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GREEN FUNCTIONS, SEGRE NUMBERS, AND KING’S FORMULA

by Mats ANDERSSON & Elizabeth WULCAN (*)

Abstract. — Let $J$ be a coherent ideal sheaf on a complex manifold $X$ with zero set $Z$, and let $G$ be a plurisubharmonic function such that $G = \log |f| + \mathcal{O}(1)$ locally at $Z$, where $f$ is a tuple of holomorphic functions that defines $J$. We give a meaning to the Monge-Ampère products $(dd^c G)^k$ for $k = 0, 1, 2, \ldots$, and prove that the Lelong numbers of the currents $M^J_k := 1_Z (dd^c G)^k$ at $x$ coincide with the so-called Segre numbers of $J$ at $x$, introduced independently by Tworzewski, Gaffney-Gassler, and Achilles-Manaresi. More generally, we show that $M^J_k$ satisfy a certain generalization of the classical King formula.

1. Introduction

Let $X$ be a complex manifold of dimension $n$ and let $J \to X$ be a coherent ideal sheaf with variety $Z$. Given a point $x \in X$, Tworzewski, [24], and Gaffney and Gassler, [14], have independently introduced a list of numbers, $e_0(J, X, x), \ldots, e_n(J, X, x)$, that we, following [14], call the Segre numbers at $x$. They are a generalization of the classical local intersection number at $x$ in case the ideal $J_x$ is a complete intersection. The definition

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in both papers is based on a local variant of the Stückrad-Vogel procedure, [23]. In [1, 2] is given an algebraic definition of these numbers generalizing the classical Hilbert-Samuel multiplicity of \( J \) at \( x \).

In this paper we show that if \( J \) is generated by global bounded functions there is a canonical global representation of the Segre numbers of \( J \) as the Lelong numbers (of restrictions to \( Z \)) of Monge-Ampère masses of the Green function \( G = G_J \) with poles along \( J \). This function was introduced by Rashkovskii-Sigurdsson in [20, Definition 2.2] as a generalization of the classical Green function \( G_a \) with pole at a point \( a \in X \). It is defined as the supremum over the class \( \mathcal{F}_J \) of all negative psh (plurisubharmonic) functions \( u \) on \( X \) that locally satisfy \( u \leq \log |f| + C \), where \( f = (f_1, \ldots, f_m) \) is a tuple of local generators of \( J \) and \( C \) is a constant.

Note that even if \( X \) is hyperconvex there might not exist non-trivial functions in \( \mathcal{F}_J \). For example, if \( X \) is the ball in \( \mathbb{C} \), and \( J \) is the radical ideal of functions vanishing at points \( a_1, a_2, \ldots, a_j \in X \), then there are negative psh functions with poles at \( a_j \) if and only if \( a_j \) satisfy the the Blaschke condition. However, if \( J \) is globally generated by bounded functions \( f_j \), then \( \log |f| + C \) is itself in \( \mathcal{F}_J \) for some constant \( C \). Then locally \( G \) is of the form

\[
G = \log |f| + h,
\]

where \( h \) is locally bounded, see [20, Theorem 2.8]. In particular, the unbounded locus of \( G \) equals \( Z \) and thus the Monge-Ampère type products

\[
(dd^c G)^k, \quad k \leq p := \text{codim } Z
\]

are well-defined, see, e.g., [9, Theorem III.4.5]. Here and throughout \( d^c = (i/2\pi)(\bar{\partial} - \partial) \). By Demailly’s comparison formula for Lelong numbers, [11, Theorem 5.9],

\[
\ell_x(dd^c G)^k = \ell_x(dd^c \log |f|)^k
\]

for \( x \in X \), where \( \ell_x \) denotes the Lelong number at \( x \). Moreover, recall that King’s formula, [15], asserts that \( (dd^c \log |f|)^p \) admits the Siu decomposition, [21],

\[
(dd^c \log |f|)^p = \sum \beta_j [Z_j^p] + R,
\]

