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GREEN FUNCTIONS, SEGRE NUMBERS, AND KING'S FORMULA

by Mats ANDERSSON & Elizabeth WULCAN (*)

Abstract. — Let $\mathcal{J}$ be a coherent ideal sheaf on a complex manifold X with zero set Z, and let G be a plurisubharmonic function such that $G = \log|f| + \mathcal{O}(1)$ locally at Z, where f is a tuple of holomorphic functions that defines $\mathcal{J}$. We give a meaning to the Monge-Ampère products $(dd^cG)^k$ for $k = 0, 1, 2, \ldots$, and prove that the Lelong numbers of the currents $M_k^{\mathcal{J}} := \mathbf{1}_Z(dd^cG)^k$ at x coincide with the so-called Segre numbers of $\mathcal{J}$ at x, introduced independently by Tworzewski, Gaffney-Gassler, and Achilles-Manaresi. More generally, we show that $M_k^{\mathcal{J}}$ satisfy a certain generalization of the classical King formula.

Résumé. — Soit $\mathcal{J}$ un faisceau cohérent d'ideaux sur un variété complexe lisse X, et soit Z la variété de $\mathcal{J}$. Soit G une fonction plurisousharmonique telle que $G = \log|f| + \mathcal{O}(1)$ localement sur Z, où f est un n-uple de fonctions holomorphes qui définit $\mathcal{J}$. Nous donnons un sens au produit de Monge-Ampère $(dd^cG)^k$ pour $k = 0, 1, 2, \ldots$, et nous montrons que les nombres de Lelong des courants $M_k^{\mathcal{J}} := \mathbf{1}_Z(dd^cG)^k$ en x coïncident avec les nombres de Segre de $\mathcal{J}$ en x, introduits indépendemment par Tworzewski, Gaffney-Gassler et Achilles-Manaresi. Plus généralement, nous montrons que les $M_k^{\mathcal{J}}$ satisfont une certaine généralisation de la formule de King.

1. Introduction

Let X be a complex manifold of dimension n and let $\mathcal{J} \to X$ be a coherent ideal sheaf with variety Z. Given a point $x \in X$, Tworzewski, [24], and Gaffney and Gassler, [14], have independently introduced a list of numbers, $e_0(\mathcal{J}, X, x), \ldots, e_n(\mathcal{J}, X, x)$, that we, following [14], call the *Segre numbers* at x. They are a generalization of the classical local intersection number at x in case the ideal $\mathcal{J}_x$ is a complete intersection. The definition

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in both papers is based on a local variant of the Stückrad-Vogel procedure, [23]. In [1, 2] is given an algebraic definition of these numbers generalizing the classical Hilbert-Samuel multiplicity of $\mathcal{J}$ at x.

In this paper we show that if $\mathcal{J}$ is generated by global bounded functions there is a canonical global representation of the Segre numbers of $\mathcal{J}$ as the Lelong numbers (of restrictions to Z) of Monge-Ampère masses of the *Green function* $G = G_{\mathcal{J}}$ *with poles along* $\mathcal{J}$. This function was introduced by Rashkovskii-Sigurdsson in [20, Definition 2.2] as a generalization of the classical Green function G_a with pole at a point $a \in X$. It is defined as the supremum over the class $\mathcal{F}_{\mathcal{J}}$ of all negative psh (plurisubharmonic) functions u on X that locally satisfy $u \leqslant \log|f| + C$, where $f = (f_1, \dots, f_m)$ is a tuple of local generators of $\mathcal{J}$ and C is a constant.

Note that even if X is hyperconvex there might not exist non-trivial functions in $\mathcal{F}_{\mathcal{J}}$. For example, if X is the ball in $\mathbb{C}$, and $\mathcal{J}$ is the radical ideal of functions vanishing at points $a_1, a_2, \ldots \in X$, then there are negative psh functions with poles at a_j if and only if a_j satisfy the the Blaschke condition. However, if $\mathcal{J}$ is globally generated by bounded functions f_j, then $\log|f| + C$ is itself in $\mathcal{F}_{\mathcal{J}}$ for some constant C. Then locally G is of the form

$$G = \log|f| + h, \tag{1.1}$$

where h is locally bounded, see [20, Theorem 2.8]. In particular, the unbounded locus of G equals Z and thus the Monge-Ampère type products

$$(dd^c G)^k, \quad k \leqslant p := \operatorname{codim} Z \tag{1.2}$$

are well-defined, see, e.g., [9, Theorem III.4.5]. Here and throughout $d^c = (i/2\pi)(\bar\partial - \partial)$. By *Demailly's comparison formula for Lelong numbers*, [11, Theorem 5.9],

$$\ell_x (dd^c G)^k = \ell_x (dd^c \log|f|)^k \tag{1.3}$$

for $x \in X$, where ℓ_x denotes the Lelong number at x. Moreover, recall that *King's formula*, [15], asserts that $(dd^c \log|f|)^p$ admits the Siu decomposition, [21],

$$(dd^c \log|f|)^p = \sum \beta_j [Z_j^p] + R, \tag{1.4}$$

cf. [11, Section 6]. Here $[Z_j^p]$ are the currents of integration along the irreducible components Z_j^p of codimension p of Z, β_j are the generic Hilbert-Samuel multiplicities of f along Z_j^p, see, e.g. [13, Chapter 4.3]. In fact, the remainder term R has integer Lelong numbers, see, e.g. [4, Theorem 1.1], and therefore the set where R has positive Lelong numbers is an analytic set of codimension $> p$. From (1.3) and (1.4) one deduces that

$$(dd^cG)^p = \sum \beta_j [Z_j^p] + R, \tag{1.5}$$

where β_j and Z_j^p are as above, and R has the same Lelong numbers as R in (1.4), cf. the proof of Theorem 2.8 in [20]. In particular, if Z is a point a, then $(dd^cG)^n = \sum \beta[a] + R$, where $[a]$ is the point evaluation at a and β is the Hilbert-Samuel multiplicity of $\mathcal{J}$. This generalizes the fact that $(dd^cG_a)^n = [a]$, [10, page 520]. The (Lelong numbers of the) Monge-Ampère products (1.2) are related to the integrability index of G (and thus the log-canoncial threshold of $\mathcal{J}$), see, e.g., [12, 19, 22]; in particular, Demailly-Pham [12] recently gave a sharp estimate of the integrability index of G in terms of the Lelong numbers of (1.2) for all $k \leqslant p$.

