ANNALES

DE

L'INSTITUT FOURIER

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Tome 65, nº 3 (2015), p. 1201-1250.

<http://aif.cedram.org/item?id=AIF_2015__65_3_1201_0>

Ann. Inst. Fourier, Grenoble
65, 3 (2015) 1201-1250

ON THE EQUIVARIANT COHOMOLOGY OF HILBERT SCHEMES OF POINTS IN THE PLANE

by Pierre-Emmanuel CHAPUT & Laurent EVAIN (*)

Abstract. — Let S be the affine plane regarded as a toric variety with an action of the 2-dimensional torus T. We study the equivariant Chow ring $A^*_K(S^{[n]})$ of the punctual Hilbert scheme $S^{[n]}$ with equivariant coefficients inverted. We compute base change formulas in $A^*_K(S^{[n]})$ between the natural bases introduced by Nakajima, Ellingsrud and Str mme, and the classical basis associated to the fixed points. We compute the equivariant commutation relations between creation/annihilation operators. We express the class of the small diagonal in $S^{[n]}$ in terms of the equivariant Chern classes of the tautological bundle. We prove that the nested Hilbert scheme $S_0^{[n,n+1]}$ parametrizing nested punctual subschemes of degree n and $n+1$ is irreducible.

Résumé. — Soit S le plan affine muni de sa structure de variété torique via l'action du tore T de dimension deux. Nous étudions l'anneau de Chow équivariant $A^*_K(S^{[n]})$ du schéma de Hilbert $S^{[n]}$. Nous calculons les formules de changement de base entre les bases naturelles introduites par Nakakjima, Ellingsrud et Strømme, et la base classique associée aux points fixes. Nous calculons les relations de commutation quivariantes entre les opérateurs de création/destruction. Nous exprimons la classe de la petite diagonale de $S^{[n]}$ en fonction des classes de Chern équivariante du fibré tautologique. Nous montrons que le schéma de Hilbert imbriqué paramétrant les couples de schémas ponctuels imbriqués de degrés respectifs n et $n+1$ est irréductible.

Introduction

Relation to previous work

If S is a quasi-projective smooth surface, let $S^{[n]}$ be the Hilbert scheme parameterizing the zero dimensional subschemes of degree n in S. Following

Keywords: equivariant cohomology, Hilbert schemes, Chow ring.
Math. classification: 14C05,14C15.
(*) We thank the Institut Henri Poincaré in Paris and the Mathematisches Forschungsinstitut Oberwolfach where part of this reaserch took place. We thank the developers of the Sage project for their software.

Nakajima and Grojnowski, a first tool to study the Chow ring $A^*(S^{[n]},\mathbb{Q})$ is to consider the direct sum $\oplus_{n\in\mathbb{N}}A^*(S^{[n]},\mathbb{Q})$ and operators acting linearly on this direct sum. Then, a lot of structure and information lies in the commutation relations between the various operators. In the case $S=\mathbb{A}^2$, this approach yields a basis of $A^*(S^{[n]},\mathbb{Q})$ that we call Nakajima's basis and a description of the ring structure on it [16, 15].

When $S=\mathbb{A}^2$, another approach is the use of the equivariant Chow rings. The 2-dimensional torus T acts on $S^{[n]}$. The equivariant Chow ring with respect to the action of the full torus T has been computed in [9] in the case $S=\mathbb{P}^2$, but this is a purely equivariant approach independent of Nakajima's framework. Similarly Bialynicki-Birula's theorem [1] yields a basis of the classical and equivariant Chow rings which has been studied in [8] and which we call Ellingsrud-Strømme's basis.

There are equivariant analogues of the operators introduced by Nakajima et al which act on the equivariant Chow ring. Following Vasserot [18], it is natural to compute these equivariant operators. In his paper, Vasserot does not consider the full action of the torus T, but the action of a non-generic one-dimensional subtorus $T'\subset T$. He computes several operators in T'-equivariant Chow rings and their commutators. As a consequence, he obtains a description of the T'-equivariant and of the classical Chow ring of the Hilbert scheme of $\mathbb{A}^2$.

On the other hand, Schiffmann and Vasserot study an algebra of operators acting on the equivariant K-theory of Hilbert schemes [17]. Since the correspondences defining q_i for $i>1$ are singular, they do not define operators on the K-theory, this is the reason why the authors only consider the operator algebra generated by q_1,q_{-1} and multiplication by some tautological bundles.

In this work we consider the action of the operators q_i for all i on the T-equivariant Chow rings.

Our approach and the main results

Each of the above approaches comes with its own formalism. Vasserot's approach relies on the representation theory of symmetric groups. Lehn's work relies on Virasoro operators. In our approach, we use restriction to fixed points, so the geometric analysis is done in the tensored equivariant Chow ring $A_T^*(S^{[n]})\otimes_{A_T^*(pt)}K$, where K is the fraction field of $A_T^*(pt)$, the equivariant Chow ring of a point. As an algebraic counterpart, we manipulate linear combinations of Young diagrams with rational functions in two variables as coefficients. Addition and multiplication of cohomology classes is straightforward in this formalism, whereas creation and destruction operators is described through a combinatorial formalism on Young diagrams.

Using this formalism we recover several equivariant analogs of formulas proved by Lehn, Vasserot, Nakajima... For instance, the analog of the commutation relation between the operators q_i, q_j is the following.

Theorem 4.25. — *Let i and j be any integers. We have*

$$[q_i, q_{-j}] \;=\; \begin{cases} 0 & \text{if } i \neq j \\ \frac{i(-1)^{i+1}}{UV} Id & \text{if } i = j \end{cases}$$

We also provide equivariant formulas which are really new, in the sense that their projection in the non equivariant context is new. We illustrate this claim with the following two theorems.

Theorem 6.7. — *The basis introduced by Nakajima and Ellingsrud-Strømme are equal up to sign and a normalizing constant in the Chow ring $A^*(S^{[n]})$.*

An important role is played by an operator $\rho : \oplus_n A^*_T(S^{[n]}) \to \oplus_n A^{*+1}_T(S^{[n+1]})$ which adds a point to a family of zero dimensional subschemes without changing the support. Its dual $\rho : \oplus_n A^*_T(S^{[n]}) \to \oplus_n A^{*-1}_T(S^{[n-1]})$ removes a point without changing the support.

Theorem 4.16. — *We have:*

$$[\rho, \rho^\vee] = \bigoplus_{n \geqslant 0} 2n \; Id_{A^*_K(S^{[n]})}.$$

The following theorem, together with the description of q_1 and q_{-1} (Proposition 4.1 and 4.2), gives a full inductive description of the creation operators in our context and explain the central role played by the operator ρ. In next theorem ∂ denotes the class of the set of non reduced schemes.

Theorem 4.8. — *We have*

$$\begin{array}{rcll} (i-1)q_i & = & \rho q_{i-1} - q_{i-1}\rho & \text{for } i > 1 \\ (i+1)q_i & = & \rho^\vee q_{i+1} - q_{i+1}\rho^\vee & \text{for } i < -1 \\ 2\rho & = & \partial q_1 - q_1 \partial & \\ 2\rho^\vee & = & q_{-1}\partial - \partial q_{-1} & \end{array}$$

The induction suggested by the theorem is easily programmed on a computer. This makes it possible to experiment and to check results (most of the results of this article have been computer checked). Note however that the computations are very tricky and impossible to follow by hand in general. For instance, we have not been able to give a purely algebraic proof of the above formulas: instead we use geometric arguments.

To prove our results, a nested analog of the irreducibility theorem of Briançon was needed. Let $S_0^{[n]}$ denote the set of subschemes z_n of length n supported as the origin, and let $S_0^{[p,q]}$ denote the similar set of couples of nested subschemes $(z_p \subset z_q)$.

Recall that the irreducibility of the Hilbert scheme $S^{[n]}$ was proved by Fogarty and Hartshorne. The irreducibility of the punctual analog $S_0^{[n]}$ is not a consequence of the irreducibility of $S^{[n]}$. It was first proved by Briançon [2] in a difficult proof with many technical steps. In the nested context, Cheah proved the irreducibility of $S^{[n,n+1]}$ [6]. We prove the punctual analog of Cheah's result.

Theorem 2.9. — *The incidence $S_0^{[n,n+1]}$ is irreducible of dimension n. At the generic point (s,b), the subschemes s and b are curvilinear.*

We remark that $S_0^{[p,q]}$ is not irreducible in general. An example is given in Proposition 2.8 and it is proved in [4] that the only irreducible cases are $p = 0, 1, q-1$ or q.

General strategy for the computations

The apparent difficulty coming from the non projectivity of $\mathbb{A}^2$ is not severe : we have all the standard constructions and properties of intersection theory that we need (pushforward, correspondences, composition of correspondences...) provided that we work in the tensored equivariant Chow ring $A_T^*(S^{[n]}) \otimes_{A_T^*(pt)} K$, instead of $A_T^*(S^{[n]})$ (Section 1).

However this construction also has its drawback: the pushforward of a contractant non proper morphism does not need to vanish (see Lemma 4.13 for an example) and some key arguments of the classical situation are not valid in our equivariant context. When one wants to compute the composition of two correspondences, the ubiquitous local situation that one has to understand are the classes $\pi_*[C]$, where C is some subvariety in $S^{[p,q]}$ and π is the projection to $S^{[p]}$ or to $S^{[q]}$. The geometry is under control when both z_p and z_q are curvilinear for the generic pair $(z_p, z_q) \in C$. In the other cases, the restriction of π to C is contractant and therefore $\pi_*[C]$ is zero in the classical Chow ring [16, 15]. However $\pi_*[C]$ need not vanish in the equivariant Chow ring. Our remedy is to prove the analog of the theorem of Briançon quoted above: $S_0^{[n,n+1]}$ is irreducible and the generic pair (z_n, z_{n+1}) parametrizes curvilinear subschemes. Then follows our construction to compute the commutators: we use algebraic arguments to reduce to the case when one of the operators adds only one point.

Contents of the paper

The first three sections develop the technical material useful in the paper : pushforward with non proper morphisms, computations of equivariant tangent spaces, basis of the Chow ring, and the analog of Briançon's theorem.

In Section 4, we consider the classical operators acting on $A_K := \bigoplus_{n\in\mathbb{N}} A_T^*(S^{[n]}) \otimes_{A_T^*(pt)} K$, namely the creation/destruction operators q_i, the boundary operator ∂, and an auxiliary operator ρ. All these operators are defined by a correspondence. Provided that the correspondence is smooth, the computation is easily done with the Bott formula. This is the strategy to compute q_1 and q_{-1} in the fixed point basis (Proposition 4.1 and 4.2). All the other correspondences are singular at some points and a turnaround is needed to compute the corresponding operators.

Computing restriction to fixed points, we prove the formula $\partial = -2c_1(\mathcal{O}_X[n])$, where $\mathcal{O}_X[n]$ denotes the tautological bundle. Following Lehn and Schiffmann-Vasserot's ideas, we consider various commutators starting with q_1, q_{-1} and ∂. We end up with the recursion formulas for the q_i's, $|i| > 1$ (Theorem 4.8). In particular this yields base change formulas between the fixed point basis and Nakajima's basis.

To compute the commutation relations between the q_i's (Theorem 4.25), using once again the same general idea as in [17], we use algebraic computations to reduce to the case of operators of conformal degree one. After the algebraic reduction, it remains to prove Theorem 4.16. Most of the technical difficulty lies in this theorem, since excess intersection components appear in the computation.

The class $\delta_n \in A^{n-1}(S^{[n]})$ of the small diagonal $\Delta_n \subset S^{[n]}$ parameterizing the subschemes supported on a single point has an expression in terms of the equivariant Chern classes of the tautological bundle: $\delta_n = (-1)^{n-1} n c_{n-1}(\mathcal{O}_X[n])$. The originial proof by Lehn [15, Theorem 4.6] remains true in our context. We give a new proof which relies on an algebraic expression for the operator q_n (Theorem 5.3).

Finally, we give an application of our equivariant computations at the level of classical Chow rings and we prove Theorem 6.7 which identifies

Nakajima's basis and Ellingsrud and Strømme's basis. Our strategy of proof is to interpret the Bialynicki-Birula cells in terms of operators: we introduce new creation operators $q_{i,X}$ such that the basis introduced by Ellingsrud and Strømme is obtained applying these operators on the vacuum. We express the $q_{i,X}$ in terms of the creation operators q_i and we get a base change formula in the equivariant Chow ring. Projecting this relation in the usual Chow ring gives the asserted formula.

1. Pushforward with non proper morphisms

We work over an algebraically closed field k of any characteristic. Let T be a 2-dimensional torus. The T-equivariant Chow ring $A_T^*(pt)$ of a point is isomorphic to a polynomial ring in two variables U, V. We denote by $K = \mathbb{Q}(U, V)$ the field of fractions of $A_T^*(pt)$. Moreover, if X is any T-variety, we denote by $A_K^*(X)$ the tensor product $A_T^*(X) \otimes_{\mathbb{Z}[U,V]} K$. We denote the product of two classes x, y in a Chow ring indifferently by $x.y$ or by $x \cup y$.

In this section $f : X \to Y$ is an equivariant morphism between smooth varieties. Moreover, we assume that X and Y are *filtrable*, in the sense of Definition 3.2 in [3].

When $f : X \to Y$ is a proper equivariant morphism, there is a well defined pushforward $f_* : A_T^*(X) \to A_T^*(Y)$. Since we shall work with the affine plane, we are in a non projective setting and we have to deal with non proper morphisms.

The goal of this section is to explain that a good notion of pushforward f_*^K exists, when f is a non proper morphism, provided that the restriction to fixed points $f^T : X^T \to Y^T$ is proper. This notion is applied to define correspondences. These results are standard and proofs of them can be found in [5].

Definition 1.1. — *If f is as above, $f^T : X^T \to Y^T$ will denote the restriction of f to T-fixed points. When f is proper, the morphisms $f_*^K : A_K^*(X) \to A_K^{*+\dim Y - \dim X}(Y)$ and $f_K^* : A_K^*(Y) \to A_K^*(X)$ are derived from the standard morphisms $f_*^T : A_T^*(X) \to A_T^{*+\dim Y - \dim X}(Y)$ and $f_T^* : A_T^*(Y) \to A_T^*(X)$ after tensorisation over $\mathbb{Z}[U, V]$ by K.*

Let $f : X \to Y$ be any T-equivariant morphism, and consider the following commutative diagram:

$$\begin{array}{ccc} X^T & \stackrel{i}{\hookrightarrow} & X \\ \downarrow f^T & & \downarrow f \\ Y^T & \stackrel{j}{\hookrightarrow} & Y \end{array} \quad . \tag{1.1}$$

Since i_*^K is an isomorphism by [7, Theorem 1], the following definition is meaningful:

Definition 1.2. — *If f^T is proper, define*

$$f_*^{T,K} = j_*^K (f^T)_*^K (i_*^K)^{-1} : A_K^*(X) \to A_K^{*+\dim Y - \dim X}(Y) \ .$$

If f is proper, then $f_*^{T,K} = f_*^K$, by the functoriality of the pushforward in the proper case. Since there is therefore no possible confusion, we will denote $f_*^{T,K}$ simply by f_*^K.

Theorem 1.3. — *The morphism f_*^K satisfies the following properties:*

(1) *Functoriality: if we have T-equivariant morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that f^T and g^T are proper, then $(g \circ f)_*^K = g_*^K \circ f_*^K$.*
(2) *Projection formula: assume here that X and Y are smooth, so that $A_K^*(X)$ and $A_K^*(Y)$ are rings. For any $\alpha \in A_K^*(X)$ and $\beta \in A_K^*(Y)$, we have the equality $f_*^K(\alpha) \cdot \beta = f_*^K(\alpha \cdot f_K^*(\beta))$.*
(3) *We have the equality $g_*^K f_K^* = l_K^* h_*^K$ if f, g, h, l are as in the following diagram:*

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{g} & Y \times Z \\ \downarrow f & & l \downarrow \\ X \times Y & \xrightarrow{h} & Y \end{array}$$

In practice, f_*^K can be computed by a "Bott formula", as in the proper case. Assume that X is smooth. Since X^T is smooth, $A_T^*(X^T) = \oplus A_T^*(X_i)$ where the sum runs through the irreducible components X_i of X^T. We denote by $c_{top}(N_{X^T,X})$ the operator which acts on $A_T^*(X^T)$ through multiplication by the equivariant Chern class c_{d_i} of the normal bundle $N_{X_i,X}$ on the component $A_T^*(X_i)$, where d_i is the codimension of X_i in X. Similarly, there is a class $c_{top}(N_{Y^T,Y})$. In $A_K^*(X)$ (or $A_K^*(Y)$), the Chern class c_{d_i} is equal to the sum of an invertible element and a nilpotent element, according to the proof of [3, Proposition 3.2(i)]. Therefore it is invertible and $c_{top}(N_{X^T,X})$ is an invertible operator.

Lemma 1.4. — *Assume that X is smooth. Let $i : X^T \to X$ be the natural inclusion. The pullback i_K^* is invertible with inverse $i_*^K \frac{1}{c_{top}(N_{X^T,X})}$.*

Theorem 1.5 (Bott Formula). — *Recall the diagram (1.1) and assume that X and Y are smooth. Let $\alpha \in A_T^*(X)$. Then*

$$j_K^* f_*^K(\alpha) = c_{top}(N_{Y^T,Y})\ (f^T)_*^K \left(\frac{1}{c_{top}(N_{X^T,X})} i_K^*(\alpha) \right).$$

In particular, when both X and Y have a finite number of fixed points $x_1, \dots, x_n$, $y_1, \dots, y_p$, the formula expresses f_*^K in terms of the localization at these fixed points:

$$(f_*^K \alpha)(y_k) = \sum_{f(x_i)=y_k} \frac{c_{top}(T_{y_k,Y})}{c_{top}(T_{x_i,X})} \alpha_{x_i}\ ,$$

where $T_{y_k,Y}$ and $T_{x_i,X}$ are the tangent T-representations.

Definition 1.6. — *An equivariant correspondence is a closed T-stable subvariety $C \subset X \times Y$ such that $C^T \to Y^T$ is a proper morphism. Let π_X and π_Y be the projections from C to X and Y respectively. The classes of such varieties C generate a subspace in $A_K^*(X \times Y)$ and we still call equivariant correspondence a class in this subspace. An equivariant correspondence C yields a morphism $f : A_K^*(X) \to A_K^*(Y)$ defined by $f(\alpha) = (\pi_Y)_*^K (\pi_X)_K^*(\alpha)$.*

Proposition 1.7. — *Assume that X, Y, Z are smooth varieties. Let $C \subset X \times Y$ and $D \subset Y \times Z$ be two equivariant correspondences, and f and g the associated morphisms. Let $\pi_{12}, \pi_{13}, \pi_{23}$ the projections from $X \times Y \times Z$ to $X \times Y, X \times Z$ and $Y \times Z$ respectively. Then the class $(\pi_{13})_*^K((\pi_{12})_K^*[C] \cup (\pi_{23})_K^*[D])$ is an equivariant correspondence with associated morphism $g \circ f$.*

Definition 1.8. — *Suppose that $\pi : X^T \to \operatorname{Spec} k$ is proper. K-bilinear product on $A_K^*(X)$ defined by $\langle \alpha, \beta \rangle_X = \pi_*^K(\alpha \cup \beta)$.*

When $\langle ., . \rangle_X$ and $\langle ., . \rangle_Y$ are both non degenerate (for instance when X and Y have a finite number of fixed points), then every map $f : A_K^*(X) \to A_K^*(Y)$ admits a dual map $f^\vee : A_K^*(Y) \to A_K^*(X)$.

