ANNALES

DE

L'INSTITUT FOURIER

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Tome 65, nº 4 (2015), p. 1469-1491.

<http://aif.cedram.org/item?id=AIF_2015__65_4_1469_0>

Ann. Inst. Fourier, Grenoble
65, 4 (2015) 1469-1491

INVARIANT SUBSPACES WITH NO GENERATOR AND A PROBLEM OF H. HELSON

by Jun-ichi TANAKA (*)

Dedicated to the memory of Henry Helson

Abstract. — In the almost-periodic context, the H_0^2−space cannot be generated by one of its elements. Together with a cocycle argument, this implies that there exist all kinds of invariant subspaces without a single generator, from which we answer some questions on invariant subspace theory.

Résumé. — Dans le contexte presque périodique, aucun espace H_0^2 ne peut être engendré par un de ses éléments. En tenant compte d'un argument faisant intervenir les cocycles, on peut en déduire qu'il existe de nombreux types de sous-espaces invariants qui ne peuvent pas être engendrés par un seul de leurs éléments; ceci permet de répondre à quelques questions de la théorie des sous-espaces invariants.

1. Introduction

The theory of invariant subspaces has been developed in the context of compact abelian groups with ordered duals, which is a natural generalization of such a theory on the unit circle $\mathbb{T}$. Many classical results extend to these cases, nevertheless, one also meets new difficulties. The purpose of this paper is to resolve a longstanding problem formulated by H. Helson in the 1950s.

Let Γ be a countable dense subgroup of the real line $\mathbb{R}$, endowed with the discrete topology. Then the dual group K of Γ is a compact abelian group that is metrizable. For λ in Γ, it is customary to denote by χ_λ the character on K defined by $\chi_\lambda(x) = x(\lambda)$. Let σ be the normalized Haar

Keywords: Compact groups with ordered duals, Invariant subspaces, Cocycles, Single generators.
Math. classification: 43A17, 46J10, 46J15, 28D10.
(*) Partially supported by NSF grant no. 0649765.

measure on K. A function ϕ in $L^1(\sigma)$ is *analytic* if its Fourier coefficients

$$a_\lambda(\phi) = \int_K \phi\,\overline{\chi_\lambda}\,d\sigma \tag{1.1}$$

vanish for all negative λ in Γ. The *Hardy space* $H^p(\sigma), 1 \leqslant p \leqslant \infty$, is defined to be the space of all analytic functions in $L^p(\sigma)$. For technical reasons, it is useful to define $H_0^p(\sigma)$ as the subspace of all ϕ in $H^p(\sigma)$ with $a_0(\phi) = 0$. A (weak*-, if $p = \infty$) closed subspace $\mathfrak{M}$ of $L^p(\sigma)$ is *invariant* if $\mathfrak{M}$ contains $\chi_\lambda\mathfrak{M}$ for all positive λ in Γ. When the inclusion is strict, $\mathfrak{M}$ is said to be *simply* invariant. Of course, both $H^p(\sigma)$ and $H_0^p(\sigma)$ are simply invariant subspaces of $L^p(\sigma)$. If ϕ is in $L^p(\sigma)$, and if $\mathfrak{M}[\phi]$ denotes the smallest invariant subspace of $L^p(\sigma)$ containing ϕ, then ϕ is called a *single generator* of $\mathfrak{M}[\phi]$. Recall that a function of modulus one is said to be *unitary* and an analytic unitary function is called an *inner* function. We say a function ϕ in $H^p(\sigma)$ is *outer* if it satisfies that

$$\log |\, a_0(\phi)\,| = \int_K \log |\,\phi\,|\, d\sigma \; > \; -\infty\,.$$

Let $1 \leqslant q \leqslant p \leqslant \infty$, and let $\mathfrak{M}$ be a simply invariant subspace of $L^p(\sigma)$. It follows from the properties of outer functions that $[\mathfrak{M} \cap L^\infty(\sigma)]_q \cap L^p(\sigma) = \mathfrak{M}$, where $[\mathfrak{M} \cap L^\infty(\sigma)]_q$ is the closure of $\mathfrak{M} \cap L^\infty(\sigma)$ in $L^q(\sigma)$ (see [3, Chapter V, Section 6] for details). This fact assures that there is a one-to-one correspondence between the invariant subspaces in $L^p(\sigma)$ and those in $L^q(\sigma)$. Therefore, in dealing with invariant subspaces, we may restrict our attention to the case of $p = 2$, in which Hilbert space theory works well. It follows from Szegö's theorem that ϕ is a single generator of $H^2(\sigma)$ if and only if ϕ is outer in $H^2(\sigma)$. However, it has been unknown for a long time whether every simply invariant subspace is singly generated or not. In the literature this has come to be known as the *single generator problem* (refer to [4, §5.4], [2, Remark, p. 158] and [3, p. 138 and p. 177]). The difficulty seems to center on the case of invariant subspace $H_0^2(\sigma)$. In [6, p. 183], it is raised in an equivalent form in connection with stochastic processes.

Our objective in this note is to show a negative answer to this problem in the almost periodic settings:

THEOREM. — *The invariant subspace $H_0^2(\sigma)$ cannot be generated by one of its elements.*

To the best of author's knowledge, $H_0^2(\sigma)$ is the first known example of invariant subspace which cannot be singly generated. On the other hand, by [4, §5.3, Theorem 33], it was shown that every invariant subspace is generated by two of its elements. In more general setting, we can artificially make H^2_0-spaces to have a single generator.

For each t in $\mathbb{R}$, let us denote by e_t the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for λ in Γ. The map sending t to e_t embeds $\mathbb{R}$ continuously onto a dense subgroup of K. Define a one-parameter group $\{T_t\}_{t\in\mathbb{R}}$ of homeomorphisms on K by

$$T_t\, x \;=\; x + e_t\,, \qquad x \in K. \tag{1.2}$$

Then the pair $(K, \{T_t\}_{t\in\mathbb{R}})$ is a strictly ergodic flow, for which σ is the unique invariant probability measure. The flow $(K, \{T_t\}_{t\in\mathbb{R}})$ is called an *almost periodic flow*, because if ϕ is continuous on K, then $t \to \phi(x + e_t)$ is a uniformly almost periodic function with exponents in Γ. Let $H^\infty(dt/\pi(1+t^2))$ be the space of all boundary functions of bounded analytic functions in the upper half-plane $\mathcal{H}$, and let $H^p(dt/\pi(1+t^2)), 1 \leqslant p < \infty$, be the closure of $H^\infty(dt/\pi(1+t^2))$ in $L^p(dt/\pi(1+t^2))$. For a function $u(x,t)$ on $K \times \mathbb{R}$, the assertion " $t \to u(x,t)$ for $\sigma - a.e.\, x$ in K " is sometimes abbreviated to "*almost every* $t \to u(x,t)$ ". Then ϕ in $L^p(\sigma)$ lies in $H^p(\sigma)$ if and only if almost every $t \to \phi(x + e_t)$ lies in $H^p(dt/\pi(1+t^2))$. This fact enables us to define Hardy spaces on every ergodic flow (see the end of the next section).

Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$. Set $\mathfrak{M}_\lambda = \chi_\lambda \mathfrak{M}$ for each λ in Γ. Define

$$\mathfrak{M}_+ \;=\; \bigwedge_{\lambda<0} \mathfrak{M}_\lambda \qquad \text{and} \qquad \mathfrak{M}_- \;=\; \bigvee_{\lambda>0} \mathfrak{M}_\lambda .$$

Since these spaces are at most one dimension apart, $\mathfrak{M}$ coincides with either or both its versions $\mathfrak{M}_+$ and $\mathfrak{M}_-$. When $\mathfrak{M} = \mathfrak{M}_+$, $\mathfrak{M}$ is said to be *normalized*. For ϕ in $L^2(\sigma)$, the subspace $\mathfrak{M}[\phi]$ is simply invariant if and only if

$$\int_{-\infty}^{\infty} \log|\phi(x + e_t)|\, \frac{dt}{1+t^2} \;>\; -\infty\,, \qquad \sigma - a.e.\; x \in K, \tag{1.3}$$

(see [4, §3.3, Theorem 22]). It is well-known that there is a function ϕ in $L^2(\sigma)$ satisfying the inequality (1.3), while $\log|\phi|$ does not belong to $L^1(\sigma)$. Our Theorem asserts that any such function ϕ must satisfy $\mathfrak{M}[\phi]_+ = \mathfrak{M}[\phi]_-$.

A unitary Borel function $A(x,t)$ on $K \times \mathbb{R}$ is said to be a *cocycle* on K if $A(x,t)$ satisfies the *cocycle identity*

$$A(x, t+s) \;=\; A(x,t) \cdot A(x + e_t, s)\,, \qquad (x,s,t) \in K \times \mathbb{R} \times \mathbb{R}\,.$$

We identify two cocycles which differ only on a set of $d\sigma \times dt-$measure zero in $K \times \mathbb{R}$. A one-to-one correspondence is established between normalized invariant subspaces and cocycles (as discussed in [4, §2.3]). More precisely, let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x,t)$. Then a function ϕ in $L^2(\sigma)$ lies in $\mathfrak{M}_+$ if and only if almost every $t \to A(x,t)\phi(x+e_t)$ lies in $H^2(dt/\pi(1+t^2))$ (see [4, §3.2]). It is easy to see that $\mathfrak{M}_+ \neq \mathfrak{M}_-$ if and only if $\mathfrak{M}_+ = qH^2(\sigma)$ for some unitary function q on K. Then the cocycle of $\mathfrak{M}$ has the form $q(x) \cdot \overline{q(x+e_t)}$, which is called a *coboundary*. If a cocycle is a coboundary multiplied by $\exp(i\alpha t)$ for some α in $\mathbb{R}$, then such a cocycle is said to be *trivial*. A trivial cocycle $\exp(i\alpha t)$ is not a coboundary only if α lies in $\mathbb{R} \setminus \Gamma$.

