodesic flow on T^*X is the Hamiltonian system associated to the principal symbol of the Laplace-Beltrami operator:

$$p(x, \xi, \theta, \eta) = \xi^2 + A^{-2}(x)\eta^2.$$

A fixed energy level $p =$ const. consists of all the geodesics of that constant "speed". For the case of the geodesic Hamiltonian system on T^*X, there are two conserved quantities, the total energy and the angular momentum η^2. The *moment map* is the map sending points of T^*X to their associated conserved quantities, that is

$$M(x, \xi, \theta, \eta) = \begin{pmatrix} \xi^2 + A^{-2}(x)\eta^2 \\ \eta^2 \end{pmatrix}.$$

When the gradient of M has rank 2, then M defines a submersion, so each connected component of the preimage is a 2-manifold; a Liouville torus. Points in T^*X where M has rank 1 or 0 are called critical points, and points $(P, Q) \in \mathbb{R}^2$ such that $\{M = (P, Q)\}$ contains critical points are called critical values.

If M has rank 0, then the total energy $p = 0$. At points where M has rank 1 there are two possibilities. The first is $\eta = 0$ and $\xi \neq 0$, where there is uniform lateral propagation. The second possibility is $\xi = 0$, $A'(x) = 0$ but $\eta \neq 0$. These critical points have no lateral propagation but do have angular momentum, so correspond to *longitudinal* periodic geodesics. These longitudinal periodic geodesics can also carry quasimode mass, and critical *values* have preimages which may have infinitely many longitudinal periodic geodesics, if $A' \equiv 0$ in some neighbourhood. One way to measure the localization of quasimodes in phase space is by describing their semiclassical wavefront set. Roughly speaking, the semiclassical wavefront set is where a function is non-trivial in phase space (i.e. roughly the complement of the set where the quasimode is $\mathcal{O}(\lambda^{-\infty})$). See Section 2.1 below for a brief review, or [12] for a comprehensive overview of semiclassical analysis. The semiclassical wavefront set is always a closed invariant subset of the energy surface, so our definition of irreducible quasimode will be one which has wavefront mass confined to the closure of one of these two kinds of sets, distinguished by rank of M.

Definition 1.1. — *A* graded *quasimode is a quasimode whose semiclassical wavefront set is contained in a set where M has constant rank.*

An irreducible *quasimode is a graded quasimode whose semiclassical wavefront set is contained in the closure of a single connected component of a level set of M in T^*X.*

In Section 2.3 we discuss how this definition is related to a more intuitive, but more restrictive definition.

We also will require a limit on the geodesic complexity by assuming there are only a finite number of connected regions of longitudinal periodic geodesics. This will not preclude having infinitely many periodic longitudinal geodesics, but merely having accumulation points of connected components of longitudinal geodesics. We therefore will assume that the moment map has a finite number of critical values, each of which has a preimage of finitely many non-empty connected components. Note this allows intervals of longitudinal periodic geodesics, but does not allow accumulation of such sets. For an example, see Figure 1.1.

Finally, we will require a certain 0-Gevrey regularity on the manifold, which in a sense says our manifold is not too far from being analytic. Such a 0-Gevrey assumption nevertheless allows for non-trivial functions which are constant on intervals, so this is a very general class of manifolds. Of course this includes analytic manifolds, for which we have a stronger estimate. See Subsection 2.2 for the precise definitions.

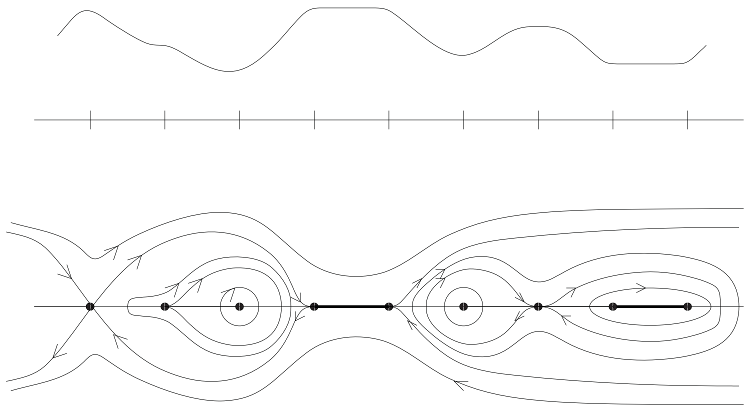

Figure 1.1. A surface of revolution with many longitudinal periodic geodesics, corresponding to critical points of the generating curve $A(x)$. Below is the reduced phase space (at a fixed angular momentum). The longitudinal geodesics are where $A'(x) = \xi = 0$, or where the Hamiltonian flow in the reduced phase space is stationary.

Theorem 1.2. — *Let X be as above, for a generating curve in the 0-Gevrey class $A(x) \in \mathcal{G}^0_\tau(\mathbb{R})$ for some $\tau < \infty$. Assume the moment map has finitely many critical values, with preimages consisting of finitely many connected components. Suppose u is a (weak) irreducible quasimode satisfying $\|u\| = 1$ and*

$$(-\Delta - \lambda^2)u = \mathcal{O}(\lambda^{-\beta_0}),$$

for some fixed $\beta_0 > 0$. Let $\Omega \subset X$ be a rotationally invariant neighbournood, $\Omega = (a,b)_x \times \mathbb{S}^1_\theta$. Then either

(1)
$$\|u\|_{L^2(\Omega)} = \mathcal{O}(\lambda^{-\infty}),$$
or

(2) *for any $\epsilon > 0$, there exists $C = C_{\epsilon,\Omega,\beta_0} > 0$ such that*

$$\|u\|_{L^2(\Omega)} \geqslant C\lambda^{-1-\epsilon}. \tag{1.1}$$

In the analytic category, we have a significant improvement. Of course in the case of an analytic manifold, there can be no infinitely degenerate critical elements, nor can there be any accumulation points of sets of longitudinal periodic geodesics, so we do not need to make the assumption about finite geodesic complexity.

COROLLARY 1.3. — *Let X be as above, and assume X is analytic. Suppose u is a (weak) irreducible quasimode satisfying $\|u\| = 1$ and*

$$(-\Delta - \lambda^2)u = \mathcal{O}(1).$$

Then for any open rotationally invariant neighbourhood $\Omega \subset X$, either

(1)

$$\|u\|_{L^2(\Omega)} = \mathcal{O}(\lambda^{-\infty}),$$

or

(2) *there exists a fixed $\delta > 0$ and a constant $C = C_\Omega > 0$ such that*

$$\|u\|_{L^2(\Omega)} \geqslant C\lambda^{-1+\delta}.$$

Remark 1.4. — The $\delta > 0$ appearing in Corollary 1.3 can be computed explicitly. The proof of both Theorem 1.2 and Corollary 1.3 proceed by separating variables to reduce to a one dimensional semiclassical Schrödinger operator, and then describing the behaviour of quasimodes near cricital points of the potential. For an analytic manifold, there are necessarily finitely many critical points of finite degeneracy. If $2m$ is the maximum degeneracy of the local maxima, and $2m_2 + 1$ is the maximum degeneracy of the inflection points, then

$$\delta = \min\left\{\frac{2}{m+1}, \frac{4}{2m_2+3}\right\}.$$

In the case of one non-degenerate maximum ($m = 1$) and no inflection points, the lower bound is $1/\log(\lambda)$, which was already known in much greater generality [3, 5, 6].

The spectral estimates for finitely degenerate critical points are all sharp, so the lower bounds for a particular analytic manifold are also sharp.

Remark 1.5. — The assumption that $\Omega \subset X$ is a rotationally invariant neighbourhood of the form $\Omega = (a, b)_x \times \mathbb{S}^1_\theta$ is necessary for this level of generality. See Section 2.3 for an example illustrating this point.

2. Preliminaries

In this section we review some of the definitions and preliminary computations necessary for Theorem 1.2, as well as recall the spectral estimates we will be using.

2.1. A brief review of semiclassical analysis

We collect for future reference a brief review of the standard symbol classes and corresponding pseudodifferential operators used later on (see also [12, Section 4.4]). Throughout this subsection, let X be a compact Riemannian manifold without boundary. We denote $x \in X$ and $(x,\xi) \in T^*X$ as variables on the manifold and the cotangent bundle respectively.