Recall that (1.2) can be defined inductively as

$$dd^c(G(dd^cG)^{k-1}). \tag{1.6}$$

In this paper we give meaning to $(dd^cG)^k$ for any k if G is any psh function of the form (1.1): Inductively we show that

$$G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}$$

has locally finite mass and define

$$(dd^cG)^k := dd^c(G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}),$$

see Proposition 4.1. When $k \leqslant p$ it follows from the dimension principle for closed positive currents, cf. Lemma 3.1 below, that $\mathbf{1}_Z(dd^cG)^{k-1} = 0$ and so our definition coincides with the classical one for $k \leqslant p$. Our definition is modeled on the paper [3] by the first author, in which currents $(dd^c \log|f|)^k$ are defined for all k inductively as above. In fact, $(dd^c \log|f|)^k$ can also be defined as a certain limit of smooth forms coming from regularizations of $\log|f|$:

$$\lim_{\epsilon\to 0}(dd^c \log(|f|^2+\epsilon)^{1/2})^k = (dd^c \log|f|)^k \tag{1.7}$$

for any k, see [3, Proposition 4.4]. However, one cannot hope for such a suggestive definition of $(dd^cG)^k$ in general, cf. Example 4.2. Also, our definition of $(dd^cG)^k$ does not coincide with the *non-pluripolar product* of dd^cG, as introduced in [6, 8], since our $(dd^cG)^k$ charges pluripolar sets in general, cf. the text after the proof of Proposition 4.1.

Our main result is the following generalization of (1.5). Let $\pi^+ \colon X^+ \to X$ be the normalization of the blow-up of X along $\mathcal{J}$ and let W_j be the various irreducible components of the exceptional divisor in X^+. Recall that the *(Fulton-MacPherson) distinguished varieties* of $\mathcal{J}$ are the subvarieties $\pi^+(W_j)$ of X, see, e.g., [16, Chapter 10.5]. In particular, the distinguished varieties of codimension p are precisely the irreducible components of Z of codimension p.

Theorem 1.1. — *Let X be an n-dimensional complex manifold, let $\mathcal{J}$ be a coherent ideal sheaf on X generated by global bounded functions, and let G be the Green function with poles along $\mathcal{J}$. Moreover, let Z be the variety of $\mathcal{J}$ and Z_j^k the Fulton-MacPherson distinguished varieties of $\mathcal{J}$ of codimension k. Then*

$$M_k^{\mathcal{J}} := \mathbf{1}_Z(dd^cG)^k = \sum_j \beta_j^k[Z_j^k] + N_k^{\mathcal{J}} =: S_k^{\mathcal{J}} + N_k^{\mathcal{J}}, \tag{1.8}$$

where the β_j^k are positive integers and the $N_k^{\mathcal{J}}$ are positive closed currents. The numbers $n_k(\mathcal{J}, X, x) := \ell_x(N_k^{\mathcal{J}})$ are nonnegative integers that only depend on the integral closure class of $\mathcal{J}$ at x, and the set where $n_k(\mathcal{J}, X, x) \geqslant 1$ has codimension at least $k+1$.

The Lelong numbers at x of $M_k^{\mathcal{J}}$ and $\mathbf{1}_{X\setminus Z}(dd^cG)^k$ are precisely the Segre number $e_k(\mathcal{J}, X, x)$ and the polar multiplicity $m_k(\mathcal{J}, X, x)$, respectively, of $\mathcal{J}_x$.

For the notion of polar multiplicities see Section 2. Notice that $M_k^{\mathcal{J}} = 0$ if $k < \operatorname{codim} Z$ and that $N_p^{\mathcal{J}} = 0$, cf., Lemma 3.1 below. Also, notice that (1.8) is the Siu decomposition, [21], of $M_k^{\mathcal{J}}$.

Remark 1.2. — If $\mathcal{J}$ is generated by a global tuple f, then Theorem 1.1 holds with G replaced by any psh function of the form (1.1).

The analogous statement to Theorem 1.1 when G is replaced by $\log|f|$, where f is a tuple of global generators, was proved by the authors and Samuelsson Kalm and Yger in [4, Theorem 1.1]. The case $k = p$ corresponds to the classical King formula, (1.4). The main idea in the proof of Theorem 1.1 is to prove that for any psh G of the form (1.1),

$$\begin{aligned} \ell_x(\mathbf{1}_Z(dd^cG)^k) &= \ell_x(\mathbf{1}_Z(dd^c\log|f|)^k), \\ \ell_x(\mathbf{1}_{X\setminus Z}(dd^cG)^k) &= \ell_x(\mathbf{1}_{X\setminus Z}(dd^c\log|f|)^k) \end{aligned} \tag{1.9}$$

for $x \in X$, see Lemma 6.1 below. Using this the theorem follows from the corresponding result in [4]. In some sense, (1.9) can be seen as a generalization of Demailly's comparison formula, (1.3), to higher k, but for the very special class of psh functions of the form (1.1).

In [4], X is allowed to be singular. Given that there is a proper definition of G when X is singular so that (1.1) still holds, the results in this paper will extend as well.

Theorem 1.1 gives us a canonical representation of the Segre numbers of $\mathcal{J}$ in the case when $\mathcal{J}$ is generated by global bounded functions. Let X be a, say hyperconvex, domain in $\mathbb{C}^n$, and let $\mathcal{J}$ be a coherent ideal sheaf on X. If we exhaust X by reasonable relatively compact subsets X_ℓ, for each ℓ we then have currents $M_k^{\mathcal{J}_\ell}$, $\mathcal{J}_\ell = \mathcal{J}|_{X_\ell}$, whose Lelong numbers at each point are the Segre numbers. If for some reason these currents converge to currents $M_k^{\mathcal{J}}$, we would have a canonical representation of the Segre numbers of $\mathcal{J}$ on X, cf. Remark 4.3.