We already know from [5] and [10] that some singly generated subspaces have nontrivial cocycles, but we can strengthen this fact by noting the following:

Corollary 1.1. — *Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$. If the cocycle of $\mathfrak{M}$ is trivial, then $\mathfrak{M}_-$ has no single generator. In other words, if $\mathfrak{M}_-$ is singly generated, then the cocycle of $\mathfrak{M}$ is always nontrivial, so that $\mathfrak{M}_+ = \mathfrak{M}_-$.*

A cocycle with values in $\{-1, 1\}$ is called a *real* cocycle. It follows from [7] that there exist real cocycles which are nontrivial.

Corollary 1.2. — *Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ with real cocycle. Then $\mathfrak{M}_-$ has no single generator.*

A cocycle $A(x,t)$ is said to be *analytic* if almost every $t \to A(x,t)$ lies in $H^\infty(dt/\pi(1+t^2))$. Then a normalized invariant subspace with analytic cocycle contains always $H^2(\sigma)$. We say that an analytic cocycle $A(x,t)$ is a *Blaschke* or a *singular* cocycle, if almost every $t \to A(x,t)$ is an inner function of that type in $H^\infty(dt/\pi(1+t^2))$. Two cocycles are called *cohomologous* if one is a coboundary times the other. It is known that every cocycle is cohomologous to a Blaschke cocycle in some restricted class (see [4, §4.6, Theorem 26] and [15]). This fact makes Blaschke cocycles so important for the subject. Using our Theorem, we may answer some questions on analytic cocycles:

Corollary 1.3. — *In the class of analytic cocycles, the following properties hold:*

(a) *There is a Blaschke cocycle not being cohomologous to any singular cocycle.*
(b) *There is a Blaschke cocycle not having exactly the same zeros as any function in $H^2(\sigma)$.*

It would be helpful to understand the basic idea behind the proof of our Theorem. On the one hand, we claim that if ϕ is a single generator of $H_0^2(\sigma)$, then ϕ must have a very special form. Assume that Γ is the smallest group determined by the nonzero Fourier coefficients of ϕ (see below for details). Similarly, let Λ be the smallest group determined by the nonzero coefficients of $|\phi|$. Since Λ is a subgroup of Γ, the dual group of Λ is represented as K/H, where H is the annihilator of Λ in K. Let τ be the normalized Haar measure on K/H, and fix an element α in Γ with $a_\alpha(\phi) \neq 0$. Then it can be shown that $\overline{\chi}_\alpha \phi$ lies in $L^2(\tau)$ and generates the simply invariant subspace of $L^2(\tau)$ with trivial cocycle $\exp(i\alpha t)$. We also see that α is independent of Λ, meaning that $n\alpha$ lies in Λ only for $n = 0$ in the integer group $\mathbb{Z}$. This implies that K and $d\sigma$ are respectively identified with $K/H \times \mathbb{T}$ and $d\tau \times d\theta/2\pi$, since H is regarded as $\mathbb{T}$. Thus, for each single generator ϕ of $H_0^2(\sigma)$, we derive that $\Gamma \neq \Lambda$. On the other hand, if $H_0^2(\sigma)$ is singly generated, we may construct a generator ϕ of $H_0^2(\sigma)$ with the property that $\Gamma = \Lambda$, which contradicts the existence of single generator of $H_0^2(\sigma)$.

In the next section, we establish some notation and elementary facts about invariant subspaces in the almost periodic setting. Using group characters, we develop certain properties of single generators of H_0^2-spaces in Section 3. In Section 4, the proof of our Theorem is provided and then Corollaries are proved by using a lemma on cocycles. We conclude the paper with some remarks in Section 5.

We refer the reader to [9], [3, Chapter VII], [4] and [14, Chapter VIII] for further details on analyticity on compact abelian groups. Basic results concerning the Hardy space theory based on uniform algebras can be found in [3, Chapter IV] and [11].

Part of this work was done while the author was visiting the University of North Carolina at Chapel Hill, and he would like to acknowledge the hospitality of the Department of Mathematics. Especially, he would like to express his sincere gratitude to Professors Joe Cima and Karl Petersen for helpful discussions. Thanks are due to the referee as well for his valuable suggestions, which improved the first version of this paper so much.

2. Extension of almost periodic functions

It is easy to show that a function ϕ in $H^2(\sigma)$ is outer if and only if $a_0(\phi) \neq 0$ and almost every $t \to \phi(x + e_t)$ is outer in $H^2(dt/\pi(1 + t^2))$. A weak version of this fact stated below is often used in what follows:

Lemma 2.1. — *Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x,t)$. A function ϕ in $L^2(\sigma)$ generates $\mathfrak{M}_-$ if and only if* $\log|\phi|$ *does not lie in $L^1(\sigma)$ and almost every $t \to A(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. In particular, $H^2_0(\sigma)$ is singly generated by ϕ if and only if $a_0(\phi) = 0$ and almost every $t \to \phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$.*

Proof. — Suppose that $\mathfrak{M}[\phi] = \mathfrak{M}_-$ for ϕ in $L^2(\sigma)$. If $\log|\phi|$ lies in $L^1(\sigma)$, then there is a unitary function q on K such that $\mathfrak{M}[\phi] = qH^2(\sigma)$ by Szegö's theorem. This implies that $\mathfrak{M}[\phi] \neq \mathfrak{M}_-$, so $\log|\phi|$ cannot lie in $L^1(\sigma)$. Let $B(x,t)$ be the analytic cocycle defined by the inner part of $t \to A(x,t)\phi(x+e_t)$. Let $\mathfrak{N}$ be the invariant subspace with cocycle $A\overline{B}(x,t)$. By [4, §3.2, Theorem 21], we see that $\mathfrak{N}_-$ is contained in $\mathfrak{M}_-$. On the other hand, since almost every $t \to A\overline{B}(x,t)\psi(x+e_t)$ lies in $H^2(dt/\pi(1+t^2))$ for each ψ in $\mathfrak{M}[\phi]$, $\mathfrak{N}_+$ includes $\mathfrak{M}[\phi]$. This shows that $\mathfrak{N}_+ = \mathfrak{M}_+$, so $B(x,t) \equiv 1$. Then almost every $t \to A(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$.

Conversely, suppose that $\mathfrak{M}[\phi]$ is contained strictly in $\mathfrak{M}_-$. Then there is a nonzero function q in $\mathfrak{M}_-$ such that

$$\int_K \psi\phi\overline{q}\,d\sigma = 0, \qquad \psi \in H^\infty(\sigma).$$

This shows that $\phi\overline{q}$ lies in $H^1(\sigma)$, so almost every $t \to \phi\overline{q}(x+e_t)$ lies in $H^1(dt/\pi(1+t^2))$. Notice that $t \to A(x,t)q(x+e_t)$ is in $H^2(dt/\pi(1+t^2))$. Since

$$\phi(x+e_t)\overline{q(x+e_t)} = A(x,t)\phi(x+e_t)\overline{A(x,t)q(x+e_t)},$$

and since $t \to A(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$, we see that almost every $t \to \overline{A(x,t)q(x+e_t)}$ is also in $H^2(dt/\pi(1+t^2))$. This shows that $t \to A(x,t)q(x+e_t)$ is constant for $\sigma - a.e.\, x$ in K, and so is $t \to |q(x+e_t)|$. It follows from the ergodic theorem that $|q(x)|$ is constant. We then assume q is a unitary function on K. Therefore, $A(x,t)$ is the coboundary $q(x)\overline{q(x+e_t)}$ and $\mathfrak{M}_- = qH^2_0(\sigma)$. Thus q does not lie in $\mathfrak{M}_-$, which is a contradiction.

The last part of assertion follows from the fact that the cocycle of $H^2(\sigma)$ equals 1. Under the assumption that almost every $t \to \phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$, we see easily $a_0(\phi) = 0$ if and only if $\log|\phi|$ does not lie in $L^1(\sigma)$. Then $\mathfrak{M}[\phi] = H^2_0(\sigma)$, so the proof is complete. □

Let $L^1(dt)$ be the usual Lebesgue space on $\mathbb{R}$. Using $\{T_t\}_{t\in\mathbb{R}}$, one may convolve a function ϕ in $L^p(\sigma), 1 \leqslant p < \infty$, with a function f in $L^1(dt)$ by setting

$$(\phi * f)(x) \;=\; \int_{-\infty}^{\infty} \phi(x+e_t)f(-t)\,dt \;=\; \int_{-\infty}^{\infty} \phi(x-e_t)f(t)\,dt\,,$$

where the integral is a Bochner integral. When $p=\infty$, the convolution $\phi * f$ is defined in the same way as the weak*-convergent integral. Under the operation of convolution, $L^p(\sigma)$ becomes an $L^1(dt)$-module such that

$$\|\phi * f\|_p \;\leqslant\; \|\phi\|_p \|f\|_1\,, \qquad \phi \in L^p(\sigma)\,,$$

for f in $L^1(dt)$. The Fourier transform $\hat{f}$ of f is defined by the formula

$$\hat{f}(\lambda) \;=\; \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\,dt, \qquad \lambda \in \mathbb{R}\,, \tag{2.1}$$

as usual. We see easily $a_\lambda(\phi * f) = a_\lambda(\phi)\hat{f}(\lambda)$, if λ is in Γ. The Poisson kernel $P_{ir}(t)$ for $\mathcal{H}$ is given by $P_{ir}(t) = r/\pi(t^2+r^2)$ for an $r>0$. If ϕ is in $L^1(\sigma)$, then the convolution $\phi * P_{ir}$ is considered as the Poisson integral of $t \to \phi(x+e_t)$, that is,

$$(\phi * P_{ir})(x+e_s) \;=\; \int_{-\infty}^{\infty} \phi(x+e_t)P_{ir}(s-t)\,dt\,.$$

Lemma 2.2. — *Suppose that $H_0^2(\sigma)$ is singly generated. Then we obtain the following properties:*

(a) *There is a single generator of $H_0^2(\sigma)$ that is bounded.*
(b) *If ϕ is a bounded generator of $H_0^2(\sigma)$, then so is each of the functions $\phi * P_{ir}$ with $r>0$ and ϕ^n for $n=1,2,\cdots$.*

Proof. — Let ψ be a single generator of $H_0^2(\sigma)$. Then there is an outer function h in $H^2(\sigma)$ such that $|h| = \min(1,|\psi|^{-1})$. From Lemma 2.1, we deduce that the bounded function ψh generates $H_0^2(\sigma)$, thus we obtain (a).