cf. [11, Section 6]. Here \([Z_j^p]\) are the currents of integration along the irreducible components \( Z_j^p \) of codimension \( p \) of \( Z \), \( \beta_j \) are the generic Hilbert-Samuel multiplicities of \( f \) along \( Z_j^p \), see, e.g. [13, Chapter 4.3]. In fact, the remainder term \( R \) has integer Lelong numbers, see, e.g. [4, Theorem 1.1],
and therefore the set where $R$ has positive Lelong numbers is an analytic set of codimension $> p$. From (1.3) and (1.4) one deduces that

$$ (dd^c G)^p = \sum \beta_j[Z_j^p] + R, $$

where $\beta_j$ and $Z_j^p$ are as above, and $R$ has the same Lelong numbers as $R$ in (1.4), cf. the proof of Theorem 2.8 in [20]. In particular, if $Z$ is a point $a$, then $(dd^c G)^n = \sum \beta[a] + R$, where $[a]$ is the point evaluation at $a$ and $\beta$ is the Hilbert-Samuel multiplicity of $J$. This generalizes the fact that $(dd^c G_a)^n = [a]$, [10, page 520]. The (Lelong numbers of the) Monge-Ampère products (1.2) are related to the integrability index of $G$ (and thus the log-canonical threshold of $J$), see, e.g., [12, 19, 22]; in particular, Demailly-Pham [12] recently gave a sharp estimate of the integrability index of $G$ in terms of the Lelong numbers of (1.2) for all $k \leq p$.

Recall that (1.2) can be defined inductively as

$$ dd^c(G(dd^c G)^{k-1}). $$

In this paper we give meaning to $(dd^c G)^k$ for any $k$ if $G$ is any psh function of the form (1.1): Inductively we show that

$$ G1_{X \backslash Z}(dd^c G)^{k-1} $$

has locally finite mass and define

$$(dd^c G)^k := dd^c(G1_{X \backslash Z}(dd^c G)^{k-1}),$$

see Proposition 4.1. When $k \leq p$ it follows from the dimension principle for closed positive currents, cf. Lemma 3.1 below, that $1_Z(dd^c G)^{k-1} = 0$ and so our definition coincides with the classical one for $k \leq p$. Our definition is modeled on the paper [3] by the first author, in which currents $(dd^c \log |f|)^k$ are defined for all $k$ inductively as above. In fact, $(dd^c \log |f|)^k$ can also be defined as a certain limit of smooth forms coming from regularizations of $\log |f|$:  

$$ \lim_{\epsilon \to 0}(dd^c \log(|f|^2 + \epsilon)^{1/2})^k = (dd^c \log |f|)^k $$

for any $k$, see [3, Proposition 4.4]. However, one cannot hope for such a suggestive definition of $(dd^c G)^k$ in general, cf. Example 4.2. Also, our definition of $(dd^c G)^k$ does not coincide with the non-pluripolar product of $dd^c G$, as introduced in [6, 8], since our $(dd^c G)^k$ charges pluripolar sets in general, cf. the text after the proof of Proposition 4.1.

Our main result is the following generalization of (1.5). Let $\pi^+: X^+ \to X$ be the normalization of the blow-up of $X$ along $J$ and let $W_j$ be the various irreducible components of the exceptional divisor in $X^+$. Recall that
the (Fulton-MacPherson) distinguished varieties of $J$ are the subvarieties $\pi^+(W_j)$ of $X$, see, e.g., [16, Chapter 10.5]. In particular, the distinguished varieties of codimension $p$ are precisely the irreducible components of $Z$ of codimension $p$.

**Theorem 1.1.** — Let $X$ be an $n$-dimensional complex manifold, let $J$ be a coherent ideal sheaf on $X$ generated by global bounded functions, and let $G$ be the Green function with poles along $J$. Moreover, let $Z$ be the variety of $J$ and $Z^k_j$ the Fulton-MacPherson distinguished varieties of $J$ of codimension $k$. Then

$$M^J_k := \mathbf{1}_Z(\ddc G)^k = \sum_j \beta^k_j[Z^k_j] + N^J_k =: S^J_k + N^J_k,$$

where the $\beta^k_j$ are positive integers and the $N^J_k$ are positive closed currents. The numbers $n_k(J, X, x) := \ell_x(N^J_k)$ are nonnegative integers that only depend on the integral closure class of $J$ at $x$, and the set where $n_k(J, X, x) \geq 1$ has codimension at least $k + 1$.