Definition 1.9. — *If $C \subset X \times Y$ is a correspondence, the dual correspondence $C^\vee$ is the correspondence in $Y \times X$ which is canonically identified with C under the natural isomorphism $X \times Y \simeq Y \times X$. In particular, if $C^\vee$ is an equivariant correspondence, it yields a map $A_K^*(Y) \to A_K^*(X)$.*

Proposition 1.10. — *Assume that X and Y are smooth. Let $C \subset X \times Y$ be an equivariant correspondence and $f : A_K^*(X) \to A_K^*(Y)$ the* *associated morphism. Suppose that $C^\vee \subset Y \times X$ is an equivariant correspondence and that $\langle .,. \rangle_X$ and $\langle .,. \rangle_Y$ are non degenerate. Then the dual map $f^\vee$ is defined by the dual correspondence $C^\vee$.*

Remark 1.11. — Restriction to fixed points does not commute with the bilinear product (see Lemma 3.4). This remark is important when one wants to compute $f^\vee$ on fixed points.

2. Tangent space to $S^{[n,n+1]}$

We denote by S the affine plane $\mathbb{A}^2$. Let $n \geqslant 0$ be an integer, we denote by $S^{[n]}$ the Hilbert scheme parameterizing length n subschemes of S [13] ($S^{[0]} = \operatorname{Spec} k$). Given $z \in S^{[n]}$, we denote by $I_z \subset k[X,Y]$ the corresponding ideal, of codimension n. Given p, q integers with $0 \leqslant p < q$, we denote by $S^{[p,q]}$ the "nested" Hilbert scheme, namely the subscheme of $S^{[p]} \times S^{[q]}$ consisting of pairs (s,b) such that $I_s \supset I_b$. The torus $(k^*)^2$ will be denoted by T. It acts on the plane S: we use the convention that an element $(u,v) \in T$ acts on a monomial $X^aY^b \in k[X,Y]$ by $(u,v) \cdot X^aY^b = (uX)^a(vY)^b$. This induces an action of T on each Hilbert scheme $S^{[n]}$ and $S^{[p,q]}$. We will denote by $aU + bV$ the weight on T defined by $(aU+bV)(u,v) = u^av^b$. Given a monomial $m = X^aY^b$, we denote by $wt(m) = aU + bV$ its weight. Any character of T defines naturally an element in $A^1(pt)$, thus our notation is compatible with the notation $A^*(pt) = \mathbb{Z}[U,V]$ in Section 1.

Several arguments in the present paper rely on a tangent space argument. In fact, at a T-fixed point $z \in S^{[n]}$, the tangent space $T_zS^{[n]}$ has several combinatorial descriptions. One [10] is in terms of significant cleft pairs and another [16] in terms of boxes of the corresponding staircase. We recall in this section the necessary material to be comfortable with these two notions. We give two applications. First, we compute the tangent space at a toric point in $S^{[n,n+1]}$ as a representation of T. Then Theorem 2.9 proves the irreducibility of $S_0^{[n,n+1]} \subset S^{[n,n+1]}$ parameterizing the pairs of subschemes $s \subset b$ with the support of b equal to the origin.

2.1. Tangent space to the Hilbert schemes

First, observe that a T-fixed point z in $S^{[n]}$ is defined by an ideal

$$I_z = \bigoplus_{(a,b) \notin E} k \cdot X^aY^b \quad ,$$

where $E \subset \mathbb{N}^2$ satisfies $(\mathbb{N}^2 \setminus E) + \mathbb{N}^2 \subset (\mathbb{N}^2 \setminus E)$. Such (finite) subsets $E \subset \mathbb{N}^2$ will be called *staircases*. A partition $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_l > 0)$ is by definition a finite sequence of non increasing positive natural numbers, l is called the length of λ, and $|\lambda| = \sum_{i=1}^{l} \lambda_i$ is the weight of λ. We denote by $\mathcal{P}_n$ the set of partitions of weight n. If E is a finite staircase associated with a T-fixed point $z \in S^{[n]}$, there exists a unique partition λ with weight n such that $(a,b) \in E \Leftrightarrow a+1 \leqslant l, b < \lambda_{a+1}$.

We begin by recalling the description given in [10] of $T_z S^{[n]}$ when $z \in S^{[n]}$ is a T-fixed point.

- A monomial $c \in I_z$ is called a *cleft* whenever $X^{-1} \cdot c \notin I_z$ and $Y^{-1} \cdot c \notin I_z$.
- A Laurent monomial is called *positive* (resp., *negative*) if it belongs to $Y^{-1}k[X, Y^{-1}]$ (resp., $X^{-1}k[X^{-1}, Y]$).
- A weight $aU + bV$ with $a \geqslant 0$ and $b < 0$ resp. $a < 0$ and $b \geqslant 0$ will be called *positive* resp. *negative*.
- A *cleft pair* is a pair (c, m) such that c is a cleft, m is a monomial not belonging to I_z, and m/c is either positive or negative (in which case, we say that (c, m) is positive or negative, respectively).

Now let $\mathcal{C} := \{c_1, \ldots, c_l\}$ denote the set of clefts, which we order following the convention that c_{i+1}/c_i be positive for $1 \leqslant i \leqslant l-1$.

For each positive (resp., negative) cleft pair (c_i, m), let $s := s_i$ denote the least common multiple of c_i and c_{i+1} (resp., c_i and c_{i-1}). We say that (c_i, m) is *significant* if $ms/c_i \in I_z$. To (c_i, m) we associate the vector $\varphi = \varphi_{(c_i,m)}$ in $T_z S^{[n]} \simeq Hom_{k[X,Y]}(I_z, k[X,Y]/I_z)$ defined by

$$\begin{cases} \varphi(c_j) = mc_j/c_i \text{ if } j \leqslant i \\ \varphi(c_j) = 0 \text{ if } j > i \end{cases} \quad \text{resp.} \quad \begin{cases} \varphi(c_j) = 0 \text{ if } j < i \\ \varphi(c_j) = mc_j/c_i \text{ if } j \geqslant i \end{cases}$$

According to [10, Theorem 3], the set of elements $\varphi_{(c,m)}$ for all significant cleft pairs (c, m) is a basis of $T_z S^{[n]}$.

On the other hand, Nakajima gives a combinatorial description of the weights occurring in $T_z S^{[n]}$ [16, Proposition 5.8]. Given a staircase E and $e \in E$, let $a(e) := \max\{i \mid X^i \cdot e \in E\}$ and let $b(e) := \max\{j \mid Y^j \cdot e \in E\}$. The set of positive weights in $T_z S^{[n]}$, counted with multiplicities, is the set of weights of the form $w_+(e) := a(e)U - (b(e)+1)V$, and the set of negative weights is the set of weights the form $w_-(e) := -(a(e)+1)U + b(e)V$.

We now give a bijection h_+ resp. h_- between the staircase E and the set of positive resp. negative significant cleft couples, preserving the weights, meaning that $wt(\varphi_{(h_\pm(e))}) = w_\pm(e)$. The bijection h_+ resp. h_- is defined as follows: given $e \in E$, we denote $c_1 = Y^b \cdot e$ resp. $c_1 = X^a \cdot e$, where b resp.

a is the minimal integer such that $Y^b \cdot e \notin E$ resp. $X^a \cdot e \notin E$. We denote $c_2 = X^a \cdot e$ resp. $c_2 = Y^b \cdot e$, where a resp. b is the maximal integer such that $X^a \cdot e \in E$ resp. $Y^b \cdot e \in E$. We let i resp. j be the maximal integer such that $X^{-i} \cdot c_1 \notin E$ resp. $Y^{-j} \cdot c_1 \notin E$ (thus $X^{-i} \cdot c_1$ resp. $Y^{-j} \cdot c_1$ is a cleft). Finally we set $h_+(e) = (X^{-i} \cdot c_1, X^{-i} \cdot c_2)$ resp. $h_-(e) = (Y^{-j} \cdot c_1, Y^{-j} \cdot c_2)$: by construction of (c_1, c_2) this is a significant cleft couple.

Proposition 2.1. — *The map h_+ resp. h_- is a bijection between E and the set of positive resp. negative significant cleft couples, and we have $wt(\varphi_{(h_+(e))}) = w_+(e)$ and $wt(\varphi_{(h_-(e))}) = w_-(e)$.*

Proof. — By symmetry, we give the proof only in the positive case. With the notations before the proposition, we have $wt(\varphi_{(h_+(e))}) = wt(c_2 c_1^{-1})$, which is readily $w_+(e)$. Thus to prove the proposition it suffices to describe the inverse of h_+. Given a positive significant cleft couple (c, m), let i be the maximal integer such that $X^i \cdot m \in E$. The inverse of h_+ maps (c, m) to the greatest common divisor of $X^i \cdot c$ and $X^i \cdot m$. □

2.2. Computation of the tangent space of $S^{[n,n+1]}$

Let $(s, b) \in (S^{[n,n+1]})^T$ (s and b stand for small and big). We denote by E_s, E_b their staircases. Let $q : S^{[n,n+1]} \to S^{[n+1]}$ denote the natural projection, and let dq denote its differential at (s, b). There is a natural exact sequence

$$0 \to \ker dq \to T_{(s,b)} S^{[n,n+1]} \xrightarrow{dq} T_b S^{[n+1]} .$$

The following is immediate:

Lemma 2.2. — *The tangent space $T_{(s,b)} S^{[p,q]}$ of $S^{[p,q]}$ at (s, b) is the set of couples of homomorphisms $(\varphi, \psi) \in Hom_{k[X,Y]}(I_s, k[X,Y]/I_s) \times Hom_{k[X,Y]}(I_b, k[X,Y]/I_b)$ such that for all $f \in I_b$, we have*

$$\varphi(f) = \psi(f) \ mod\ I_s.$$

Proof. — This tangent space is included in the tangent space $T_{(s,b)}(S^{[p]} \times S^{[q]})$, which is the direct sum $Hom_{k[X,Y]}(I_s, k[X,Y]/I_s) \oplus Hom_{k[X,Y]}(I_b, k[X,Y]/I_b)$. Consider the ring $k[\epsilon]$, where $\epsilon^2 = 0$. Identifying the tangent bundle of $S^{[p]}$ resp. $S^{[q]}$ with the $k[\epsilon]$-points of $S^{[p]}$ resp. $S^{[q]}$, we see that $(\varphi, \psi) \in T_{(s,b)} S^{[p,q]}$ if and only if the restriction of φ to I_b is equal to the quotient of ψ modulo I_s. □

The following propositions 2.3 and 2.4 describe the infinitesimal deformations of b that admit a lift to a deformation of (s,b). There are two cases, depending on the geometry of the staircases involved.

If m is a monomial, we denote by $x(m)$ its exponent for the variable X. In other words $x(X^aY^b)=a$. Similarly $y(X^aY^b)=b$. We denote by $c_1,\dots,c_l$ the clefts of s, and by k the index such that $c_k\in E_b$.

Proposition 2.3. — *Assume that $y(c_{k-1})>y(c_k)+1$, resp. $x(c_{k+1})>x(c_k)+1$. Then the positive, resp. negative part of* Im dq *is the subspace of $T_bS^{[n+1]}$ generated by those $\varphi_{(c,m)}$ with $(c,m)\neq(Yc_k,X^{-1}c_i), i>k$, resp. $(c,m)\neq(Xc_k,Y^{-1}c_i), i<k$.*

Proof. — By symmetry, it suffices to describe the positive part of Im dq when $y(c_{k-1})>y(c_k)+1$.

The fact that many cleft pairs of s are also cleft pairs of b is a potential source of confusion. Consequently, given a pair (c,m) of both s and b, we will use $\varphi^n_{(c,m)}$ resp. $\varphi^{n+1}_{(c,m)}$ to denote the corresponding tangent vector in $T_sS^{[n]}$ resp. $T_bS^{[n+1]}$. Moreover we will use the convention that if $m\notin E_s$, then $\varphi^n_{(c,m)}=0$. Let (c,m) be a positive significant cleft pair of b.

First note that if $c=c_i$ with $i\neq k$, then $\varphi^{n+1}_{(c,m)}\in$ Im dq. In fact in this case it is clear that $(\varphi^n_{(c,m)},\varphi^{n+1}_{(c,m)})$ is a tangent vector of $S^{[n,n+1]}$ (recall our convention that $\varphi^n_{(c,m)}$ is 0 if $m=c_k$).

Next consider the case where $c=Xc_k$. We also see that $\varphi^{n+1}_{(c,m)}\in$ Im dq since the pair $(\varphi^n_{(c_k,X^{-1}m)},\varphi^{n+1}_{(Xc_k,m)})$ is a tangent vector of $S^{[n,n+1]}$ by Lemma 2.2.

Now, if $c=Yc_k$, the fact that (c,m) is a significant cleft pair implies that $Xm\in I_b$. If $m\neq X^{-1}c_i$ for any $i>k$, then $XY^{-1}m\in I_b$ and thus $\varphi^n_{(c_k,Y^{-1}m)}(Xc_k)=0$, so that $(\varphi^n_{(c_k,Y^{-1}m)},\varphi^{n+1}_{(Yc_k,m)})$ is a tangent vector of $S^{[n,n+1]}$.

It remains to show that the $\varphi^{n+1}_{(Yc_k,X^{-1}c_i)}$-coefficient of any vector in Im dq vanishes, for all $i>k$. To this end, let $(\varphi^n,\varphi^{n+1})$ be a tangent vector to the incidence variety. Considering φ^n resp. φ^{n+1} as an element of $Hom_{k[X,Y]}(I_s,k[X,Y]/I_s)$ resp. $Hom_{k[X,Y]}(I_b,k[X,Y]/I_b)$, we see that the coefficient of $Y^{-1}c_i$ in $\varphi^n(Xc_k)$ is equal to the coefficient of $X^{-1}c_i$ in $\varphi^n(Yc_k)$ (namely, those coefficients equal the coefficient of $X^{-1}Y^{-1}c_i$ in $\varphi^n(c_k)$). It follows from Lemma 2.2 that φ^{n+1} has the same property. On the other hand, among all the basis vectors $\left(\varphi^{n+1}_{(c,m)}\right)$, $\varphi^{n+1}_{(Yc_k,X^{-1}c_i)}$ is the only vector for which these coefficients are not equal. Thus the $\varphi^{n+1}_{(Yc_k,X^{-1}c_i)}$-coefficient in φ^{n+1} vanishes. $\square$

Proposition 2.4. — *Assume that $y(c_{k-1}) = y(c_k) + 1$ resp. $x(c_{k+1}) = x(c_k) + 1$. Then the positive resp. negative part of* Im dq *is the subspace of $T_b S^{[n+1]}$ generated by those $\varphi_{(c,m)}$ with $(c, m) \neq (c_{k-1}, X^{x(c_{k-1})-x(c_k)-1} \cdot c_i), i > k$ resp. $(c, m) \neq (c_{k+1}, Y^{y(c_{k+1})-y(c_k)-1} \cdot c_i), i < k$.*

Proof. — The argument is similar to that used in the proof of the preceding proposition. For any positive significant cleft pair (c_i, m) with $i > k$ or $i < k-1$, the vector $(\varphi^n_{(c_i,m)}, \varphi^{n+1}_{(c_i,m)})$ belongs to $T_{(s,b)} S^{[n,n+1]}$; whence, $\varphi^{n+1}_{(c,m)}$ belongs to the image of dq.

Similarly, if (c, m) is a positive cleft pair and $c = Xc_k$, then the pair $(\varphi^n_{(c_k, X^{-1}m)}, \varphi^{n+1}_{(c,m)})$ belongs to $T_{(s,b)} S^{[n,n+1]}$; we deduce $\varphi^{n+1}_{(c,m)} \in$ Im dq.

We now consider positive significant cleft pairs of the form (c_{k-1}, m). Assume that $i \geqslant k$ is such that $y(c_{i-1}) > y(m) \geqslant y(c_i)$. Since (c_{k-1}, m) is significant, we have

$$x(m) \geqslant x(c_i) + x(c_{k-1}) - x(c_k) - 1.$$

When $x(m) \geqslant x(c_i) + x(c_{k-1}) - x(c_k)$, the pair $(\varphi^n_{(c_{k-1},m)}, \varphi^{n+1}_{(c_{k-1},m)})$ belongs to the tangent space $T_{(s,b)} S^{[n,n+1]}$, so $\varphi^{n+1}_{(c_{k-1},m)} \in$ Im dq. When $x(m) = x(c_i) + x(c_{k-1}) - x(c_k) - 1$ and $y(m) > y(c_i)$, the pair $(\varphi^n_{(c_k, X^{x(c_k)-x(c_{k-1})} Y^{-1} m)}, \varphi^{n+1}_{(c_{k-1},m)})$ belongs to $T_{(s,b)} S^{[n,n+1]}$, so $\varphi^{n+1}_{(c_{k-1},m)} \in$ Im dq.

Finally it remains to show that the $\varphi^{n+1}_{(c_{k-1}, X^{x(c_{k-1})-x(c_k)-1} c_i)}$-coordinate of any vector in Im dq vanishes. To this end, note that $\varphi^{n+1}_{(c_{k-1}, X^{x(c_{k-1})-x(c_k)-1} c_i)}$ is the only vector φ in our basis of $T_b S^{[n+1]}$ for which

$$[Y^{-1} c_i] \varphi(X c_k) \neq [X^{-1} c_i] \varphi(Y c_k),$$

while

$$[Y^{-1} c_i] \psi(X c_k) = [X^{-1} c_i] \psi(Y c_k)$$

for any $\psi \in Hom(I_s, k[X,Y]/I_s)$, where $[m]\varphi(n)$ denotes the coefficient of m in $\varphi(n)$. □

While the description of Im (dq) given in Propositions 2.3 and 2.4 depends on the shape of the Young tableau corresponding to s, the weights of the T-representation $T_b S^{[n+1]}/\text{Im } dq$ have a more uniform description. This space measures the obstructions to lift a deformation of b to a deformation of (s, b).

Proposition 2.5. — *The weights of the T-representation $T_bS^{[n+1]}/\mathrm{Im}\ dq$ have multiplicity one and are given by*

$$(x(c_i) - x(c_k) - 1)U + (y(c_i) - y(c_k) - 1)V, i \neq k.$$

These weights are the weights of $\frac{c_i}{XYc_k}$, $i \neq k$.

Proof. — By symmetry, it suffices to consider only the positive part of the quotient $T_bS^{[n+1]}/\mathrm{Im}\ dq$. Assume first that $y(c_{k-1}) > y(c_k) + 1$. By Proposition 2.3, the positive weights of $T_bS^{[n+1]}/\mathrm{Im}\ dq$ are the weights of $\frac{X^{-1}c_i}{Yc_k}$. Thus the proposition is proved in this case. Assume now that $y(c_{k-1}) = y(c_k) + 1$. By Proposition 2.4, the positive weights of $T_bS^{[n+1]}/\mathrm{Im}\ dq$ are the weights of $\frac{X^{x(c_{k-1})-x(c_k)-1}c_i}{c_{k-1}}$. Since $c_{k-1} = X^{x(c_{k-1})-x(c_k)}Yc_k$, the proposition follows in this case too. □

Proposition 2.6. — *The weights of* $\ker dq$ *have multiplicity one and are the following:*

$$\begin{array}{ll} -(y(c_{i-1}) - y(c_k) + 1)V + (x(c_i) - x(c_k) - 1)U & i > k \\ -(x(c_k) - x(c_{i+1}) + 1)U + (y(c_i) - y(c_k) - 1)V & i < k \end{array}$$

These weights are the weights of the arrows from c_k to the corners of the partition corresponding to s.