To show (b), we observe that $t \to (\phi * P_{ir})(x+e_t)$ as well as $t \to \phi^n(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$ for $\sigma - a.e.\ x$ in K. Since $a_0(\phi * P_{ir}) = a_0(\phi^n) = 0$, (b) follows from Lemma 2.1 immediately. □

We next introduce a local product decomposition of K, which is useful for studying analytic functions on K. Fix a positive γ in Γ, and let K_γ be the closed subgroup of all x in K such that $\chi_\gamma(x)=1$. Then $K_\gamma \times [0,2\pi/\gamma)$ is identified with K via the map $(y,s) \to y+e_s$. Let σ_1 be the normalized Haar measure on K_γ. Then the probability measure $(\gamma/2\pi)d\sigma_1 \times dt$ on $K_\gamma \times [0,2\pi/\gamma)$ is carried by the map to $d\sigma$ on K. The one-parameter group $\{T_t\}_{t\in\mathbb{R}}$ given by (1.2) is represented as

$$T_t(y,s) \;=\; (y + [(t+s)\gamma/2\pi]e_{2\pi/\gamma}, t+s-[(t+s)\gamma/2\pi]2\pi/\gamma)$$

on $K_\gamma \times [0, 2\pi/\gamma)$, where $[t]$ is the largest integer not exceeding t. Define the homeomorphism T on K_γ by $Ty = y + e_{2\pi/\gamma}$. We denote by $\mathcal{O}(\omega, T)$ the orbit of a point ω in (K_γ, T), that is, the set of all $T^n\omega$ for n in $\mathbb{Z}$. Since $\mathcal{O}(\omega, T)$ is dense in K_γ, the discrete flow (K_γ, T) is also a strictly ergodic flow, on which σ_1 is the unique invariant probability measure. Since Γ is countable, K_γ is metrizable (see [14, 2.2.6]).

A function ϕ on K has the *automorphic extension* $\phi^\sharp$ to $K_\gamma \times \mathbb{R}$ defined by

$$\phi^\sharp(y,t) \;=\; \phi(y + [t\gamma/2\pi]e_{2\pi/\gamma}\,, t - [t\gamma/2\pi]2\pi/\gamma).$$

Since a function f in $H^1(dt/\pi(1+t^2))$ extends analytically to $\mathcal{H}$ by $f(s + ir) = (f * P_{ir})(s)$, we write

$$\phi^\sharp(y,z) \;=\; (\phi^\sharp * P_{ir})(y,s), \qquad z = s + ir \in \mathcal{H}\,,$$

for each ϕ in $H^1(\sigma)$. It is clear that $(\phi^\sharp * P_{ir})(y,s) = (\phi * P_{ir})^\sharp(y,s)$ on $K_\gamma \times \mathbb{R}$.

The following is due to a property of Lebesgue sets.

Lemma 2.3. — *If E_1 is a compact subset of K_γ with $\sigma_1(E_1) > 0$, then there is a closed subset E of E_1 with $\sigma_1(E_1) = \sigma_1(E)$ such that $\mathcal{O}(\omega, T) \cap E$ is dense in E, for $\sigma_1 - a.e.\,\omega$ in K_γ.*

Proof. — Recall that the *metric density* of E_1 is 1 at $\sigma_1 - a.e.\,\omega$ in E_1, meaning that

$$\lim_{\delta \to 0} \frac{\sigma_1(E_1 \cap B(\omega,\delta))}{\sigma_1(B(\omega,\delta))} \;=\; 1\,,$$

where $B(\omega,\delta)$ is the open ball with center ω and radius $\delta > 0$. Define E to be the closure of the set of points of E_1 at which the metric density of E_1 is 1. Clearly, we have $\sigma_1(E_1) = \sigma_1(E)$, since E_1 is closed. If $\sigma_1(E) = 1$, then $E = K_\gamma$. Since (K_γ, T) is strictly ergodic every orbit $\mathcal{O}(\omega, T)$ is dense in E. Assume that $0 < \sigma_1(E) < 1$. It follows from the ergodic theorem that there is a σ_1−null set N in K_γ outside which

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} I_E(T^j\omega) \;=\; \sigma_1(E)\,,$$

where I_E denotes the characteristic function of E. Let H_ω be the closure of $\mathcal{O}(\omega, T) \cap E$ in K_γ. We claim that if $E \neq H_\omega$, then ω lies in N. Indeed, we see that $\sigma_1(E \setminus H_\omega) > 0$, since the metric density of E does not vanish identically on $E \setminus H_\omega$. Let p be a continuous function on K_γ such that $0 \leqslant p \leqslant 1$, $p \equiv 1$ on H_ω, and $\int_{K_\gamma} p\, d\sigma < \sigma_1(E)$. Since $I_E(T^j\omega) = I_{H_\omega}(T^j\omega)$

for j in $\mathbb{Z}$ and since (K_γ, T) is strictly ergodic, we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} I_E(T^j\omega) \leqslant \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} p(T^j\omega) = \int_{K_\gamma} p\, d\sigma_1 < \sigma_1(E)$$

by [13, §4.2, Proposition 2.8]. Thus ω has to lie in the null set N. $\square$

For each ϕ in $H^\infty(\sigma)$, there is a σ_1-null set of K_γ outside which $z \to \phi^\sharp(y,z)$ is analytic and uniformly bounded on the upper half plane $\mathcal{H}$. Recall that if a family of analytic functions is uniformly bounded, then it forms a normal family. The next proposition may be regarded as a strengthened form of Lusin's theorem for analytic functions on K, so that it has some interest of its own. Here we denote by $cl(\mathcal{H})$ the closure of $\mathcal{H}$ in $\mathbb{R}^2$.

Proposition 2.4. — *Let ϕ be a function in $H^\infty(\sigma)$, and let $\epsilon > 0$. Then there is a closed subset E of K_γ with $\sigma_1(E) > 1-\epsilon$ having the following properties:*

(a) *The convolution $(\phi^\sharp * P_{ir})(y,t)$ is continuous on $E\times\mathbb{R}$, for a given $r > 0$.*

(b) *For $\sigma_1 - a.e.\, \omega$ in K_γ, the function $(\phi^\sharp * P_{ir})(T^j\omega, z)$ on $(\mathcal{O}(\omega,T)\cap E)\times cl(\mathcal{H})$ extends to $(\phi^\sharp * P_{ir})(y,z)$ on $E\times cl(\mathcal{H})$.*

Proof. — Since $\phi * P_{ir}$ lies in $H^\infty(\sigma)$, Lusin's theorem asserts that there is a compact subset F of K with $\sigma(F) > 1-\epsilon^2$ on which $\phi * P_{ir}$ is continuous. Regarding F as a subset of $K_\gamma \times [0, 2\pi/\gamma)$, we choose a compact subset E of K_γ with $\sigma_1(E) > 1-\epsilon$ such that E satisfies the property of Lemma 2.3 and

$$\frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} I_F(y,s)\, ds > 1-\epsilon\,, \qquad y\in E\,. \tag{2.2}$$

In addition, we assume that $z \to (\phi^\sharp * P_{ir/2})(y,z)$ is analytic on $\mathcal{H}$ and

$$|(\phi^\sharp * P_{ir/2})(y,z)| \leqslant \|\phi\|_\infty\,, \qquad y\in E\,.$$

Then the family

$$\mathcal{F} = \left\{ (\phi^\sharp * P_{ir/2})(y,z)\,;\, y\in E\right\}$$

forms a normal family on $\mathcal{H}$. Let $\{y_n\}$ be a sequence in E tending to y. Since $\mathcal{F}$ is normal, there is a subsequence $\{y_j\}$ of $\{y_n\}$ such that $(\phi^\sharp * P_{ir/2})(y_j, z)$ converges uniformly on compact subsets of $\mathcal{H}$ to a bounded analytic function $f(z)$ on $\mathcal{H}$. Let us show that $f(z) = (\phi^\sharp * P_{ir/2})(y,z)$. Indeed, we observe by (2.2) that $F\cap(\{y\}\times[0,2\pi/\gamma))$ contains an infinite compact set of the form $\{y\}\times J$. Since

$$(\phi^\sharp * P_{ir})(y,t) = (\phi^\sharp * P_{ir/2})(y, t+ir/2) = f(t+ir/2), \qquad t\in J\,,$$

it follows from the uniqueness principle that $f(z) = (\phi^\sharp * P_{ir/2})(y,z)$. This shows that if (y_n, t_n) tends to (y,t), then $(\phi^\sharp * P_{ir})(y_n,t_n)$ tends to $(\phi^\sharp * P_{ir})(y,t)$. Thus (a) holds. We notice that $(\phi^\sharp * P_{ir/2})(y,z)$ is also continuous on $E \times \mathcal{H}$.