The standard homogeneous symbol spaces that are relevant here are

$$S^m_{\rho,\delta}(T^*X) = \{a(x,\xi) \in C^\infty(T^*X \setminus 0); |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| = \mathcal{O}_{\alpha,\beta}(\langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}), \} \tag{2.1}$$

with $1 \geqslant \rho > \delta$. As for the semiclassical symbols, the relevant symbol classes for our purposes are

$$S^{m,k}_{cl}(T^*X \times (0,h_0]) = \{a(x,\xi;h) \in C^\infty(T^*X \times (0,h_0]); \quad a(x,\xi;h) \sim \sum_{j=0}^\infty a_{m-j}(x,\xi) h^{-m+j},\ a_{m-j} \in S^k_{0,0}\}, \tag{2.2}$$

$$S^m_\delta(T^*X \times (0,h_0]) = \{a(x,\xi;h) \in C^\infty(T^*X \times (0,h_0]); \quad |\partial_x^\alpha \partial_\xi^\beta a| = \mathcal{O}_{\alpha,\beta}(h^{-m} h^{-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{-\infty})\}, \tag{2.3}$$

with $\delta \in [0,1/2]$. Consequently, the semiclassical symbol classes S^m_δ are most relevant. In the special case where $\delta = 0$,

$$S^m_0(T^*X \times (0,h_0]) = \{a \in C^\infty(T^*X \times (0,h_0]); |\partial_x^\alpha \partial_\xi^\beta a| = \mathcal{O}_{\alpha,\beta}(h^{-m} \langle \xi \rangle^{-\infty})\}.$$

When the context is clear, we sometimes just write S^m_δ instead of $S^m_\delta(T^*X \times (0,h_0])$. The case where $\delta = 0$ is sometimes denoted by $S^m(1)$ in the literature.

The corresponding h-Weyl pseudodifferential operators have Schwartz kernels that are sums of the local integrals of the form

$$Op_h^w(a)(x,y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle/h} a\left(\frac{x+y}{2},\xi;h\right) d\xi. \tag{2.4}$$

We will use $Op_h^w(a)$, a_h^w and $a^w(x,hD_x)$ interchangeably to denote h-Weyl quantizations of $a(x,\xi;h)$ since each has its advantages. We write Ψ^m_δ for the algebra of semiclassical operators so quantized. It is standard that for $a \in S^{m_1,k_1}_{cl}, b \in S^{m_2,k_2}_{cl}$,

$$a^w(x,hD_x) \circ b^w(x,hD_x) = e^{ih\sigma(D_x,D_\xi,D_y,D_\eta)/2} a(x,\xi) b(y,\eta)|_{y=x,\eta=\xi}$$

$$= c^w(x, hD_x) \in Op_h^w(S_{cl}^{m_1+m_2,k_1+k_2}(T^*X))$$

with $c(x,\xi;h) = a(x,\xi;h)\#b(x,\xi;h)$ and $\sigma(x,\xi,y,\eta) = y\xi - x\eta$. Similarly, for $a \in S_\delta^{m_1}, b \in S_\delta^{m_2}$ with $\delta \in [0, 1/2)$,

$$a^w(x, hD_x) \circ b^w(x, hD_x) = c^w(x, hD_x) \in Op_h^w(S_\delta^{m_1+m_2}(T^*X))$$

with $c(x,\xi;h) = a(x,\xi;h)\#b(x,\xi;h)$.

We recall the definition of semiclassical wavefront set, which measures where in phase space a distribution is nontrivial. Let $u(h)$, $0 < h \leqslant h_0$ be a family of h-tempered distributions. The semiclassical wavefront set of $u(h)$ is defined by its complement:

$$\begin{aligned}\mathrm{WF}_h(u) = \complement\{(x,\xi) \in T^*X : \exists a \in S_0^0 \text{ with } a(x,\xi) \neq 0 \\ \text{and } \|a^w u\|_{H^k(X)} = \mathcal{O}(h^\infty)\}\end{aligned}$$

for every Sobolev space H^k, $k \geqslant 0$.

The key facts to remember about the h-wavefront set are that it is a closed subset of T^*X and it is not increased by application of pseudodifferential operators. Further, suppose u is a family of quasimodes for an elliptic pseudodifferential operator

$$(P - z)u = f,$$

for $z \sim 1$ and $f \in L^2$. Then, if p is the principal symbol of P and $\partial p \neq 0$ on $\{p = z\}$, then propagation of singularities (see Lemma 3.2 below) implies that $\mathrm{WF}_h(u) \setminus \mathrm{WF}_h(f)$ is invariant under the Hamiltonian flow of p. In particular, $\mathrm{WF}_h(u) \subset \mathrm{WF}_h(f) \cup \{p = z\}$.

Since eigenfunctions and quasimodes have compact h-wavefront sets (see for example [12, Section 8.4]) we are interested here in only the case where the ξ variables are in a compact set. Hence it is the algebra $Op_h(S_\delta^*)$ that is most relevant here.

2.2. The 0-Gevrey class of functions

For this paper, we use the following 0-Gevrey classes of functions with respect to order of vanishing, introduced in [2].

Definition 2.1. — *For $0 \leqslant \tau < \infty$, let $\mathcal{G}_\tau^0(\mathbb{R})$ be the set of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ such that, for each $x_0 \in \mathbb{R}$, there exists a neighbourhood $U \ni x_0$ and a constant C such that, for all $0 \leqslant s \leqslant k$,*

$$|\partial_x^k f(x) - \partial_x^k f(x_0)| \leqslant C(k!)^C |x - x_0|^{-\tau(k-s)} |\partial_x^s f(x) - \partial_x^s f(x_0)|,$$

as $x \to x_0$ in U.

This definition says that the order of vanishing of derivatives of a function is only polynomially worse than that of lower derivatives. Every analytic function is in one of the 0-Gevrey classes $\mathcal{G}^0_\tau$ for some $\tau < \infty$, but many more functions are as well. For example, the function

$$f(x) = \begin{cases} \exp(-1/x^p), & \text{for } x > 0, \\ 0, & \text{for } x \leqslant 0 \end{cases}$$

is in $\mathcal{G}^0_{p+1}$, but

$$f(x) = \begin{cases} \exp(-\exp(1/x)), & \text{for } x > 0, \\ 0, & \text{for } x \leqslant 0 \end{cases}$$

is not in any 0-Gevrey class for finite τ.

2.3. On the definition of "irreducible quasimode"

There is a more intuitive but more restrictive definition of irreducible quasimode, which may be useful in other situations. In this section we discuss this other definition and show it is strictly stronger than the definition we have chosen.

The intuitive definition says that a reducible quasimode is one which can be broken up as a sum of two nontrivial quasimodes. We will call this *strong irreducibility*.

Definition 2.2. — *Let u be a quasimode satisfying*

$$(-\Delta - \lambda^2)u = \mathcal{O}(\lambda^{-\beta_0})\|u\|$$

for some $\beta_0 > 0$. Assume $\|u\| = 1$. We say u is (strongly) reducible if there exist pseudodifferential operators $B_1, B_2 \in \Psi^0$ such that $(B_1 + B_2)u = u + \mathcal{O}(\lambda^{-\beta_0})$, $B_1 B_2 = \mathcal{O}(\lambda^{-\infty})$,

$$(-\Delta - \lambda^2)B_j u = \mathcal{O}_{B_j}(\lambda^{-\beta_0})\|B_j u\|,$$

for $j = 1, 2$, and

$$\min\{\|B_1 u\|, \|B_2 u\|\} \geqslant \lambda^{-N}$$

for some $N < \infty$. Here $\beta_0 > 0$ is the order of the quasimode as in the introduction.

We now show that in our situation, the notion of irreducible defined in Definition 2.2 is strictly stronger than that in Definition 1.1.

Proposition 2.3. — *Suppose u is a strongly irreducible quasimode as in Definition 2.2. Then u is an irreducible quasimode as defined in Definition 1.1.*

Proof. — This is equivalent to showing *reducibility* in Definition 1.1 implies *strong reducibility* In Definition 2.2. If u is a reducible quasimode according to Definition 1.1, then u has semiclassical wavefront set on the closures of at least two different connected components of a level set of M in T^*X where M has constant rank. Let K_1, K_2 be two distinct sets carrying wavefront mass of u. This means that K_1 and K_2 are the closures of flow invariant sets. The Hamiltonian flow preserves the total energy p and the angular momentum η, so K_1 and K_2 lie in level sets of both, that is, a level set of the moment map.

First, suppose K_1 and K_2 lie in different level sets of the moment map. That means they have either different energy p or different angular momenta η (or both). Suppose K_1 has angular momentum η_1 and K_2 has angular momentum $\eta_2 \neq \eta_1$. Then one can construct a microlocal partition of unity $B_1 + B_2$ as functions of η alone, localizing near η_1 or η_2 respectively. As the B_j, $j = 1, 2$ are functions of η alone, their quantizations commute with $-\Delta$, which shows u is strongly irreducible.