This paper is organized as follows. In Section 2 we recall the construction of Vogel cycles and Segre numbers. In Section 4 we show that the currents $(dd^cG)^k$ are well-defined and discuss some properties. The proof of Theorem 1.1 occupies Section 6. In Sections 3 and 5 we give some background on psh functions and positive currents needed for the proofs.

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2. Segre numbers

We will briefly recall the construction of Segre numbers from [24, 14]. Throughout we will assume that X is a complex manifold of dimension n and that $\mathcal{J}$ is a coherent ideal sheaf on X with variety Z. Fix a point $x \in X$. A sequence $h = (h_1, h_2, \ldots, h_n)$ in the local ideal $\mathcal{J}_x$ is called a *Vogel sequence of* $\mathcal{J}$ *at* x if there is a neighborhood $\mathcal{U} \subset X$ of x where the h_j are defined, such that

$$\text{codim}\left[(\mathcal{U} \setminus Z) \cap (|H_1| \cap \cdots \cap |H_k|)\right] = k \text{ or } \infty, \; k = 1, \ldots, n; \tag{2.1}$$

here $|H_\ell|$ are the supports of the divisors H_ℓ defined by h_ℓ. Notice that if $f_1, \ldots, f_m$ generate $\mathcal{J}_x$, any generic sequence of n linear combinations of the f_j is a Vogel sequence at x. Set $X_0 = X$, let X_0^Z denote the irreducible components of X_0 that are contained in Z, and let $X_0^{X\setminus Z}$ be the remaining components[(1)] so that

$$X_0 = X_0^Z + X_0^{X\setminus Z}.$$

By the Vogel condition (2.1), H_1 intersects $X_0^{X\setminus Z}$ properly. Set

$$X_1 = H_1 \cdot X_0^{X\setminus Z}$$

and decompose analogously X_1 into the components X_1^Z contained in Z and the remaining components $X_1^{X\setminus Z}$, so that $X_1 = X_1^Z + X_1^{X\setminus Z}$. Define inductively $X_{k+1} = H_{k+1} \cdot X_k^{X\setminus Z}$, X_{k+1}^Z, and $X_{k+1}^{X\setminus Z}$. Then

$$V^h := X_0^Z + X_1^Z + \cdots + X_n^Z$$

is the *Vogel cycle*[(2)] associated with the Vogel sequence h. Let V_k^h denote the components of V^h of codimension k, i.e., $V_k^h = X_k^Z$. The irreducible components of V^h that appear in any Vogel cycle, associated with a generic Vogel sequence at x, are called *fixed* components in [14]. The remaining ones are called *moving*. It turns out that the fixed Vogel components of $\mathcal{J}$ coincide with the distinguished varieties of $\mathcal{J}$, see, e.g., see [14] or [4].

It is proved in [14] and in [24] that the multiplicities $e_k(\mathcal{J}, X, x) := \text{mult}_x V_k^h$ and $m_k(\mathcal{J}, X, x) := \text{mult}_x X_k^{X\setminus Z}$ are independent of h for a generic h, where however "generic" depends on x, cf., Remark 2.1; these numbers are called the *Segre numbers* and *polar multiplicities*, respectively.

Remark 2.1. — Recall that if W is an analytic cycle in X, then the Lelong number at $x \in X$ of the current of integration $[W]$ along W is precisely the multiplicity $\text{mult}_x W$ of W at x.

Assume that x is a point for which $n_k(\mathcal{J}, X, x) \geqslant 1$ for some k, where we use the notation from Theorem 1.1. Moreover, let V^h be a generic Vogel cycle such that $\text{mult}_x V_k^h = e_k(x)$. Then $V_k^h = S_k^{\mathcal{J}} + W$, where we have identified $S_k^{\mathcal{J}}$ in Theorem 1.1 with the corresponding cycle and W is a positive cycle of codimension k, such that $\text{mult}_x W = n_k(\mathcal{J}, X, x)$. Since $n_k(\mathcal{J}, X, y) \geqslant 1$ only on a set of codimension $\geqslant k+1$, at most points y on V_k^h we have that $e_k(\mathcal{J}, X, y) = \text{mult}_y(S_k^{\mathcal{J}})$ and hence $\text{mult}_y V_k^h > e_k(\mathcal{J}, X, y)$. As soon as there is a moving component at x it is thus impossible to find a Vogel cycle that realizes the Segre numbers in a whole neighborhood of x.

(1) Since we assume X is smooth and connected, X_0^Z is empty unless $\mathcal{J} = 0$, in which case it equals X.

(2) If $\mathcal{J}$ is the pullback to X of the radical sheaf of an analytic set A, this is precisely Tworzewski's algorithm, [24]. The notion Vogel cycle was introduced by Massey [17, 18]. For a generic choice of Vogel sequence the associated Vogel cycle coincides with the *Segre cycle* introduced by Gaffney-Gassler, [14], see Lemma 2.2 in [14].

In [4] Theorem 1.1 with G replaced by $\log|f|$ was proved by showing that $M_k^f := \mathbf{1}_Z(dd^c \log|f|)^k$ can be seen as a certain average (of currents of integration) of Vogel cycles. The fixed Vogel components then appear as the leading part $S_k^{\mathcal{J}}$ in the Siu decomposition of M_k^f, whereas the remainder term N_k^f is a mean value of the moving parts.

3. Preliminaries

Let μ be a positive closed current on X. Recall that if W is any subvariety, then $\mathbf{1}_W\mu$ and $\mathbf{1}_{X\setminus W}\mu$ are positive closed currents as well; this is the Skoda-El Mir theorem, see, e.g., [9, Chapter III.2.A].