On the other hand, by Lemma 2.3, $\mathcal{O}(\omega,T) \cap E$ is dense in E for $\sigma_1 - a.e.\,\omega$ in K_γ. Since $(\mathcal{O}(\omega,T)\cap E) \times cl(\mathcal{H})$ is dense in $E \times cl(\mathcal{H})$ and since $(\phi^\sharp * P_{ir})(y,z)$ is continuous on $E \times cl(\mathcal{H})$, the function $(\phi^\sharp * P_{ir})(T^j\omega,t)$ on $(\mathcal{O}(\omega,T)\cap E)\times cl(\mathcal{H})$ extends to $(\phi^\sharp * P_{ir})(y,t)$ on $E\times\mathcal{H}$. Thus (b) follows immediately. □

We make some remarks on Proposition 2.4. Since $t \to \phi^\sharp(y,t)$ lies in $H^\infty(dt/\pi(1+t^2))$ for each y in E, we see that $(\phi^\sharp * P_{ir})(y, t+2\pi/\gamma) = (\phi^\sharp * P_{ir})(Ty,t)$. Then $E \cup TE \cup \cdots \cup T^nE$ also satisfies the properties (a) and (b) and $\sigma_1(E\cup TE\cup\cdots\cup T^nE)$ converges to 1, as $n\to\infty$, by the recurrence theorem (see [13, §2.3, Theorem 3.2]). However, to obtain ϕ itself, we need a version of Fatou's theorem as discussed in [12, Theorem II]. Denote by $\mathcal{O}(x,\{T_t\}_{t\in\mathbb{R}})$ the *orbit* of x in $(K,\{T_t\}_{t\in\mathbb{R}})$. With the notation above, when $x=(y,s)$ in $K_\gamma\times[0,2\pi/\gamma)$, we see that $\mathcal{O}(x,\{T_t\}_{t\in\mathbb{R}}) = \mathcal{O}(y,T)\times[0,2\pi/\gamma)$. For x in K, we say that $t\to(\phi * P_{ir})(x+e_t)$ *extends to* $\phi * P_{ir}$ if, for each $\epsilon>0$, there is a compact subset $F=F(\epsilon,\phi)$ of K with $\sigma(F) > 1-\epsilon$ such that $\phi * P_{ir}$ is continuous on F and $\mathcal{O}(x,\{T_t\}_{t\in\mathbb{R}})\cap F$ is dense in F. The above proof may be modified so as to apply to functions in $H^1(\sigma)$ as well.

The next lemma is an immediate consequence of Proposition 2.4.

Lemma 2.5. — *Let ϕ be a function in $H^\infty(\sigma)$, and let $r>0$. Then there is an invariant $\sigma-$null set $N=N(\phi)$ in K outside which $t\to(\phi * P_{ir})(x+e_t)$ extends to $\phi * P_{ir}$.*

Proof. — For a given $\epsilon>0$, let E be a closed subset of K_γ with $\sigma_1(E) > 1-\epsilon$ which has the property (a) and (b) of Proposition 2.4. Putting $F = E\times[0,2\pi/\gamma]$, we regard F as a compact subset of K. By (b) of Proposition 2.4, we choose an invariant null set $N' = N'(\phi)$ in (K_γ, T) outside which $\mathcal{O}(\omega,T)\cap E$ is dense in E. If we set $N = N'\times[0,2\pi/\gamma)$, then the $\sigma-$null set N satisfies the desired property. □

Let Ω be a compact metric space on which $\mathbb{R}$ acts as a Borel transformation group. This means that there is a one-parameter group $\{U_t\}_{t\in\mathbb{R}}$ of Borel isomorphisms on Ω such that the map $(\omega,t)\to U_t\omega$ of $\Omega\times\mathbb{R}$ to Ω is a Borel map. The pair $(\Omega,\{U_t\}_{t\in\mathbb{R}})$ is referred to a *Borel flow*. Especially, $(\Omega,\{U_t\}_{t\in\mathbb{R}})$ is called a *continuous flow*, if U_t is a homeomorphism on Ω and the map $(\omega,t)\to U_t\omega$ is continuous on $\Omega\times\mathbb{R}$. We often write $\omega+t$

for the translate $U_t\omega$ of ω by t. Let μ be an invariant probability measure on $(\Omega, \{U_t\}_{t\in\mathbb{R}})$ which is *ergodic*, meaning that $\mu(E) = 1$ or 0 for each invariant subset E of Ω. A function ϕ in $L^1(\mu)$ is *analytic* if $t \to \phi(\omega + t)$ lies in $H^1(dt/\pi(1+t^2))$ for $\mu - a.e.\,\omega$ in Ω. Then the *ergodic Hardy space* $H^p(\mu), 1 \leqslant p \leqslant \infty$, is defined to be the space of all analytic functions in $L^p(\mu)$. It follows from [11, Theorem I] that μ is a representing measure for $H^\infty(\mu)$, for which $H^\infty(\mu)$ is a weak*-Dirichlet algebra in $L^\infty(\mu)$. This fundamental result enables us to apply the Hardy space theory based on uniform algebras, and most of the machinery of invariant subspaces on an almost periodic flow $(K, \{T_t\}_{t\in\mathbb{R}})$ can be reconstructed (see [1], [11] and [12] for related topics). As we mentioned earlier, the H_0^2-spaces may be singly generated in the situation of ergodic flows other than almost periodic flows (see [16] and §5 (b)).

Let $A(x,t)$ be a cocycle on an almost periodic flow $(K, \{T_t\}_{t\in\mathbb{R}})$ and define the Borel flow $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ by

$$S_t(x, e^{i\theta}) = (T_t x, A(x,t)e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T}, \tag{2.3}$$

which is called the *skew product* of K and $\mathbb{T}$ induced by $A(x,t)$. Then $d\sigma \times d\theta/2\pi$ is an invariant probability measure on $K \times \mathbb{T}$. Observe that each function f in $L^2(d\sigma \times d\theta/2\pi)$ is represented as

$$f(x, e^{i\theta}) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{in\theta},$$

where the coefficients ϕ_n are in $L^2(\sigma)$. From this fact, it follows easily that $d\sigma \times d\theta/2\pi$ is ergodic on $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ if and only if $A(x,t)^n$ is a coboundary only for $n = 0$ (see [4, §6.2] for details).

3. Approximation to generators

We now turn to the structure of compact group K, under the assumption that $H_0^2(\sigma)$ is singly generated by ϕ in $H_0^2(\sigma)$. By multiplying by a suitable outer function, if necessary, we can assume that ϕ is a function in $L^\infty(\sigma)$ with $1 \leqslant \|\phi\|_\infty < \infty$. Furthermore, we also assume that Γ is the smallest group containing all λ such that $a_\lambda(\phi) \neq 0$, that is, the smallest group over which Fourier series,

$$\phi(x) \sim \sum_{\Gamma\ni\lambda>0} a_\lambda(\phi)\chi_\lambda(x),$$

holds. Similarly, denote by Λ the smallest group containing all λ such that $a_\lambda(|\phi|) \neq 0$. We observe that the Fourier series of

$$\left(|\phi|^2 + \epsilon\right)^{1/2} \;=\; \exp\left\{\frac{1}{2}\,\log\left(\phi\overline{\phi} + \epsilon\right)\right\}, \qquad \epsilon > 0\,,$$

is represented on Γ, by considering the Taylor series of $z \to \log z$ at a large positive. This shows that Λ is a subgroup of Γ, since

$$a_\lambda(|\phi|) \;=\; \lim_{\epsilon\to+0}\, a_\lambda\left(\left(|\phi|^2+\epsilon\right)^{1/2}\right)$$

by (1.1). Since $\log|\phi|$ does not lie in $L^1(\sigma)$, the generator ϕ cannot be periodic in $(K, \{T_t\}_{t\in\mathbb{R}})$. Then Γ as well as Λ is a countable dense subgroup of $\mathbb{R}$, endowed with discrete topology. Let H be the annihilator of Λ, meaning that H is the closed subgroup of all x in K such that $\chi_\lambda(x) = 1$ for all λ in Λ. Then the dual group of Λ is identified with the quotient group K/H (see [14, 2.1]). We denote by τ the normalized Haar measure on K/H. Let π be the canonical homomorphism of K onto K/H. For each x in K, we write $\bar{x}$ for $\pi(x) = x + H$. When a function ψ on K is represented as $\psi = \tilde{\psi}\circ\pi$ for a function $\tilde{\psi}$ on K/H, we usually identify ψ with $\tilde{\psi}$, so that $\psi(x) = \psi(\bar{x})$. Then we say descriptively that ψ *is generated by a function on* K/H. If $1 \leqslant p \leqslant \infty$, then $L^p(\tau)$ and $H^p(\tau)$ are subspaces of $L^p(\sigma)$ and $H^p(\sigma)$, respectively.