If the two sets belong to the same energy and angular momentum, then they are closed connected components in a compact set in T^*X, hence compact. Either $K_1 \cap K_2 \neq \emptyset$ or $K_1 \cap K_2 = \emptyset$. In the latter case, there is a positive distance between K_1 and K_2 so again a microlocal partition of unity $B_1 + B_2$ is easy to construct. If $K_1 \cap K_2 \neq \emptyset$, since the angular momentum is the same, both sets project into the same *reduced* phase space in the (x, ξ) variables (for an example, see Figure 1.1). Let η_0 be the angular momentum of K_1 and K_2, and let $\pi : \{(x, \xi, \theta, \eta_0)\} \to \{(x, \xi)\}$ be the projection, and let $\tilde{K}_j = \pi(K_j)$ for $j = 1, 2$ be the projections. Since the K_j are flow invariant in T^*X, the $\tilde{K}_j$ are flow invariant in the reduced (x, ξ) phase space, and $\tilde{K}_1 \cap \tilde{K}_2 \neq \emptyset$ as well. Then either $\tilde{K}_1 = \tilde{K}_2$ (since they are flow invariant), or they intersect at a point p where the rank of M changes. For fixed η_0, the rank of M changes when moving from sets where $A'(x) = 0$ to sets where $A'(x) \neq 0$. Either there exists a partition of unity as in the definition of strong reducibility 2.2, or for every $b \in S^0_0$ with $b(p) \neq 0$, $b^w u$ is nontrivial. But then u has h-wavefront set both at p and where the rank of M is larger. Hence u was not a graded quasimode. □

On the other hand, there are quasimodes which can be decomposed as a sum of nontrivial quasimodes which are nevertheless irreducible according to Definition 1.1. This also explains why our main theorem requires a rotationally invariant neighbourhood. Consider the case where X has part of a 2-sphere embedded in it. The longitudinal geodesic at the thickest part is elliptic, in the sense of being stable as a dynamical system. In particular, there are many periodic geodesics close to the longitudinal one. Select a stable periodic geodesic near the longitudinal one, for example a great circle on the spherical part transversal to the longitudinal periodic geodesic. Call this geodesic γ_0. By rotating γ_0 by a fixed angle α in the θ direction, a continuous family γ_α of distinct stable periodic geodesics is obtained (see Figure 2.1) without changing the angular momentum. Each one of these is elliptic and can carry a Gaussian beam type quasimode (see, for example, [6] and the references therein). That means there is a continuous family (one for each $0 \leqslant \alpha < 2\pi$) of quasimodes, with each quasimode highly concentrated on a single elliptic periodic geodesic. Hence one can create an irreducible quasimode as a continuous superposition of these Gaussian beams. That is, for each angle α, there is a Gaussian beam quasimode associated to the periodic geodesic rotated around the sphere by angle α, say $u_\alpha(x,\theta)$. The function u_α has wavefront set confined to (the lift of) this single periodic geodesic. We take a superposition of such quasimodes. Let $f(\alpha)$ be a continuous positive function; f represents the profile of weight for each quasimode u_α, and can be chosen to be small or large wherever we choose. Then the function

$$u(x,\theta) = \int_0^{2\pi} f(\alpha) u_\alpha(x,\theta) d\alpha$$

is also a quasimode, and can have mass in “bands” with arbitrary weight, according to the profile function f. These resulting bands of quasimodes need not have nontrivial mass except in a rotationally invariant neighbourhood, which will automatically see the tallest of the profile function f. See Figure 2.1.

2.4. Conjugation to a flat problem

We observe that we can conjugate Δ by an isometry of metric spaces and separate variables so that spectral analysis of Δ is equivalent to a one-variable semiclassical problem with potential. That is, let

$$T : L^2(X, d\mathrm{Vol}) \to L^2(X, dx d\theta)$$

be the isometry given by

$$Tu(x,\theta) = A^{1/2}(x) u(x,\theta).$$

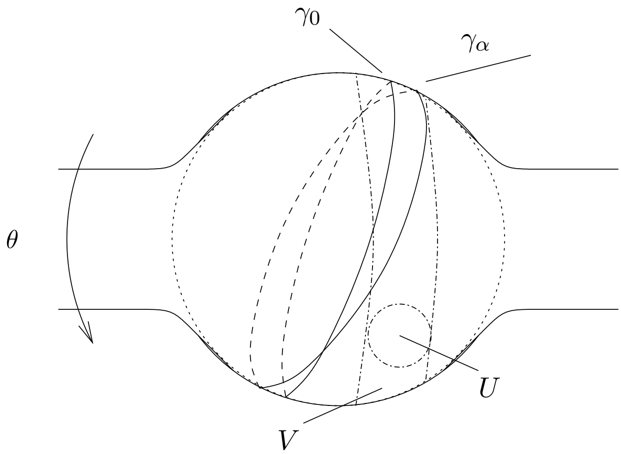

Figure 2.1. *A surface of revolution with a piece of $\mathbb{S}^2$ embedded. Also sketched are two "isoenergetic" periodic geodesics γ_0 and γ_α, which are rotations of each other by a fixed angle α in the θ direction. One can construct pathological quasimodes which are continuous, compactly supported superpositions of isoenergetic quasimodes associated to such geodesics. The neighbourhood U is not rotationally invariant, hence may not see the full L^2 mass of this superposition. On the other hand, the neighbourhood V is rotationally invariant, so passes through every γ_α. In other words, V must see the full L^2 mass of this superposition.*

Then $\widetilde{\Delta} = T\Delta T^{-1}$ is essentially self-adjoint on $L^2(X, dxd\theta)$. A simple calculation gives

$$-\widetilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x))f,$$

where the potential

$$V_1(x) = \frac{1}{2}A''A^{-1} - \frac{1}{4}(A')^2A^{-2}.$$

If we now separate variables and write $\psi(x,\theta) = \sum_k \varphi_k(x)e^{ik\theta}$, we see that

$$(-\widetilde{\Delta} - \lambda^2)\psi = \sum_k e^{ik\theta}P_k\varphi_k(x),$$

where

$$P_k\varphi_k(x) = \left(-\frac{d^2}{dx^2} + k^2A^{-2}(x) + V_1(x) - \lambda^2\right)\varphi_k(x).$$

Setting $h = |k|^{-1}$ and rescaling, we have the semiclassical operator

$$P(z,h)\varphi(x) = \left(-h^2 \frac{d^2}{dx^2} + V(x) - z \right) \varphi(x), \tag{2.5}$$

where the potential is

$$V(x) = A^{-2}(x) + h^2 V_1(x)$$

and the spectral parameter is $z = h^2\lambda^2$. In Section 3 we will at first let $h = \lambda^{-1}$ be our semiclassical parameter for the whole quasimode, but then switch to $h = |k|^{-1}$ to estimate the parts of the quasimode microsupported where the critical elements are located. The relevant microlocal estimates near critical elements are summarized in the following Subsection.

2.5. Spectral estimates for weakly unstable critical sets

In this subsection we summarize the spectral estimates we will use for weakly unstable critical elements obtained in [3, 5, 6, 8, 7, 2].

Definition 2.4. — *Let (P, Q) be a critical value of the moment map. Then there are points in $M^{-1}(P, Q)$ where the moment map has rank 1 (or 0, but these points are easy to handle (see below)). For these points, there are longitudinal periodic geodesics. If the principal part of the potential, $A^{-2}(x)$, for the reduced Hamiltonian $\xi^2 + A^{-2}(x)$ has an "honest" minimum at x_0 in the sense that if $[a, b]$ is the maximal closed interval containing x_0 with $A^{-2}(x) = A^{-2}(x_0)$ on it, then $(A^{-2})' < 0$ for $x < a$ in some small neighbourhood, and $(A^{-2})' > 0$ for $x > b$ in some other small neighbourhood, then we say this critical element is* weakly stable. *In all other cases, we say the critical element is weakly unstable.*

In the following subsections, we review the microlocal estimates from [2] for weakly unstable critical elements. Taken together, they imply the following theorem.

Theorem 2.5. — *Let Λ be a weakly unstable critical element in the reduced phase space $T^*\mathbb{S}^1_x$, and assume u has h-wavefront set sufficiently close to Λ. Then for any $\eta > 0$, there exists $C = C_\eta$ such that*

$$\|u\| \leqslant C h^{-2-\eta} \|((hD)^2 + V(x) - z)u\|,$$

for any $z \in \mathbb{R}$.