Proof. — This kernel consists of those morphisms $\varphi \in Hom_{k[X,Y]}(I_s, k[X,Y]/I_s)$ for which $\varphi(I_b) = 0$, by Lemma 2.2. If follows that if (c, m) is a cleft pair with $c \neq c_k$, we have

$$[\varphi^n_{(c,m)}]\varphi = 0$$

for every $\varphi \in \ker dq$. On the other hand, if (c_k, m) is a cleft pair and φ has a non-vanishing $\varphi^n_{(c_k,m)}$-coefficient, the fact that

$$\varphi(Xc_k) = X\varphi(c_k) = 0 \text{ and } \varphi(Yc_k) = Y\varphi(c_k) = 0 \bmod I_s$$

implies that m is a corner of s. Conversely, for m a corner of s, it is clear that $\varphi^n_{(c_k,m)} \in \ker dq$. Thus $\ker dq$ is generated by those elements $\varphi^n_{(c_k,m)}$ for which m is a corner of s. The proposition now follows immediately. □

The last two propositions and the exact sequence together describe the tangent space $T_{(s,b)}S^{[n,n+1]}$ as a linear T-representation. A similar description of the tangent space is already present in [17], in the context of equivariant K-theory.

This will be useful later on to compute equivariant Chern classes. Considering only the dimensions of these spaces, we recover the following well-known result of Cheah [6, Theorem 3.2.2] about the smoothness of $S^{[n,n+1]}$:

Proposition 2.7. — *The incidence $S^{[n,n+1]}$ is a smooth irreducible subvariety of $S^{[n]} \times S^{[n+1]}$.*

2.3. Application to the irreducibility of $S_0^{[n,n+1]}$

We consider the subvariety $S_0^{[p,q]}$ of $S^{[p,q]}$ parameterizing incident schemes $(s \subset b)$ with respective length p and q both supported at the origin. Recall Briançon's theorem which asserts the irreducibility of the variety $S_0^{[n]} \subset S^{[n]}$ parameterizing the subschemes of length n and support the origin. The corresponding theorem for pairs of incident schemes is not true, as shown by the following example.

Proposition 2.8. — *The scheme $S_0^{[2,4]}$ is not irreducible.*

Proof. — As $S_0^{[2,4]}$ is a strict subscheme of the irreducible 4-dimensional product $S_0^{[4]} \times S_0^{[2]}$, any component has dimension at most 3 and any irreducible $F \subset S_0^{[2,4]}$ of dimension 3 is an irreducible component. One irreducible component of dimension 3 is birational to $S_0^{[4]}$: its generic point parameterizes the couple (s, b) with b the generic curvilinear subscheme of length 4 and s the unique subscheme of b with length 2. An other 3-dimensional family is constructed as follows. Let (f, g) be two distinct linear forms and b the subscheme of S with equation (f^2, g^2). Let s be any scheme of length 2 supported at the origin. Since $s \subset b$, the set of such (s, b) describe a subvariety $F \subset S_0^{[2,4]}$ of dimension $1 + 2 = 3$. □

Although the general incidence $S_0^{[p,q]}$ is wild and difficult to describe, the case $(p, q) = (n, n + 1)$ behaves nicely. The rest of this section is devoted to the proof of the following theorem:

Theorem 2.9. — *The incidence $S_0^{[n,n+1]}$ is irreducible of dimension n. At the generic point (s, b), the subschemes s and b are curvilinear.*

We start the proof with a weaker version of the theorem, in the next proposition.

Proposition 2.10. — *We have $\dim S_0^{[n,n+1]} = n$ and there is only one irreducible component H_0 in $S_0^{[n,n+1]}$ of dimension n. At a generic point $(s, b) \in H_0$, the subschemes s and b are curvilinear.*

Proof. — Once again this result is a consequence of the detailed study of $S_0^{[n,n+1]}$ performed by Cheah [6, Proposition 3.4.11]. We give a short proof.

To apply Bialynicki-Birula's decomposition theorem in $S^{[n,n+1]}$ which is not compact, we first compactify $S^{[n,n+1]}$. So consider the inclusion $S^{[n,n+1]} \subset (\mathbb{P}^2)^{[n,n+1]}$. Since $S^{[i,i+1]}$ is smooth for all values of i, $(\mathbb{P}^2)^{[n,n+1]}$ is smooth. Consider the action of k^* induced by the action on the affine plane defined by $t \cdot X = t^\alpha X$, $t \cdot Y = tY$, where α is any integer strictly greater than $n+1$.

Let O denote the origin of $\mathbb{A}^2 \subset \mathbb{P}^2$. Let Z be a subscheme of $\mathbb{P}^2$. If Z is not supported on O, then the limit at $t = 0$ of $t \cdot Z$ is also not supported on O. For (s,b) a k^*-fixed point in $(\mathbb{P}^2)^{[n,n+1]}$ let $C_{(s,b)}$ be the corresponding Bialynicki-Birula cell: we have

$$(\mathbb{P}^2)_0^{[n,n+1]} = \coprod_{(s,b)\in \left(S_0^{[n,n+1]}\right)^T} C_{(s,b)} \ .$$

Thus to prove the proposition it is enough to show that all the cells $C_{(s,b)}$ with s and b supported at the origin have dimension at most n and that exactly one has dimension n.

Let $(s,b) \in S_0^{[n,n+1]}$ be a given k^*-fixed point. Let us say that a tangent vector $x \in T_{(s,b)}S^{[n,n+1]}$ is contractant if it is an eigenvector for the k^*-action of positive weight. It is well-known that the dimension of $C_{(s,b)}$ is the number of independent contractant tangent vectors.

Let $x \in T_{(s,b)}S^{[n,n+1]}$ be a T-weight vector which is contractant and let $w = aU + bV$ be its weight. Its weight for k^* is $a\alpha + b$ and this is a positive integer. Since $\alpha > n+1$, we have $a \geqslant 0$ so w is a positive weight (recall the definition of positive weights in Subsection 2.1). In particular, the vector space W generated by such contractant tangent vectors x satisfies $\dim W \leqslant \dim T^+ = n+1$, where $T^+ \subset T_{(s,b)}S^{[n,n+1]}$ is the vector space generated by tangent vectors with positive weight. Recall the description of $T_{(s,b)}S^{[n,n+1]}$ given in terms of the projection $q : S^{[n,n+1]} \to S^{[n+1]}$ and its differential dq. By Proposition 2.5, all the tangent vectors of $T_bS^{[n+1]}$ of positive weight $-V$ are in the image of dq but are not contractant. Since there is at least one such vector, we get $\dim W \leqslant \dim T^+ - 1 = n$. Moreover, if we have equality, there is exactly one vector in $T_bS^{[n+1]}$ of weight $-V$. This implies that the partition corresponding to b is a rectangle: $\lambda = (m, m, \dots, m)$. But, if $m > 1$, the eigenspace of weight $-V$ in $\operatorname{Im} dq$ has dimension 1, as well as the eigenspace of weight $-(m-1)V$ in $\ker dq$, by Proposition 2.6. Since these vectors are not contractant, we get $\dim W \leqslant n-1$ and a contradiction. Thus the only possibility is $\lambda = (1, 1, \dots, 1)$, and the proposition is proved. $\square$

The next proposition describes the Bialynicki-Birula cells of dimension $n-1$ introduced in the proof of the previous proposition.

Proposition 2.11. — *A Bialynicki-Birula cell $C_{(s,b)}$ has dimension $n-1$ if and only if the height of the partition λ associated to b is 2.*

Proof. — We keep the notations of the previous proposition. If the height h of λ is 1, then $\dim C_{(s,b)} = n$ by the above. If $h \geqslant 3$, since there are at least h independent vectors in $\mathrm{Im}(dq)$ of weight a positive multiple of $-V$, there are at least three tangent vectors in $T_{(s,b)}S^{[n,n+1]}$ which are positive non contractant, hence $\dim C_{(s,b)} \leqslant \dim T^+ - h \leqslant n-2$. If $h=2$, then $\lambda = (2^\alpha, 1^\beta)$ and the partition μ of s is $\mu_0 = (2^{\alpha-1}, 1^{\beta+1})$ or $\mu_1 = (2^\alpha, 1^{\beta-1})$. Let us denote by T_{cont} the subspace of a vector space T generated by the contractant tangent vectors. Then $\dim T_b S^{[n+1]}_{cont} = n-1$.

Our description of the kernel $\ker(dq)$ and the coimage $\mathrm{coIm}(dq)$ of dq (Propositions 2.4 and 2.6) show that when $\mu = \mu_1$ or $\beta = 0$, then $\dim \mathrm{coIm}(dq)_{cont} = \dim \ker(dq)_{cont} = 0$. When $\mu = \mu_0$ and $\beta \neq 0$, then $\dim \mathrm{coIm}(dq)_{cont} = \dim \ker(dq)_{cont} = 1$. Summing up, $\dim C_{(s,b)} = \dim(T_b)_{cont} - \dim \mathrm{coIm}(dq)_{cont} + \dim \ker(dq)_{cont} = n-1$ as required. □

Proposition 2.12. — *Let $L_n \subset S^{[n,n+1]}$ the set of (s,b) such that s is punctual. Then every component of L_n has dimension at least $n+3$.*

Proof. — Following Gaffney and Lazarsfeld, if $f : X \to Y$ is a finite morphism between irreducible varieties we define the ramification locus $R_l \subset X$ containing the points x for which $f^{-1}(f(x))$ is a scheme whose support on x has length at least $l+1$. When X is normal, Y non-singular and f surjective, then the components of R_l have codimension at most l [12, p.58],[14].

We apply this theorem with $X = U_n$ the universal family over $Y = S^{[n,n+1]}$ whose fiber over (s,b) is the scheme s. It suffices to prove that U_n is normal. We shall prove that U_n is Cohen-Macaulay and smooth in codimension one, which implies normality according to Serre's criteria.

U_n is Cohen-Macaulay as it is flat over the smooth base $S^{[n,n+1]}$.

For any $\lambda = (\lambda_1, \dots, \lambda_k)$ ordered k-tuple with $\sum \lambda_i = n+1$, we denote by $\Delta_\lambda \subset S^{[n+1]}$ the stratum of subschemes z of type λ, ie. $z = z_1 \amalg \cdots \amalg z_k$ with $length(z_k) = \lambda_k$ and z_k punctual. Since any punctual $z_i(p)$ supported by p is the translation of a subscheme $z_i(0)$ supported by the origin, $\dim \Delta_\lambda = \dim(S^k \times S_0^{[\lambda_1]} \times \cdots \times S_0^{[\lambda_k]})$

For $i \leqslant k$, let $\mu_i(\lambda) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$. For $\mu \subset \lambda$ with $\Sigma \mu_i = n$ let $D_{\mu,\lambda} \subset S^{[n,n+1]}$ be the image of the generically well defined quasi-finite map

$$\begin{array}{rcl} S^k \times S_0^{[\mu_1,\lambda_1]} \times \cdots \times S_0^{[\mu_k,\lambda_k]} & \to & S^{[n,n+1]} \\ ((p_1,\dots,p_k),(t_1,w_1),\dots,(t_k,w_k)) & \mapsto & (\amalg t_i(p_i), \amalg w_i(p_i)). \end{array}$$

Let $D_\lambda \subset S^{[n,n+1]}$ denote the inverse image of Δ_λ by the natural projection $S^{[n,n+1]} \to S^{[n+1]}$. Then $D_\lambda = \cup_{i\leqslant k} D_{\mu_i(\lambda),\lambda}$. For $\lambda \neq (2,1,\dots 1)$ and $\lambda \neq (1,\dots,1)$, the codimension of D_λ in $S^{[n,n+1]}$ is at least 2 according to Proposition 2.10. In particular, no smoothness condition is required for the universal family U_n over D_λ. When $\lambda = (1,\dots,1)$, the smoothness of U_n is obvious.

We consider now the case $\lambda = (2,1,\dots 1)$. Let $(s,b) \in D_\lambda$.

If $(s,b) \in D_{\mu,\lambda}$ with $\mu = (1,\dots,1)$, then locally around (s,b), $S^{[n,n+1]}$ is isomorphic to $S^{[1,2]} \times S^{n-1}$. The universal family U_n over $D_{\mu,\lambda}$ is locally a disjoint union of sheets. The sheets coming from the universal families over S are obviously smooth. The last sheet Z coming from the factor $S^{[1,2]}$ is such that the projection $Z \to S^{[1,2]}$ is an isomorphism (the fiber is zero dimensional with length 1), so this last sheet is smooth too.

If $(s,b) \in D_{\mu,\lambda}$ with $\mu = (2,1,\dots,1,0,1,\dots,1)$, then locally around (s,b), $S^{[n,n+1]}$ is isomorphic to $S^{[2]} \times S^{n-1}$. The universal family U_n is smooth since it is the disjoint union of the pullback of the smooth universal families over $S^{[2]}$ and S.

□

Corollary 2.13. — *Every irreducible component of $S_0^{[n,n+1]}$ has dimension n or $n-1$.*

Proof. — Moving the subschemes of $S_0^{[n,n+1]}$ with translations, the product $S_0^{[n,n+1]} \times S$ parameterizes the set L of pairs $(s,b) \in S^{[n,n+1]}$ with s,b punctual with any support $p \in S$. We need to prove that the components of L have dimension $n+1$ or $n+2$. Consider the residual morphism $Res : S^{[n,n+1]} \to S$ that sends a pair (s,b) to the point q defined by the ideal $(I_s : I_b)$. Let $\Delta_x : L_n \to k$, $(s,b) \to x(s) - x(Res(s,b))$, where $x(s)$ denotes the x coordinate of the punctual subscheme s. Define similarly Δ_y. The components of L_n have dimension at least $n+3$ by Proposition 2.12. From the equality $L = L_n \cap \Delta_x^{-1}(0) \cap \Delta_y^{-1}(0)$, we conclude that any component of L has dimension at least $n+1$. The components of L have dimension at most $n+2$ by Proposition 2.10. □

We now conclude the proof of Theorem 2.9. For the proof, we need to produce some universal families over Bialynicki-Birula cells. They are constructed from the description of the tangent space by a procedure similar to the one used in [10].

Proof. — By Corollary 2.13, the components of $S_0^{[n,n+1]}$ have dimension n or $n-1$. The Bialynicki-Birula decomposition of $S_0^{[n,n+1]}$ is a partition into irreducible sets. It follows that the irreducible components of $S_0^{[n,n+1]}$

are the maximal sets for the inclusion among the closure of the Bialynicki-Birula cells. Since we already proved that there is a unique maximal component of dimension n, it remains to prove that the closure of the cells of dimension $n-1$ described by Proposition 2.11 are not irreducible components of $S_0^{[n,n+1]}$.

Let $C_{s_0,b_0} \subset S_0^{[n,n+1]}$ be a Bialynicki-Birula cell of dimension $n-1$, $(s_0,b_0) \in S_0^{[n,n+1]}$ the corresponding fixed point, $\lambda = (2^\alpha, 1^\beta)$ and μ be the partitions of b_0 and s_0. Let $\mu_0 = (2^{\alpha-1}, 1^{\beta+1})$ and $\mu_1 = (2^\alpha, 1^{\beta-1})$. We have $\mu = \mu_0$ or $\mu = \mu_1$.

First, we remark that the irreducible components of $S_0^{[n,n+1]}$ are invariant under GL_2, the group of linear automorphisms of the plane. In particular, if we prove that the generic point of C_{s_0,b_0} is not invariant under GL_2, it follows that the closure $\overline{C_{s_0,b_0}}$ is not a component of $S_0^{[n,n+1]}$. Moreover we will use the following notation: we denote by $k[X,Y]_d$ the space of homogeneous polynomials of degree d and by $\pi_d : k[X,Y] \to k[X,Y]_d$ the natural projection. Given an ideal I, I_d will denote the subspace $\pi_d(I) \subset k[X,Y]_d$. Let us finally say that an admissible cleft couple for I_b is liftable if the corresponding infinitesimal deformation of I_b can be lifted to a infinitesimal deformation of the pair (I_s, I_b).

- If $\alpha > 1$ and $\beta = 0$, let $I_b = (X^\alpha, Y^2 + \sum_{0<i<\alpha, j\leqslant 1} c_{ij} X^i Y^j)$ and $I_s = I_b + (YX^{\alpha-1})$. The variables c_{ij} are the $(n-1)$ coordinates on the Bialynicki-Birula cell C and I_s,I_b are the corresponding universal ideals. If (I_s, I_b) is the generic element in this cell, we have $(I_b)_1 = k \cdot X$, thus this generic point is not GL_2-invariant.

- If $\alpha > 1$, $\beta = 1$ and $\mu = \mu_1$, we again have $(I_b)_1 = k \cdot X$ for the generic pair (I_s, I_b), and this is not GL_2-invariant.

- If $\alpha = 1$ and $\beta = 0$, then $I_b = (X, Y^2)$ and $I_s = (X, Y)$, so this point is not GL_2-invariant. Similarly, if $\alpha = 1, \beta = 1$ and $\mu = \mu_1$, then $I_s = (X, Y^2)$, and this is not GL_2-invariant.

- If $\alpha > 1$ and $\beta > 2$ then the cleft couple $((2,0),(0,2))$ is not admissible for I_b. It follows that the generic element (b,s), $(I_b)_2$ has dimension one and is generated by a polynomial divisible by Y. In particular, the generic element of the cell is not GL_2-invariant.

- If $\alpha > 1$, $\beta = 2$ and $\mu = \mu_0$, then the cleft couple $((2,0)(0,2))$ is admissible, but not liftable, by Proposition 2.5. Thus for the same reason the cell is not GL_2-invariant.

It remains to consider the cases ($\beta = 1$ and $\mu = \mu_0$), ($\alpha = 1, \beta > 1, \mu = \mu_0$), ($\alpha = 1, \beta > 1, \mu = \mu_1$), and ($\alpha > 1, \beta = 2$ and $\mu = \mu_1$). For these cases, we will see that the closure of the Bialynicki-Birula cells are invariant under GL_2, and thus we cannot apply the same arguments as above. Instead, we will prove that the closure $\overline{C_{s_0,b_0}}$ of the cell under consideration is not an irreducible component of $S_0^{[n,n+1]}$ as it is included in the unique ("curvilinear") component of dimension n. To this end, we apply a change of coordinates to obtain simple equations for the generic point (s,b) of the cell C_{s_0,b_o} and we express (s,b) as the limit of $(s(t),b(t)) \in S_0^{[n,n+1]}$ with $s(t)$ and $b(t)$ curvilinear.

• Consider the case $\beta = 1$ and $\mu = \mu_0$. The universal families over C_{s_0,b_0} are described by coordinates c_{ij}, d and universal ideals $I_b = (X^{\alpha+1}, YX^{\alpha}, Y^2 + \sum_{i+j\geqslant 2,(i,j)\in\lambda} c_{ij}X^iY^j)$, $I_s = I_b + (YX^{\alpha-1} + dX^{\alpha})$. Note that I_b contains all the monomials of degree $\alpha+1$. The element in I_b with initial term Y^2 vanishes on a curve locally reducible around 0 as a union of two distinct smooth curves when the coefficients are generic. Up to a change of coordinates, one may suppose that the two branches have equations $X = 0$ and $Y = 0$. Then I_b contains XY and all the monomials of degree $\alpha+1$. Thus $I_b = (X^{\alpha+1}, Y^{\alpha+1}, XY)$ because of the inclusion, and both ideals have the same colength. The ideal I_s has codimension one in I_b thus the general element has the form $I_s = I_b + (X^{\alpha} + dY^{\alpha})$. Up to a linear change of coordinates of the form $Y \mapsto c.Y$, $X \mapsto X$, one may suppose that $I_s = I_b + (X^{\alpha} + Y^{\alpha})$. For a generic pair (s,b), b is the union of two curvilinear schemes of length $\alpha+1$ supported at the origin, and s a colength one subscheme of b: this cell is invariant under automorphisms.

For $t \neq 0$ let $I_b(t) = (XY + t(X+Y), X^{2\alpha+1}, Y^{2\alpha+1})$ and $I_s(t) = (XY + t(X+Y), X^{2\alpha}, Y^{2\alpha})$. Let $(I_s(0), I_b(0))$ be the limit at $t = 0$ of $(I_s(t), I_b(t))$. Obviously $XY = \lim_{t\to 0} XY + t(X+Y) \in I_b(0)$.