Since almost every $t \to \phi(x + e_t)$ is outer in $H^\infty(dt/\pi(1+t^2))$ by Lemma 2.1, we see that

$$-\infty \;<\; \log|(\phi * P_{ir})(x)| \;=\; (\log|\phi| * P_{ir})(x)$$

for a given $r > 0$. Since $\log|\phi|$ is not in $L^1(\sigma)$ and $\log|\phi| \leqslant \|\phi\|_\infty$, Fubini's theorem shows that

$$\int_K \log|\phi * P_{ir}|\,d\sigma \;=\; \int_K (\log|\phi| * P_{ir})\,d\sigma \;=\; \int_K \log|\phi|\,d\sigma \;=\; -\infty\,.$$

Let $g = \phi * P_{ir}$. Then Lemma 2.1 shows that g is also a bounded generator of $H^2_0(\sigma)$. Since $\hat{P}_{ir}(\lambda) = e^{-r|\lambda|}$ by (2.1), we obtain $a_\lambda(g) = a_\lambda(\phi * P_{ir}) = a_\lambda(\phi)e^{-r|\lambda|}$, hence $a_\lambda(\phi) \neq 0$ if and only if $a_\lambda(g) \neq 0$. Thus the generator g plays the same role as ϕ. For $n = 1, 2, \ldots$, we then denote by ϕ_n the outer function in $H^\infty(\tau)$ with $|\phi_n| = \max(1/n, |\phi|)$. Since $-\log n \leqslant \log|\phi_n| \leqslant \|\phi\|_\infty$, each ϕ_n^{-1} is also an outer function in $H^\infty(\tau)$. Putting $g_n = \phi_n * P_{ir}$, we obtain a sequence $\{g_n\}$ of outer functions in $H^\infty(\tau)$ with $\|g_n\|_\infty \leqslant \|\phi\|_\infty$. Notice that $t \to g(x + e_t)$ and $t \to g_n(x + e_t)$ extend analytically up to $\{Re\, z > -r\}$. Let us look into the relation between g and g_n. Since

$$|g_n(x)| \;=\; \exp\{(\log|\phi_n| * P_{ir})(x)\}\,,$$

we obtain

$$|g_1(x)| \geqslant |g_2(x)| \geqslant \cdots \geqslant |g_n(x)| \longrightarrow |g(x)|, \qquad n \to \infty, \tag{3.1}$$

for $\sigma - a.e.\, x$ in K. Although g may not be in $L^\infty(\tau)$, we observe that $|g_n(x)| = |g_n(\bar{x})|$ and $|g(x)| = |g|(\bar{x})$. By (3.1), it is easy to see that almost every $t \to |(g/g_n)(x+e_t)|$ converges pointwise to 1 on $\mathbb{R}$. Put $G_n^x(t) = g_n(x+e_t)$ and $G^x(t) = g(x+e_t)$. Let N_0 be an invariant null set in K outside which the property of Lemma 2.5 holds simultaneously for ϕ and all ϕ_n. Moreover, for x in $K \setminus N_0$, we may assume $G_n^x(t)$ and $G^x(t)$ are outer functions in $H^\infty(dt/\pi(1+t^2))$. Then the family of all analytic extensions $G_n^x(z)$ of $G_n^x(t)$ to $\{Re\, z > -r\}$ forms a normal family, since $|G_n^x(z)| \leqslant \|\phi\|_\infty$.

The following lemma is crucial in our proof of the Theorem.

Lemma 3.1. — *For a bounded generator ϕ of $H_0^2(\sigma)$, let Λ, H and τ be as above. Choose an α in Γ with $a_\alpha(\phi) \neq 0$. Then $\overline{\chi_\alpha}\phi$ is generated by a function on K/H, so lies in $L^\infty(\tau)$. Consequently, Γ is generated by Λ and α.*

Proof. — Let $\{\delta_k\}$ be a decreasing sequence tending to 0. Then there is a sequence $\{f_k\}$ in $L^1(dt)$ such that $\hat{f}_k(\alpha) = 1$, $\|f_k\|_1 = 1$ and $\hat{f}_k = 0$ outside $(\alpha - \delta_k, \alpha + \delta_k)$, by modifying the function $t \to (1/\pi)\sin^2 t/t^2$ in $L^1(dt)$. Since $a_\lambda(g) = a_\lambda(\phi)e^{-r|\lambda|}$, we see that $\overline{\chi_\alpha}\phi$ lies in $L^2(\tau)$ if and only if so does $\overline{\chi_\alpha}g$. Thus we may replace ϕ with g in our argument. Since $a_\lambda(g * f_k) = a_\lambda(g)\hat{f}_k(\lambda)$, we observe that

$$\|g * f_k - a_\alpha(g)\chi_\alpha\|_2^2 = \sum_{0<|\lambda|<\delta_k} |a_{\alpha+\lambda}(g)\hat{f}_k(\alpha+\lambda)|^2 \to 0, \qquad k \to \infty,$$

by the Parseval theorem and that

$$\|\overline{(g * f_k)}g - \overline{a_\alpha(g)}(\overline{\chi_\alpha}g)\|_2 \leqslant \|g * f_k - a_\alpha(g)\chi_\alpha\|_2\, \|g\|_\infty .$$

From these facts, we conclude that if each $\overline{(g * f_k)}g$ lies in $L^\infty(\tau)$, then so does $\overline{\chi_\alpha}g$. Since the outer function ϕ_n lies in $L^\infty(\tau)$, so do g_n and $g_n * f_k$. Then each $\overline{(g_n * f_k)}g_n$ lies in $L^\infty(\tau)$. Let us show that the sequence $\{\overline{(g_n * f_k)}g_n\}$ converges to $\{\overline{(g * f_k)}g\}$ in $L^2(\sigma)$, from which we obtain that $\overline{(g * f_k)}g$ lies in $L^\infty(\tau)$. Indeed, in the notation above, if we fix an x in $K \setminus N_0$, there is a subsequence $\{g_m\}$ of $\{g_n\}$ such that $\{G_m^x(t)\}$ converges pointwise to $e^{i\gamma}G^x(t)$ in $H^\infty(dt/\pi(1+t^2))$ with $0 \leqslant \gamma < 2\pi$, where γ depends on x and $\{g_m\}$. This implies that

$$\overline{(g_m * f_k)}(x+e_t) \to e^{-i\gamma}\overline{(g * f_k)}(x+e_t), \qquad m \to \infty,$$

pointwise in $L^\infty(dt/\pi(1+t^2))$. Note that every subsequence of $\{g_n\}$ contains such a subsequence $\{g_m\}$. Since $e^{-i\gamma}e^{i\gamma} = 1$, the sequence $\{g_n\}$ itself satisfies

$$\overline{(g_n * f_k)}\, g_n(x+e_t) \to \overline{(g * f_k)}\, g(x+e_t)\,, \qquad n\to\infty,$$

pointwise in $L^\infty(dt/\pi(1+t^2))$. Since

$$\|\overline{(g_n * f_k)}\, g_n\|_\infty \ \leqslant\ \|g_n\|_\infty^2\|f_k\|_1 \ \leqslant\ \|\phi\|_\infty^2\|f_k\|_1,$$

it follows from the bounded convergence theorem that

$$\|\overline{(g_n * f_k)}\, g_n - \overline{(g * f_k)}\, g\|_2 \to 0\,, \qquad n\to\infty,$$

so that $\overline{(g * f_k)}\, g$ lies in $L^\infty(\tau)$. Therefore, $\overline{\chi_\alpha}\, g$ as well as $\overline{\chi_\alpha}\,\phi$ is generated by a function on K/H. On the other hand, by the property of Γ, each element in Γ has the form $\lambda + n\alpha$ for λ in Λ and n in $\mathbb{Z}$, thus the proof is complete. □

Recall that K/H coincides with the dual group of Λ. Let α be as in Lemma 3.1 and let $C(\bar{x},t)$ be the trivial cocycle on K/H defined by $C(\bar{x},t) = \exp(i\alpha t)$. Since α is positive, $C(\bar{x},t)$ is an analytic cocycle. We denote by $(K/H\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ the skew product of K/H and $\mathbb{T}$ induced by $C(\bar{x},t)$, which is the continuous flow obtained by

$$S_t(\bar{x}, e^{i\theta}) \ =\ (T_t\,\bar{x}, C(\bar{x},t)e^{i\theta}), \qquad (\bar{x}, e^{i\theta}) \in K/H\times\mathbb{T}\,.$$

Then $d\tau\times d\theta/2\pi$ is the invariant probability measure on $K/H\times\mathbb{T}$ (see the end of the preceding section). Let us represent the generator g and all the limits of subsequences of $\{g_n\}$ on $K/H\times\mathbb{T}$, which is the smallest product group with such property. Each function ψ on K/H extends naturally to the one on $K/H\times\mathbb{T}$ by setting $\psi(\bar{x}, e^{i\theta}) = \psi(\bar{x})$. Since $|g|$ and g_n are functions on K/H, they belong to $L^\infty(d\tau\times d\theta/2\pi)$.