Remark 2.6. — We observe that the set Λ is a subset of a fixed energy for the semiclassical principal symbol $\Lambda \subset \{p = \xi^2 + V(x) = E\}$. For values of z far from E, $(hD)^2 + V(x) - z$ is microlocally elliptic on Λ, so the estimate is trivial in that case. The interest then lies in values of z close to E, which will range over critical values of the potential $V(x)$. Hence the microlocal spectral estimates catalogued below will be stated as valid in a small spectral window near the values of V at a critical point of specific type.

2.5.1. Unstable nondegenerate critical elements

A nondegenerate unstable critical element exists where the principal part of the potential $V_0(x) = A^{-2}(x)$ has a nondegenerate maximum. To say that $x = 0$ is a nondegenerate maximum means that $x = 0$ is a critical point of $V_0(x)$ satisfying $V_0'(0) = 0$, $V_0''(0) < 0$.

The following result as stated can be read off from [3, 6], and has also been studied in slightly different contexts in [10, 11] and [1], amongst many others.

Lemma 2.7. — *Suppose $x = 0$ is a nondegenerate local maximum of the principal part of the potential V_0, $V_0(0) = 1$. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|P(z,h)\varphi^w u\| \geqslant C_\epsilon \frac{h}{\log(1/h)} \|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{2.6}$$

Remark 2.8. — This estimate is known to be sharp, in the sense that the logarithmic loss cannot be improved (see, for example, [10]).

2.5.2. Unstable finitely degenerate critical elements

In this subsection, we consider an isolated critical point at an unstable but finitely degenerate maximum. That is, we now assume that $x = 0$ is a degenerate maximum for the function $V_0(x) = A^{-2}(x)$ of order $m \geqslant 2$. If we again assume $V_0(0) = 1$, then this means that near $x = 0$, $V_0(x) \sim 1 - x^{2m}$. Critical points of this form were first studied in [8].

This Lemma and the proof are given in [8, Lemma 2.3].

Lemma 2.9. — *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|P(z,h)\varphi^w u\| \geqslant C_\epsilon h^{2m/(m+1)} \|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{2.7}$$

Remark 2.10. — This estimate is known to be sharp, in the sense that the exponent $2m/(m+1)$ cannot be improved (see [8]).

2.5.3. Finitely degenerate inflection transmission critical elements

We next study the case when the principal part of the potential has an inflection point of finitely degenerate type. That is, let us assume the point $x = 0$ is a finitely degenerate inflection point, so that locally near $x = 0$, the potential $V_0(x) = A^{-2}(x)$ takes the form

$$V_0(x) \sim 1 - c_2 x^{2m_2+1}, \ m_2 \geqslant 1$$

with $c_2 > 0$. We remark that c_2 could be negative as well without changing the following estimate. This Lemma and the proof are in [7].

Lemma 2.11. — *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|P(z,h)\varphi^w u\| \geqslant C_\epsilon h^{(4m_2+2)/(2m_2+3)} \|\varphi^w u\|, \ \ z \in [1-\epsilon, 1+\epsilon]. \tag{2.8}$$

Remark 2.12. — This estimate is also known to be sharp in the sense that the exponent $(4m_2+2)/(2m_2+3)$ cannot be improved (see [7]).

2.5.4. Unstable infinitely degenerate and cylindrical critical elements

In this subsection, we study the case where the principal part of the potential $V(x) = A^{-2}(x) + h^2 V_1(x)$ has an infinitely degenerate maximum, say, at the point $x = 0$. Let $V_0(x) = A^{-2}(x)$. As usual, we again assume that $V_0(0) = 1$, so that

$$V_0(x) = 1 - \mathcal{O}(x^\infty)$$

in a neighbourhood of $x = 0$. Of course this is not very precise, as V_0 could be constant in a neighbourhood of $x = 0$ and still satisfy this. So let us first assume that $V_0(0) = 1$, and $V_0'(x)$ vanishes to infinite order at $x = 0$, however, $\pm V_0'(x) < 0$ for $\pm x > 0$. That is, the critical point at $x = 0$ is infinitely degenerate but isolated.

Lemma 2.13. — *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C_{\epsilon,\eta} > 0$ such that*

$$\|P(z,h)\varphi^w u\| \geqslant C_{\epsilon,\eta} h^{2+\eta} \|\varphi^w u\|, \ \ z \in [1-\epsilon, 1+\epsilon]. \tag{2.9}$$

For our next result, we consider the case where there is a whole interval at a local maximum value. That is, we assume the principal part of the effective potential $V_0(x)$ has a maximum $V_0(x) \equiv 1$ on an interval, say $x \in [-a, a]$, and that $\pm V_0'(x) < 0$ for $\pm x > a$ in some neighbourhood.

Lemma 2.14. — *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|x| \leqslant a + \epsilon, |\xi| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C_{\epsilon,\eta} > 0$ such that*

$$\|P(z,h)\varphi^w u\| \geqslant C_{\epsilon,\eta} h^{2+\eta} \|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{2.10}$$

2.5.5. Infinitely degenerate and cylindrical inflection transmission critical elements

In this subsection, we assume the effective potential has a critical element of infinitely degenerate or cylindrical inflection transmission type. This is very similar to Subsection 2.5.4, but now the potential is assumed to be monotonic in a neighbourhood of the critical value.

We begin with the case where the potential has an isolated infinitely degenerate critical point of inflection transmission type. As in the previous subsection, we write $V(x) = A^{-2}(x) + h^2 V_1(x)$ and denote $V_0(x) = A^{-2}(x)$ to be the principal part of the potential. Let us assume the point $x = 0$ is an infinitely degenerate inflection point, so that locally near $x = 0$, the potential takes the form

$$V_0(x) \sim 1 - (x-1)^\infty.$$

Let us assume that our potential satisfies $V_0'(x) \leqslant 0$ near $x = 0$, with $V_0'(x) < 0$ for $x \neq 0$ in some neighbourhood so that the critical point $x = 0$ is isolated.

Lemma 2.15. — *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C = C_{\epsilon,\eta} > 0$ such that*

$$\|P(z,h)\varphi^w u\| \geqslant C_\epsilon h^{2+\eta} \|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{2.11}$$

On the other hand, if $V_0'(x) \equiv 0$ on an interval, say $x \in [-a, a]$ with $V_0'(x) < 0$ for $x < -a$ and $x > a$, we do not expect anything better than Lemma 2.15. The next lemma says that this is exactly what we do get. To fix an energy level, assume $V_0 \equiv 1$ on $[-a, a]$.

Lemma 2.16. — *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|x| \leqslant a + \epsilon,\ |\xi| \leqslant \epsilon\}$. Then for any $\eta > 0$, there exists $C = C_{\epsilon,\eta} > 0$ such that*

$$\|P(z,h)\varphi^w u\| \geqslant C_\epsilon h^{2+\eta}\|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{2.12}$$

3. Proof of Theorem 1.2 and Corollary 1.3

Recall the conjugated Laplacian is

$$-\widetilde{\Delta} = -\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x),$$

where $V_1(x)$ has been computed above. We will do some analysis and reductions now *before* separating variables. If we are considering quasimodes

$$(-\widetilde{\Delta} - \lambda^2)u = E(\lambda)\|u\|,$$

where

$$E(\lambda) = \mathcal{O}(\lambda^{-\beta_0})$$

for some $\beta_0 > 0$, then we begin by rescaling. Set $h = \lambda^{-1}$ so that

$$(-h^2\partial_x^2 - h^2 A^{-2}(x)\partial_\theta^2 + h^2 V_1(x) - 1)u = \widetilde{E}(h)\|u\|,$$

where $\widetilde{E}(h) = h^2 E(h^{-1}) = \mathcal{O}(h^{2+\beta_0})$. With ξ, η the dual variables to x, θ as usual, the semiclassical symbol of this operator is

$$p = \xi^2 + A^{-2}(x)\eta^2 + h^2 V_1(x) - 1,$$

and the semiclassical principal symbol is

$$p_0 = \xi^2 + A^{-2}(x)\eta^2 - 1.$$

It is worthwhile to point out that at this point our semiclassical parameter is $h = \lambda^{-1}$. After separating variables later in the proof, we will let $h = |k|^{-1}$, where k is the angular momentum parameter. However, in the regime where we so take h, $|k|$ and λ will be comparable, so it is merely a choice of convenience.

It is important to keep in mind for the remainder of this paper what the various parameters represent. Here, the variable η represents hD_θ. As we will eventually be decomposing in Fourier modes in the θ direction, this means that the variable η takes values in $h\mathbb{Z}$.