LEMMA 3.1. — *Let μ be a positive closed current of bidegree (p,p) that has support on a subvariety of codimension k. If $k > p$ then $\mu = 0$. If $k = p$, then $\mu = \alpha_1[W_1] + \cdots + \alpha_\nu[W_\nu]$ where W_j are the irreducible components of W and $\alpha_j \geqslant 0$.*

We refer to the first part of Lemma 3.1 as the *dimension principle*. A proof can be found in [9, Chapter III.2.C].

If b is psh and locally bounded and T is any positive closed current, then $T\wedge(dd^c b)^k$ is a well-defined positive current for any k, and if b_j is a decreasing sequence of bounded psh functions converging pointwise to b, then

(3.1)
$$T\wedge(dd^c b)^k = \lim_j T\wedge(dd^c b_j)^k, \quad T\wedge b(dd^c b)^k = \lim_j T\wedge b_j(dd^c b_j)^k, \ k \leqslant n.$$

See, e.g., [9, Theorem III.3.7]. The case $T \equiv 1$ was first proved by Bedford and Taylor, [5].

PROPOSITION 3.2. — *Assume that v, b are psh and that b is (locally) bounded.*
(i) For $k \leqslant n-1$,
$$v(dd^c b)^k$$
has locally finite mass; more precisely, for any compact sets L, K, such that $L \subset int(K)$, we have

$$\|v(dd^c b)^k\|_L \leqslant C_{K,L}\|v\|_K(\sup_K |b|)^k. \tag{3.2}$$

(ii) Moreover, if the unbounded locus of v has Hausdorff dimension $< 2n-1$, then

$$dd^c(v(dd^c b)^k) = dd^c v\wedge(dd^c b)^k. \tag{3.3}$$

If v_j is a decreasing sequence of psh functions converging pointwise to v, then

$$v_j(dd^cb)^k \to v(dd^cb)^k, \tag{3.4}$$

and

$$dd^cv_j\wedge(dd^cb)^k \to dd^cv\wedge(dd^cb)^k \tag{3.5}$$

in the current sense.

The first part of Proposition 3.2 follows immediately from Proposition 3.11 in [9, Chapter III]. Moreover, Proposition 4.9 in loc. cit. applied to $u_1 = v$ and $u_j = b$ implies (3.4) and (3.5). If we choose v_j smooth, then

$$dd^c(v_j(dd^cb)^k) = dd^cv_j\wedge(dd^cb)^k.$$

Thus (3.3) follows from (3.4) and (3.5). In fact, the assumption about the Hausdorff dimension is not necessary; an elegant and quite direct argument has been communicated to us by Z. Błocki, [7].

Corollary 3.3. — *If b is psh and (locally) bounded on X and W is an analytic variety of positive codimension, then for each $k \geqslant 0$,*

$$\mathbf{1}_W(dd^cb)^k = 0. \tag{3.6}$$

Proof. — It is enough to consider the case when W is a smooth hypersurface. The general case follows by stratification. Since it is a local statement, we may choose coordinates $z = (z', w)$ so that $W = \{w = 0\}$. Notice that in a set $|w| \leqslant r, |z'| \leqslant r'$, we have that $\mathbf{1}_W(dd^cb)^k$ is the value at $\lambda = 0$ of

$$-(|w|^{2\lambda} - 1)(dd^cb)^k.$$

Since $|w|^{2\lambda}-1$ is psh, (3.6) follows from (3.2) since the total mass of $|w|^{2\lambda}-1$ tends to 0 when $\lambda \to 0$. □

Lemma 3.4. — *If b is psh and (locally) bounded on X and $i\colon Y \to X$ is a smooth submanifold, then for $k \leqslant n$,*

$$[Y]\wedge(dd^cb)^k = i_*(dd^ci^*b)^k, \quad [Y]\wedge b(dd^cb)^k = i_*\big(i^*b(dd^ci^*b)^k\big). \tag{3.7}$$

Proof. — First assume that b is smooth. Then

$$\int_X [Y]\wedge(dd^cb)^k\wedge\xi = \int_Y (dd^ci^*b)^k\wedge i^*\xi = \int_X i_*\big((dd^ci^*b)^k\big)\wedge\xi$$

and similarly

$$\int_X [Y]\wedge b(dd^cb)^k\wedge\xi = \int_X i_*\big(i^*b(dd^ci^*b)^k\big)\wedge\xi,$$

so that (3.7) holds in this case. Now let b be bounded and psh and let b_j be a decreasing sequence of smooth psh functions converging pointwise to b. Now (3.7) follows from the smooth case and (3.1). □

4. Higher Monge-Ampère products

Let G be a psh function of the form (1.1). We will give meaning to

$$(dd^cG)^k \tag{4.1}$$

by inductively defining it as $(dd^cG)^0 = 1$ and

$$(dd^cG)^k := dd^c\big(G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}\big), \quad k \geqslant 1. \tag{4.2}$$

Proposition 4.1 below asserts that this definition makes sense and that $(dd^cG)^k$ are positive and closed. As pointed out in the introduction this definition coincides with the iterative definition (1.6) for $k \leqslant p$.

Proposition 4.1. — *Let X be a complex manifold of dimension n, let f be a tuple of global functions of X, let G be a psh function of the form* (1.1), *and let G_j be a decreasing sequence of smooth psh functions in X converging pointwise to G. Assume that* (4.1) *is inductively defined via* (4.2) *for a fixed k. Then*

$$G\mathbf{1}_{X\setminus Z}(dd^cG)^k := \lim_j G_j\mathbf{1}_{X\setminus Z}(dd^cG)^k$$

has locally finite mass and does not depend on the choice of sequence G_j. Moreover $(dd^cG)^{k+1} = dd^c(G\mathbf{1}_{X\setminus Z}(dd^cG)^k)$ is positive and closed.

The proof below relies heavily on the fact that G is of the form (1.1). It could be interesting to investigate whether Proposition 4.1 holds for a wider class of psh functions G with unbounded locus Z.

Proof. — Let $\pi\colon \widetilde{X} \to X$ be a smooth modification such that $\pi^*\mathcal{J}$ is principal and its divisor is of the form

$$D = \sum \alpha_j D_j, \tag{4.3}$$

where D_j are smooth hypersurfaces with normal crossings. In particular, then $\pi^*f = f^0 f'$, where f^0 is a section of the line bundle L_D that defines D and f' is a non-vanishing tuple of sections of L_D^{-1}.