With the above notation, we fix a w in $K\setminus N_0$. Since $G_n^w(t)$ and $G^w(t)$ are outer functions in $H^2(dt/\pi(1+t^2))$ which extend analytically to $\{Re\, z > -r\}$, we may assume that $G_n^w(t)$ converges pointwise to $G^w(t)$ on $\mathbb{R}$, by multiplying each g_n by a suitable constant of modulus one. By regarding Lemma 2.5, the functions $G_n^w(t)$ and $G^w(t)$ extend to g_n and g, respectively. However, we obtain the following:

Lemma 3.2. — *For $\sigma-a.e.\ x$ in K, $G_n^x(t)$ never converges pointwise on $\mathbb{R}$. Consequently, we find two subsequences $\{g_m\}$ and $\{g_k\}$ of $\{g_n\}$ such that $G_m^x(t)$ and $G_k^x(t)$ converge to $e^{i\beta}G^x(t)$ and $e^{i\gamma}G^x(t)$ with $0\leqslant\beta<\gamma<2\pi$, respectively.*

Proof. — Since $1/n \leqslant |g_n(x)| \leqslant \|\phi\|_\infty$, each g_n^{-1} is also an outer function in $H^\infty(\sigma)$. This implies that almost every $t \to (g/g_n)(x+e_t)$ is an outer function in $H^\infty(dt/\pi(1+t^2))$. Furthermore, since

$$a_0(g/g_n) \;=\; \int_K g/g_n\, d\sigma \;=\; \int_K g\, d\sigma \int_K g_n^{-1}\, d\sigma \;=\; 0\,,$$

Lemma 2.1 assures that each g/g_n is also a single generator of $H_0^2(\sigma)$.

Denote by F the invariant set of all x in K for which $\{G_n^x(t)\}$ itself converges. Suppose that F has positive measure. By (3.1) and the ergodic theorem, $(g/g_n)(x)$ converges to an invariant function on F, so to a constant of modulus one on K. Then the bounded convergence theorem shows that $a_0(g/g_n) \neq 0$ for large n. Such g/g_n cannot be a single generator of $H_0^2(\sigma)$, which contradicts the above observation. □

Let us mention a few remarks derived from Lemma 3.2. When $0 \leqslant \beta < 2\pi$, $\mathcal{Z}(\beta)$ denotes the subgroup of $\mathbb{T}$ generated by $e^{i\beta}$, that is,

$$\mathcal{Z}(\beta) \;=\; \left\{e^{ij\beta}\,;\, j\in\mathbb{Z}\right\}\,.$$

If $\beta/2\pi$ is rational, then the order of $\mathcal{Z}(\beta)$ is finite. Fix two points w and x in $K\setminus N_0$. We assume by Lemma 3.2 that a subsequence $\{g_k\}$ of $\{g_n\}$ satisfies that $G_k^w(t)$ and $G_k^x(t)$ converge respectively to $e^{ij\beta}G^w(t)$ and $e^{i(j+1)\beta}G^x(t)$ for j in $\mathbb{Z}$, by multiplying each g_k by a suitable constant of modulus one. Denote by $\mathcal{O}(\bar{w})$ the orbit $\mathcal{O}(\bar{w}, \{T_t\}_{t\in\mathbb{R}})$ of $\bar{w}$ in $(K/H, \{T_t\}_{t\in\mathbb{R}})$. Then g is determined naturally on $\mathcal{O}(\bar{w})\times\mathcal{Z}(\beta)$ and $\mathcal{O}(\bar{x})\times\mathcal{Z}(\beta)$ to represent the limits of the subsequence $\{g_k\}$ of $\{g_n\}$ on them. For each m in $\mathbb{Z}$, we see also that every limit of $\{g_k^m\}$ is represented on these product subsets.

If ℓ is a positive integer, then g^ℓ as well as ϕ^ℓ is also a bounded generator of $H_0^2(\sigma)$ by Lemma 2.2. We choose an invariant null set $N(\ell)$ including N_0 outside which a subsequence $\{G_j^x(t)^\ell\}$ of $\{G_n^x(t)^\ell\}$ converges to $e^{i\gamma}G^x(t)^\ell$ with $0<\gamma<2\pi$. Define the invariant null set N_1 by $N_1 \;=\; \cup_{\ell=1}^\infty N(\ell)\,.$ When $\ell \;=\; m\,!$, we take again a subsequence $\{G_k^x(t)\}$ of $\{G_j^x(t)\}$ converging to $e^{i\beta(m)}G^x(t)$ with $e^{i\beta(m)\ell} \;=\; e^{i\gamma}$. Then the order of $\mathcal{Z}(\beta(m))$ is larger than m, so $\cup_{m=1}^\infty \mathcal{Z}(\beta(m))$ is dense in $\mathbb{T}$. Therefore, to represent g and all the limits of subsequences of $\{g_n\}$ on each orbit, the product group $K/H\times\mathbb{T}$ is the smallest one. Let us explain the meaning more precisely. Under the assumption of Lemma 3.1, we put $h_\alpha = \overline{\chi_\alpha} g$. Then h_α lies in $L^2(\tau)$. Define the group character $\mathcal{P}_\alpha$ of $K/H\times\mathbb{T}$ by the projection $\mathcal{P}_\alpha(\bar{x}, e^{i\theta}) \;=\; e^{i\theta}$. Since

$$(h_\alpha\mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta})) \;=\; h_\alpha(\bar{x}+e_t)C(\bar{x},t)e^{i\theta} \;=\; h_\alpha(\bar{x}+e_t)e^{i\alpha t}e^{i\theta},$$

the function $t \to (h_\alpha \mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta}))$ is an outer function in $H^\infty(dt/\pi(1+t^2))$ for $d\tau \times d\theta/2\pi - a.e.\,(\bar{x}, e^{i\theta})$ in $K/H \times \mathbb{T}$. Then the outer function $G^x(t)$ equals $t \to (h_\alpha \mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta}))$ for some θ with $0 \leqslant \theta < 2\pi$. In order to represent consistently all kinds of limits of subsequences $\{G_k^x(t)\}$, we require the family of all outer functions $t \to (h_\alpha \mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta}))$ with $0 \leqslant \theta < 2\pi$.

Lemma 3.3. — *Let Γ and Λ be as above. Then Λ cannot be equal to Γ.*

Proof. — Let α be as in Lemma 3.1. Then α lies in Λ if and only if $\Lambda = \Gamma$. We suppose, on the contrary, that α lies in Λ. Since $K/H = K$, let us consider the skew product $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ of K and $\mathbb{T}$ induced by the cocycle $C(x,t) = e^{i\alpha t}$. We use freely the notation above. Since

$$\mathcal{F}(x, e^{i\theta}) = (\overline{\chi_\alpha}\, \mathcal{P}_\alpha)(x, e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T},$$

is an invariant function that is not constant, $d\sigma \times d\theta/2\pi$ is not an ergodic measure on $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. Now K is represented as the local product decomposition $K_\alpha \times [\,0, 2\pi/\alpha)$, in which K_α is the closed subgroup of all x in K such that $\chi_\alpha(x) = 1$. If we put

$$\mathcal{G}(x, e^{i\theta}) = h_\alpha(x)\, \mathcal{P}_\alpha(x, e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T},$$

then, for each $x = (y, s)$ in $K_\alpha \times [0, 2\pi/\alpha)$, the equation

$$\mathcal{G}(S_t(x, e^{i\theta})) = e^{i(\theta+\alpha t)} h_\alpha(x + e_t) = e^{i(\theta-\alpha s)} g(x + e_t) \tag{3.2}$$

holds, since $e^{i(\theta+\alpha t)}\overline{\chi_\alpha}(y+e_s+e_t) = e^{i(\theta-\alpha s)}$ and $h_\alpha = \overline{\chi_\alpha} g$. By regarding $\mathbb{T}$ as the interval $[\,0, 2\pi/\alpha)$, $K \times \mathbb{T}$ is identified with $K_\alpha \times [\,0, 2\pi/\alpha) \times [\,0, 2\pi/\alpha)$. Let E be the subset of $K \times \mathbb{T}$ defined by

$$E = K_\alpha \times \{(s, s)\,; 0 \leqslant s < 2\pi/\alpha\}.$$

Then E is a closed invariant set in $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$, for which $(K, \{T_t\}_{t\in\mathbb{R}})$ is isomorphic to $(E, \{S_t\}_{t\in\mathbb{R}})$ via the map $(y, s) \to (y, s, s)$. We see also that the ergodic measure $d\sigma$ is carried to $(\alpha/2\pi) d\sigma_1 \times ds$ on E by this map, where σ_1 is the normalized Haar measure on K_α. We regard g_n, g and h_α as the functions on $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. Recall that almost every $G_n^x(t)$ and $G^x(t)$ are outer functions in $H^\infty(dt/\pi(1+t^2))$.

Let x be in $K \setminus N_1$ and let $\{g_k\}$ be a subsequence of $\{g_n\}$ such that $G_k^x(t)$ converges pointwise to $t \to e^{i\alpha\beta} e^{i\alpha t} h_\alpha(x + e_t)$ with $0 \leqslant \beta < 2\pi/\alpha$. Notice that $t \to e^{i\alpha\beta} e^{i\alpha t} h_\alpha(x + e_t)$ is an outer function in $H^\infty(dt/\pi(1+t^2))$ and that $|h_\alpha(x + e_t)| = |g(x + e_t)|$. Let $x = (y, s)$ in $K_\alpha \times [\,0, 2\pi/\alpha)$ as above.