The Lelong numbers at $x$ of $M^J_k$ and $1_{X\setminus Z}(\ddc G)^k$ are precisely the Segre number $e_k(J, X, x)$ and the polar multiplicity $m_k(J, X, x)$, respectively, of $J_x$.

For the notion of polar multiplicities see Section 2. Notice that $M^J_k = 0$ if $k < \text{codim } Z$ and that $N^J_p = 0$, cf., Lemma 3.1 below. Also, notice that (1.8) is the Siu decomposition, [21], of $M^J_k$.

**Remark 1.2.** — If $J$ is generated by a global tuple $f$, then Theorem 1.1 holds with $G$ replaced by any psh function of the form (1.1).

The analogous statement to Theorem 1.1 when $G$ is replaced by $\log |f|$, where $f$ is a tuple of global generators, was proved by the authors and Samuelsson Kalm and Yger in [4, Theorem 1.1]. The case $k = p$ corresponds to the classical King formula, (1.4). The main idea in the proof of Theorem 1.1 is to prove that for any psh $G$ of the form (1.1),

$$\ell_x(\mathbf{1}_Z(\ddc G)^k) = \ell_x(\mathbf{1}_Z(\ddc \log |f|)^k),$$

$$\ell_x(1_{X\setminus Z}(\ddc G)^k) = \ell_x(1_{X\setminus Z}(\ddc \log |f|)^k)$$

for $x \in X$, see Lemma 6.1 below. Using this the theorem follows from the corresponding result in [4]. In some sense, (1.9) can be seen as a generalization of Demailly’s comparison formula, (1.3), to higher $k$, but for the very special class of psh functions of the form (1.1).
In [4], $X$ is allowed to be singular. Given that there is a proper definition of $G$ when $X$ is singular so that (1.1) still holds, the results in this paper will extend as well.

Theorem 1.1 gives us a canonical representation of the Segre numbers of $J$ in the case when $J$ is generated by global bounded functions. Let $X$ be a, say hyperconvex, domain in $\mathbb{C}^n$, and let $J$ be a coherent ideal sheaf on $X$. If we exhaust $X$ by reasonable relatively compact subsets $X_\ell$, for each $\ell$ we then have currents $M^{J_\ell}_k$, $J_\ell = J|_{X_\ell}$, whose Lelong numbers at each point are the Segre numbers. If for some reason these currents converge to currents $M^J_k$, we would have a canonical representation of the Segre numbers of $J$ on $X$, cf. Remark 4.3.

This paper is organized as follows. In Section 2 we recall the construction of Vogel cycles and Segre numbers. In Section 4 we show that the currents $(dd^c G)^k$ are well-defined and discuss some properties. The proof of Theorem 1.1 occupies Section 6. In Sections 3 and 5 we give some background on psh functions and positive currents needed for the proofs.

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2. Segre numbers

We will briefly recall the construction of Segre numbers from [24, 14]. Throughout we will assume that $X$ is a complex manifold of dimension $n$ and that $J$ is a coherent ideal sheaf on $X$ with variety $Z$. Fix a point $x \in X$. A sequence $h = (h_1, h_2, \ldots, h_n)$ in the local ideal $J_x$ is called a Vogel sequence of $J$ at $x$ if there is a neighborhood $\mathcal{U} \subset X$ of $x$ where the $h_j$ are defined, such that

$$\text{codim} \left[ (\mathcal{U} \setminus Z) \cap (|H_1| \cap \cdots \cap |H_k|) \right] = k \text{ or } \infty, \ k = 1, \ldots, n;$$