Locally on $\widetilde{X}$ we can choose a frame for L_D and in this frame we have, cf. (1.1),

$$\pi^*G = \log|f^0| + \log|f'| + \pi^*h =: \log|f^0| + b. \tag{4.4}$$

Since $\log|f^0|$ is pluriharmonic outside

$$|D| := \cup_j D_j$$

it follows that

$$b = \log|f'| + \pi^* h$$

is psh there; furthermore it is locally bounded at $|D|$. By a standard argument b has a unique (bounded) psh extension B across $|D|$. Notice that dd^cB is a global positive closed current on $\widetilde{X}$ and

$$dd^c\pi^*G = [D] + dd^cB.$$

Let G_j be a decreasing sequence of smooth psh functions converging pointwise to G. Since

$$dd^cG_j = \pi_*(dd^c\pi^*G_j) \to \pi_*\big(dd^c\pi^*G\big) = \pi_*\big([D] + dd^cB\big)$$

it follows that

$$dd^cG = \pi_*\big([D] + dd^cB\big).$$

Let us now assume that we have proved Proposition 4.1 as well as the equality

$$\big(dd^cG\big)^\ell = \pi_*\big([D]\wedge(dd^cB)^{\ell-1} + (dd^cB)^\ell\big) \tag{4.5}$$

for $\ell \leqslant k$. We are to see that then:

(i) $G\mathbf{1}_{X\setminus Z}(dd^cG)^k := \lim_j G_j\mathbf{1}_{X\setminus Z}(dd^cG)^k$ *has locally finite mass.*

(ii) *If*

$$(dd^cG)^{k+1} := dd^c\big(G\mathbf{1}_{X\setminus Z}(dd^cG)^k\big),$$

then (4.5) *holds for* $\ell = k+1$.

As soon as (i) and (ii) are verified, Proposition 4.1 follows.

Notice that if μ is a closed positive current, then

$$\mathbf{1}_Z\pi_*\mu = \pi_*(\mathbf{1}_{|D|}\mu). \tag{4.6}$$

In view of Corollary 3.3 we have that

$$\mathbf{1}_{|D|}(dd^cB)^k = 0. \tag{4.7}$$

From the induction hypothesis (4.5), (4.6) and (4.7) we get

$$\mathbf{1}_{X\setminus Z}\big(dd^cG\big)^k = \pi_*(dd^cB)^k. \tag{4.8}$$

By Proposition 3.2, $(\pi^*G)(dd^cB)^k$ has locally finite mass, and

$$(\pi^*G_j)(dd^cB)^k \to (\pi^*G)(dd^cB)^k$$

if G_j is any decreasing sequence of psh functions that tends to G. If G_j are smooth we have by (4.8) that

$$G_j \mathbf{1}_{X\setminus Z}(dd^cG)^k = \pi_*\big((\pi^*G_j)(dd^cB)^k\big),$$

which tends to

$$G\mathbf{1}_{X\setminus Z}(dd^cG)^k = \pi_*\big((\pi^*G)(dd^cB)^k\big), \tag{4.9}$$

which has locally finite mass. Thus (i) is verified.

We now consider (ii). We claim that

$$dd^c\big(\pi^*G\wedge(dd^cB)^k\big) = [D]\wedge(dd^cB)^k + (dd^cB)^{k+1}. \tag{4.10}$$

Recall that locally $\pi^*G = v + B$, where $v = \log|f^0|$ and B is psh and bounded. Take smooth psh v_j that decrease to v. Then $v_j + B$ are psh and decrease to $v + B$ and thus, by Proposition 3.2,

$$v_j(dd^cB)^k + B(dd^cB)^k = (v_j + B)(dd^cB)^k \to (v + B)(dd^cB)^k.$$

It follows that

$$(v + B)(dd^cB)^k = v(dd^cB)^k + B(dd^cB)^k.$$

From Proposition 3.2 we get that

$$dd^c\big(v(dd^cB)^k\big) = [D]\wedge(dd^cB)^k,$$

which proves the claim. In view of (4.9) and (4.10) the statement (ii) now follows. □

For future reference we notice that

$$M_k^{\mathcal{J}} = \pi_*\big([D]\wedge(dd^cB)^{k-1}\big), \quad \mathbf{1}_{X\setminus Z}(dd^cG)^k = \pi_*(dd^cB)^k. \tag{4.11}$$

In fact $\mathbf{1}_{X\setminus Z}(dd^cG)^k$ equals the *non-pluripolar product* $\langle dd^cG\rangle^k$ as defined in [6, 8].

It follows from the proof above and Proposition 3.2 that if G_j is any decreasing sequence of psh functions converging pointwise to G, then $G_j\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1} \to G\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}$ and

$$dd^c(G_j\wedge\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1}) = dd^cG_j\wedge\mathbf{1}_{X\setminus Z}(dd^cG)^{k-1} \to (dd^cG)^k.$$

Recall that if G_j are psh functions that decrease to G, then

$$\lim_j (dd^cG_j)^k = (dd^cG)^k, \quad k \leqslant p,$$

see, e.g., [9, Proposition III.4.9]. However, for $k > p$ one cannot hope for a definition of $(dd^cG)^k$ that is robust in this sense. In fact, even if G_j and $\widetilde{G}_j$ are sequences of smooth psh functions decreasing to G and $(dd^cG_j)^k$

and $(dd^c\widetilde{G}_j)^k$ converge to positive closed currents T and $\widetilde{T}$, respectively, T might be different from $\widetilde{T}$, as is illustrated by the following example.