Since x may be replaced by any point in the orbit $\mathcal{O}(x)$ of x, we consider x as a function of s on $[0, 2\pi/\alpha)$. It follows from (3.2) that

$$e^{i\alpha\beta}e^{i\alpha t}h_\alpha(y+e_s+e_t) \;=\; e^{i\alpha(\beta-s)}\mathcal{G}(S_t(y+e_s, e^{i\alpha s}))\,, \qquad (s,t)\in[0,2\pi/\alpha)\times\mathbb{R}.$$

Putting $t=0$ and replacing y with $y+e_{[s\alpha/2\pi]}$, if necessary, we observe that

$$e^{i\alpha(\beta-s)}\mathcal{G}(y+e_s, e^{i\alpha s}) \;=\; e^{i\alpha\beta}e^{-i\alpha s}G^y(s)\,, \qquad s\in\mathbb{R}.$$

This shows that $G^y_k(s)$ converges pointwise to $s\to e^{i\alpha\beta}(\overline{\chi_\alpha}\, g)(y+e_s)$, which cannot be an outer function in $H^\infty(dt/\pi(1+t^2))$. Hence any subsequence of $\{G^x_n(t)\}$ cannot converge to an outer function in $H^\infty(dt/\pi(1+t^2))$ for $\sigma-a.e.\, x$ in K. Thus we have a contradiction. □

In view of Lemma 3.3, we know that there are two possibilities in relation to α and Λ. Either $n\alpha$ lies in Λ only for $n=0$ or $\ell\alpha$ lies in Λ for an integer $\ell\geqslant 2$. We claim that the latter case cannot occur, meaning that α is independent to Λ.

Lemma 3.4. — *Let Λ, H and α be as above. Then $n\alpha$ lies in Λ if and only if $n=0$ in $\mathbb{Z}$. Consequently, H is isomorphic to $\mathbb{T}$, so that K and $d\sigma$ are identified with $K/H\times\mathbb{T}$ and $d\tau\times d\theta/2\pi$, respectively.*

Proof. — Suppose that $\ell\alpha$ lies in Λ for some $\ell\geqslant 2$. By Lemma 2.2, ϕ^ℓ is also a bounded generator of $H^2_0(\sigma)$. It follows from Lemma 3.2 that $\chi_{\ell\alpha}$ and $(\overline{\chi_\alpha}\phi)^\ell$ lie in $L^2(\tau)$, so does ϕ^ℓ itself. Let Γ_ℓ and Λ_ℓ be the smallest groups determined by the nonzero Fourier coefficients of ϕ^ℓ and $|\phi^\ell|$ as above. Then they both are subgroups of Λ. On the other hand, since

$$a_\lambda(|\phi|) \;=\; \lim_{\epsilon\to+0}\, a_\lambda\left((|\phi|^\ell+\epsilon)^{1/\ell}\right)\,,$$

each λ in Λ with $a_\lambda(|\phi|)\neq 0$ lies in Λ_ℓ. This implies that $\Lambda=\Lambda_\ell=\Gamma_\ell$. By replacing ϕ with ϕ^ℓ in Lemma 3.3, this gives a contradiction. Thus $n\alpha$ lies in Λ if and only if $n=0$.

Since $C(\bar{x},t)^n$ is a coboundary only for $n=0$, the measure $d\tau\times d\theta/2\pi$ is ergodic on $(K/H\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. Define the isomorphism of $\Lambda\times\mathbb{Z}$ onto Γ by

$$\varrho(\lambda,n) \;=\; \lambda+n\alpha\,, \qquad (\lambda,n)\in\Lambda\times\mathbb{Z}.$$

Then the conjugate map ϱ^* of ϱ is given by $\varrho^*(x)=(\bar{x},e^{i\theta})$ on K, where $\chi_\alpha(x)=e^{i\theta}$. Indeed, we observe that

$$\chi_\lambda(\bar{x})e^{in\theta} \;=\; \langle(\lambda,n),(\bar{x},e^{i\theta})\rangle \;=\; \chi_{\lambda+n\alpha}(x) \;=\; \chi_\lambda(\bar{x})\chi_\alpha(x)^n\,,$$

for each (λ, n) in $\Lambda \times \mathbb{Z}$. Via the map ϱ^*, K is identified with $K/H \times \mathbb{T}$, and $d\tau \times d\theta/2\pi$ is carried by the map to $d\sigma$ on K. $\square$

We notice that the annihilator H of Λ is isomorphic to $\mathbb{T}$, and $|g(x)|$ as well as $|\phi(x)|$ is constant on almost every coset $\bar{x} = x + H$ in K/H.

4. Contradiction to existence

We may now offer our proof of the main result stated in Section 1.

Proof of the Theorem. — Suppose, on the contrary, that a bounded function ϕ generates $H_0^2(\sigma)$. Let Γ and Λ be the dense subgroups of $\mathbb{R}$ defined as in Section 3 with respect to ϕ and $|\phi|$, respectively. Choose an α in Γ with $a_\alpha(\phi) \neq 0$. It follows from Lemma 3.4 that α is independent of Λ and Γ is generated by α and Λ. Let $0 < \beta < 1$. Since the function

$$(1 + \beta\chi_\alpha)^{-1} = \sum_{k=0}^{\infty} (-\beta)^k \chi_{k\alpha}$$

lies in $H^\infty(\sigma)$, $(1 + \beta\chi_\alpha)^2$ is an outer function in $H^\infty(\sigma)$. Define $\phi_1 = (1 + \beta\chi_\alpha)^2\,\phi$. In view of Lemma 2.1, ϕ_1 is also a bounded generator of $H_0^2(\sigma)$. As above, let Γ_1 and Λ_1 be the smallest groups determined by the nonzero Fourier coefficients of ϕ_1 and $|\phi_1|$, respectively. Notice that Γ_1 is a subgroup of Γ. We claim that the generator ϕ_1 cannot satisfy the property of Lemma 3.3. Indeed, since $|\phi_1| = (1 + \beta^2 + \beta\overline{\chi_\alpha} + \beta\chi_\alpha)\,|\phi|$, we obtain by (1.1) that

$$a_\lambda(|\phi_1|) = (1 + \beta^2)a_\lambda(|\phi|) + \beta a_{\lambda+\alpha}(|\phi|) + \beta a_{\lambda-\alpha}(|\phi|).$$

Since α does not lie in Λ, if λ is in Λ, then $a_{\lambda+\alpha}(|\phi|) = a_{\lambda-\alpha}(|\phi|) = 0$. Then we have

$$a_\lambda(|\phi_1|) = (1 + \beta^2)a_\lambda(|\phi|) \quad \text{and} \quad a_{\lambda+\alpha}(|\phi_1|) = \beta a_\lambda(|\phi|),$$

for each λ in Λ. These facts imply that Λ_1 contains Λ and α, so that $\Gamma = \Lambda_1 = \Gamma_1$, which contradicts Lemma 3.3. $\square$

The next proof is of independent interest, because it suggests that our Theorem is regarded essentially as the converse to Corollary 1.1.

Proof of Corollary 1.1. — We consider the case where the cocycle $C(x,t)$ of $\mathfrak{M}$ has the form $C(x,t) = e^{i\alpha t}$. Then $\mathfrak{M}_-$ is the space of all ψ in $L^2(\sigma)$ satisfying that

$$\psi(x) \sim \sum_{\Gamma \ni \lambda > -\alpha} a_\lambda(\psi)\chi_\lambda(x)\,.$$

Suppose that $\mathfrak{M}_-$ has a generator ϕ. Then $\log|\phi|$ does not lie in $L^1(\sigma)$ and we may assume that ϕ is bounded. If $\ell\alpha$ is in Γ for a positive integer ℓ, then the bounded function $(\chi_\alpha \phi)^\ell$ is a single generator of $H_0^2(\sigma)$ by Lemma 2.1, which is contrary to Theorem. We next consider the case that

$$\alpha \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (1/n)\Gamma\,.$$

Since $C(x,t)^n$ is a coboundary only for $n = 0$, the measure $d\sigma \times d\theta/2\pi$ is ergodic on the skew product $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ induced by $C(y,t)$, that is,

$$S_t(x, e^{i\theta}) \;=\; (x + e_t\,,\, e^{i\alpha t} e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T}.$$

Let Γ_1 be the discrete group generated by Γ and α, and let K_1 be the dual group of Γ_1. Since $\varrho(\lambda, n) \;=\; \lambda + \alpha n$ is an isomorphism of $\Gamma \times \mathbb{Z}$ onto Γ_1, the almost periodic flow on K_1 is identified with $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. Then, via the dual map ϱ^* of ϱ, the normalized Haar measure $d\mu$ on K_1 is identified with $d\sigma \times d\theta/2\pi$. Define the function ϕ_1 in $L^2(\mu)$ by $\phi_1(x, e^{i\theta}) \;=\; \phi(x)e^{i\theta}$. Since $\log|\phi|$ does not lie in $L^1(\sigma)$, neither does $\log|\phi_1|$ in $L^1(\mu)$. Since $t \to \phi_1 \circ S_t(x, e^{i\theta})$ is outer in $H^2(dt/\pi(1+t^2))$ for $\mu - a.e.\,(x, e^{i\theta})$ in $K \times \mathbb{T}$, Lemma 2.1 implies that ϕ_1 is a single generator of $H_0^2(\mu)$, which contradicts our Theorem. $\square$

Proof of Corollary 1.2. — Denote by $C(x,t)$ the real cocycle of $\mathfrak{M}$. Suppose that $\mathfrak{M}_-$ has a generator ϕ, for which $\log|\phi|$ does not lie in $L^1(\sigma)$. It follows from Lemma 2.1 that almost every $t \to C(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. We may assume that ϕ is bounded. Since $C(x,t)^2 \;\equiv\; 1$, ϕ^2 is a single generator of $H_0^2(\sigma)$ by Lemma 2.1, which contradicts our Theorem. $\square$

By the same way as above, we may show that if $C(x,t)$ takes only finite values, then $\mathfrak{M}_-$ cannot be singly generated. Indeed, by the cocycle identity, the set of values of $C(x,t)$ forms a group of order k,

$$\mathcal{Z}(2\pi/k) \;=\; \left\{ e^{i2\pi j/k}\;;\; j = 0, \ldots, k-1 \right\}.$$

Then if ϕ generates $\mathfrak{M}_-$, then ϕ^k is a generator of $H_0^2(\sigma)$.