where $|H_\ell|$ are the supports of the divisors $H_\ell$ defined by $h_\ell$. Notice that if $f_1, \ldots, f_m$ generate $J_x$, any generic sequence of $n$ linear combinations of the $f_j$ is a Vogel sequence at $x$. Set $X_0 = X$, let $X^Z_0$ denote the irreducible
components of $X_0$ that are contained in $Z$, and let $X_0^{X\setminus Z}$ be the remaining components\(^{(1)}\) so that

$$X_0 = X_0^Z + X_0^{X\setminus Z}.$$  

By the Vogel condition (2.1), $H_1$ intersects $X_0^{X\setminus Z}$ properly. Set

$$X_1 = H_1 \cdot X_0^{X\setminus Z}$$

and decompose analogously $X_1$ into the components $X_1^Z$ contained in $Z$ and the remaining components $X_1^{X\setminus Z}$, so that $X_1 = X_1^Z + X_1^{X\setminus Z}$. Define inductively $X_{k+1} = H_{k+1} \cdot X_k^{X\setminus Z}$, $X_k^Z$, and $X_{k+1}^{X\setminus Z}$, so that

$$X_k^Z = X_k^Z + X_{k+1}^{X\setminus Z}.$$  

Define inductively $X_{k+1} = H_{k+1} \cdot X_k^{X\setminus Z}$, $X_k^Z$, and $X_{k+1}^{X\setminus Z}$. Then

$$V^h := X_0^Z + X_1^Z + \cdots + X_n^Z$$

is the Vogel cycle\(^{(2)}\) associated with the Vogel sequence $h$. Let $V_k^h$ denote the components of $V^h$ of codimension $k$, i.e., $V_k^h = X_k^Z$. The irreducible components of $V^h$ that appear in any Vogel cycle, associated with a generic Vogel sequence at $x$, are called fixed components in [14]. The remaining ones are called moving. It turns out that the fixed Vogel components of $J$ coincide with the distinguished varieties of $J$, see, e.g., see [14] or [4].

It is proved in [14] and in [24] that the multiplicities $e_k(J, X, x) := \text{mult}_x V_k^h$ and $m_k(J, X, x) := \text{mult}_x X_k^{X\setminus Z}$ are independent of $h$ for a generic $h$, where however “generic” depends on $x$, cf., Remark 2.1; these numbers are called the Segre numbers and polar multiplicities, respectively.

**Remark 2.1.** — Recall that if $W$ is an analytic cycle in $X$, then the Lelong number at $x \in X$ of the current of integration $[W]$ along $W$ is precisely the multiplicity $\text{mult}_x W$ of $W$ at $x$.

Assume that $x$ is a point for which $n_k(J, X, x) \geq 1$ for some $k$, where we use the notation from Theorem 1.1. Moreover, let $V^h$ be a generic Vogel cycle such that $\text{mult}_x V_k^h = e_k(x)$. Then $V_k^h = S_k^J + W$, where we have identified $S_k^J$ in Theorem 1.1 with the corresponding cycle and $W$ is a positive cycle of codimension $k$, such that $\text{mult}_x W = n_k(J, X, x)$. Since $n_k(J, X, y) \geq 1$ only on a set of codimension $\geq k+1$, at most points $y$ on $V_k^h$ we have that $e_k(J, X, y) = \text{mult}_y S_k^J$ and hence $\text{mult}_y V_k^h > e_k(J, X, y)$. As soon as there is a moving component at $x$ it is thus impossible to find a Vogel cycle that realizes the Segre numbers in a whole neighborhood of $x$.

\(^{(1)}\) Since we assume $X$ is smooth and connected, $X_0^Z$ is empty unless $J = 0$, in which case it equals $X$.

\(^{(2)}\) If $J$ is the pullback to $X$ of the radical sheaf of an analytic set $A$, this is precisely Tworzewski’s algorithm, [24]. The notion Vogel cycle was introduced by Massey [17, 18]. For a generic choice of Vogel sequence the associated Vogel cycle coincides with the Segre cycle introduced by Gaffney-Gassler, [14], see Lemma 2.2 in [14].
In [4] Theorem 1.1 with $G$ replaced by $\log |f|$ was proved by showing that $M_k^f := 1_Z(dd^c \log |f|)^k$ can be seen as a certain average (of currents of integration) of Vogel cycles. The fixed Vogel components then appear as the leading part $S^f_k$ in the Siu decomposition of $M_k^f$, whereas the remainder term $N_k^f$ is a mean value of the moving parts.