Since $b(t)$ (resp. $s(t)$) has length $2\alpha+1$ (resp. 2α) with support the origin, every monomial of degree $2\alpha+1$ (resp. 2α) is in $I_b(t)$ (resp $I_s(t)$). Since $X + Y = \frac{-XY}{t}$ modulo $I_b(t)$, we obtain $X(X+Y)^{\alpha} \in I_b(t)$ and $Y(X+Y)^{\alpha} \in I_b(t)$. Summing up, $I_b(0) \supset (XY, X(X+Y)^{\alpha}, Y(X+Y)^{\alpha}) = (XY, X^{\alpha+1}, Y^{\alpha+1})$. This inclusion is an equality since the two ideals have colength $2\alpha+1$. The same reasoning with the curvilinear $s(t)$ instead of $b(t)$ shows that $I_s(0) \supset (XY, X^{\alpha+1}, Y^{\alpha+1})$. Modulo $I_s(t)$, $(X+Y)^{\alpha} = (\frac{-XY}{t})^{\alpha} = 0$. Thus $I_s(0) \supset (XY, X^{\alpha+1}, Y^{\alpha+1}, (X+Y)^{\alpha})$ and the equality follows by length considerations. We have proved $I_b = I_b(0)$ and $I_s = I_s(0)$, as expected.

• If $\alpha > 1, \beta = 2, \mu = \mu_1$, we can perform as above a change of coordinates in order to reduce to the case $I_s = (XY, X^{\alpha+1}, Y^{\alpha+1})$ and $I_b = ((XY, X^{\alpha+2}, Y^{\alpha+2}, X^{\alpha+1} + Y^{\alpha+1})$. For a generic pair (b, s), b is a colength one subscheme in the union $c_1 \cup c_2$ of two curvilinear subschemes of length $\alpha + 2$, and s is the union $c_1' \cup c_2'$, where $c_i' \subset c_i$ is the unique colength one subscheme: this cell is invariant under automorphisms. The same computation as above now shows that $I_s = \lim I_s(t)$, $I_b = \lim I_b(t)$ with $I_s(t) = (XY + t(X+Y), X^{2\alpha+1}, Y^{2\alpha+1})$ and $I_b(t) = (XY + t(X+Y), X^{2\alpha+2}, Y^{2\alpha+2})$.

• If $\alpha = 1, \beta > 1, \mu = \mu_0$, then $I_b = (X^{1+\beta}, XY + \sum_{2\leqslant j\leqslant\beta} a_j X^j, Y^2 + \sum_{2\leqslant j\leqslant\beta} a_j Y X^{j-1})$ and $I_s = I_b + (Y + \sum_{2\leqslant j\leqslant\beta} a_j X^{j-1} + dX^\beta)$. Up to the coordinate change $X \mapsto X$, $Y \mapsto Y + \sum_{2\leqslant j\leqslant\beta} a_j X^{j-1} + dX^\beta$, one may suppose that $I_b = (X^{1+\beta}, XY, Y^2)$ and $I_s = (X^{1+\beta}, Y)$. It follows that for a generic pair (b, s), s is a curvilinear scheme and b the union of s and the 2-fat point: this cell is invariant under automorphisms. Consider the curvilinear ideals $c = (X^{2+\beta}, Y)$, and the automorphism $\phi_t : X \mapsto X, Y \mapsto tY + X^{\beta+1}$. The ideals $I_b(t) = \phi_t(c)$ and $I_s(t) = \phi_t(I_s) = I_s$ are such that $\lim_{t\to 0} I_s(t) = I_s$ and $\lim_{t\to 0} I_b(t) = I_b$.

• If $\alpha = 1, \beta > 1, \mu = \mu_1$, then $I_b = (X^{1+\beta}, XY + \sum_{2\leqslant j\leqslant\beta} a_j X^j, Y^2 + \sum_{2\leqslant j\leqslant\beta} a_j Y X^{j-1} + dX^\beta)$ and $I_s = I_b + (X^\beta)$. Up to the two coordinate changes $X \mapsto X$, $Y \mapsto Y + \sum_{2\leqslant j\leqslant\beta} a_j X^{j-1}$, and then $X \mapsto X, Y \mapsto \lambda Y$, one may suppose that $I_b = (X^{1+\beta}, XY, Y^2 + X^\beta)$ and $I_s = (X^\beta, XY, Y^2)$.

For a generic pair (b, s), b is a colength one subscheme in the union $c_1 \cup c_2$ of two curvilinear subschemes of length 3 and $n-1$, and s is the union $c_1' \cup c_2'$, where $c_i' \subset c_i$ is the unique colength one subscheme: this cell is invariant under automorphisms.

For $t \neq 0$ let $I_b(t) = (XY - t^2 Y + t^\beta X, X^{\beta+2}, Y^{\beta+2})$ and $I_s(t) = (XY - t^2 Y + t^\beta X, X^{\beta+1}, Y^{\beta+1})$. Let $(I_s(0), I_b(0))$ be the limit at $t = 0$ of $(I_s(t), I_b(t))$. Obviously $XY = \lim_{t\to 0} XY - t^2 Y + t^\beta X \in I_b(0) \cap I_s(0)$.

Since $b(t)$ (resp. $s(t)$) has length $\beta + 2$ (resp. $\beta + 1$) with support the origin, all the monomials in $k[X, Y]$ of degree $\beta+2$ (resp. $\beta+1$) are in $I_b(t)$ (resp $I_s(t)$). A straigthforward induction shows that

$$\forall k \geqslant 1,\ Y = t^{\beta-2} X + t^{\beta-4} X^2 + \cdots + t^{\beta-2k} X^k + t^{-2k} X^k Y \ mod\ I_b(t) \cap I_s(t).$$

In particular, $e(t) := Y - t^{\beta-2}X - \cdots - t^{-\beta-2}X^{\beta+1} \in I_b(t)$, $f(t) := Y - t^{\beta-2}X - \cdots - t^{-\beta}X^{\beta} \in I_s(t)$, and

$$X^{\beta+1} = \lim_{t\to 0} t^{\beta+2}e(t) \in I_b(0),$$
$$X^{\beta} = \lim_{t\to 0} t^{\beta}f(t) \in I_s(0).$$

Since $Y^2 = (Y - e(t))^2 \ \ mod\ I_b(t)$ and $Y^2 = (Y - f(t))^2 \ \ mod\ I_s(t)$, we get

$$g(t) := Y^2 - t^{2\beta-4}X^2 - \cdots - (\beta-1)X^{\beta} - \beta t^{-2}X^{\beta+1} \ \in I_b(t),$$
$$h(t) := Y^2 - t^{2\beta-4}X^2 - \cdots - (\beta-1)X^{\beta} \ \in I_s(t).$$

It follows that

$$Y^2 + X^{\beta} = \lim_{t\to 0} g(t) - \beta t^{\beta}e(t) \in I_b(0)$$
$$Y^2 = \lim_{t\to 0} h(t) - (\beta-1)t^{\beta}f(t) \in I_s(0)$$

Summing up, these limits prove that $I_s(0) \supset I_s$ and $I_b(0) \supset I_b$. The equalities follows from the inclusions by length considerations. $\square$

3. Bases of the equivariant Chow ring

We now present three natural bases $\mathrm{fix}(\lambda), \mathrm{nak}(\lambda), \mathrm{es}(\lambda)$ of the K-vector space $A^*_K(S^{[n]})$. Our three bases of $A^*_K(S^{[n]})$ are naturally parameterized by the set $\mathcal{P}_n$ of partitions λ of weight n.

Let $n \geqslant 0$ and let $i > 0$ be integers. We define some correspondences following Nakajima [16]:

Definition 3.1. — *Let $Q^n_i \subset S^{[n]} \times S^{[n+i]}$ be the closure of the set of pairs (z_n, z_{n+i}) where $z_n \in S^{[n]}$ is arbitrary and $z_{n+i} \in S^{[n+i]}$ is the disjoint union of z_n and a punctual scheme of length i.*

The T-invariant correspondence $\amalg_n Q^n_i$ induces an operator (called "creation operator") $q_i : \oplus_n A^*_T(S^{[n]}) \to \oplus_n A^{*+i-1}_T(S^{[n+i]})$ on Chow groups. Assume now that $i < 0$. The "destruction operator" q_i is defined either as the dual of q_{-i} or with the correspondence $Q^n_i \subset S^{[n]} \times S^{[n+i]}$ which is dual to the correspondence Q^{n+i}_{-i}, in the sense of Definition 1.9. By Proposition 1.10, both definitions lead to the same operator. For any i, q_i has conformal degree i and cohomological degree $i-1$. We make the convention that $q_0 = 0$.

Remark 3.2. — There is a morphism $s : Q_i \to S$ mapping the pair (z_n, z_{n+i}) to the support of $\mathcal{O}_{z_{n+i}}/\mathcal{O}_{z_n}$. Let $\alpha \in A^*_T(S)$. One may define operators $q_i(\alpha)$ by the incidence $Q_i \cup s^*\alpha$ as in [15]. These operators have the geometric meaning "adding a punctual scheme of length i whose support lies in α". However, since $A_T^*(S) = A_T^*(pt)$, we have by linearity $q_i(\alpha) = \alpha q_i$, therefore these operators do not yield new operators.

Given a partition λ of length l, we will denote by $\mathrm{nak}(\lambda)$ the equivariant class obtained applying $q_{\lambda_l} \circ \cdots \circ q_{\lambda_1}$ to the vacuum ϕ, where the vacuum is the fundamental class on $S^{[0]}$. Since by [16], the classes $\mathrm{nak}(\lambda)$ for $\lambda \in \mathcal{P}_n$ restrict to a basis of the non equivariant Chow group of $S^{[n]}$, $\{\mathrm{nak}(\lambda)\ ,\ \lambda \in \mathcal{P}_n\}$ is a basis of $A_K^*(S^{[n]})$ over K.

Recall the classes introduced by Ellingsrud and Strømme in [8]. These classes are introduced for $\mathbb{P}^2$ but we can consider the same classes for $\mathbb{A}^2$. We choose an injection $k^* \to T, t \mapsto (t^{-1}, t^{-d})$ where d is large. The action of T on $S^{[n]}$ induces an action of k^*. With the assumption that d is large enough, the k^*-fixed points are the T-fixed points; in particular there is a finite number of them and they are parameterized by partitions. More precisely, if $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition, we denote by x_λ the subscheme with ideal I_{x_λ} generated by the $l+1$ polynomials $X^{k-1}Y^{\lambda_k}$, where k varies from 1 to $l+1$, with the convention that $\lambda_{l+1} = 0$.

To each partition λ of weight n corresponds a Bialynicki-Birula cell containing the points $p \in S^{[n]}$ such that $\lim_{t\to 0} t.p = x_\lambda$. We denote $ES_\lambda \subset S^{[n]}$ the closure of this cell. Let l be the length of the partition λ. Geometrically, the Bialynicki-Birula cell associated to λ parameterizes the subschemes $Z \subset S$ for which there exist $x_1, \dots, x_l \in k$ such that each intersection $Z \cap \{X = x_i\}$ has length λ_i. The equivariant class of ES_λ in the Chow ring will be denoted es_λ. Since by definition $S^{[n]}$ has the cellular decomposition $S^{[n]} = \amalg_\lambda ES_\lambda$, where $\lambda \in \mathcal{P}_n$, the classes es_λ for $\lambda \in \mathcal{P}_n$ form a basis of $A_T^*(S^{[n]})$.

Finally, the classes $\mathrm{fix}(\lambda) \in A_K^*(S^{[n]})$ are defined using the localization theorem [7, Theorem 1]. The set $\left(S^{[n]}\right)^T$ contains the points x_λ parameterized by $\lambda \in \mathcal{P}_n$. Let $1_\lambda \in A_T^*(\left(S^{[n]}\right)^T)$ be the class corresponding to x_λ. Let $i : \left(S^{[n]}\right)^T \to S^{[n]}$ denote the inclusion. By Lemma 1.4, $i_K^* : A_K^*(S^{[n]})^T \to A_K^*(S^{[n]})$ is an isomorphism.

Definition 3.3. — *Let* $\mathrm{fix}(\lambda)$ *be the unique element in* $A_K^*(S^{[n]})$ *such that* $i_K^*(\mathrm{fix}(\lambda)) = 1_\lambda$.

Let us denote by $\mathrm{Tan}(\lambda) \in \mathbb{Z}[U, V]$ the product of the weights of the tangent space $T_{x_\lambda} S^{[n]}$. According to the self-intersection formula, we have

$$\mathrm{fix}(\lambda) = i_*(1_\lambda)/Tan(\lambda)\ . \tag{3.1}$$

Recall Definition 1.8. We deduce from (3.1) the following lemma:

Lemma 3.4. — *We have* $\langle \mathrm{fix}(\lambda), \mathrm{fix}(\lambda)\rangle_{S^{[n]}} = 1/\mathrm{Tan}(\lambda)$.

Proof. — By (3.1), we have

$$\langle \mathrm{fix}(\lambda), \mathrm{fix}(\lambda)\rangle_{S^{[n]}} = \frac{\langle i_*(1_\lambda), i_*(1_\lambda)\rangle_{S^{[n]}}}{\mathrm{Tan}(\lambda)^2}.$$

By Definition 1.8, this is $\pi_*^K(i_*(1_\lambda) \cup i_*(1_\lambda))/\mathrm{Tan}(\lambda)^2$, if $\pi : S^{[n]} \to \mathrm{Spec}\, k$ denotes the projection to a point. Since $i_*(1_\lambda) \cup i_*(1_\lambda) = \mathrm{Tan}(\lambda) \cdot i_*(1_\lambda)$, the lemma follows. □

4. Classical Operators

Let us denote by A the direct sum $\bigoplus_n A_T^*(S^{[n]})$ and $A_K := \bigoplus_n A_K^*(S^{[n]})$. In this section, we consider the classical operators acting on A_K, namely the creation/destruction operators q_i and the boundary operator ∂, and an auxiliary operator ρ. We compute them in the basis $\mathrm{fix}(\lambda)$. We also compute the commutators of these operators.

The operators ∂, ρ, q_i for $i > 0$ are naturally defined on A and they are naturally extended to A_K. We use freely the same notation for the operators on A and on A_K. On the contrary, the operators q_i for $i < 0$ are defined on A_K but not on A. This is because their definition involves non proper morphisms.

In Theorem 4.8 we give an explicit algorithm to compute all operators q_i in the basis $\mathrm{fix}(\lambda)$. With the help of this result, we checked on a computer our formulas for commutators, such as Theorem 4.25.

4.1. The operators q_1 and q_{-1}

Given a partition λ, we denote by $\lambda[1]$ the set of partitions μ with $\lambda \subset \mu$ and $|\mu| = |\lambda| + 1$. Given two partitions λ, μ with $\mu \in \lambda[1]$, we denote by $\mathrm{Coker}(\lambda,\mu) \in \mathbb{Z}[U,V]$ the product of the weights of Proposition 2.5 and by $\mathrm{Ker}(\lambda,\mu) \in \mathbb{Z}[U,V]$ the product of the weights of Proposition 2.6. With these notations we have the following proposition:

Proposition 4.1. — *We have the following formula:*

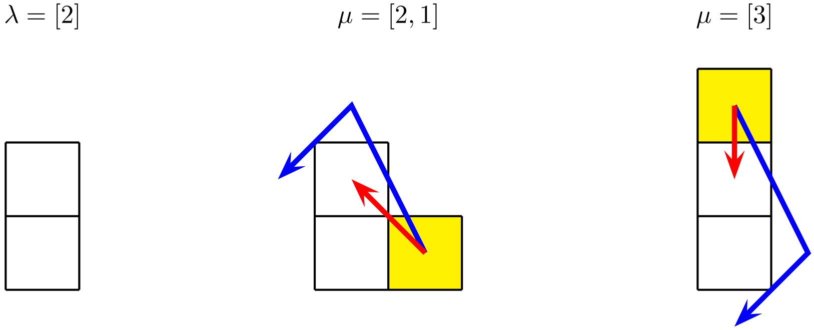
$$q_1(\mathrm{fix}(\lambda)) = \sum_{\mu \in \lambda[1]} \frac{\mathrm{Coker}(\lambda,\mu)}{\mathrm{Ker}(\lambda,\mu)} \mathrm{fix}(\mu)\ .$$

Proof. — Let $S^{[n,n+1]}$ denote the incidence, with projections π_n, π_{n+1}. By definition, we have $q_1(\mathrm{fix}(\lambda)) = \pi^K_{n+1,*}\left(\pi^*_{n,K}(\mathrm{fix}(\lambda)) \cup [S^{[n,n+1]}]\right)$. It follows from Propositions 2.7, 2.5 and 2.6 that

$$[S^{[n,n+1]}] = \sum_{\lambda,\mu:\mu\in\lambda[1]} \frac{\mathrm{Tan}(\lambda)\mathrm{Coker}(\lambda,\mu)}{\mathrm{Ker}(\lambda,\mu)} \ \mathrm{fix}(\lambda) \otimes \mathrm{fix}(\mu).$$

We conclude thanks to Bott Formula (Theorem 1.5). □

For example, we have $q_1(\mathrm{fix}([2])) = \frac{-2U+V}{-U+V}\mathrm{fix}([2,1]) + 3\mathrm{fix}([3])$. This is illustrated as follows, where the weights of the blue resp. red arrows are the numerators resp. denominators of the coefficients:



We deduce a formula for q_{-1}. Given a partition μ, let $\mu[-1]$ denote the set of partitions λ with $\lambda \subset \mu$ and $|\lambda| = |\mu| - 1$. By Lemma 3.4, since q_{-1} is the adjoint of q_1, we get :

Proposition 4.2. — *We have the following formula:*

$$q_{-1}(\mathrm{fix}(\mu)) = \sum_{\lambda\in\mu[-1]} \frac{\mathrm{Coker}(\lambda,\mu)}{\mathrm{Ker}(\lambda,\mu)} \cdot \frac{\mathrm{Tan}(\lambda)}{\mathrm{Tan}(\mu)} \ \mathrm{fix}(\lambda).$$

4.2. Class of the boundary and derivatives

We turn to the problem of determining the equivariant class of the divisor Δ_2 of non-reduced schemes. In the non equivariant setting on $S^{[n]}$, recall the following formula of Lehn which expresses the class $[\Delta_2]_{cla}$ of Δ_2 in terms of the classical Chern class c^{cla} of the tautological bundle: $[\Delta_2]_{cla} = -2c_1^{cla}(\mathcal{O}^{[n]})$. We prove an equivariant analog in the equivariant Chow ring: $[\Delta_2] = -2c_1(\mathcal{O}^{[n]})$, where the Chern class considered is the equivariant Chern class. Our method involves equivariant techniques and does not rely on Lehn's ideas.

Denote $\partial \in A_T^*(S^{[n]})$ the class of Δ_2, and let $p : \mathrm{Spec}\ \ k \to S^{[n]}$ be a T-fixed point. We'd like to compute $p^*\partial$. To this end, assume that p corresponds to the partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of weight n. We let $l(\lambda)$ denote the number of non-vanishing parts of λ, and $h(\lambda) = \lambda_1$. Let $\lambda^\vee$ denote the partition dual to λ.