Example 4.2. — Let $\varphi = (w, zw)$. Then

$$dd^c \log|\varphi| = dd^c \log|w| + dd^c \log(1+|z|^2)^{1/2} = [w=0] + dd^c\alpha,$$

where $[w=0]$ denotes the current of integration along $\{w=0\}$ and $\alpha = \log(1+|z|^2)^{1/2}$. Thus by (4.2),

$$(dd^c \log|\varphi|)^2 = [w=0] \wedge dd^c\alpha.$$

Let $G_\epsilon = \log(|\varphi|^2+\epsilon)^{1/2}$ and $\widetilde{G}_\epsilon = \log(|w|^2+\epsilon)^{1/2} + \alpha$. Then G_ϵ and $\widetilde{G}_\epsilon$ are smooth psh functions that decrease towards $\log|\varphi|$ as ϵ tends to 0. On the one hand, by (1.7),

$$\lim_{\epsilon\to 0}(dd^c G_\epsilon)^2 = (dd^c \log|\varphi|)^2.$$

On the other hand, again using (1.7), but now for $(dd^c \log|w|)^2$,

$$(dd^c\widetilde{G}_\epsilon)^2 = (dd^c \log(|w|^2+\epsilon)^{1/2})^2 + 2dd^c \log(|w|^2+\epsilon)^{1/2} \wedge dd^c\alpha \longrightarrow 2[w=0]\wedge dd^c\alpha.$$

Remark 4.3. — Assume that X_ℓ is an exhaustion of X by relatively compact subsets such that the restriction $\mathcal{J}_\ell$ of $\mathcal{J}$ to X_ℓ is generated by global bounded functions. It would be interesting to know whether, or under what assumptions, the currents $M_k^{\mathcal{J}_\ell}$ then converge. Convergence would give us a global canonical representation of the Segre numbers of $\mathcal{J}$.

Assume that $\mathcal{J}$ is indeed generated by global bounded functions and let G_ℓ denote the Green function with poles along $\mathcal{J}_\ell$. Then, arguing as in the proof of Proposition 4.1 and using the notation from that proof,

$$\pi^* G_\ell = \log|f^0| + B_\ell,$$

where B_ℓ is psh and bounded, and moreover

$$(dd^c G_\ell)^k = \pi_*([D]\wedge(dd^c B_\ell)^{k-1} + (dd^c B_\ell)^k).$$

Assume that G_ℓ decrease towards G. Then B_ℓ decrease towards B, as defined in (4.4), and thus $\lim_\ell (dd^c G_\ell)^k = (dd^c G)^k$ in light of (3.1) and (4.5).

5. Lelong numbers

Let T be a positive closed (k,k)-current. If $k=n$, following [4, Section 5], we let

$$M_0^\xi \wedge T := \mathbf{1}_{\{x\}} T.$$

Otherwise

$$M^\xi_{n-k}\wedge T := \mathbf{1}_{\{x\}}\big((dd^c\log|\xi|)^{n-k}\wedge T\big);$$

here we inductively define

$$(dd^c\log|\xi|)^\ell\wedge T :=$$
$$dd^c\big(\log|\xi|\wedge(dd^c\log|\xi|)^{\ell-1}\wedge T\big) = \lim_j dd^c\big(v_j\wedge(dd^c\log|\xi|)^{\ell-1}\wedge T\big),$$

where v_j is a decreasing sequence of smooth psh functions converging pointwise to $\log|\xi|$. Because of the dimension principle it is not necessary to insert $\mathbf{1}_{X\setminus\{x\}}$ in this definition, cf., Section 4. See Remark 5.1 below for another possible definition of $M^\xi_{n-k}\wedge T$. Clearly $M^\xi_{n-k}\wedge T$ is an (n,n)-current with support at x, and it is in fact equal to $\alpha[x]$, where α is the Lelong number of T at x, see, e.g, [4, Lemma 2.1].

Remark 5.1. — As is pointed out in [4, Section 5] one can define $M^\xi\wedge T$ as the value at $\lambda=0$ of the current-valued analytic function

$$\lambda\mapsto \frac{\bar\partial|\xi|^{2\lambda}\wedge\partial|\xi|^2}{2\pi i|\xi|^2}\wedge(dd^c\log|\xi|)^{n-k-1}\wedge T.$$

6. Proof of Theorem 1.1

We will prove the slightly more general formulation of Theorem 1.1 stated in Remark 1.2, i.e., we let G be any psh function of the form (1.1).

We still assume that $\pi\colon \widetilde X\to X$ is a smooth modification and use the notation from the proof of Proposition 4.1. Notice that L_D has a Hermitian metric such that $|f^0|_{L_D}=|\pi^*f|$. By the Poincaré-Lelong formula,

$$dd^c\log|\pi^*f| = [D]+\omega_f, \tag{6.1}$$

where ω_f is the first Chern form for L_D^{-1}.

Let us fix a local holomorphic frame so that $\log|f'|$ is a well-defined function as above. Since

$$\log|\pi^*f| = \log|f^0| + \log|f'|,$$

from (6.1) we have that

$$\omega_f = dd^c\log|f'|. \tag{6.2}$$

Let b be the psh bounded function outside $|D|$ defined in (4.4). If we choose another local frame for L_D, then $\log|f'|$ is changed to $\log|f'|+\alpha$ where α is pluriharmonic, and b is thus changed to $\tilde b := b+\alpha$. Moreover $\widetilde B := B+\alpha$

is the unique psh extension of $\tilde{b}$ across $|D|$, cf. the proof of Proposition 4.1. It follows that A, locally defined as

$$A := B - \log|f'|, \tag{6.3}$$

is a global upper semicontinuous extension of $\pi^* h$ across $|D|$. Notice also that $A(dd^c B)^\ell$ is well-defined on $\widetilde{X}$ and, in light of (6.2) and (6.3), that

$$(dd^c B)^{k-1} - \omega_f^{k-1} = dd^c\Big(A\sum_{\ell=0}^{k-2}(dd^c B)^\ell \wedge \omega_f^{k-2-\ell}\Big).$$

Assume now that $Y \subset \widetilde{X}$ is a smooth submanifold and that $i\colon Y \to \widetilde{X}$ is the natural inclusion. Then $i^* B$ is psh and bounded, $i^* \log|f'|$ is smooth, and, in the same way as above, $i^* A$ is a global upper semi-continuous function on Y and

$$(dd^c i^* B)^{k-1} - i^*\omega_f^{k-1} = dd^c\Big(i^* A\sum_{\ell=0}^{k-2}(dd^c i^* B)^\ell \wedge i^*\omega_f^{k-2-\ell}\Big). \tag{6.4}$$