3. Preliminaries

Let $\mu$ be a positive closed current on $X$. Recall that if $W$ is any subvariety, then $1_W \mu$ and $1_{X \setminus W} \mu$ are positive closed currents as well; this is the Skoda-El Mir theorem, see, e.g., [9, Chapter III.2.A].

**Lemma 3.1.** — Let $\mu$ be a positive closed current of bidegree $(p,p)$ that has support on a subvariety of codimension $k$. If $k > p$ then $\mu = 0$. If $k = p$, then $\mu = \alpha_1[W_1] + \cdots + \alpha_v[W_v]$ where $W_j$ are the irreducible components of $W$ and $\alpha_j \geq 0$.

We refer to the first part of Lemma 3.1 as the *dimension principle*. A proof can be found in [9, Chapter III.2.C].

If $b$ is psh and locally bounded and $T$ is any positive closed current, then $T \wedge (dd^c b)^k$ is a well-defined positive current for any $k$, and if $b_j$ is a decreasing sequence of bounded psh functions converging pointwise to $b$, then

$$T \wedge (dd^c b)^k = \lim_j T \wedge (dd^c b_j)^k, \quad T \wedge b (dd^c b)^k = \lim_j T \wedge b_j (dd^c b_j)^k, \quad k \leq n.$$  

See, e.g., [9, Theorem III.3.7]. The case $T \equiv 1$ was first proved by Bedford and Taylor, [5].

**Proposition 3.2.** — Assume that $v, b$ are psh and that $b$ is (locally) bounded.

(i) For $k \leq n - 1$,

$$v(dd^c b)^k$$

has locally finite mass; more precisely, for any compact sets $L, K$, such that $L \subset \text{int}(K)$, we have

$$\|v(dd^c b)^k\|_L \leq C_{K,L} \|v\|_K (\sup_K |b|)^k.$$  

(ii) Moreover, if the unbounded locus of $v$ has Hausdorff dimension $< 2n - 1$, then

$$dd^c (v(dd^c b)^k) = dd^c v \wedge (dd^c b)^k.$$
If \( v_j \) is a decreasing sequence of psh functions converging pointwise to \( v \), then
\[
(3.4) \quad v_j(\dd c b)^k \to v(\dd c b)^k,
\]
and
\[
(3.5) \quad \dd c v_j \wedge (\dd c b)^k \to \dd c v \wedge (\dd c b)^k
\]
in the current sense.

The first part of Proposition 3.2 follows immediately from Proposition 3.11 in [9, Chapter III]. Moreover, Proposition 4.9 in loc. cit. applied to \( u_1 = v \) and \( u_j = b \) implies (3.4) and (3.5). If we choose \( v_j \) smooth, then
\[
\dd c (v_j(\dd c b)^k) = \dd c v_j \wedge (\dd c b)^k.
\]
Thus (3.3) follows from (3.4) and (3.5). In fact, the assumption about the Hausdorff dimension is not necessary; an elegant and quite direct argument has been communicated to us by Z. Błocki, [7].

**Corollary 3.3.** — If \( b \) is psh and (locally) bounded on \( X \) and \( W \) is an analytic variety of positive codimension, then for each \( k \geq 0 \),
\[
(3.6) \quad 1_W (\dd c b)^k = 0.
\]

**Proof.** — It is enough to consider the case when \( W \) is a smooth hypersurface. The general case follows by stratification. Since it is a local statement, we may choose coordinates \( z = (z', w) \) so that \( W = \{ w = 0 \} \). Notice that in a set \( |w| \leq r, |z'| \leq r' \), we have that \( 1_W (\dd c b)^k \) is the value at \( \lambda = 0 \) of
\[
-(|w|^{2\lambda} - 1)(\dd c b)^k.
\]
Since \( |w|^{2\lambda} - 1 \) is psh, (3.6) follows from (3.2) since the total mass of \( |w|^{2\lambda} - 1 \) tends to 0 when \( \lambda \to 0 \).