Proposition 4.3. — *We have*

$$p^*\partial = -(\Sigma_{j=1}^{h(\lambda)} \lambda_j^\vee(\lambda_j^\vee - 1))\, U - (\Sigma_{i=1}^{l(\lambda)} \lambda_i(\lambda_i - 1))\, V.$$

Proof. — We treat first the case $n = 2$. Then $S^{[2]}$ is the blow-up of $S \times S$ along the diagonal. Assume, moreover, that $\lambda = (2)$. Then $T_p S^{[2]}$ contains 4 eigenlines, of weight $-U, -U+V, -V, -2V$. In this case Δ is smooth, and the tangent space $T_p\Delta$ contains the three eigenlines of weight $-U, -V$ and $-U+V$: in fact, the first two lines are obtained by translating the double point p, and $-U+V$ is the weight of the deformation obtained with schemes supported at the origin. We deduce that $p^*\partial = -2V$.

We now consider the general case. Let $l = l(\lambda)$ and $h = h(\lambda)$. Given $\mathbf{x} = (x_1, \dots, x_l)$ and $\mathbf{y} = (y_1, \dots, y_h)$ tuples of elements in k, we let $I_{(\mathbf{x},\mathbf{y})}$ denote the ideal generated by the $l+1$ polynomials $\prod_{i=1}^{m-1}(X - x_i) \cdot \prod_{j=1}^{\lambda_m}(Y - y_j)$, where m varies from 1 to $l+1$. When all the x_i and all the y_j are distinct, $k[X,Y]/I_{(\mathbf{x},\mathbf{y})}$ is reduced and the corresponding set of points is the set of (x_i, y_j) where $i \leqslant l$ and $j \leqslant \lambda_i$. Thus $I_{(\mathbf{x},\mathbf{y})}$ has length n. On the other hand, when $\mathbf{x} = (0, \dots, 0)$ and $\mathbf{y} = (0, \dots, 0)$, the ideal $I_{(\mathbf{x},\mathbf{y})}$ is monomial and generated by the elements the $X^m Y^{\lambda_m}$, and thus also has length n. Since the length of this family of ideals is upper-semicontinuous, it follows that it is constant, and this family is flat.

In this way, we obtain a T-equivariant morphism $\varphi : k^{l+h} \to S^{[n]}$ with respect to the natural action on k^{l+h}. We now compute $\varphi^*\partial$. If $\{i_1, i_2\} \subset \{1, \dots, l\}$ is a subset with two elements, where we assume $i_1 < i_2$, we denote by $\Delta_{\{i_1,i_2\}} \subset k^{l+h}$ the class of the variety of tuples $(\mathbf{x}, \mathbf{y})$ with $x_{i_1} = x_{i_2}$ and $\partial_{\{i_1,i_2\}}$ its class in the equivariant Chow ring of k^{l+h}. Let $z : \mathrm{Spec}\ k \to k^{h+l}$ be the origin of k^{h+l}; since $\partial_{\{i_1,i_2\}}$ is defined by one equation of weight U, it follows that $z^*\partial_{\{i_1,i_2\}} = -U$. Similarly, if $j_1 < j_2$, let $\Delta_{\{j_1,j_2\}}$ be the divisor defined by $y_{j_1} = y_{j_2}$, and let $\partial_{\{j_1,j_2\}}$ denote its class. We have $z^*\partial_{\{i_1,i_2\}} = -V$.

We claim that

$$\varphi^*\partial = \Sigma_{\{i_1,i_2\} \subset \{1,\dots,l\}} 2\lambda_{i_2} \partial_{\{i_1,i_2\}} + \Sigma_{\{j_1,j_2\} \subset \{1,\dots,h\}} 2\lambda_{j_2}^\vee \partial_{\{j_1,j_2\}}\ .$$

Clearly, we have an equality of sets

$$\varphi^{-1}(\Delta) = \bigcup_{\{i_1,i_2\}\subset\{1,\dots,l\}} \Delta_{\{i_1,i_2\}} \cup \bigcup_{\{j_1,j_2\}\subset\{1,\dots,h\}} \Delta_{\{j_1,j_2\}} ,$$

and we claim that the multiplicity of $\Delta_{\{i_1,i_2\}}$ is $2\lambda_{i_2}$. To see why, let $(\mathbf{x},\mathbf{y})$ be a generic point in $\Delta_{\{i_1,i_2\}}$: we have $x_{i_1} = x_{i_2}$ but no other equality among the x_i's and the y_j's. Thus the scheme represented by $\varphi(\mathbf{x},\mathbf{y})$ is a union of λ_{i_2} double points and $n-2\lambda_{i_2}$ other distinct points. Near the point $\varphi(\mathbf{x},\mathbf{y})$, $S^{[n]}$ is isomorphic to $(S^{[2]})^{\lambda_{i_2}} \times S^{n-2\lambda_{i_2}}$. Thus the multiplicity of our component may be deduced from the case of $S^{[2]}$: in this case the multiplicity was 2 in view of the computation we made at the beginning of the proof. Thus the multiplicity is $2\lambda_{i_2}$ as claimed.

Since $z^*\varphi^*\partial = p^*\partial$, it remains only to show that $2\Sigma_{\{i_1,i_2\}\subset\{1,\dots,l\}}\lambda_{i_2} = \Sigma_{j=1}^{h(\lambda)}\lambda_j^\vee(\lambda_j^\vee - 1)$. The first sum is equal to $\Sigma_{1\leqslant i_1<i_2\leqslant l, 1\leqslant j\leqslant \lambda_{i_2}} 2$. In this sum, when $j = j_0$ is fixed, i_2 is such that $\lambda_{i_2} \geqslant j_0$, which forces $i_2 \leqslant \lambda_{j_0}^\vee$. Thus $\Sigma_{1\leqslant i_1<i_2\leqslant l, j\leqslant\lambda_{i_2}, j=j_0} 2 = \lambda_{j_0}^\vee(\lambda_{j_0}^\vee - 1)$. Our proof is now complete. □

Corollary 4.4. — *The equivariant class of Δ_2 in $A_T^*(S^{[n]})$ is $\partial = -2c_1(\mathcal{O}_X^{[n]})$.*

Proof. — By Proposition 4.3, the two classes have the same restriction on the T-fixed points of $S^{[n]}$ and the restriction morphism is injective. □

If $f : A \to A$ is any operator, we now give a formula for the commutator $[\partial, f]$. To express this formula, let us introduce the following notation:

Notation 4.5. — If $f : A \to A$ is an endomorphism, define $\Delta_{f,\lambda,\mu} \in K$ for λ, μ partitions by the formula

$$f(\mathrm{fix}(\lambda)) = \sum_\mu \Delta_{f,\lambda,\mu}\mathrm{fix}(\mu).$$

For $c = (a, b) \in \mathbb{N}^2$, let $w(c) = aU+bV$ be the weight of the corresponding monomial. Corollary 4.4 immediately implies:

Corollary 4.6. — *Let $f : A \to A$ be any operator and let $\lambda \subset \mu$ be two partitions. We have*

$$\Delta_{[\partial,f],\lambda,\mu} = -2\Delta_{f,\lambda,\mu} \sum_{c\in\mu\setminus\lambda} w(c) .$$

4.3. Computation of the operator q_i for all i

In the previous sections, we computed q_1, q_{-1} and ∂ on the basis $\mathrm{fix}(\lambda)$. We introduce an auxiliary operator ρ and give formulas for higher q_i's in terms of q_1, q_{-1} and ρ. This yields an inductive procedure to compute q_i on the basis $\mathrm{fix}(\lambda)$.

Definition 4.7. — *Let $R^n \subset S^{[n,n+1]}$ be the closure of the set of pairs of schemes (z_n, z_{n+1}) with z_n reduced, $z_n \subset z_{n+1}$ and $z_n = (z_{n+1})_{red}$.*

Let $\rho : \oplus_n A_T^*(S^{[n]}) \to \oplus_n A_T^{*+1}(S^{[n+1]})$ be the morphism associated with the correspondence $\amalg_n [R^n]$. It has conformal and cohomological degree 1.

The following theorem gives a complete computation of the operators q_i.

Theorem 4.8. — *We have*

$$\begin{array}{rcll} (i-1)q_i & = & \rho q_{i-1} - q_{i-1}\rho & \text{for } i>1 \\ (i+1)q_i & = & \rho^\vee q_{i+1} - q_{i+1}\rho^\vee & \text{for } i<-1 \\ 2\rho & = & \partial q_1 - q_1 \partial & \\ 2\rho^\vee & = & q_{-1}\partial - \partial q_{-1} & \end{array}$$

Proof. — The non equivariant version of the first statement is proved in [15, Theorem 3.5]. Our formula can be proved geometrically as follows. Let π_1, π_2, π_3 be the projections of $S^{[n]} \times S^{[n+i-1]} \times S^{[n+i]}$ on each factor and, for $a, b \in \{1,2,3\}$, let π_{ab} be the projection on two factors. To compute the composition ρq_{i-1}, we have to understand the intersection $\pi_{12}^{-1}(Q_{i-1}^n) \cap \pi_{23}^{-1}(R^{n+i-1})$. There are two irreducible components in this intersection. One, say E_1, is the closure of the set of triples of the form $(z_n, z_n \amalg w_{i-1}, z_{n+1} \amalg w_{i-1})$, where z_n is a reduced subscheme of length n, w_{i-1} is a punctual subscheme of length $i-1$ with support not belonging to z_n, and z_{n+1} is a subscheme of length $n+1$ containing z_n and having the same support as z_n.

Another component denoted E_2 is the closure of the set of triples of the form $(z_n, z_n \amalg w_{i-1}, z_n \amalg w_i)$, where z_n is again a reduced subscheme of length n and w_{i-1} resp. w_i are punctual subschemes of length $i-1$ resp. i with common support not belonging to z_n. The component E_2 has multiplicity $i-1$ and $\pi_{13}(E_2) = Q_i^n$. We claim that these are all the components of the intersection $\pi_{12}^{-1}(Q_i^n) \cap \pi_{23}^{-1}(R^{n+i-1})$. This can be seen using arguments similar to the detailed proof of Proposition 4.11; details will be skipped here.

Consider now the composition $q_{i-1}\rho$ and the product $S^{[n]} \times S^{[n+1]} \times S^{[i]}$. The intersection $\pi_{12}^{-1}(R^n) \cap \pi_{23}^{-1}(Q_{i-1}^{n+1})$ has only one component E_1' which is the closure of the set of triples $(z_n, z_{n+1}, z_{n+1} \amalg w_{i-1})$, with the same notations as for the component E_1. In the commutator $\rho q_{i-1} - q_{i-1}\rho$ the components E_1 and E_1' cancel each other, and we get the formula.

The third statement is proved by a similar argument. The correspondences in $S^{[n]} \times S^{[n+1]}$ corresponding to both compositions ∂q_1 and $q_1\partial$ contain the closure of the set of pairs $(z_n, z_n \amalg w_1)$ where z_n is a non-reduced subscheme of length n, and these cancel each other. The composition ∂q_1 moreover contains the closure of the set of pairs (z_n, z_{n+1}) with z_n reduced and $\mathrm{supp}(z_{n+1}) = \mathrm{supp}(z_n)$, namely, the correspondence R^n, with mutliplicity 2.

The second and the fourth equalities are obtained from the first and the third equalities using duality and the fact that ∂ is self-dual. □

Applying this theorem and Corollary 4.6, we deduce the following formula for the operator ρ:

Corollary 4.9.

$$\rho(\mathrm{fix}(\lambda)) = -\sum_{\mu\in\lambda[1]} \frac{\mathrm{Coker}(\lambda,\mu)}{\mathrm{Ker}(\lambda,\mu)}\, w(\mu\setminus\lambda)\,\mathrm{fix}(\mu) \tag{4.1}$$

4.4. Commutation relations

In this subsection, we compute the commutators between the different q_i's.

We note that it is not possible to keep the proof by Nakajima. Indeed, the equivariant pushforward of a class under a non proper contracting morphism is not zero and the vanishing arguments of Nakajima are not valid in our context. This non vanishing feature is crucial for us because this is precisely the contribution of such contracting morphisms that will give the non commutativity $[q_{-1}, q_1] = \frac{1}{UV} Id$.

4.4.1. Commutation with q_1

Our first goal is to study the commutator $[q_1, q_i]$. This will follow from a geometric argument studying directly the correspondences.

Recall (Definition 3.1) that we denoted by $Q_i^n \subset S^{[n]} \times S^{[n+i]}$ Nakajima's correspondence. Consider the product $S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]}$ and for $a, b \in \{n, n+1, n+i+1\}$ the projection $\pi_{a,b} : S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]} \to S^{[a]} \times S^{[b]}$. Let us denote by $\mathcal{I}_{1i} \subset S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]}$ the intersection $\pi_{n,n+1}^{-1}(Q_1^n) \cap \pi_{n+1,n+i+1}^{-1}(Q_i^{n+1})$.

Let us introduce some piece of notation:

Notation 4.10. — Let $w, z \subset S$ be two subschemes. Assume that $w \subset z$ or $w \supset z$. If $w \subset z$ assume moreover that the support of $\mathcal{O}_z/\mathcal{O}_w$ is a point: in this case we denote by $\mathrm{supp}(w \neq z)$ this point. If $w \supset z$ assume that the support of $\mathcal{O}_w/\mathcal{O}_z$ is a point: we denote by $\mathrm{supp}(w \neq z)$ this point.

Moreover, given a subscheme w and a point x, we denote by w_x the largest punctual subscheme of w whose support is x.

We denote by $l(w)$ the length of w.

Let $E_1 = \{(w,z,t) \in \mathcal{I}_{1i}, z \cap t \text{ reduced}, \mathit{supp}(w \neq z) \neq \mathit{supp}(z \neq t)\}$ and denote by E_2 the set $\{(w,z,t) \in \mathcal{I}_{1i}, z \cap t \text{ reduced}, \mathit{supp}(w \neq z) = \mathit{supp}(z \neq t)\}$.

Proposition 4.11. — *The intersection*

$$\mathcal{I}_{1i} = \pi^{-1}_{n,n+1}(Q^n_1) \cap \pi^{-1}_{n+1,n+i+1}(Q^{n+1}_i)$$

is proper. If $i > 0$, then $\mathcal{I}_{1i} = \overline{E_1}$ and $\mathcal{I}_{1i}$ is reduced irreducible of dimension $2n+i+3$. If $i < 0$, then $\mathcal{I}_{1i} = \overline{E_1} \cup \overline{E_2}$ a union of two reduced subschemes of dimension $2n+i+3$.

Proof. — By Proposition 2.7, $\pi^{-1}_{n,n+1}(Q^n_1)$ is smooth and thus locally a complete intersection. Therefore each irreducible component of $\mathcal{I}_{1i}$ has codimension at most $4n+i+1$ in $S^{[n]} \times S^{[n+1]} \times S^{[n+i+1]}$, and so has dimension at least $2n+i+3$.

If $i < 0$, let $e = 2$ and if $i > 0$, let $e = 1$. To prove that $\mathcal{I}_{1i}$ has exactly e reduced components and the other claims of the proposition, it suffices to describe a set of subschemes $E(p,q) \subset \mathcal{I}_{1i}$ and $E(p) \subset \mathcal{I}_{1i}$ with the following conditions:

- $\mathcal{I}_{1i} = \coprod_{p,q} E(p,q) \amalg \coprod_p E(p)$ realizes $\mathcal{I}_{1i}$ as a disjoint union.
- Exactly e elements among the subschemes $E(p,q)$ and $E(p)$ have the expected dimension $2n+i+3$.
- These e strata are reduced.
- The other strata have dimension less than $2n+i+3$.

The components in the intersection will then be the closures of the maximal strata. For $p \geqslant 0$, $q \geqslant 0$, $i \neq 0$, $q+i \geqslant 0$, let $E(p,q)$ be the set

$$\{(w,z,t) \in \mathcal{I}_{1i}, \mathrm{supp}(w \neq z) \neq \mathrm{supp}(z \neq t), \\ l(w_{supp(w \neq z)}) = p, l(w_{supp(z \neq t)}) = q\}.$$

For $p \geqslant 0$, $i \neq 0$, $p+1+i \geqslant 0$, let

$$E(p) := \{(w,z,t) \in \mathcal{I}_{1i}, \mathrm{supp}(w \neq z) = \mathit{supp}(z \neq t) = x, l(w_x) = p\}.$$

Let (w,z,t) in $E(p,q)$. Let $x = supp(w \neq z)$ and $y = supp(z \neq t)$. Let $w_1 \subset w$ the largest subscheme whose support does not contain x nor y. Since $w = w_1 \cup w_x \cup w_y$, $z = w_1 \cup z_x \cup w_y$, $t = w_1 \cup z_x \cup t_y$, the triple (w,z,t) is characterized by the data $w_1, (w_x, z_x), w_y, t_y$. Since $l(w_1) = n-p-q$, w_1 moves in dimension $2n-2p-2q$. The pair (w_x, z_x) with $w_x \subset z_x$, $l(w_x) = l(z_x) - 1 = p$ moves in dimension $p+2$ by Proposition 2.10. The scheme w_y with $l(w_y) = q$ moves in dimension q if $q=0$ and $q+1$ if $q>0$. Given w_y, the scheme t_y with $l(t_y) = q+i$, $t_y \supset w_y$ (case $i>0$), $t_y \subset w_y$ (case $i<0$) moves in dimension $q+i$ if $q+i=0$, $q+i+1$ if $q+i>0$ and $q=0$, at most $q+i-1$ if $q+i>0$ and $q>0$.

Summing up, in any case, the dimension of $E(p,q)$ is at most $2n+i+3$, and the equality $\dim E(p,q) = 2n+i+3$ is realized only when $i>0$, $p=0$, $q=0$, and when $i<0, p=0, q=-i$.

Let (w,z,t) in $E(p)$. Let $x = supp(w \neq z)$. Let $w_1 \subset w$ the largest subscheme whose support does not contain x. Since $w = w_1 \cup w_x$, $z = w_1 \cup z_x$, $t = w_1 \cup t_x$, the triple (w,z,t) is characterized by the data $w_1, (w_x, z_x), t_x$. Since $l(w_1) = n-p$, w_1 moves in dimension $2n-2p$. The pair w_x, z_x with $w_x \subset z_x$ and $l(w_x) = l(z_x) - 1 = p$ moves in dimension $p+2$. The scheme t_x with $l(t_x) = p+1+i$, $t_x \supset z_x$ (case $i>0$), $t_x \subset z_x$ (case $i<0$) moves in dimension $p+1+i$ if $p+1+i=0$, at most $p+i$ if $p+1+i>0$.

Summing up, in any case, the dimension of $E(p)$ is at most $2n+i+3$, and the equality $\dim E(p) = 2n+i+3$ is realized only when $i<0, p+1+i=0$.

By construction, a point (w,z,t) in a stratum of maximal dimension is such that $z \cap t$ is reduced. The result follows. □

We now consider the product $S^{[n]} \times S^{[n+i]} \times S^{[n+i+1]}$ and the three projections $\pi_{n,n+i}$, $\pi_{n,n+i+1}$, $\pi_{n+i,n+i+1}$ defined as above. We denote by $\mathcal{I}_{i1}$ the intersection $\pi_{n,n+i}^{-1}(Q_i^n) \cap \pi_{n+i,n+i+1}^{-1}(Q_1^{n+i})$.

Let $E' = \{(w,z,t) \in \mathcal{I}_{1i}, z \cap t \text{ reduced}, supp(w \neq z) \neq supp(z \neq t)\}$.