In view of Lemma 3.4, (6.4) implies that

$$[Y]\wedge\Big((dd^c B)^{k-1} - \omega_f^{k-1}\Big) = dd^c i_*\Big(i^* A\sum_{\ell=0}^{k-2}(dd^c i^* B)^\ell \wedge i^*\omega_f^{k-2-\ell}\Big).$$

The currents $(dd^c \log|f|)^k$ and M_k^f are defined in a completely analogous way as $(dd^c G)^k$ and $M_k^{\mathcal{J}}$, just replacing G by $\log|f|$, cf., the introduction and the end of Section 2 and also [4]. Arguing as in the proof of Proposition 4.1, we get, cf., (4.11), that

$$M_k^f = \pi_*([D]\wedge\omega_f^{k-1}), \quad \mathbf{1}_{X\setminus Z}(dd^c \log|f|)^k = \pi_*\omega_f^k$$

Lemma 6.1. — *The currents $M_k^{\mathcal{J}}$ and M_k^f have the same Lelong number at each point $x \in X$. Moreover, the currents $\mathbf{1}_{X\setminus Z}(dd^c G)^k$ and $\mathbf{1}_{X\setminus Z}(dd^c \log|f|)^k$ have the same Lelong number at each point $x \in X$.*

Proof. — Let us fix a point $x \in X$ and let ξ be a tuple of functions that defines the maximal ideal $\mathfrak{m}_x$ at x. We can choose the modification $\pi\colon \widetilde{X} \to X$ so that also $\pi^*\mathfrak{m}_x$ is principal, i.e., $\pi^*\xi = \xi^0\xi'$, where ξ^0 is a section of a line bundle L_E that defines the exceptional divisor E, and ξ' is a non-vanishing tuple of sections of L_E^{-1}. Let us assume that

$$E = \sum_\kappa \beta_\kappa E_\kappa, \tag{6.5}$$

where E_κ are irreducible with simple normal crossings and β_κ are integers. We may also assume that, for each j, cf., (4.3), either $D_j \subset |E|$ or all E_κ intersect D_j properly and that

$$E_\kappa^{D_j} := E_\kappa \cap D_j$$

are smooth. Let ω_ξ be the first Chern form of L_E^{-1} with respect to the metric induced by ξ, so that

$$\omega_\xi = dd^c \log |\xi'|,$$

cf., (6.2), and

$$dd^c \log |\pi^* \xi| = [E] + \omega_\xi.$$

Let $i_j \colon D_j \to \widetilde{X}$ be the injection of D_j as a submanifold of $\widetilde{X}$. It follows from (4.3), (4.11) and Lemma 3.4 that

$$M_k^{\mathcal{J}} = \sum_j \alpha_j \pi_* (i_j)_* \big((dd^c (i_j)^* B)^{k-1} \big). \tag{6.6}$$

In order to prove the first part of the lemma, it is enough to consider one single term in (6.6) and verify that

$$T_k^{\mathcal{J}} := \pi_* i_* \big((dd^c i^* B)^{k-1} \big)$$

and

$$T_k^f := \pi_* i_* \big(i^* \omega_f^{k-1} \big)$$

have the same Lelong numbers, where we write $D = D_j$ and $i = i_j$ for simplicity.

Let us first assume that $k = n$. If $D \subset |E|$, then $T_n^{\mathcal{J}}$ and T_n^f both have support at x. In view of (6.4), with $Y = D$, we have that

$$T_k^{\mathcal{J}} - T_k^f = dd^c \pi_* i_* \Big(i^* A \sum_{\ell=1}^{k-2} (dd^c i^* B)^\ell \wedge i^* \omega_f^{k-2-\ell} \Big) =: dW,$$

where W has support at x. By Stokes' theorem thus

$$\int (T_n^{\mathcal{J}} - T_n^f) = \int dW = 0,$$

which means that $T_n^{\mathcal{J}}$ and T_n^f have the same Lelong number at x. If D is not contained in E, then $i^{-1}E$ has positive codimension in D and therefore,

$$\mathbf{1}_{\{x\}} T_n^{\mathcal{J}} = \pi_* i_* (\mathbf{1}_{|i^{-1}E|} (dd^c i^* B)^{n-1}) = 0$$

by Corollary 3.3. In the same way we see that $\mathbf{1}_{\{x\}} T_n^f = 0$.

Let us now assume that $k < n$. If $D \subset |E|$, then $T_k^{\mathcal{J}}$ and T_k^f are positive closed (k,k)-currents with support at x, so by the dimension principle they both vanish. We can therefore assume that $i^* \pi^* \xi$ does not vanish identically on D; by assumption it then defines a smooth divisor E^D on D. Locally on D,

$$\log|i^*\pi^*\xi| = \log|i^*\xi^0| + \log|i^*\xi'|,$$

and thus

$$dd^c \log|i^*\pi^*\xi| = [E^D] + i^*\omega_\xi, \tag{6.7}$$

where $[E^D]$ is the Lelong current on D associated to E^D. If v_j are as in Section 5, then

$$dd^c i^*\pi^* v_j \to [E^D] + i^*\omega_\xi.$$

Now

$$dd^c(v_j T_k^{\mathcal{J}}) = \pi_* i_*\big(dd^c i^*\pi^* v_j \wedge (dd^c i^* B)^{k-1}\big)$$

so that

$$dd^c \log|\xi| \wedge T_k^{\mathcal{J}} = \pi_* i_*\big(([E^D] + i^*\omega_\xi)\wedge (dd^c i^* B)^{k-1}\big)$$

by Proposition 3.2 and (6.7). Moreover, since $\pi_* i_*\big([E^D]\wedge(dd^c i^* B)^{k-1}\big)$ has support at x, by the dimension principle,

$$dd^c \log|\xi| \wedge T_k^{\mathcal{J}} = \pi_* i_*\big(i^*\omega_\xi \wedge (dd^c i^* B)^{k-1}\big).$$