**Lemma 3.4.** — If \( b \) is psh and (locally) bounded on \( X \) and \( i \): \( Y \rightarrow X \) is a smooth submanifold, then for \( k \leq n \),
\[
(3.7) \quad [Y] \wedge (\dd c b)^k = i_*(\dd c i^* b)^k, \quad [Y] \wedge b(\dd c b)^k = i_*(i^* b(\dd c i^* b)^k).
\]

**Proof.** — First assume that \( b \) is smooth. Then
\[
\int_X [Y] \wedge (\dd c b)^k \wedge \xi = \int_Y (\dd c i^* b)^k \wedge i^* \xi = \int_X i_*(\dd c i^* b)^k \wedge \xi
\]
and similarly
\[
\int_X [Y] \wedge b(\dd c b)^k \wedge \xi = \int_X i_*(i^* b(\dd c i^* b)^k) \wedge \xi,
\]
so that (3.7) holds in this case. Now let $b$ be bounded and psh and let $b_j$ be a decreasing sequence of smooth psh functions converging pointwise to $b$. Now (3.7) follows from the smooth case and (3.1).

\[ \square \]

4. Higher Monge-Ampère products

Let $G$ be a psh function of the form (1.1). We will give meaning to

\begin{equation}
(dd^c G)^k
\end{equation}

by inductively defining it as $(dd^c G)^0 = 1$ and

\begin{equation}
(dd^c G)^k := dd^c(G1_{X \setminus Z}(dd^c G)^{k-1}), \quad k \geq 1.
\end{equation}

Proposition 4.1 below asserts that this definition makes sense and that $(dd^c G)^k$ are positive and closed. As pointed out in the introduction this definition coincides with the iterative definition (1.6) for $k \leq p$.

**Proposition 4.1.** — Let $X$ be a complex manifold of dimension $n$, let $f$ be a tuple of global functions of $X$, let $G$ be a psh function of the form (1.1), and let $G_j$ be a decreasing sequence of smooth psh functions in $X$ converging pointwise to $G$. Assume that (4.1) is inductively defined via (4.2) for a fixed $k$. Then

\[
G1_{X \setminus Z}(dd^c G)^k := \lim_j G_j 1_{X \setminus Z}(dd^c G)^k
\]

has locally finite mass and does not depend on the choice of sequence $G_j$. Moreover $(dd^c G)^{k+1} = dd^c(G1_{X \setminus Z}(dd^c G)^k)$ is positive and closed.

The proof below relies heavily on the fact that $G$ is of the form (1.1). It could be interesting to investigate whether Proposition 4.1 holds for a wider class of psh functions $G$ with unbounded locus $Z$.

**Proof.** — Let $\pi: \tilde{X} \to X$ be a smooth modification such that $\pi^* J$ is principal and its divisor is of the form

\begin{equation}
D = \sum \alpha_j D_j,
\end{equation}

where $D_j$ are smooth hypersurfaces with normal crossings. In particular, then $\pi^* f = f^0 f'$, where $f^0$ is a section of the line bundle $L_D$ that defines $D$ and $f'$ is a non-vanishing tuple of sections of $L_D^{-1}$.

Locally on $\tilde{X}$ we can choose a frame for $L_D$ and in this frame we have, cf. (1.1),

\begin{equation}
\pi^* G = \log |f^0| + \log |f'| + \pi^* h =: \log |f^0| + b.
\end{equation}
Since \( \log |f^0| \) is pluriharmonic outside

\[ |D| := \bigcup_j D_j \]

it follows that

\[ b = \log |f'| + \pi^*h \]

is psh there; furthermore it is locally bounded at \( |D| \). By a standard argument \( b \) has a unique (bounded) psh extension \( B \) across \( |D| \). Notice that \( dd^cB \) is a global positive closed current on \( \tilde{X} \) and

\[ dd^c\pi^*G = [D] + dd^cB. \]