We next record that a standard h-parametrix argument tells us that any quasimode is concentrated on the energy surface where $\{p_0 = 0\}$. The proof is standard.

Lemma 3.1. — *Suppose u satisfies*

$$(-h^2\partial_x^2 - h^2A^{-2}(x)\partial_\theta^2 + h^2V_1(x) - 1)u = \widetilde{E}(h)\|u\|,$$

where $\widetilde{E}(h) = h^2E(h^{-1}) = \mathcal{O}(h^{2+\beta_0})$, and $\Gamma \in \mathcal{S}^0$ satisfies $\Gamma \equiv 1$ in a small fixed neighbourhood of $\{p_0 = 0\}$. Then

$$(1 - \Gamma^w)u = \mathcal{O}(h^{2+\beta_0}).$$

Hence we will restrict our attention to the characteristic surface where $\{p_0 = 0\}$. Using our moment map idea, we know that η is invariant under the classical flow. Hence if η is very large, our operator will be elliptic, while if η is very small, the parameter ξ will be bounded away from zero, and hence we will have uniform propagation estimates. Let us make this more precise. Let $A_0 = \min(A(x))$ and $A_1 = \max(A(x))$, and let

$$1 = \psi_0(\eta) + \psi_1(\eta) + \psi_2(\eta)$$

be a partition of unity satisfying

$$\psi_0 \equiv 1 \text{ on } \{|\eta|^2 \leqslant \frac{1}{2}A_0^2\}$$

with support in $\{|\eta|^2 \leqslant \frac{3}{4}A_0^2\}$;

$$\psi_2 \equiv 1 \text{ on } \{|\eta|^2 \geqslant 2A_1^2\}$$

with support in $\{|\eta|^2 \geqslant \frac{3}{2}A_1^2\}$. Then, on $\operatorname{supp}\psi_0$, we have

$$\eta^2A^{-2}(x) \leqslant \eta^2A_0^{-2} \leqslant \frac{3}{4},$$

and on $\operatorname{supp}\psi_2$, we have

$$\eta^2A^{-2}(x) \geqslant \eta^2A_1^{-2} \geqslant \frac{3}{2}.$$

Now for our quasimode u, write

$$u = u_0 + u_1 + u_2 + u_3 := \psi_0^w\Gamma^wu + \psi_1^w\Gamma^wu + \psi_2^w\Gamma^wu + (1 - \Gamma^w)u.$$

The derivative hD_θ commutes with $-\widetilde{\Delta}$ and we can choose $\Gamma = \Gamma(p_0)$ so that $[p_0^w, \Gamma^w] = \mathcal{O}(h^3)$. This means that each of these u_j are also quasimodes, but with an error in terms of u. That is, each u_j satisfies

$$Pu_j = \mathcal{O}(h^{2+\beta_1})\|u\|.$$

3.1. Some reductions

In this section, we make a few reductions. Lemma 3.1 shows us that $\|u_3\| = \mathcal{O}(h^{2+\beta_0})$.

On the other hand, on the support of ψ_2, we have the principal symbol satisfies

$$|p_0| \geqslant \frac{1}{2},$$

so we will use an elliptic argument.

That is, since $|p_0| \geqslant \frac{1}{2}$ on support of ψ_2, there is an h-parametrix for P there: there exists Q such that

$$QP\psi_2^w = \psi_2^w + \mathcal{O}(h^\infty),$$

and further Q has bounded L^2 norm. Hence

$$\begin{aligned}\|u_2\| &= \|QPu_2\| + \mathcal{O}(h^\infty)\|u_2\| \\ &\leqslant C\|Pu_2\| + \mathcal{O}(h^\infty)\|u_2\| \\ &= \mathcal{O}(h^{2+\beta_0})\|u\|.\end{aligned}$$

This implies $u_2 = \mathcal{O}(h^{2+\beta_0})$.

This tells us that our original quasimode u satisfies

$$u = u_0 + u_1 + \mathcal{O}(h^{2+\beta_0}).$$

Since we are proving *lower* bounds instead of upper bounds, we need to show that u is controlled from below by u_0 and u_1. We compute

$$\begin{aligned}\|u\|^2 &= \|u_0\|^2 + \|u_1\|^2 + \langle u_0, u_1\rangle + \langle u_1, u_0\rangle + \mathcal{O}(h^{2(2+\beta_0)}) \\ &= \|u_0\|^2 + \|u_1\|^2 + \langle \psi_0\Gamma u, \psi_1\Gamma u\rangle + \langle \psi_1\Gamma u, \psi_0\Gamma u\rangle + \mathcal{O}(h^{2(2+\beta_0)}) \\ &= \|u_0\|^2 + \|u_1\|^2 + \langle \Gamma\psi_1\psi_0\Gamma u, u\rangle + \langle \Gamma\psi_0\psi_1\Gamma u, u\rangle + \mathcal{O}(h^{2(2+\beta_0)}) \\ &= \|u_0\|^2 + \|u_1\|^2 + 2\,\langle \Gamma\psi_1\psi_0\Gamma u, u\rangle + \mathcal{O}(h^{2(2+\beta_0)}) \\ &\geqslant \|u_0\|^2 + \|u_1\|^2 - Ch\|u\|^2 - \mathcal{O}(h^{2(2+\beta_0)}).\end{aligned} \tag{3.1}$$

Equation (3.1) follows from the sharp Gårding inequality since the principal symbol of $\Gamma\psi_1\psi_0\Gamma$ is non-negative. The previous line follows since $[\psi_1, \psi_0] = 0$. Rearranging (3.1), we have

$$\|u\| \geqslant \frac{1}{2}(\|u_0\| + \|u_1\|) - \mathcal{O}(h^{2+\beta_0}).$$

This means we need to prove lower bounds on u_0 and u_1 in terms of u. On the other hand, we also have

$$\|u\| \leqslant \|u_0\| + \|u_1\| + \mathcal{O}(h^{2+\beta_0}).$$

This means that, in particular, since u is normalized, for $h > 0$ sufficiently small, we must have either

$$\|u_0\| \geqslant \frac{1}{3},$$

or

$$\|u_1\| \geqslant \frac{1}{3}.$$

3.2. Estimation of u_0

In this subsection we assume $\|u_0\| \geqslant 1/3$, so that in particular $\|u_0\|$ is comparable to $\|u\|$. Observe that on the support of ψ_0, since η is invariant, we have $|\xi|^2 \geqslant 1/4 - \mathcal{O}(h^2)$, which means the propagation speed in the x-direction is bounded below. We claim this implies

$$\|u_0\|_{L^2_{x,\theta}} \leqslant c_0 \|u_0\|_{L^2([a,b]_x \times \mathbb{S}_\theta)}$$

for some $c_0 > 0$. In other words, u_0 is *uniformly* distributed in the sense that the mass cannot be vanishing in h on any set, unless it is vanishing in h everywhere.

The claim follows by propagation of singularities. The standard propagation of singularities result applies whenever the classical flow propagates singularities from one region to another in phase space. It says that if all points in a region in phase space flow into a second region under the classical flow, then the mass of a function in the first region is controlled by the mass in the second, modulo a term with the operator. Since we are analyzing the region where $\xi \neq 0$, we have uniform propagation in the x direction. A general statement is given in the following Lemma (a refinement of Hörmander's original result [9]). For a proof in this context, see, for example, [4, Lemma 2.4] and [1, Lemma 4.1].