Proposition 4.12. — *The intersection $\pi_{n,n+i}^{-1}(Q_i^n) \cap \pi_{n+i,n+i+1}^{-1}(Q_1^{n+i})$ is proper. More precisely $\mathcal{I}_{i1} = \overline{E'}$ and $\mathcal{I}_{1i}$ is reduced irreducible of dimension $2n+i+3$.*

Proof. — The proof is similar to the proof of Proposition 4.11. We introduce a stratification of $\mathcal{I}_{i1}$ in the form $\mathcal{I}_{i1} = \coprod_{p,q} E(q,p) \amalg \coprod_q E(q)$. For $p \geqslant 0$, $q \geqslant 0$, $i \neq 0$, $q+i \geqslant 0$, let $E(p,q)$ be the set

$$\{(w,z,t) \in \mathcal{I}_{i1}, \operatorname{supp}(w \neq z) \neq \operatorname{supp}(z \neq t), \qquad l(w_{supp(w \neq z)}) = q, l(w_{supp(z \neq t)}) = p\}.$$

The only stratum of expected dimension $2n+i+3$ is $E(0,0)$ when $i>0$ and $E(-i,0)$ when $i<0$. The study of the strata $E(p,q)$ is rigorously similar to the mentioned Proposition 4.11 and we skip it. For $q\geqslant 0$, $i\neq 0$, $q+i\geqslant 0$, let

$$E(q):=\{(w,z,t)\in\mathcal{I}_{i1},\operatorname{supp}(w\neq z)=\mathit{supp}(z\neq t)=x,l(w_x)=q\}.$$

Let (w,z,t) in $E(q)$. Let $x=\mathit{supp}(w\neq z)$. Let $w_1\subset w$ the largest subscheme whose support does not contain x. Since $w=w_1\cup w_x$, $z=w_1\cup z_x$, $t=w_1\cup t_x$, the triple (w,z,t) is characterized by the data $w_1,(z_x,t_x),w_x$. Since $l(w_1)=n-q$, w_1 moves in dimension $2n-2q$. The pair (z_x,t_x) with $z_x\subset t_x$, , $l(z_x)=l(t_x)-1=q+i$ moves in dimension $q+i+2$ by Proposition 2.10. Given z_x, the scheme w_x with $l(w_x)=q$, $w_x\subset z_x$ (case $i>0$), $w_x\supset z_x$ (case $i<0$) moves in dimension q if $q=0$, at most $q-1$ if $q>0$.

Summing up, in any case, the dimension of $E(q)$ is at most $2n+i+2$. There is no stratum $E(q)$ of the expected dimension $2n+i+3$. □

In the proof of the next proposition we will use the following easy lemma:

Lemma 4.13. — *Let $\pi:S\to pt$ be the projection of S to a point. Then $\pi_*1=1/UV$.*

Proof. — This is a direct application of Theorem 1.5. □

Proposition 4.14. — *We have $[q_{-1},q_1]=\frac{1}{UV}Id$. Moreover, for $i\neq -1$, we have $[q_i,q_1]=0$.*

Proof. — The composition q_iq_1 (resp. q_1q_i) corresponds to

$$(\pi_{n,n+i+1})_*(\pi^*_{n,n+1}[Q^n_1]\cup\pi^*_{n+1,n+i+1}[Q^{n+1}_i])$$
$$\text{(resp. }(\pi_{n,n+i+1})_*(\pi^*_{n,n+i}[Q^n_i]\cup\pi^*_{n+i,n+i+1}[Q^{n+1}_1])).$$

By Propositions 4.11 and 4.12, the intersection

$$\pi^{-1}_{n,n+1}(Q^n_1)\cap\pi^{-1}_{n+1,n+i+1}(Q^{n+1}_i)\text{ (resp. }\pi^{-1}_{n,n+i}(Q^n_i)\cap\pi^{-1}_{n+i,n+i+1}(Q^{n+i}_i))$$

is proper and therefore the cup product

$$\pi^*_{n,n+1}[Q^n_1]\cup\pi^*_{n+1,n+i+1}[Q^{n+1}_i]\text{ (resp. }\pi^*_{n,n+i}[Q^n_1]\cup\pi^*_{n+i,n+i+1}[Q^{n+1}_i])$$

is equal to the class of $\mathcal{I}_{1i}$ resp. $\mathcal{I}_{i1}$.

Moreover, when $i>0$, these two propositions show that $\mathcal{I}_{1i}$ and $\mathcal{I}_{i1}$ are birational to $S^{[n]}\times S\times S^{[i]}_{punc}$, where the indice *punc* refers to the punctual

Hilbert scheme. So there is a commutative T-equivariant diagram:

$$\begin{tikzcd} \mathcal{I}_{1i} \arrow[rr, dashed] \arrow[dr, "\pi_{n,n+i+1}"'] & & \mathcal{I}_{i1} \arrow[dl, "\pi_{n,n+i+1}"] \\ & S^{[n]} \times S^{[n+i+1]} & \end{tikzcd}$$

From this it follows that, when $i > 0$, the correspondence

$$(\pi_{n,n+i+1})_*(\pi^*_{n,n+1}[Q^n_1] \cup \pi^*_{n+1,n+i+1}[Q^{n+1}_i]) \\ - (\pi_{n,n+i+1})_*(\pi^*_{n,n+i}[Q^n_i] \cup \pi^*_{n+i,n+i+1}[Q^{n+1}_1])$$

defining the commutator morphism $[q_1, q_i]$ is zero.

When $i < 0$, there is an extra component in $\mathcal{I}_{1i}$, namely $\overline{E(-i-1)}$ with the notations of Proposition 4.11. Let $c = [\overline{E(-i-1)}]$ denote the class of this component. The commutator $[q_i, q_1]$ is then defined by the correspondence $(\pi_{n,n+i+1})_*(c)$.

If $i < -1$, the morphism $\pi_{n,n+i+1} : \overline{E(-i-1)} \to S^{[n]} \times S^{[n+i+1]}$ is a proper morphism with fibers of positive dimension. Indeed, with the notations of Proposition 4.11, the projection of a general point is $\pi_{n,n+i+1} : (w, z, t) = (w, t)$ with $w = w_x \cup w_1$, $z = z_x \cup w_1$, $t = w_1$. The couple (w_x, z_x) moves in dimension $-i+1$ whereas w_x moves in dimension $-i$, hence the morphism contracts $\overline{E(-i-1)}$. The fibers are proper since the support of z_x is x, the support of w_x. It follows that the correspondence $(\pi_{n,n+i+1})_*(c)$ defining the morphism is equal to zero.

If $i = -1$, the morphism $\pi_{n,n+i+1} : \overline{E(-i-1)} \to S^{[n]} \times S^{[n+i+1]}$ is not proper any more. It is birational to the morphism $\varphi : \Delta \times S \to S^{[n]} \times S^{[n]}$ where $\Delta \subset S^{[n]} \times S^{[n]}$ is the diagonal and $\varphi(w, w, x) = (w, w)$. It follows that the correspondence $(\pi_{n,n+i+1})_*(c)$ defining the morphism is equal to $\frac{1}{UV}[\Delta]$ and the proposition follows. □

Proposition 4.15. — *Let i, j be positive integers. Then $q_i q_j = q_j q_i$.*

Proof. — By Theorem 4.8, we have $(i-1)q_i = \rho q_{i-1} - q_{i-1}\rho$ and $j q_{j+1} = \rho q_j - q_j \rho$. From this it follows that

$$(i-1)[q_i, q_j] = [\rho, [q_{i-1}, q_j]] - j[q_{i-1}, q_{j+1}] \quad .$$

By Proposition 4.14 and induction on i, we may assume that $[q_{i-1}, q_j] = 0$ and $[q_{i-1}, q_{j+1}] = 0$. Thus the proposition is proved. □

4.4.2. Commuting ρ and $\rho^\vee$

We now compute the commutator $[\rho, \rho^\vee]$. This is the technical key point of the computation of the commutation relations involving higher q_i's.

Theorem 4.16. — *We have:*

$$[\rho,\rho^\vee]=\bigoplus_{n\geqslant 0}2n\ Id_{A^*_K(S^{[n]})}.$$

The heart of the proof is to get rid of an excess intersection component. To this aim, we use some standard intersection theory formulas to break up the initial intersection product into several pieces. After this rewriting, some of the intersections that show up are transverse and easy to compute. The other pieces (responsible for the excess intersections) are intersections with Cartier divisors. They can be handled with Chern class formalism.

Proof. — Let us first compute the correspondence $\rho\rho^\vee$ in the equivariant Chow ring of $S^{[n]}\times S^{[n]}$.

On the product $S^{[n]}\times S^{[n-1]}\times S^{[n]}$ we denote by π_i and π_{ij} $(i,j\in\{1,2,3\})$ the natural projections. Let $C=\pi_{12}^{-1}((R^\vee)^n)$ and $D=\pi_{23}^{-1}(R^{n-1})$. Let E_1 resp. E_2 in $S^{[n]}\times S^{[n-1]}\times S^{[n]}$ be the closure of the set of triples (z_n,z_{n-1},z'_n) with z_{n-1} reduced, z_n and z'_n non reduced, $z_{n-1}\subset z_n$, $z_{n-1}\subset z'_n$ and $\mathrm{supp}(z_n\neq z_{n-1})=\mathrm{supp}(z'_n\neq z_{n-1})$ resp. $\mathrm{supp}(z_n\neq z_{n-1})\neq\mathrm{supp}(z'_n\neq z_{n-1})$. Let $F_i:=\pi_{13}(E_i)$: a generic element in F_1 resp. F_2 is a couple (z_n,z'_n) where z_n and z'_n have the same support, both have exactly one double point and the double points have the same resp. different support. The generic elements in E_1,E_2 are depicted in the following picture:

Proposition 4.17. — *The intersection $C\cap D$ is generically transverse and equal to the union $E_1\cup E_2$.*

Proof. — The codimension of C and D in the product $S^{[n]}\times S^{[n-1]}\times S^{[n]}$ is $2n-1$. It follows that the components of $C\cap D$ have dimension at least $2n$.

Moreover $C\cap D\subset L$, where L parametrizes the triples (z_n,z_{n-1},z'_n) with $z_n\supset z_{n-1}, z'_n\supset z_{n-1}, supp(z_n)=supp(z_{n-1}), supp(z'_n)=supp(z_{n-1})$. The locus $L_k\subset L$ parametrizing the triples (z_n,z_{n-1},z'_n) with z_{n-1} supported by k points is such that $\overline{L_{n-1}}=E_1\cup E_2$ has pure dimension $2n$. For $k<n-1$, $\dim L_k<2n$. Thus the generic point of any component of $C\cap D$ is in L_{n-1}. The reverse inclusion $L_{n-1}\subset C\cap D$ is obvious, so that $C \cap D = \overline{L_{n-1}} = E_1 \cup E_2$. The intersection is proper since both E_1 and E_2 have dimension $2n$. The intersection is transverse along L_{n-1}, thus generically transverse. □

Since the restrictions of π_{13} to E_1 and E_2 are birational on their image, it follows from the proposition that

$$\rho\rho^\vee = [F_1] + [F_2]\ . \tag{4.2}$$

Now we compute $\rho^\vee\rho$. We use similar notations for π_i, π_{ij} on $S^{[n]} \times S^{[n+1]} \times S^{[n]}$, and moreover we denote by η_1, η_2 the two projections from $S^{[n]} \times S^{[n]}$ to $S^{[n]}$. First of all we consider the variety

$$Q' = \pi_{12}^{-1}(Q_1^n) \cap \pi_{23}^{-1}(Q_{-1}^{n+1}).$$

We want to prove that Q' admits two irreducible components.

Lemma 4.18. — *Let $L_k \subset S^{[k-1]} \times S^{[k]} \times S^{[k-1]}$ be the locus parametrizing the triples (z_{k-1}, z_k, z'_{k-1}) with $z_{k-1} \subset z_k$, $z'_{k-1} \subset z_k$ and z_k supported at the origin. Then for $k \geqslant 2$, $\dim L_k \leqslant 2k-3$.*

Proof. — The pair (z_{k-1}, z_k) moves in dimension at most $k-1$. When z_{k-1} and z_k are fixed, z'_{k-1} moves in dimension at most $k-2$. □

An element in Q' is a triple (z_n, z_{n+1}, z'_n) with $z_n \subset z_{n+1}$ and $z_{n+1} \supset z'_n$. Let $Q'_2 \subset Q'$ the closed locus defined by the condition $z_n = z'_n$ and $\underline{Q'_1}$ the open locus defined by the condition $z_n \neq z'_n$. Let Q'_1 be the closure of $\underline{Q'_1}$.

Proposition 4.19. — *The varieties Q'_1 and Q'_2 are irreducible of dimension $\dim Q'_1 = \dim Q'_2 = 2n+2$. The irreducible components of Q' are Q'_1 and Q'_2. Moreover, the intersection $Q' = \pi_{12}^{-1}(Q_1^n) \cap \pi_{23}^{-1}(Q_{-1}^{n+1})$ is generically transverse.*

Proof. — The claims concerning Q'_2 are true since Q'_2 is isomorphic to $S^{[n,n+1]}$ by projection on the first two factors.

As for $\underline{Q'_1}$, let us denote by $p = z_{n+1} \setminus z_n$ and $p' = z_{n+1} \setminus z'_n$ the natural residual points defined by a triple $(z_n, z_{n+1}, z'_n) \in \underline{Q'_1}$.

The irreducibility of $S_0^{[k,k+1]}$ implies that the locus $L_0 \subset \underline{Q'_1}$ with $p \neq p'$ is irreducible of dimension $2n+2$.

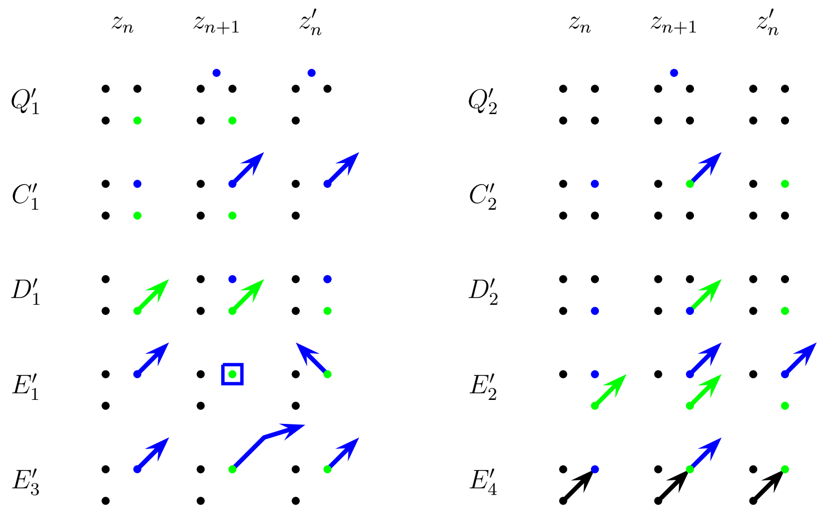
For $m \geqslant 1$, we define the locus L_m in $\underline{Q'_1}$ by the conditions $p = p'$ and $length(z_{n+1})_p = m$. It follows from Lemma 4.18 that it has dimension at most $(2m-1) + 2(n+1-m) = 2n+1$ when $m \geqslant 2$. When $m = 1$, $L_1 = \emptyset$.

Since the codimension of the intersection is bounded by the sum of the codimensions, the components of Q' have dimension at least $2n+2$.

By construction $\underline{Q'_1} = \cup_{m \geqslant 2} L_m \cup L_0$ and $Q' = \underline{Q'_1} \cup Q'_2$. The dimensions computed above show that the generic points of Q' coincide with the generic points of L_0 and Q'_2. Moreover, for $m \geqslant 2$, $L_m \subset \overline{L_0}$ otherwise there would be in Q' a component of dimension less than $2n+2$.

The transversality of the intersection $\pi_{12}^{-1}(Q_1^n) \cap \pi_{23}^{-1}(Q_{-1}^{n+1})$ is easily verified at the generic points of Q'_1 and Q'_2. $\square$

We denote by C'_1 resp. C'_2 the closures of the sets of triples (z_n, z_{n+1}, z'_n) where $\mathrm{supp}(z_{n+1} \neq z_n) \in z_n$ and $\mathrm{supp}(z_{n+1} \neq z_n) \neq \mathrm{supp}(z_{n+1} \neq z'_n)$ resp. $\mathrm{supp}(z_{n+1} \neq z_n) = \mathrm{supp}(z_{n+1} \neq z'_n)$. We denote by D'_1 resp. D'_2 the closures of the sets of triples (z_n, z_{n+1}, z'_n) where $\mathrm{supp}(z_{n+1} \neq z'_n) \in z'_n$ and $\mathrm{supp}(z_{n+1} \neq z_n) \neq \mathrm{supp}(z_{n+1} \neq z'_n)$ resp. $\mathrm{supp}(z_{n+1} \neq z_n) = \mathrm{supp}(z_{n+1} \neq z'_n)$. The varieties $Q'_1, Q'_2, C'_1, C'_2, D'_1, D'_2$ (as well as the following varieties E'_1, E'_2, E'_3, E'_4) are depicted in the following array :



Intersecting with Chern classes of line bundles, in particular with Cartier divisors, commutes with the intersection product in the Chow ring ([11], Example 8.1.6). It follows that if X and Y are smooth in Z smooth, if $\Delta_X \subset X$ and $\Delta_Y \subset Y$ are Cartier divisors with restrictions R_X and R_Y on the (not necessarily smooth) generically transverse intersection $X \cap Y$, then the intersection $[\Delta_X].{}_Z[\Delta_Y]$ computed in the Chow ring $A^*(Z)$ is equal to $i_*(R_X ._{X\cap Y} R_Y)$ where $i : X \cap Y \to Z$ is the natural injection. Since we are working with divisors, one can replace smoothness of X and Y with smooth in codimension one. Moreover, there are equivariant analogs of these statements.

According to Proposition 4.19, one can apply the above with $Z = S^{[n]} \times S^{[n+1]} \times S^{[n]}$, $X = \pi_{12}^{-1}(Q_1^n)$, $Y = \pi_{23}^{-1}(Q_{-1}^{n+1})$, $X \cap Y = Q' = Q'_1 \cup Q'_2$,

$\Delta_X = \pi_{12}^{-1}(R^n)$, $\Delta_Y = \pi_{23}^{-1}((R^\vee)^{n+1})$. For the restrictions of the divisors, we use the notation $R_{Xi} = \Delta_X \cap Q_i$ and $R_{Yi} = \Delta_Y \cap Q_i$. We obtain:

$$\rho^\vee \rho = (\pi_{13})_* i_*((R_{X1} + R_{X2}).(R_{Y1} + R_{Y2}))$$

where the intersection product takes place in Q'.

Proposition 4.20. — $R_{X1} = [C'_1]$, $R_{X2} = 2[C'_2]$, $R_{Y1} = [D'_1]$, $R_{Y2} = 2[D'_2]$.

Proof. — This is clear set theorically. The multiplicities are computed in local coordinates at a generic point. □

Since R^n is a divisor on the smooth variety $S^{[n,n+1]}$, Δ_X is a Cartier divisor, and so are its restrictions R_{Xi}. Thus $[C'_1]$ and $2[C'_2]$ are Cartier divisors, and similarly for D'_1 and D'_2. Thus our task now is to compute the product of the divisors $([C'_1] + 2[C'_2]) \cdot ([D'_1] + 2[D'_2])$ in $A^*_T Q'$. Note however that $C'_2 = D'_2$, so that the corresponding intersection is certainly not proper. In fact we compute $2[C'_2] \cdot ([D'_1] + 2[D'_2])$ by another method. We know that $\rho^\vee = \frac{1}{2}(q_{-1}\delta - \delta q_{-1})$. So in $A^*_T(Q')$ we have $2([D'_1] + 2[D'_2]) = \pi_2^* \partial_{n+1} - \pi_3^* \partial_n$. The pushforward $\pi_{13,*}([C'_2] \cup \pi_3^* \partial_n)$ can be computed thanks to the projection formula: this is $\pi_{13,*}[C'_2] \cup \eta_2^* \partial_n$. But since π_{13} is proper and contractant when restricted to C'_2, $\pi_{13,*}\,[C'_2] = 0$.