Let \( G_j \) be a decreasing sequence of smooth psh functions converging pointwise to \( G \). Since

\[ dd^cG_j = \pi_* (dd^c\pi^*G_j) \to \pi_* (dd^c\pi^*G) = \pi_* ([D] + dd^cB) \]

it follows that

\[ dd^cG = \pi_* ([D] + dd^cB). \]

Let us now assume that we have proved Proposition 4.1 as well as the equality

\[ (dd^cG)^{\ell} = \pi_* ([D] \wedge (dd^cB)^{\ell-1} + (dd^cB)^{\ell}) \]

for \( \ell \leq k \). We are to see that then:

(i) \( G_1 \chi_{X \setminus Z} (dd^cG)^k := \lim_j G_j \chi_{X \setminus Z} (dd^cG)^k \) has locally finite mass.

(ii) If

\[ (dd^cG)^{k+1} := dd^c (G_1 \chi_{X \setminus Z} (dd^cG)^k), \]

then (4.5) holds for \( \ell = k + 1 \).

As soon as (i) and (ii) are verified, Proposition 4.1 follows.

Notice that if \( \mu \) is a closed positive current, then

\[ 1 \chi Z \pi_* \mu = \pi_* (1|_D| \mu). \]

In view of Corollary 3.3 we have that

\[ 1|_D| (dd^cB)^k = 0. \]

From the induction hypothesis (4.5), (4.6) and (4.7) we get

\[ 1 \chi_{X \setminus Z} (dd^cG)^k = \pi_* (dd^cB)^k. \]

By Proposition 3.2, \( (\pi^*G)(dd^cB)^k \) has locally finite mass, and

\[ (\pi^*G_j)(dd^cB)^k \to (\pi^*G)(dd^cB)^k \]
if $G_j$ is any decreasing sequence of psh functions that tends to $G$. If $G_j$ are smooth we have by (4.8) that

$$G_j 1_{X \setminus Z}(dd^c G)^k = \pi_*((\pi^* G_j)(dd^c B)^k),$$

which tends to

(4.9) $$G 1_{X \setminus Z}(dd^c G)^k = \pi_*((\pi^* G)(dd^c B)^k),$$

which has locally finite mass. Thus (i) is verified.

We now consider (ii). We claim that

(4.10) $$dd^c (\pi^* G \wedge (dd^c B)^k) = [D] \wedge (dd^c B)^k + (dd^c B)^{k+1}.$$

Recall that locally $\pi^* G = v + B$, where $v = \log |f^0|$ and $B$ is psh and bounded. Take smooth psh $v_j$ that decrease to $v$. Then $v_j + B$ are psh and decrease to $v + B$ and thus, by Proposition 3.2,

$$v_j (dd^c B)^k + B (dd^c B)^k = (v_j + B)(dd^c B)^k \rightarrow (v + B)(dd^c B)^k.$$

It follows that

$$(v + B)(dd^c B)^k = v(dd^c B)^k + B(dd^c B)^k.$$

From Proposition 3.2 we get that

$$dd^c (v(dd^c B)^k) = [D] \wedge (dd^c B)^k,$$

which proves the claim. In view of (4.9) and (4.10) the statement (ii) now follows.

For future reference we notice that

(4.11) $$M^J_k = \pi_*([D] \wedge (dd^c B)^{k-1}), \quad 1_{X \setminus Z}(dd^c G)^k = \pi_*(dd^c B)^k.$$

In fact $1_{X \setminus Z}(dd^c G)^k$ equals the non-pluripolar product $\langle dd^c G \rangle^k$ as defined in [6, 8].