Lemma 3.2. — *Let $V_1, V_2 \Subset T^*X$, p the symbol of a semiclassical pseudodifferential operator P with principal symbol p_0. Suppose A, B are pseudodifferential operators with $A \equiv 1$ on V_2, and the semiclassical wavefront set of B is in V_1. Suppose there exists $T > 0$ such that*

$$\begin{cases} \forall \rho \text{ in a neighbourhood of } V_1, \\ \exp(tH_{p_0})(\rho) \in V_2 \text{ for } t \sim T. \end{cases} \tag{3.2}$$

Then

$$\|Bv\| \leqslant C \left(h^{-1} \|Pv\| + \|Av\|\right) + \mathcal{O}(h^\infty)\|v\|. \tag{3.3}$$

Fix two non-empty intervals in the x direction, (a,b) and (c,d) and assume u_0 is L^2 normalized (this is possible since $\|u_0\| \sim \|u\|$). Now using that $Pu_0 = \mathcal{O}(h^{2+\beta_0})\|u_0\|$, we have

$$\begin{aligned}\|u_0\|_{L^2((c,d)\times\mathbb{S}^1)} &\leqslant Ch^{-1}\|Pu_0\| + C_2\|u_0\|_{L^2((a,b)\times\mathbb{S}^1)} \\ &\leqslant Ch^{1+\beta_0}\|u_0\|_{L^2(\mathbb{S}^1\times\mathbb{S}^1)} + C_2\|u_0\|_{L^2((a,b)\times\mathbb{S}^1)},\end{aligned}$$

for some $C_2 > 0$. For $h > 0$ sufficiently small, this implies if u_0 has mass bounded below independent of h in any x neighbourhood (c,d) (and in particular also for $\mathbb{S}^1$), then

$$\|u_0\|_{L^2((a,b)\times\mathbb{S}^1)} \geqslant c' > 0$$

independent of h. Rescaling in terms of u_0 if u_0 is not normalized, we recover

$$\|u_0\|_{L^2((a,b)\times\mathbb{S}^1)} \geqslant c'\|u_0\|.$$

Since the interval (a,b) is arbitrary, we have shown that the L^2-mass on any rotationally invariant neighbourhood is positive independent of h. Thus (1.1) holds with a lower bound independent of $h = \lambda^{-1}$.

3.3. Estimation of u_1

We next analyze u_1, assuming that

$$\|u_1\| \geqslant \frac{1}{3} = \frac{1}{3}\|u\|.$$

The quasimode u_1 is microsupported where all the critical points of $A(x)$ are. Let us recall that we are trying to prove that for any open interval (a,b) in the x variable, that either

$$\|u_1\|_{L^2((a,b)_x\times\mathbb{S}^1_\theta)} = \mathcal{O}(\lambda^{-\infty}),$$

or for all $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$\|u_1\|_{L^2((a,b)\times\mathbb{S}^1)} \geqslant C_\epsilon\lambda^{-1-\epsilon}\|u_1\|_{L^2(\mathbb{S}^1_x\times\mathbb{S}^1_\theta)}.$$

We also have yet to use the fact that $u = u_0 + u_1 + u_2 + u_3$ was assumed to be irreducible. We will use that in two ways in this subsection. First, we use this assumption to decompose the Fourier series of u_1 into finer "bands" in phase space, each associated to distinct angular momenta. An irreducible quasimode can live on at most one of these finer bands, so that we require fewer Fourier modes in u_1 than at first impression. Second, we use this assumption together with the geodesic complexity assumption to describe the possible behaviour of u_1 in each possible invariant set in a fixed level set of the moment map.

Since u_1 is microsupported in a region where $|\eta|$ is bounded between two constants, say, $a_0 \leqslant |\eta| \leqslant a_1$, and $\eta = hk$ for some integer k, a priori the number of angular momenta k in the wavefront set of u_1 is comparable to h^{-1}. We can do better than that. Using the semiclassical calculus, we will next show that there exists $k_0 \in \mathbb{Z}$ such that for any $\epsilon > 0$, we have

$$u_1 = \sum_{|k-k_0|\leqslant h^{-\epsilon}} e^{ik\theta}\varphi_k(x) + \mathcal{O}(h^\infty)\|u_1\|.$$

That is, we claim that the Fourier decomposition of u_1 can actually only have $\mathcal{O}(h^{-\epsilon})$ non-trivial modes. To prove this claim, fix $k_0 \in \mathbb{Z}$ and any $\epsilon > 0$, and choose a $k_1 \in \mathbb{Z}$ satisfying

$$|k_1 - k_0| \geqslant h^{-\epsilon}.$$

We will show that we can decompose u_1 into (at least) two pieces with disjoint microsupport, one near hk_0 and one near hk_1. Evidently, these two pieces correspond to different angular momenta η, so have wavefront sets associated to different level sets of the moment map. Of course, level sets sufficiently close (in an h-dependent set) may contribute to a single irreducible quasimode, but the point is to quantify how far away from a single level set one needs to go before leaving the microsupport of an irreducible quasimode.

In order to make this rigorous, let $\eta_j = hk_j$ for $j = 0, 1$, and choose $\chi(r) \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfying

$$\chi(r) \equiv 1 \text{ for } |r| \leqslant 1,$$

with support in $\{|r| \leqslant 2\}$. For $j = 0, 1$, let

$$\chi_j(\eta, h) = \chi\left(\frac{\eta - \eta_j}{h^{1-\epsilon/2}}\right).$$

As semiclassical symbols, the χ_j are in a harmless $S^0_{1/2-\epsilon/4}$ symbol class (after rescaling the local Weyl integral), and moreover they only depend on η (not on θ) and commute with $-\widetilde{\Delta}$. On the support of each of the χ_j, we have

$$\left|\frac{\eta - \eta_j}{h^{1-\epsilon/2}}\right| = \left|\frac{hk - hk_j}{h^{1-\epsilon/2}}\right| = \left|\frac{k - k_j}{h^{-\epsilon/2}}\right| \leqslant 2.$$

This implies

$$|k - k_j| \leqslant 2h^{-\epsilon/2},$$

so as $h \to 0+$, χ_0 and χ_1 have disjoint supports. This means the functions $\chi_1 u_1$ and $\chi_2 u_1$ have disjoint h-wavefront sets, so they are almost orthgonal:

$$\langle \chi_1 u_1, \chi_2 u_1 \rangle = \mathcal{O}(h^\infty).$$

Hence if each of these functions has nontrivial L^2 mass, then u was not an irreducible quasimode.

Finally, we analyze the function u_1, but spread over at most $\mathcal{O}(h^{-\epsilon})$ Fourier modes. Fix k_0 as above. Recalling again the separated equation (2.5) with the potential

$$V(x) = A^{-2}(x) + h^2 V_1(x),$$

let A_0 and A_1 again be the min/max respectively of $A(x)$. Our spectral parameter now is $z = h^2\lambda^2$, where h will be allowed to vary in the values $h \in \{|k|^{-1} : |k - k_0| \leqslant k_0^\epsilon\}$. We are localized where

$$\frac{1}{2} A_0^2 \leqslant (\lambda^{-1} k)^2 \leqslant 2A_1^2,$$

or

$$\frac{1}{2} A_1^{-2} \leqslant z \leqslant 2A_0^{-2}.$$

This of course implies that λ and k are comparable, and hence both comparable to h^{-1}. Write

$$u_1 = \sum_{|k-k_0| \leqslant k_0^\epsilon} \varphi_k(x) e^{ik\theta} + \mathcal{O}(h^\infty). \tag{3.4}$$

Throughout the remainder of this section, let λ be large and fixed. Let $(a,b) \subset \mathbb{S}^1$ be a non-empty interval, and assume that u_1 is nontrivial in $(a,b) \times \mathbb{S}^1$:

$$\|u_1\|_{L^2((a,b)\times\mathbb{S}^1)} \geqslant |\lambda|^{-N}$$

for some N. Let $\Gamma \subset T^*X$ be the set

$$\Gamma = (a,b)_x \times \mathbb{R}_\xi.$$

If $\varphi_k(x)$ is one of the Fourier modes in the Fourier series (3.4) for u_1, then either $\|\varphi_k\|_{L^2((a,b))} = \mathcal{O}(|k|^{-\infty})$, or $\|\varphi_k\|_{L^2((a,b))} \geqslant |k|^{-N_k}$ for some $N_k \geqslant 0$. In the first case, $\mathrm{WF}_{|k|^{-1}}(\varphi_k) \cap \Gamma = \emptyset$, so these modes have disjoint wavefront set from u_1. Hence these Fourier modes would contribute to a reducible quasimode, so must not be in the sum. Hence let

$$A = \{k : |k - k_0| \leqslant |k_0|^\epsilon \text{ and } \|\varphi_k\|_{L^2((a,b))} \geqslant |k|^{-N_k} \text{ for some } N_k \geqslant 0\},$$

and write

$$u_1 = \sum_{k\in A} \varphi_k(x) e^{ik\theta} + \mathcal{O}(h^\infty). \tag{3.5}$$

The key point is that, by orthonormality, there is at least one of the φ_k which is relatively large. We have

$$\|u_1\|^2_{L^2(\mathbb{S}^1\times\mathbb{S}^1)} = \sum_{k\in A} \|\varphi_k\|^2_{L^2(\mathbb{S}^1)} + \mathcal{O}(h^\infty),$$

so at least one of the φ_ks satisfies

$$c_0 k_0^{-\epsilon_0} \|u_1\|^2_{L^2(\mathbb{S}^1\times\mathbb{S}^1)} \leqslant \|\varphi_k\|^2_{L^2(\mathbb{S}^1)}$$

for some small constant $c_0 > 0$. Fix this φ_k. We have for this k and $h = |k|^{-1}$ as usual

$$((hD)^2 + V(x) - z)\varphi_k = \mathcal{O}(h^{2+\beta_0})\|u_1\| = \mathcal{O}(h^{2+\beta_0-\epsilon})\|\varphi_k\|,$$

so that φ_k is a quasimode of order $h^{2+\beta_0/2}$ if $\epsilon < \beta_0/2$ is sufficiently small. We now need to show that if u is a weak irreducible quasimode,

$$((hD)^2 + V(x) - z)u = \mathcal{O}(h^{2+\beta_0/2})\|u\|,$$

with $\|u\| = 1$, then either $\|u\|_{L^2(a,b)} = \mathcal{O}(h^\infty) = \mathcal{O}(\lambda^{-\infty})$, or $\|u\|_{L^2(a,b)} \geqslant C_\epsilon h^{1+\epsilon}$ for any $\epsilon > 0$. If u is the φ_k selected above, then we have already assumed that u is nontrivial in (a,b), so that $\|u\|_{L^2(a,b)} \geqslant ch^N$ for some N. For the remainder of this section, we replace $\beta_0/2$ with β_0 to avoid excessive notation.