By induction we get

$$(dd^c \log|\xi|)^{n-k}\wedge T_k^{\mathcal{J}} = \pi_* i_*\big(([E^D] + i^*\omega_\xi)\wedge i^*\omega_\xi^{n-k-1}\wedge (dd^c i^* B)^{k-1}\big).$$

Therefore, by Corollary 3.3,

$$\begin{aligned} M_{n-k}^{\xi}\wedge T_k^{\mathcal{J}} &= \mathbf{1}_{\{x\}}(dd^c \log|\xi|)^{n-k}\wedge T_k^{\mathcal{J}} \\ &= \pi_* i_*\big([E^D]\wedge i^*\omega_\xi^{n-k-1}\wedge (dd^c i^* B)^{k-1}\big). \end{aligned}$$

Let $\iota_\kappa\colon E_\kappa^D \to D$ be the natural injection. By (6.5) and Lemma 3.4 we have that

$$M_{n-k}^{\xi}\wedge T_k^{\mathcal{J}} = \sum_\kappa \beta_\kappa \pi_* i_* (\iota_\kappa)_*\big((\iota_\kappa)^* i^*\omega_\xi^{n-k-1}\wedge (dd^c(\iota_\kappa)^* i^* B)^{k-1}\big).$$

By analogous arguments,

$$M_{n-k}^{\xi}\wedge T_k^{f} = \sum_\kappa \beta_\kappa \pi_* i_* (\iota_\kappa)_*\big((\iota_\kappa)^* i^*\omega_\xi^{n-k-1}\wedge (\iota_\kappa)^* i^*\omega_f^{k-1}\big).$$

For simplicity in notation let us assume that E^D has just one irreducible component and let $\iota : E^D \to D$ be the natural injection. By (6.4) applied to E^D we have that

$$\begin{aligned} &M_{n-k}^{\xi}\wedge T_k^{\mathcal{J}} - M_{n-k}^{\xi}\wedge T_k^{f} = \\ &\qquad dd^c \pi_* i_* \iota_*\Big(\iota^* i^* A\ \iota^* i^*\omega_\xi^{n-k-1}\wedge \sum_{\ell=0}^{k-2}(dd^c \iota^* i^* B)^{\ell}\iota^* i^*\omega_f^{k-1-\ell}\Big) =: dW, \end{aligned}$$

where W has support at x. It follows by Stokes' theorem that the integral of this current is zero, and thus the Lelong numbers at x of $T_k^{\mathcal{J}}$ and T_k^f coincide. Thus the first part of the lemma is proved.

By analogous arguments we get that $\pi_*(dd^cB)^k$ and $\pi_*(\omega_f)^k$ have the same Lelong number at x, which proves the second part of the lemma, cf. (4.11) and (6.5). □

We can now conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. — Let D_j^ℓ be the irreducible components of D such that $\pi(D_j^\ell)$ have codimension ℓ. Then

$$M_k^{\mathcal{J}} = \pi_*\big([D]\wedge(dd^cB)^{k-1}\big) = \pi_*\Big(\sum_{\ell\leqslant k}\sum_j([D_j^\ell]\wedge(dd^cB)^{k-1}\Big)$$

since terms with $\ell > k$ vanish because of the dimension principle. We claim that

$$\begin{aligned} M_k^{\mathcal{J}} &= \pi_*\Big(\sum_j([D_j^k]\wedge(dd^cB)^{k-1}\Big) + \pi_*\Big(\sum_{\ell<k}\sum_j([D_j^\ell]\wedge(dd^cB)^{k-1}\Big) \\ &=: S_k^{\mathcal{J}} + N_k^{\mathcal{J}} \end{aligned} \tag{6.8}$$

is the Siu decomposition of $M_k^{\mathcal{J}}$. First notice that since

$$\pi_*\big([D_j^k]\wedge(dd^cB)^{k-1}\big)$$

is a (k,k)-current with support on the set $Z := \pi(D_j^k)$ of codimension k it must be of the form $\alpha[Z]$ where α is a constant, see Lemma 3.1.

It is now enough to see that if W is a subvariety of codimension k, then $\mathbf{1}_W N_k^{\mathcal{J}} = 0$, i.e.,

$$\mathbf{1}_W\pi_*\big([D_j^\ell]\wedge(dd^cB)^{k-1}\big) = 0$$

if $\ell < k$. Let $i\colon D_j^\ell \to \widetilde{X}$ be the natural injection. By Lemma 3.4 we have

$$\begin{aligned} \mathbf{1}_W\pi_*\big([D_j^\ell]\wedge(dd^cB)^{k-1}\big) &= \mathbf{1}_W\big(\pi_*i_*\big(dd^ci^*B\big)^{k-1}\big) \\ &= \pi_*i_*\big(\mathbf{1}_{(\pi\circ i)^{-1}(W)}(dd^ci^*B)^{k-1}\big). \end{aligned}$$

Notice that since $\pi(D_j^\ell)$ is irreducible and not contained in W it follows that $\pi^{-1}(W)\cap D_j^\ell$ has positive codimension in D_j^ℓ, and hence

$$\mathbf{1}_{(\pi\circ i)^{-1}(W)}(dd^ci^*B)^{k-1} = 0$$

in view of Corollary 3.3.

Thus (6.8) is the Siu decomposition. Since $M_k^{\mathcal{J}}$ and M_k^f have the same Lelong number at each point by Lemma 6.1 and the set where $N_k^{\mathcal{J}}$ and N_k^f have positive Lelong number have codimension $> k$ we conclude that $S_k^{\mathcal{J}} = S_k^f$, see Remark 2.1. Since also $\mathbf{1}_{X\setminus Z}(dd^cG)^k$ and $\mathbf{1}_{X\setminus Z}(dd^c\log|f|)^k$

have the same Lelong numbers at x by Lemma 6.1, Theorem 1.1 follows from the analogous result, Theorem 1.1, for M^f in [4]. □

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