Let $\mathfrak{M}$ be the normalized simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x,t)$. Recall that ψ lies in $\mathfrak{M}$ if and only if almost every $t \to A(x,t)\psi(x+e_t)$ lies in $H^2(dt/\pi(1+t^2))$. Denote by $\widetilde{\mathfrak{M}}$ the invariant subspace with cocycle $\overline{A(x,t)}$ (as discussed in [4, §3.2]). To prove Corollary 1.3. we need the following:

Lemma 4.1. — *Let $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ be as above. If $\mathfrak{M}$ is singly generated, then $(\widetilde{\mathfrak{M}})_-$ cannot be singly generated.*

Proof. — Since $A(x,t)\cdot\overline{A(x,t)}\equiv 1$, $H_0^2(\sigma)$ is the smallest subspace of $L^2(\sigma)$ containing all $\psi_1\psi_2$ with ψ_1 in $\mathfrak{M}\cap L^\infty(\sigma)$ and ψ_2 in $(\widetilde{\mathfrak{M}})_-\cap L^\infty(\sigma)$ (see [4, §3.2, Theorem 20]). Suppose that $(\widetilde{\mathfrak{M}})_-$ is singly generated. Then Lemma 2.1 shows that there are bounded single generators ϕ_1 and ϕ_2 of $\mathfrak{M}$ and $(\widetilde{\mathfrak{M}})_-$, respectively. Thus $\phi_1\phi_2$ is a single generator of $H_0^2(\sigma)$, which contradicts our Theorem. □

Proof of Corollary 1.3.

(a) Let $\mathfrak{M}$ be a simply invariant subspace with nontrivial cocycle $A(x,t)$. It follows from [8] that $\mathfrak{M}$ is singly generated if and only if $A(x,t)$ is cohomologous to a singular cocycle. On the other hand, by [4, §4.6, Theorem 26], every cocycle is cohomologous to a Blaschke cocycle. By virtue of Lemma 4.1, we obtain easily a desired Blaschke cocycle.

(b) From Lemma 4.1, we choose a Blaschke cocycle $B(x,t)$ such that the invariant subspace $\mathfrak{N}$ having the cocycle $\overline{B(x,t)}$ is not singly generated. We claim that $B(x,t)$ satisfies the desired property. Suppose, on the contrary, that some function ψ in $H^2(\sigma)$ has exactly the same zeros as $B(x,t)$. By multiplying by a suitable outer function, we assume that ψ is bounded. Then ψ generates the invariant subspace with cocycle $\overline{B(x,t)S(x,t)}$, where $S(x,t)$ is the singular cocycle determined by the inner part of $t\to\overline{B(x,t)}\psi(x+e_t)$ in $H^2(dt/\pi(1+t^2))$. On the other hand, it follows from [8] and Lemma 2.1 that there is a function h in $L^2(\sigma)$ such that almost every $t\to S(x,t)h(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. Observe that

$$(h\psi)(x+e_t) \;=\; B(x,t)\cdot S(x,t)h(x+e_t)\cdot\overline{B(x,t)S(x,t)}\psi(x+e_t)\,.$$

Since the inner part of $t\to(h\psi)(x+e_t)$ is $t\to B(x,t)$, the subspace $\mathfrak{N}$ is singly generated by $h\psi$, thus we have a contradiction.

□

In the proof of (b) above, if the singular cocycle $S(x,t)$ is a coboundary, then h is taken as a unitary function, otherwise $\log|h|$ does not lie in $L^1(\sigma)$.

5. Remarks

Remark A. It is sometimes useful to study the spectral measures associated with invariant subspaces. Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ and put

$$\mathfrak{M}_\lambda \;=\; \bigwedge_{\lambda\geqslant\nu}\,\chi_\nu\,\mathfrak{M}.$$

for each λ in $\mathbb{R}$. Denote by P_λ the orthogonal projection of $L^2(\sigma)$ onto $\mathfrak{M}_\lambda$. By the property that

$$\bigwedge_{-\infty<\lambda<\infty} \mathfrak{M}_\lambda = \{0\} \qquad \text{and} \qquad \bigvee_{-\infty<\lambda<\infty} \mathfrak{M}_\lambda = L^2(\sigma)\,,$$

we obtain the continuity of the spectral resolution of identity $\{I - P_\lambda\}_{\lambda\in\mathbb{R}}$ on $L^2(\sigma)$, where I is the identity map on $L^2(\sigma)$. Let $A(x,t)$ be the cocycle of $\mathfrak{M}$. By Stone's theorem, a unitary group $\{V_t\}_{t\in\mathbb{R}}$ on $L^2(\sigma)$ is defined as

$$V_t\,\phi(x) = A(x,t)T_t\,\phi(x) = -\int_{-\infty}^{\infty} e^{i\lambda t}\,dP_\lambda\phi(x)\,, \qquad \phi \in L^2(\sigma)\,,$$

where $T_t\,\phi(x) = \phi(x+e_t)$. For a nonzero function ϕ in $L^2(\sigma)$, $-d(P_\lambda\phi,\phi)$ is a finite positive measure on $\mathbb{R}$. On almost periodic flows, by comparing with Lebesgue measure $d\lambda$, the type of such measures is uniquely determined. We then say that each of $\mathfrak{M}$, $A(x,t)$ and $\{V_t\}_{t\in\mathbb{R}}$ is of *absolutely continuous*, or *singular continuous*, or *discrete* type (as discussed in [4, §2.4]). This fact plays an important role to classify invariant subspaces in this special context. It is easy to observe that $A(x,t)$ and $\overline{A(x,t)}$ have the same spectral type, so the following is an immediate consequence of Lemma 4.1.

Proposition 5.1. — *There is a simply invariant subspace of $L^2(\sigma)$ of either absolutely continuous or singular continuous type which has no single generator.*

Let w be a nonnegative function in $L^2(\sigma)$ satisfying (1.3), while $\log w$ does not lie in $L^1(\sigma)$. We know that a cocycle is trivial if and only if it is of discrete type (see [4, §2.4, Theorem 15]). It follows from Corollary 1.1 that the type of $\mathfrak{M}[w]$ has to be continuous. However, we have no idea to decide what kind of continuous spectrum $\mathfrak{M}[w]$ may have.

Remark B. Using a suitable cocycle, we may construct a skew product on which the H_0^2−space is singly generated. Indeed, let w be a bounded function as above and let $A(x,t)$ be the cocycle of $\mathfrak{M}[w]$. By Lemma 2.1 we see that almost every $t \to A(x,t)w(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. Denote by $(K\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ the skew product induced by $A(x,t)$. If $A(x,t)^n, n \geqslant 1$, is a coboundary $\overline{q(x)}q(x+e_t)$ with unitary function q on K, then qw^n is a single generator of $H_0^2(\sigma)$. It then follows from Theorem that $A(x,t)^n$ is a coboundary only for $n=0$. Hence $d\mu = d\sigma\times d\theta/2\pi$ is an ergodic measure on $(K\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. If we set

$$\phi(x,e^{i\theta}) = w(x)e^{i\theta}\,, \qquad (x,e^{i\theta}) \in K\times\mathbb{T}\,,$$

then ϕ is a single generator of $H_0^2(\mu)$, since $\log|\phi|$ does not lie in $L^1(\mu)$ and almost every $t \to \phi(S_t(x, e^{i\theta}))$ is outer in $H^2(dt/\pi(1+t^2))$ (see [16] for another construction).

Remark C. We have a bit of information on the distribution of zeros of functions in $H^2(\sigma)$ which are connected with Dirichlet series (refer to [17] for related topics). Let $\{\lambda_n\}$ be a sequence in Γ such that

$$0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n \longrightarrow \lambda, \qquad n \to \infty,$$

for some λ in Γ. Define a function ψ in $H^2(\sigma)$ by

$$\psi = \sum_{n=1}^{\infty} a_n \chi_{\lambda_n}$$

with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Observe that almost every $t \to \psi(x + e_t)$ extends to an entire function.

Proposition 5.2. — *Let ψ be as above and let $\delta > 0$. Then there is a decreasing sequence $\{m_n\}$ with $m_n \to -\infty$ such that the number of zeros of $z \to \psi(x + e_z)$ in the strip*

$$S_n = \{ z = t + iu \,;\, m_n > u > m_n - \delta \}$$

is infinite, for $\sigma - a.e.\, x$ in K.

Proof. — Putting $\nu_n = \lambda - \lambda_n$, we let $\phi = \sum_{n=1}^{\infty} \overline{a_n}\, \chi_{\nu_n}$. Since $z \to e^{i\lambda z}$ has no zero, $z \to \psi(x+e_z)$ has zero at z if and only if so does $z \to \phi(x+e_z)$ at $\bar{z}$. For each $r > 0$, $t \to \phi * P_{ir}(x + e_t)$ cannot be an outer function in $H^2(dt/\pi(1 + t^2))$, even if $\log|\phi|$ does not lie in $L^1(\sigma)$. Since ϕ has no weight at infinity, the inner part of $t \to \phi * P_{ir}(x + e_t)$ derives a Blaschke cocycle being not constant. From this fact, we may choose easily a desired decreasing sequence $\{m_n\}$. □

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Manuscrit reçu le 21 novembre 2009,
révisé le 17 janvier 2013,
accepté le 13 avril 2015.

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