There are a number of subcases to consider here. We observe that, according to Lemma 3.1, we can always microlocalize further to a set close to the energy level of interest. That is, for $P(z,h) = (hD)^2 + V(x) - z$, if $P(z,h)u = \mathcal{O}(h^{2+\beta_0})$, then if $\psi(r) \in \mathcal{C}^\infty_c(\mathbb{R})$ satisfies $\psi \equiv 1$ for r near 0, we have for any $\delta > 0$

$$\psi^w((\xi^2 + V(x) - z)/\delta)u = u + \mathcal{O}(h^{2+\beta_0}).$$

For the rest of this section, we write ψ^w for this energy cutoff.

Case 1. — Next, assume z is in a small neighbourhood of a critical energy level, and assume $A'(x) \neq 0$ somewhere on (a,b). Then let (a',b') be a non-empty interval with

$$(a',b') \Subset \{A' \neq 0\} \cap (a,b),$$

and let $(\alpha',\beta) \supset (a',b')$ be the maximal connected interval with $A'(x) \neq 0$ on (α',β). Now A' has constant sign on (α',β), so at least one of α' or β is part of a weakly unstable critical element (see Figure 3.1).

Without loss in generality, assume $A' < 0$ on (α',β) so that at least β lies in a weakly unstable critical element. That is, the principal part of the potential $A^{-2}(x)$ increases as $x \to \beta-$, and takes the value, say $A^{-2}(\beta) = A_2$. Let (α,β) be the maximal open interval containing (α',β)

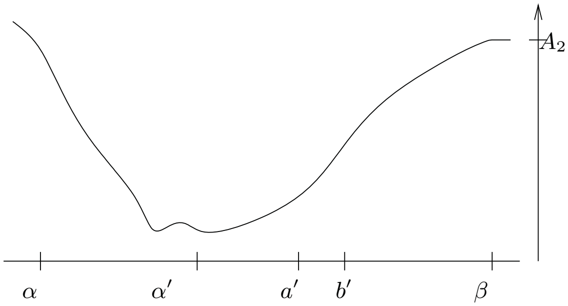

Figure 3.1. The function $A^{-2}(x)$ and the weakly unstable critical point β.

where $A^{-2}(x) < A_2$ on (α, β). As $A^{-2}(x) < A^{-2}(\alpha)$ for $x \in (\alpha, \beta)$ and $A^{-2}(\alpha) = A_2$, we have $(A^{-2}(x))' < 0$ for $x \in (\alpha, \beta)$ sufficiently close to α. That means that either α is part of a weakly unstable critical element, or $A'(\alpha) \neq 0$. We break the analysis into the two separate subsubcases, beginning with $A'(\alpha) \neq 0$.

Case 1a. — If $A'(\alpha) \neq 0$, then the weakly unstable/stable manifolds associated to $(A^{-2})'(\beta) = 0$ are homoclinic to each other (see Figure 3.2), and in particular, propagation of singularities can be used to control the mass along this whole trajectory, as long as we stay away from the right hand endpoint β. That is, propagation of singularities implies for any $\eta > 0$ independent of h,

$$\begin{aligned} \|\psi^w u\|_{L^2(\alpha, \beta - \eta)} &\leqslant C_\eta (h^{-1} \|((hD)^2 + V - z)\psi^w u\| + \|\psi^w u\|_{L^2(a', b')} \\ &\leqslant C_\eta h^{1+\beta_0} \|u\| + \|\psi^w u\|_{L^2(a', b')}. \end{aligned}$$

Hence by taking $h > 0$ sufficiently small, we need to bound $\|\psi^w u\|_{L^2(\alpha, \beta-\eta)}$ from below in terms of $\|u\|$.

Let $[\beta, \kappa]$ be the maximal connected interval containing β on which $A' = 0$ (we allow $\kappa = \beta$ if the critical point is isolated). Let $\tilde{\chi} \equiv 1$ on $[\beta, \kappa]$ with support in a small neighbourhood thereof, and let $\chi \equiv 1$ on $\operatorname{supp} \tilde{\chi}$ with support in a slightly smaller set so that $(1 - \tilde{\chi}) \geqslant (1 - \chi)$ and $(1 - \tilde{\chi}) \geqslant c|\chi'|$. Then writing $P(z, h) = (hD)^2 + V - z$, we have from Theorem 2.5 (for any

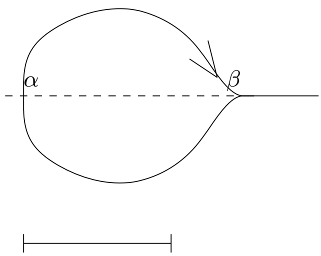

Figure 3.2. If $A'(\alpha) \neq 0$, the unstable manifold from β flows into the stable manifold at β (homoclinicity). The interval indicates a region with propagation speed uniformly bounded below.

$\epsilon > 0$)

$$\begin{aligned}\|u\| &\leqslant \|\chi u\| + \|(1-\chi)u\| \\ &\leqslant C_\epsilon h^{-2-\epsilon}\|P(z,h)\chi u\| + \|(1-\tilde{\chi})u\| \\ &\leqslant C_\epsilon h^{-2-\epsilon}(\|\chi P(z,h)u\| + \|[P(z,h),\chi]u\|) + \|(1-\tilde{\chi})u\| \\ &\leqslant C'_\epsilon(h^{\beta_0-\epsilon}\|u\| + h^{-1-\epsilon}\|(1-\tilde{\chi})u\|) + \|(1-\tilde{\chi})u\|\end{aligned}$$

Rearranging and taking $h > 0$ sufficiently small and $\epsilon < \beta_0$, we get

$$\|(1-\tilde{\chi})u\| \geqslant C_\epsilon h^{1+\epsilon}\|u\|. \tag{3.6}$$

Now either the wavefront set of u is contained in the closure of the lift of (α,β) or it isn't. In the latter case there is nothing to prove. In the former case, we conclude that $u = \mathcal{O}(h^\infty)$ on any open subset whose closure does not meet the set $[\alpha,\beta]$. We appeal to propagation of singularities one more time. Since $A'(\alpha) \neq 0$, propagation of singularities applies in a neighbourhood of α, so that (shrinking $\eta > 0$ if necessary) for some $c_1 > 0$,

$$\|u\|_{L^2(\alpha,\beta-\eta)} \geqslant c_1\|u\|_{L^2(\alpha-\eta,\beta-\eta)}.$$

Since we have assumed $u = \mathcal{O}(h^\infty)$ on $(\alpha-\eta,\beta+\eta)^c$, this estimate, together with (3.6) and (3.3) allows us to conclude

$$\|u\|_{L^2(\alpha,\beta-\eta)} \geqslant C\|(1-\tilde{\chi})u\| \geqslant C_\epsilon h^{1+\epsilon}\|u\|.$$

Case 1b. — We now consider the possibility that $A'(\alpha) = 0$ as well as $A'(\beta) = 0$ (see Figure 3.3). In this case, propagation of singularities fails at both endpoints of (α,β), so we can only conclude that for any $\eta > 0$

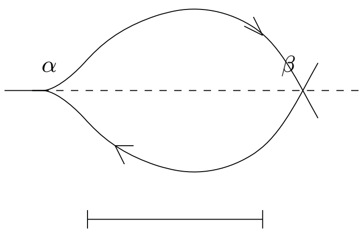

Figure 3.3. *If $A'(\alpha) = 0$, the unstable manifold from β flows into the stable manifold at α and vice versa. The interval indicates a region with propagation speed uniformly bounded below.*

independent of h,

$$\|u\|_{L^2(\alpha+\eta,\beta-\eta)} \leqslant C_\eta (h^{-1} \|((hD)^2 + V - z)u\| + \|u\|_{L^2(a',b')}.$$

Hence now it suffices to prove that for some $\eta > 0$ small but independent of h, we have the estimate

$$\|u\|_{L^2(\alpha+\eta,\beta-\eta)} \geqslant C_\epsilon h^{1+\epsilon} \|u\|$$

for any $\epsilon > 0$.