To compute $\pi_{13,*}([C'_2] \cup \pi_2^* \partial_{n+1})$ we observe that the general fibers of π_{13} over $\pi_{13}(C'_2)$ are isomorphic n copies of $\mathbb{P}^1$ and $\pi_2^* \partial_{n+1}$ restricts to a line bundle isomorphic to $\mathcal{O}(-2)$ on each $\mathbb{P}^1$ (in fact the class of the diagonal is $-2c_1^T(\mathcal{O}^{[2]})$ if $\mathcal{O}^{[2]}$ denotes the tautological bundle). Thus we get $\pi_{13,*}([C'_2] \cup \pi_3^* \partial_n) = -2nId_{S^{[n]}}$.

To compute the other products we consider geometric intersections. Let E'_1, E'_2, E'_3, E'_4 be the closures of some sets of triples (z_n, z_{n+1}, z'_n). To define these triples we use the following conventions: p_i, p'_i will be punctual subschemes of length i and w_j will be reduced subschemes of length j. Moreover, unless otherwise stated, these subschemes will be generic (among punctual subschemes) and their supports disjoint.

Let p_2, p'_2 share the same support, and let p_3 be the 2-fat point having the same support as p_2 and p'_2. A generic triple (z_n, z_{n+1}, z'_n) in E'_1 is given as follows: $z_n = w_{n-2} \amalg p_2$, $z'_n = w_{n-2} \amalg p'_2$ and $z_{n+1} = w_{n-2} \amalg p_3$.

Let $p_1 \subset p_2, p'_1 \subset p'_2$, w_{n-3} be generic. A generic triple (z_n, z_{n+1}, z'_n) in E'_2 is given as follows: $z_n = w_{n-3} \amalg p_1 \amalg p'_2, z_{n+1} = w_{n-3} \amalg p_2 \amalg p'_2, z'_n = w_{n-3} \amalg p_2 \amalg p'_1$.

Let $p_2 \subset p_3$, w_{n-2} be generic. A generic triple (z_n, z_{n+1}, z'_n) in E'_3 is given as follows: $z_n = z'_n = w_{n-2} \amalg p_2$ and $z_{n+1} = w_{n-2} \amalg p_3$.

Let $p_1 \subset p_2$, let p'_2 and w_{n-3} be generic. A generic triple (z_n, z_{n+1}, z'_n) in E'_4 is given as follows: $z_n = z'_n = w_{n-3} \amalg p_1 \amalg p'_2$, $z_{n+1} = w_{n-3} \amalg p_2 \amalg p'_2$.

Lemma 4.21. — *We have the set theoretic intersection $C'_1 \cap (D'_1 \cup D'_2) = E'_1 \cup E'_2 \cup E'_3 \cup E'_4$.*

Proof. — Let $\mathcal{I}$ denote an irreducible component in the intersection of the lemma. We know that $\mathcal{I}$ has dimension at least $2n$. Let $\xi = (z_n, z_{n+1}, z'_n)$ be a generic point in one of these components. Let $x = \mathrm{supp}(z_{n+1} \neq z_n)$ and $y = \mathrm{supp}(z_{n+1} \neq z'_n)$. Let p be the length of z_n at x and q the length of z'_n at y. Since $\mathcal{I} \subset C'_1$, $p \geqslant 1$. Since $\mathcal{I} \subset D'_1 \cup D'_2$, $q \geqslant 1$. Finally, since $\mathcal{I} \subset C'_1$, z'_n is non reduced.

Assume first that $x \neq y$. Then the triple ξ is defined by the inclusions $(z_n)_{|x} \subset (z_{n+1})_{|x}$ and $(z'_n)_{|y} \subset (z_{n+1})_{|y}$ and the intersection $z_n \cap (S \setminus \{x, y\})$ which has length $n - p - q - 1$. So the dimension of the set of such triples is $(p+2) + (q+2) + 2(n-p-q-1) = 2n + 2 - p - q$, so that $p + q = 2$. Therefore $p = 1 = q$, and so $\mathcal{I} = E'_2$.

From now on, we assume that $x = y$, so $q = p$. If $p = 1$, since z'_n is non reduced, we have $\mathcal{I} = E'_4$.

Let us see that $p \leqslant 2$. We denote by f the dimension of the set of schemes of length p included in $(z_{n+1})_{|x}$. Since the support of such a subscheme of $(z_{n+1})_{|x}$ is x we have $f \leqslant p - 1$. Let $r : \mathcal{I} \dashrightarrow S_0^{[p,p+1]}$ which maps a triple (z_n, z_{n+1}, z'_n) to the pair $((z_n)_{|x}, (z_{n+1})_{|x})$. We denote by d the dimension of $r(\mathcal{I})$. By Proposition 2.10 $\dim S_0^{[p,p+1]} = p + 2$ so we have $d \leqslant p + 2$. Moreover $\dim \mathcal{I} = 2n - 2p + d + f \geqslant 2n$. Summing up, we have

$$d + f \geqslant 2p\ ,\ \ d \leqslant p + 2\ ,\ \ f \leqslant p - 1\ . \tag{4.3}$$

If $f = p - 1$, then any scheme of length p supported at x is included in $(z_{n+1})_{|x}$, and this implies that $p = 2$ and $(z_{n+1})_{|x}$ is a 2-fat point. In this case $\mathcal{I} = E'_1$.

Let us assume that $f \leqslant p - 2$. Equation (4.3) implies that $f = p - 2$ and $d = p + 2$, so $r(\mathcal{I}) = S_0^{[n,n+1]}$. If we assume that $p > 2$, we get $f > 0$ and therefore $(z_{n+1})_{|x}$ cannot be curvilinear, contradicting $r(\mathcal{I}) = S_0^{[n,n+1]}$. Thus we have $p = 2$ and $\mathcal{I} = E'_3$. □

Lemma 4.22. — *There exist integers a, b, c, d such that $[C'_1] \cup [D'_1] = [E'_1] + [E'_2] + a[E'_3] + b[E'_4]$ and $[C'_1] \cup [D'_2] = c[E'_3] + d[E'_4]$.*

Proof. — Note that a generic point in E'_1 and E'_2 is a smooth point in Q' (this will for example be a consequence of our following parametrization of Q' near such a point). Let us first compute the intersection number of C_1' and D_1' along E_1'. A generic point in C_1' resp. D_1', E_1' can be obtained by disjoint union of $n-2$ distinct points and a generic point in the same variety in the case $n=2$, thus it is enough to consider the case where $n=2$. We consider the particular point $\xi = (z_2, z_3, z_2')$ where z_2 resp. z_3, z_2' is the subscheme of the plane defined by the equations (X, Y^2) resp. $(X^2, XY, Y^2), (X^2, Y)$. Note that the projection $S^{[2]} \times S^{[3]} \times S^{[2]} \to S^{[2]} \times S^{[2]}$ restricts to an isomorphism on its image in a neighborhood of ξ in Q', since for $\varepsilon = (y_2, y_3, y_2')$ in such a neighborhood, y_3 is the scheme-theoretic union of y_2 and y_2'. Thus Q' is locally isomorphic to the set of pairs (y_2, y_2') of subschemes of length 2 which meet. Note that for both y_2 and y_2' there is a unique line containing it. Moreover since all our intersection computations are invariant under translations, we may assume that the intersection point of these two lines is the origin.

We parameterize pairs of subschemes (y_2, y_2') near (z_2, z_2') such that these two lines meet at the origin by stating that y_2 resp. y_2' corresponds to the ideal $(X+aY, Y^2+bY+c)$ resp. $(X^2+dX+e, Y+fX)$. Then Q' is defined by the fact that the origin belongs to y_2 and y_2', namely by the equations $c=e=0$ (thus Q' is locally an affine space).

Inside this variety, C_1' resp. D_1' is defined by the fact that y_2' resp. y_2 is non reduced. Thus it is defined by the equation $d=0$ resp. $b=0$. We thus see that the intersection of C_1' and D_1' is transverse along a generic point in E_1'.

Around a generic point in E_2' things are easier because the projection $Q' \to S^{[n+1]}, \xi \mapsto z_{n+1}$ is locally an isomorphism. Thus Q' is locally isomorphic to the product $S^{[n-3]} \times S^{[2]} \times S^{[2]}$, and (z_{n-3}, z_2, z_2') in this product belongs to C_1' resp. D_1' if and only if z_2 resp. z_2' is punctual. So the intersection $C_1' \cap D_1'$ is transverse at such a point.

Now the lemma follows from Lemma 4.21. □

We have $\pi_{13,*}[E_3'] = \pi_{13,*}[E_4'] = 0$ since the restriction of π_{13} to E_3' and E_4' is proper and contractant. We have $\pi_{13,*}[E_1'] = [F_1]$ and $\pi_{13,*}[E_2'] = [F_2]$. Therefore this gives $\rho^\vee\rho = [F_1] + [F_2] - 2n\,Id$. Since by (4.2), $\rho\rho^\vee = [F_1] + [F_2]$, the proposition is proved. □

4.4.3. The commutator $[q_i, q_j]$

We can now compute the commutator $[q_i, q_j]$ for all i, j.

Lemma 4.23. — *We have* $[q_{-1}, \rho] = 0$.

Proof. — First let us compute the correspondence ρq_{-1}. Consider the product $S^{[n]} \times S^{[n-1]} \times S^{[n]}$ and the natural projections on this product. Let $C := \pi_{12}^{-1}(Q_{-1}^n) \cap \pi_{23}^{-1}(R^{n-1})$. It is the closure of the set of triples (z_n, z_{n-1}, z'_n) with z_n reduced, $z_{n-1} \subset z_n \cap z'_n$, and z'_n having a point of length 2. Let $F \subset S^{[n]} \times S^{[n]}$ be the closure of the set of pairs (z_n, z'_n) with z_n reduced, $(z'_n)_{red} \subset z_n$ and z'_n having a point of length 2 and simple points otherwise. Since the restriction of π_{13} to C is birational, the morphism ρq_{-1} is given by the correspondence F.

Now we compute the correspondence $q_{-1}\rho$. The corresponding intersection has been studied in the proof of Proposition 4.16. With these notations we have $\pi_{23}^*[Q_{-1}^{n+1}] \cup \pi_{12}^*[R^n] = [C'_1] + 2[C'_2]$. Moreover the restriction of π_{13} to C'_1 is birational with image F and the restriction of π_{13} to C'_2 is proper contractant. Thus the morphism $q_{-1}\rho$ is also given by the correspondence F, and the lemma is proved. □

Recall the convention that $q_0 = 0$.

Proposition 4.24. — *Let i be arbitrary. We have $[\rho, q_i] = |i|\, q_{i+1}$.*

Proof. — If $i \geqslant 0$ this is Theorem 4.8. If $i = -1$ this is Lemma 4.23. Let us assume that $i = -j$ with $j \geqslant 2$. The Jacobi identity reads:

$$[[q_{-j+1}, \rho^\vee], \rho] \;+\; [[\rho, q_{-j+1}], \rho^\vee] \;+\; [[\rho^\vee, \rho], q_{-j+1}] = 0.$$

Theorem 4.8 yields $[q_{-j+1}, \rho^\vee] = (j-1)q_{-j}$. We may assume by induction that $[\rho, q_{-j+1}] = (j-1)q_{-j+2}$. Finally by Proposition 4.16 we have $[[\rho^\vee, \rho], q_{-j+1}] = 2(j-1)q_{-j+1}$. Thus we get:

$$(j-1)[q_{-j}, \rho] + (j-1)(j-2)q_{-j+1} + 2(j-1)q_{-j+1} = 0\ ,$$

hence the proposition is proved. □

Theorem 4.25. — *Let i and j be any integers. We have*

$$[q_i, q_{-j}] \;=\; \begin{cases} 0 & \text{if } i \neq j \\ \frac{i(-1)^{i+1}}{UV} Id & \text{if } i = j \end{cases}$$

Proof. — Since q_{-i} is the adjoint of q_i and $[q_i, q_{-j}] = 0$ if $i \geqslant 0$ and $j \leqslant 0$ by Proposition 4.15, we may assume that $i, j \geqslant 1$. Moreover the proposition will be true if $i = 1$ or $j = 1$ by Proposition 4.14. Thus we assume $i, j \geqslant 2$. Once again we apply Jacobi identity:

$$(4.4) \qquad [[\rho, q_{i-1}], q_{-j}] \;+\; [[q_{-j}, \rho], q_{i-1}] \;+\; [[q_{i-1}, q_{-j}], \rho] \;=\; 0\ .$$

By induction we may assume that the commutator $[q_{i-1}, q_{-j}]$ is given by the proposition. Therefore it is either 0 or a scalar; in both cases it will commute with ρ, so the last term vanishes. By Proposition 4.24, $[q_{-j}, \rho] = -jq_{-j+1}$ and $[\rho, q_{i-1}] = (i-1)q_i$.

Therefore equation (4.4) reads $(i-1)[q_i, q_{-j}] = j[q_{-j+1}, q_{i-1}]$. If $i \neq j$, the second term vanishes by induction and so $[q_i, q_{-j}] = 0$. If $i = j$ we get $[q_i, q_{-i}] = i\frac{(-1)^{i+1}}{UV} Id$ as we wanted to prove. $\square$

5. Class of the small diagonal

Let Δ_i be the locus in $S^{[n]}$ where at least i points share the same support. In particular Δ_2 is the big diagonal, and Δ_n is the small diagonal. In Corollary 4.4, we proved the equivariant formula for the big diagonal $[\Delta_2] = -2c_1(\mathcal{O}^{[n]})$, which is analogous to Lehn's formula valid in the classical setting. In this section, we prove an equivariant formula for the small diagonal.

Theorem 5.1. — *The T-equivariant class of Δ_n is:*

$$[\Delta_n] = (-1)^{n-1}\, n\, c_{n-1}(\mathcal{O}^{[n]}).$$

The projection from the equivariant Chow ring to the classical Chow ring gives obviously the analogous formula in the classical setting.

Remark 5.2. — Given u an equivariant line bundle over S, let $c(u) \in \bigoplus A_T^*(S^{[n]})$ denote $([u^{[n]}])_n$, where $[\ \cdot\ ]$ denotes total equivariant Chern polynomial and $u^{[n]}$ is the bundle over $S^{[n]}$ tautologically defined by u. More generally we have the following formula:

$$c(u) = \exp\left(\sum_{m\geqslant 1} \frac{(-1)^{m-1}}{m} q_m([u^{[1]}])\right)\cdot \phi\ .$$

In this formula, ϕ denotes the fundamental class in $S^{[0]}$ and the operators $q_m([u^{[1]}]) = [u^{[1]}]\cdot q_m$ have been defined in Remark 3.2.

Proof. — This formula is Lehn's Theorem 4.6 [15] and we explain why his proof is valid in our equivariant context. Lehn introduces the operator $\mathfrak{C}(u) := c(u)\cdot q_1 \cdot c(u)^{-1}$ and shows [15, Theorem 4.2] that

$$\mathfrak{C}(u) = q_1(c(u)) + \rho\ . \tag{5.1}$$

The proof of this theorem relies on the exact sequence [15, (11)] which is equivariant and his Lemma 3.9, for which we proved an equivariant version (Corollary 4.6). Thus the relation (5.1) holds in the equivariant context. Lehn's proof of [15, Corollary 4.3] is purely algebraic and therefore we also have $c(u) = \exp(\mathfrak{C}(u)) \cdot \phi$. Finally, the proof of [15, Theorem 4.6] uses this relation together with the commutation relations of q_i and ρ, which we also proved in Theorem 4.8. □

We now give another proof of Theorem 5.1, as a straightforward consequence of an explicit expression of q_n (Theorem 5.3) which we believe is interesting in itself. The class $[\Delta_n]$ is equal to $q_n \cdot \phi$.

Recall Notation 4.5. If $\lambda \subset \mathbb{N}^2$ is a set of cardinal n and $M : \{1, \dots, n\} \to \lambda$ is a bijection, let $M^- : \{1, \dots, n-1\} \to \lambda \setminus M(n)$ be the restriction of M and $M^+ : \{1, \dots, n-1\} \to \lambda \setminus M(1)$ the bijection defined by $M^+(i) = M(i+1)$. Let $w : \lambda \to \mathbb{Q}[U, V]$ be the map sending (a, b) to the linear form $aU + bV$ corresponding to the weight of the monomial $X^a Y^b$ for the T-action. Let

$$P_M = \frac{(-1)^{n-1}}{(n-1)!} \sum_{i=1}^{i=n} (-1)^{i-1} \binom{n-1}{i-1} \\ \times w(M(1)) \cdots w(M(i-1)) w(M(i+1)) \cdots w(M(n))$$

if $n > 1$ and $P_M = 1$ if $n = 1$.

Theorem 5.3. — *We have the relation*

$$\Delta_{q_n, \lambda, \mu} = \sum_M P_M \prod_{i=0}^{n-1} \Delta_{q_1, \lambda_i, \lambda_{i+1}} \quad ,$$

where M runs through the standard skew Young diagrams of shape $\mu \setminus \lambda$, and λ_i is the partition defined by $\lambda_i = \lambda \cup \{M(1), \dots, M(i)\}$.

Note that $\sum_M \prod_{i=0}^{n-1} \Delta_{q_1, \lambda_i, \lambda_{i+1}} = \Delta_{q_1^n, \lambda, \mu}$. If λ is empty and M is a tableau of shape μ, all the terms but the first in the sum defining P_M are zero, and $P_M = \frac{(-1)^{n-1}}{(n-1)!} c_{n-1}(\mathcal{O}^{[n]})_{|\mathrm{fix}(\mu)}$. Since $S^{[n,n+1]}$ is irreducible and since the projection $S^{[n,n+1]} \to S^{[n+1]}$ is proper, with a finite generic fiber of cardinal $n+1$, it follows that the fundamental class $1_{S^{[n]}} \in A_T^*(S^{[n]})$ satisfies $q_1(1_{S^{[n]}}) = (n+1) 1_{S^{[n+1]}}$, and by induction $q_1^n(\phi) := q_1 \circ \cdots \circ q_1(\phi) = n! \in A_T^*(S^{[n]})$. Theorem 5.1 is then a consequence of Theorem 5.3.

If $M : \{1, \dots, n\} \to \lambda$ is a standard skew young diagram of shape λ, define Q_M by $Q_M = 1$ if $n = 1$ and recursively by the formula $Q_M = \frac{1}{n-1}(-w(M(n)) Q_{M^-} + w(M(1)) Q_{M^+})$.

LEMMA 5.4. — *For every standard skew Young diagram* $M:\{1,\dots,n\}\to\lambda$, $P_M = Q_M$.

Proof. — This is obvious if $n=1$ or $n=2$. To simplify the notation, we denote $w(M(k))$ by m_k. For n general, we have

$$\begin{aligned}
(n-1)Q_M &= -m_n Q_{M^-} + m_1 Q_{M^+}\\
&= -m_n P_{M^-} + m_1 P_{M^+}\\
&= -\frac{(-1)^{n-2}}{(n-2)!}(m_n \sum_{i=1}^{i=n-1} (-1)^{i-1}\binom{n-2}{i-1} m_1\dots\hat{m}_i\dots m_{n-1})\\
&\quad +m_1\sum_{i=2}^{i=n}(-1)^i\binom{n-2}{i-2} m_2\dots\hat{m}_i\dots m_n))\\
&= \frac{(-1)^{n-1}}{(n-2)!}(\sum_{i=2}^{i=n-1}(-1)^{i-1}m_1\dots\hat{m}_i\dots m_n\\
&\quad \times\left(\binom{n-2}{i-1}+\binom{n-2}{i-2}\right) + m_2\dots m_n\\
&\quad +(-1)^{n-1}m_1\dots m_{n-1})\\
&= \frac{(-1)^{n-1}}{(n-2)!}\sum_{i=1}^{i=n}(-1)^{i-1}\binom{n-1}{i-1} m_1\dots\hat{m}_i\dots m_n\\
&= (n-1)P_M
\end{aligned}$$

□

LEMMA 5.5. — *Let* λ *and* μ *be two Young diagrams of cardinal* n *and* $n+1$ *with* $\lambda\subset\mu$. *Then* $\Delta_{\rho,\lambda,\mu} = -w(\mu\setminus\lambda)\Delta_{q_1,\lambda,\mu}$.