Let $[\beta, \kappa]$ be the maximal connected interval containing β on which $A' = 0$, and let $[\omega, \alpha]$ be the maximal connected interval containing α on which $A' = 0$. Let $\tilde{\chi} \equiv 1$ on $[\beta, \kappa] \cup [\omega, \alpha]$ with support in small neighbourhoods thereof, and let $\chi \equiv 1$ on $\operatorname{supp} \tilde{\chi}$ with support in a slightly smaller set so that $(1 - \tilde{\chi}) \geqslant (1 - \chi)$ and $(1 - \tilde{\chi}) \geqslant c|\chi'|$. Since both $[\omega, \alpha]$ and $[\beta, \kappa]$ are weakly unstable, we can apply Theorem 2.5 and the same argument as above to finish this case.

Case 2. — Finally, we assume $(a, b) \subset \{A' = 0\}$. Again, if $A^{-2} \equiv A_3$ on (a, b) and $z \neq A_3$, we can use propagation of singularities to control $\|u\|_{L^2(a,b)}$ from below by its mass on the connected component in $\{p = z\}$ containing (a, b) (as in the case of u_0 above). Hence we are interested in the case where z is in a small neighbourhood of A_3.

If $u = \mathcal{O}(h^\infty)$ on (a, b) there is nothing to prove, so assume not. Then if $[\alpha, \beta] \supset (a, b)$ is the maximal connected interval where $A^{-2}(x) \equiv A_3$, the wavefront set of u is contained in a small neighbourhood of $[\alpha, \beta]$, so that for $\delta > 0$ as small as we like by taking a sufficiently localized energy cutoff, we have

$$\|u\|_{L^2([\alpha-\delta,\beta+\delta]^c)} = \mathcal{O}_\delta(h^\infty).$$

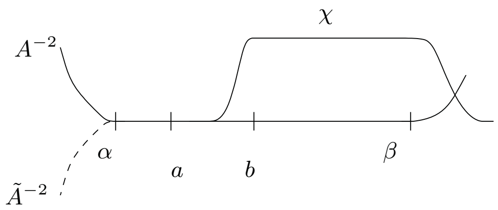

Figure 3.4. The setup for Case 2. Here if the quasimode is small in (a,b), we cut off to the right of (a,b) and modify A^{-2} to the left to be weakly unstable. We then arrive at a contradiction.

That means that, either

$$\|u\|_{L^2([a,b])} \geqslant c > 0, \ \ \|u\|_{L^2([\alpha-\delta,a])} \geqslant c > 0, \ \text{or} \ \|u\|_{L^2([b,\beta+\delta])} \geqslant c > 0.$$

If the first estimate is true, we're done, so assume without loss in generality that $\|u\|_{L^2([b,\beta+\delta])} \geqslant c > 0$. Assume for contradiction that there exists $\epsilon_0 > 0$ such that $\|u\|_{L^2(a,b)} \leqslant Ch^{1+\epsilon_0}$. Let $\chi \in \mathcal{C}_c^\infty$ be a smooth function such that $\chi \equiv 1$ on $[b, \beta+\delta]$ with support in $(a, \beta+2\delta)$. Write $\tilde{u} = \chi u$. If $[\alpha,\beta]$ is a weakly stable critical element, modify $A^{-2}(x)$ on the support of $1-\chi$ so that $[\alpha,\beta]$ is weakly unstable. That is, if $(A^{-2}(x))' < 0$ for $x < \alpha$ in some neighbourhood, replace A with a locally defined function $\tilde{A}$ satisfying $\tilde{A} \equiv A$ on $\operatorname{supp}\chi$ but $(\tilde{A}^{-2}(x))' > 0$ for $x < \alpha$ in some neighbourhood. If $[\alpha,\beta]$ is weakly unstable, then let $\tilde{A} \equiv A$ (see Figure 3.4. We apply Theorem 2.5 once again (for any $\epsilon > 0$):

$$\begin{aligned}
\|\chi u\| &\leqslant C_\epsilon h^{-2-\epsilon} \|((hD)^2 + \tilde{A}^{-2} + h^2 V_1 - z)\chi u\| \\
&= C_\epsilon h^{-2-\epsilon} \|((hD)^2 + A^{-2} + h^2 V_1 - z)\chi u\| \\
&\leqslant C_\epsilon h^{-2-\epsilon} (\|P(z,h)u\| + \|[P(z,h),\chi]u\|) \\
&\leqslant C'_\epsilon (h^{\epsilon_0-\epsilon}\|u\| + h^{-1-\epsilon}\|u\|_{L^2(a,b)}) + \mathcal{O}(h^\infty),
\end{aligned}$$

where the $\mathcal{O}(h^\infty)$ error comes from the part of the commutator $[P(z,h),\chi]$ supported outside a neighbourhood of $[\alpha,\beta]$ (the other part contributing the integral over (a,b)). But our contradiction assumption implies that the right hand side is $o(1)$ as $h \to 0$ provided $\epsilon < \epsilon_0$. As $\|\chi u\| \geqslant c > 0$, this is a contradiction.

3.4. Finishing up the proof

We now put together the estimates of u_0, u_1, u_2, u_3. Since $u_3 = \mathcal{O}(h^{2+\beta_0})$ and $u_2 = \mathcal{O}(h^\infty)$, for $h > 0$ sufficiently small, at least one of u_0 and u_1 must have L^2 mass bounded below independent of h. If u_0 has L^2 mass bounded below independent of h we're done by the propagation of singularities argument in Subsection 3.2. Hence we need to conclude Theorem 1.2 assuming u_0 is small and u_1 carries most of the L^2 mass.

Fix (a, b) as considered in Subsection 3.3 and recall we know that for any $\epsilon > 0$ there is a set $A \subset \{k : |k - k_0| \leqslant |k_0|^\epsilon\}$ of nontrivial Fourier modes such that

$$u_1 = \sum_{k \in A} e^{ik\theta} \varphi_k(x) + \mathcal{O}(h^\infty)\|u_1\|.$$

We have fixed a $k_1 \in A$ with

$$\|\varphi_{k_1}\|_{L^2((a,b))} \geqslant |k_1|^{-N}$$

for some integer N and also

$$\|\varphi_{k_1}\|_{L^2(\mathbb{S}^1)} \geqslant c_0 |k_1|^{-\epsilon} \|u_1\|_{L^2(\mathbb{S}^1 \times \mathbb{S}^1)}.$$

We use the notation $\Omega = (a, b)_x \times \mathbb{S}^1_\theta$ as in the statement of Theorem 1.2. Since φ_{k_1} is nontrivial in (a, b), it satisfies

$$\|\varphi_{k_1}\|_{L^2(a,b)} \geqslant c_2 |k_1|^{-1-\epsilon} \|\varphi_{k_1}\|_{L^2(\mathbb{S}^1_x)},$$

for any $\epsilon > 0$. We conclude

$$\begin{aligned} \|u_1\|^2_{L^2(\Omega)} &= \sum_{k \in A} \|\varphi_k(x)\|^2_{L^2(a,b)} + \mathcal{O}(h^\infty)\|u_1\|^2 \\ &\geqslant c_2' \|\varphi_{k_1}\|^2_{L^2(a,b)} - \mathcal{O}(h^\infty)\|u_1\|^2 \\ &\geqslant c_2'' k_1^{-2-2\epsilon} \|\varphi_{k_1}(x)\|^2_{L^2(\mathbb{S}^1_x)} - \mathcal{O}(h^\infty)\|u_1\|^2 \\ &\geqslant c_2''' \lambda^{-2-4\epsilon} \|u_1\|^2_{L^2(\mathbb{S}^1_x \times \mathbb{S}^1_\theta)} - \mathcal{O}(\lambda^{-\infty})\|u_1\|^2. \end{aligned}$$

This concludes the proof of Theorem 1.2. □

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