Proof. — This is a direct consequence of the formula $2\rho = \partial q_1 - q_1\partial$ and the formula for ∂ given in Proposition 4.3. □

We now prove the formula for $\Delta_{q_n,\lambda,\mu}$ from Theorem 5.3. The formula is clearly true for $n=1$. Suppose that the formula for q_{n-1} is true. Since

$(n-1)q_n = \rho q_{n-1} - q_{n-1}\rho$, we get:

$$\begin{aligned}
(n-1)\Delta_{q_n,\lambda,\mu} &= \sum_{p_n \in Corners(\mu)} \Delta_{q_{n-1},\lambda,\mu\backslash p_n}\Delta_{\rho,\mu\backslash p_n,\mu} \\
&\quad - \sum_{p_1 \in OutsideCorners(\lambda)\cap\mu} \Delta_{\rho,\lambda,\lambda\cup p_1}\Delta_{q_{n-1},\lambda\cup\{p_1\},\mu} \\
&\overset{\text{Lemma 5.5}}{=} \sum_{p_n \in Corners(\mu)} -\Delta_{q_{n-1},\lambda,\mu\backslash p_n}\Delta_{q_1,\mu\backslash p_n,\mu} w(p_n) \\
&\quad + \sum_{p_1 \in OutsideCorners(\lambda)\cap\mu} \Delta_{q_1,\lambda,\lambda\cup p_1} w(p_1)\Delta_{q_{n-1},\lambda\cup\{p_1\},\mu} \\
&\overset{\text{induction}}{=} \sum_{M \text{ standard}} (\prod_{j=0}^{n-1}\Delta_{q_1,\lambda_j,\lambda_{j+1}})(-w(M(n))P_{M^-} \\
&\qquad\qquad + w(M(1))P_{M^+}) \\
&\overset{P_M=Q_M}{=} (n-1)\sum_{M \text{ standard}} (\prod_{j=0}^{n-1}\Delta_{q_1,\lambda_j,\lambda_{j+1}})P_M
\end{aligned}$$

6. Base change formulas

The goal of this section is to compute the base change formula from es to nak and its inverse (recall Section 3 for the bases es and nak of A). In particular, we prove that in the classical setting, these two bases are equal up to a constant (Theorem 6.7).

6.1. Equivariant operators $q_{i,X}$

The basis nak(λ) is defined using creation operators. The basis es(λ) is defined via a Bialynicki-Birula stratification. However, one can introduce operators $q_{i,X}$ such that es(λ) is defined using creation operators too. The goal of this section is to introduce the operators $q_{i,X}$ and to compute a base change inductive formula between $q_{i,X}$ and q_i (Theorem 6.3).

The operator $q_{i,X}$ means "adding i points on a vertical line". More formally, $q_{i,X} : A^*_K(S^{[n]}) \to A^*_K(S^{[n+i]})$ is defined by the Fourier transform along the correspondence $Q_{i,X} \subset S^{[n]} \times S^{[n+i]}$, where $Q_{i,X}$ is the closure of the set of pairs $(z_n, z_n \amalg x_i)$ where $z_n \in S^{[n]}, x_i \in S^{[i]}$, x_i is included in the vertical line Δ_{x_0} with equation $X = x_0$ for some $x_0 \in k$, and z_n and x_i have disjoint support. We denote by $\pi_n : Q_{i,X} \to S^{[n]}$ resp. $\pi_{n+i} : Q_{i,X} \to S^{[n+i]}$ the natural projections.

First of all these operators allow the computation of the Ellingsrud-Stromme cells:

Proposition 6.1. — *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition. Then we have*

$$q_{\lambda_1,X} \circ \cdots \circ q_{\lambda_l,X}(\phi) = \prod_i (\lambda_i^\vee - \lambda_{i+1}^\vee)\,! \cdot \mathrm{es}_\lambda \ .$$

Proof. — To prove this result by induction on l, it is enough to show that $q_{i,X}(\mathrm{es}_\lambda) = k\,\mathrm{es}_\mu$, where μ is the partition obtained inserting one part equal to i in λ and k is the number of parts equal to i in μ.

To this end we apply the definition of $q_{i,X}$. For $n = |\lambda|$, we have $q_{i,X}(\mathrm{es}_\lambda) = \pi_{n+i,*}\pi_n^*(ES_\lambda)$. Recall that the Bialynicki-Birula cell decomposition by ES_λ is associated to the injection $k^* \to T, t \mapsto (t^{-1}, t^{-d})$. Let (z_n, z_{n+i}) be a point belonging to $\pi_n^{-1}(ES_\lambda)$, and assume that z_n resp. z_{n+i} belongs to the open cell corresponding to the partition λ' resp. μ'. We claim that $l(\mu') \leqslant l(\lambda) + 1$. In fact, since the whole construction is k^*-invariant, we also have $(x_{\lambda'}, x_{\mu'}) \in \pi_n^{-1}(ES_\lambda)$. On the other hand, for a generic $(z_n, z_{n+i}) \in \pi_n^{-1}(ES_\lambda)$, there exists $x \in k$ such that $(X - x) \cdot I(z_n) \subset I(z_{n+i})$ and thus we get $X \cdot I(x_{\lambda'}) \subset I(x_{\mu'})$. Therefore $l(\mu') \leqslant l(\lambda') + 1$. Since $x_{\lambda'} \in ES_\lambda$, $l(\lambda') \leqslant l(\lambda)$, thus $l(\mu') \leqslant l(\lambda) + 1$.

Now, given such μ', we have $\dim(ES_{\mu'}) \leqslant n + l(\lambda) + i + 1$. In fact, the dimension of ES_λ is equal to $n + l(\lambda)$. Let C be a component of $\pi_n^{-1}(ES_\lambda)$. The dimension of C is at least $n + l(\lambda) + i + 1$. Thus, if the restriction $C \to ES_{\mu'}$ is not dominant, it is contractant and it follows that $\pi_{n+i,*}[C] = 0$. If it is dominant, then arguing on the generic points the only possibility is that $\mu' = \mu$ and that C is the component which is the closure of the set of points (z_n, z_{n+i}) with z_n generic in ES_λ and z_{n+i} obtained adding i points on a vertical line to z_n. Let C be this component.

The morphism $\pi_n : C \to ES_\lambda$ is submersive at a generic point in C, so C is a reduced component of $\pi_n^{-1}(ES_\lambda)$, thus $\pi_{n+i,*}\pi_n^*[ES_\lambda] = \pi_{n+i,*}[C]$.

Moreover, given a generic element $z_{n+i} \in ES_\mu$, there are k vertical lines containing exactly i points. Thus there are k couples (z_n, z_{n+i}) in the fiber $q^{-1}(z_{n+i})$: z_n is obtained from z_{n+i} removing one of these lines. Thus the restriction of π_{n+i} to C has degree k with image ES_μ, which proves the claim. □

Let $\Delta := \Delta_0$ denote the vertical line with equation $X = 0$. Let $S_\Delta^{[n]}$ denote the subvariety of $S^{[n]}$ parameterizing subschemes with support included in Δ. If λ is a partition of weight n and length l, let $S_{\Delta,\lambda}^{[n]}$ denote the closure in $S^{[n]}_\Delta$ of the variety of schemes $z = z_1 \amalg \cdots \amalg z_l$, where z_i has length λ_i and is supported on one point in Δ. The varieties $S^{[n]}_{\Delta,\lambda}$ are special cases of the Lagrangians considered by Grojnowski and Nakajima [16].

Proposition 6.2. — *The varieties $S^{[n]}_{\Delta,\lambda}$ are the irreducible components of $S^{[n]}_\Delta$, which is therefore equidimensional of dimension n.*

Proof. — Let λ be a partition of weight n and length l, and let i such that $1 \leqslant i \leqslant n$. The variety parameterizing schemes of length λ_i supported on one fixed point is irreducible by [2, 8]. Thus so is the variety parameterizing schemes of length λ_i supported on one point in Δ. Thus each $S^{[n]}_{\Delta,\lambda}$ is irreducible of dimension n. Since we have $S^{[n]}_\Delta = \bigcup_\lambda S^{[n]}_{\Delta,\lambda}$, the proposition is proved. □

Theorem 6.3. — *We have the following formula:*

$$i\, q_{i,X} = (-1)^{i+1}\, q_i + U \cdot \sum_{j=1}^{i-1} (-1)^j\, q_j \circ q_{i-j,X}$$

With the help of this theorem one can compute all the operators $q_{i,X}$ by induction on i.

To prove the theorem we define auxiliary operators. For $i \geqslant 0$ and $j \geqslant 2$ let $q_{i,j,X}$ be the operator corresponding to "adding i points on a same vertical line plus one punctual scheme of length j whose support is on this line". Formally, $q_{i,j,X}$ is defined by an incidence $Q_{i,j,X}$ in $S^{[n]} \times S^{[n+i+j]}$ where a generic point in $Q_{i,j,X}$ is of the form $(z_n, z_n \amalg \{x_1, \ldots, x_i\} \amalg t_j)$ where the x_k's are distinct points on a vertical line Δ not meeting z_n and t_j is a length j punctual scheme supported on $\Delta \setminus \{x_1, \ldots, x_i\}$. Let us moreover use the convention that $q_{i,1,X} = (i+1) q_{i+1,X}$, $q_{-1,j,X} = 0$, and $q_{0,X} = -1/U$.

The theorem is a consequence of the following proposition because this proposition implies that the right hand side is equal to $q_{i-1,1,X} = i\, q_{i,X}$.

Proposition 6.4. — *For $i \geqslant 0$ and $j \geqslant 1$, we have the following relation in $A^*_T(S^{[n]} \times S^{[n+i]})$:*

$$-U\, q_j \circ q_{i,X} = q_{i,j,X} + q_{i-1,j+1,X}$$

Proof. — If $i = 0$ then the proposition is just formal thanks to the conventions we made above. Let us assume that $i > 0$. Let Δ be the line in S with equation $X = 0$. Let $Q_j(-U) \subset Q_j$ denote the divisor containing the set of pairs (z_n, z_{n+j}) in Q_j with $s(z_n, z_{n+j}) \in \Delta$, where s is the morphism of Remark 3.2. We have in $A^*_T(S^{[n]} \times S^{[n+j]})$ the relation $[Q_j(-U)] =$

$s^*[\Delta] = -U \cdot [Q_j]$. Similarly there is a morphism $x : Q_{i,X} \to \mathbb{A}^1$, mapping a pair (z_n, z_{n+i}) to the common X-coordinate of the points in $z_{n+i} \setminus z_n$. Thus we use similar notations and define $Q_{i,X}(-U) := x^{-1}(0) \subset Q_{i,X}$. In the Chow ring, $[Q_{i,X}(-U)] = -U[Q_{i,X}]$.

Consider the product $S^{[n]} \times S^{[n+i]} \times S^{[n+i+j]}$ and the projections π_a, π_{ab}. Let I be the intersection $\pi_{12}^{-1}(Q_{i,X}(-U)) \cap \pi_{23}^{-1}(Q_j(-U))$. Every proper component of I has dimension $2n+i+j$.

Let C be a component of I which contributes to the composition $q_j \circ q_{i,X}$, ie. a component with $\pi_{13,*}[C] \neq 0$. Our first task is to prove that for a generic element $(z_n, z_{n+i}, z_{n+i+j})$ of C, the support of z_n is disjoint from Δ.

If $z \subset S$ is a subscheme of dimension 0, we denote by z_Δ the union of the components of z supported on Δ. Let I_k be the locally closed set of pairs (z_n, z_{n+i+j}) in $S^{[n]} \times S^{[n+i+j]}$ such that the length of $(z_n)_\Delta$ is k, and $z_{n+i+j} \supset z_n \cap (S \setminus \Delta)$, the support of $\mathcal{O}_{z_{n+i+j}}/\mathcal{O}_{z_n} \cap (S\setminus\Delta)$ is included in Δ. Then I_k is birational to $S_\Delta^{[k]} \times S^{[n-k]} \times S_\Delta^{[k+i+j]}$, and thus has dimension $2n+i+j$.

We denote by k the integer such that for a generic triple $(z_n, z_{n+i}, z_{n+i+j})$ in C, the length of $(z_n)_\Delta$ is k. Since, $\pi_{13}(C) \subset \overline{I}_k$, $\dim \pi_{13}(C) \leqslant 2n+i+j$. Moreover, if $k > 0$, since z_{n+i+j} has to contain z_n, $\pi_{13}(C)$ cannot contain $\overline{I}_k$, and thus $\dim \pi_{13}(C) < 2n+i+j$. Since π_{13} is proper we deduce that $\pi_{13,*}[C] = 0$ in this case.

Let us now assume that $\dim \pi_{13}(C) = 2n+i+j$. We thus have $k = 0$ and $\dim C = 2n+i+j$. For a generic element $(z_n, z_{n+i}, z_{n+i+j})$ in C, $(z_{n+i+j})_\Delta$ has length $i+j$, thus we have a well-defined rational map $C \dashrightarrow S_\Delta^{[i+j]}$, with $2n$-dimensional fibers. Let D be the closure of the image of this rational map. Since $\dim D = i+j$, D is a component of $S_\Delta^{[i+j]}$; let us denote λ the partition such that $D = S_{\Delta,\lambda}^{[i+j]}$. By definition of I, λ must be dominated by the partition $(j, 1^i)$. It is clear that D can contain $S_{\Delta,\mu}^{[i+j]}$ only if $\mu = (j, 1^i)$ or $\mu = (j+1, 1^{i-1})$. Therefore I has exactly two components which are not contracted by π_{13}.

To describe these components let us consider some subschemes $z_n, x_{i-1}, x_i, p_j, p_{j+1}$ satisfying the following conditions. The lengths of these subschemes are given by their indices. The support of z_n does not meet Δ, whereas the other subschemes have support included in Δ. The subschemes p_j, p_{j+1} are punctual whereas x_{i-1} and x_i are reduced. Finally $p_j \subset p_{j+1}$, $x_{i-1} \subset x_i$, and the support of p_{j+1} is not included in x_i. With these conditions let I_1 resp. I_2 be the closure of the set of triples $(z_n, z_{n+i}, z_{n+i+j})$ where $z_{n+i} = z_n \amalg x_i$ and $z_{n+i+j} = z_n \amalg x_i \amalg p_j$ resp. $z_{n+i+j} = z_n \amalg x_{i-1} \amalg$

p_{j+1}. The restriction of π_{13} to I_2 is birational with image $Q_{i-1,j+1,X}(-U)$. We have $\pi_{13}(I_1) = Q_{i,j,X}(-U)$. If $j > 1$ then the restriction of π_{13} to I_1 is birational whereas if $j = 1$ it has degree $i+1$. In view of our convention for $q_{i,1,X}$, this proves the proposition. □

6.2. Base change formulas

Definition 6.5. — *If $\lambda \in \mathcal{P}_n$ is a partition, we define the operators $q_\lambda = \Pi_{i\in\lambda} q_i$, $q_{\lambda,X} = \Pi_{i\in\lambda} q_{i,X}$, and the constant $z_\lambda = \Pi\lambda_i \Pi(\lambda_i^\vee - \lambda_{i+1}^\vee)!$. Let $j \in \lambda$. With the notation with multiplicity $\lambda = (1^{\alpha_1}, \dots, r^{\alpha_r})$, we denote by $\lambda \setminus j$ the partition $(1^{\alpha_1}, \dots, j^{\alpha_j - 1}, \dots, r^{\alpha_r})$, with multiplicity one less for j. We let $t_\lambda = \sum_{j\in\lambda} \frac{j}{(\alpha_j-1)!}\frac{(l(\lambda)-1)!}{\Pi_{i\neq j}(\alpha_i!)}$ and $u_\lambda = \prod_i (\lambda_i^\vee - \lambda_{i+1}^\vee)!$.*

By definition of q_λ and $q_{\lambda,X}$, the base change formulas from q_λ to $q_{\lambda,X}$ are determined by the decomposition of q_n in terms of the operators $q_{\lambda,X}$ and similarly for the inverse base change. In particular, the following theorem gives a full base change at the level of operators. Since $\mathrm{es}(\lambda) = \frac{1}{u_\lambda} q_{\lambda,X}(\phi)$ and since $\mathrm{nak}(\lambda) = q_\lambda(\phi)$ the theorem applied to the vacuum also yields the corresponding base changes between $\mathrm{es}(\lambda)$ and $\mathrm{nak}(\lambda)$.

Theorem 6.6.

$$q_{i,X} = (-1)^{i+1} \sum_{|\lambda|=i} z_\lambda^{-1}\, U^{l(\lambda)-1}\, q_\lambda$$
$$q_i = (-1)^{i+1} \sum_{|\lambda|=i} t_\lambda\, U^{l(\lambda)-1}\, q_{\lambda,X}$$

Proof. — By induction, the case $i = 1$ being obvious.

$$\begin{aligned} i\, q_{i,X} &\overset{\text{Theorem 6.3}}{=} (-1)^{i+1} q_i + U \sum_{j=1}^{i-1} (-1)^j\, q_j \circ q_{i-j,X} \\ &\overset{\text{induction hypothesis}}{=} (-1)^{i+1} q_i + \sum_{j=1}^{i-1} (-1)^j\, U\, (-1)^{i-j+1} \\ &\qquad \times \sum_{|\lambda|=i-j} z_\lambda^{-1}\, U^{l(\lambda)-1}\, q_j \circ q_\lambda \\ &= (-1)^{i+1} q_i + (-1)^{i+1} \sum_{|\mu|=i,\, l(\mu)>1} c_\mu\, q_\mu \end{aligned}$$

with

$$c_\mu = \sum_{j\in\mu} z_{\mu\setminus j}^{-1}\, U^{l(\mu)-1}$$

Since $\sum_{j\in\mu}\frac{z_\mu}{z_{\mu\setminus j}}=|\mu|$, we obtain $c_\mu=|\mu|\,(-1)^{|\mu|+1}\,z_\mu^{-1}\,U^{l(\mu)-1}$, as required for the induction.

The proof of the second formula is similar : the difficulty is to guess the formula for q_i, then the induction is straightforward. Indeed, we start with the formula of theorem 6.3 $(-1)^{i+1}q_i=-iq_{i,X}+U\sum_{j=1}^{i-1}(-1)^j\,q_j\circ q_{i-j,X}$, and we replace q_j on the right hand side by the induction formula. With the value of t_λ in the definition above and the the formula for q_i, the induction follows.

□

We now project the previous theorem from the equivariant Chow ring to the classical Chow ring. All the constructions made so far in the equivariant setting can be realized in the classical setting. We denote by q_n^{cla} and $q_{n,X}^{cla}$ the corresponding operators on the classical Chow ring. Similarly, we denote by $\mathrm{nak}^{cla}(\lambda)$ and $\mathrm{es}^{cla}(\lambda)$ the bases of the classical Chow ring induced by these operators.

Theorem 6.7.

$$\begin{aligned} q_n^{cla} &= (-1)^{n+1}\,n\,q_{n,X}^{cla} \\ \mathrm{nak}^{cla}(\lambda) &= (-1)^{|\lambda|+l(\lambda)}\,(\textstyle\prod_{i\in\lambda} i)\,\mathrm{es}^{cla}(\lambda) \end{aligned}$$

Proof. — In the classical setting, $U=0$ and the first formula for the operators is the projection of the corresponding formula in the equivariant setting. Applying the operators to the vacuum yields the second formula.

□

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Manuscrit reçu le 12 septembre 2012,
révisé le 7 mars 2014,
accepté le 23 octobre 2014.

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