It follows from the proof above and Proposition 3.2 that if $G_j$ is any decreasing sequence of psh functions converging pointwise to $G$, then $G_j 1_{X \setminus Z}(dd^c G)^{k-1} \rightarrow G 1_{X \setminus Z}(dd^c G)^{k-1}$ and

$$dd^c (G_j \wedge 1_{X \setminus Z}(dd^c G)^{k-1}) = dd^c G_j \wedge 1_{X \setminus Z}(dd^c G)^{k-1} \rightarrow (dd^c G)^k.$$

Recall that if $G_j$ are psh functions that decrease to $G$, then

$$\lim_j (dd^c G_j)^k = (dd^c G)^k, \quad k \leqslant p,$$

see, e.g., [9, Proposition III.4.9]. However, for $k > p$ one cannot hope for a definition of $(dd^c G)^k$ that is robust in this sense. In fact, even if $G_j$ and $\tilde{G}_j$ are sequences of smooth psh functions decreasing to $G$ and $(dd^c G_j)^k$
and \((dd^c \tilde{G}_j)^k\) converge to positive closed currents \(T\) and \(\tilde{T}\), respectively, \(T\) might be different from \(\tilde{T}\), as is illustrated by the following example.

**Example 4.2.** — Let \(\varphi = (w, zw)\). Then
\[
\begin{align*}
dd^c \log |\varphi| &= \dd^c \log |w| + \dd^c \log(1 + |z|^2)^{1/2} = [w = 0] + \dd^c \alpha,
\end{align*}
\]
where \([w = 0]\) denotes the current of integration along \(\{w = 0\}\) and \(\alpha = \log(1 + |z|^2)^{1/2}\). Thus by (4.2),
\[
(dd^c \log |\varphi|)^2 = [w = 0] \wedge \dd^c \alpha.
\]

Let \(G_\epsilon = \log(|\varphi|^2 + \epsilon)^{1/2}\) and \(\tilde{G}_\epsilon = \log(|w|^2 + \epsilon)^{1/2} + \alpha\). Then \(G_\epsilon\) and \(\tilde{G}_\epsilon\) are smooth psh functions that decrease towards \(\log |\varphi|\) as \(\epsilon\) tends to 0. On the one hand, by (1.7),
\[
\lim_{\epsilon \to 0} (dd^c G_\epsilon)^2 = (dd^c \log |\varphi|)^2.
\]
On the other hand, again using (1.7), but now for \((dd^c \log |w|)^2\),
\[
(dd^c \tilde{G}_\epsilon)^2 = (dd^c \log(|w|^2 + \epsilon)^{1/2})^2 + 2dd^c \log(|w|^2 + \epsilon)^{1/2} \wedge dd^c \alpha
\]
\[
\to 2[w = 0] \wedge dd^c \alpha.
\]

**Remark 4.3.** — Assume that \(X_\ell\) is an exhaustion of \(X\) by relatively compact subsets such that the restriction \(J_\ell\) of \(J\) to \(X_\ell\) is generated by global bounded functions. It would be interesting to know whether, or under what assumptions, the currents \(M^k_{J_\ell}\) then converge. Convergence would give us a global canonical representation of the Segre numbers of \(J\).

Assume that \(J\) is indeed generated by global bounded functions and let \(G_\ell\) denote the Green function with poles along \(J_\ell\). Then, arguing as in the proof of Proposition 4.1 and using the notation from that proof,
\[
\pi^* G_\ell = \log |f^0| + B_\ell,
\]
where \(B_\ell\) is psh and bounded, and moreover
\[
(dd^c G_\ell)^k = \pi_* ([D] \wedge (dd^c B_\ell)^{k-1} + (dd^c B_\ell)^k).
\]
Assume that \(G_\ell\) decrease towards \(G\). Then \(B_\ell\) decrease towards \(B\), as defined in (4.4), and thus \(\lim_{\ell} (dd^c G_\ell)^k = (dd^c G)^k\) in light of (3.1) and (4.5).

## 5. Lelong numbers

Let \(T\) be a positive closed \((k, k)\)-current. If \(k = n\), following [4, Section 5], we let
\[
M_0^x \wedge T := 1_{\{x\}} T.
\]
Otherwise
\[ M_{n-k}^\xi \wedge T := 1_{\{x\}}((dd^c \log |\xi|)^{n-k} \wedge T); \]
here we inductively define
\[ (dd^c \log |\xi|)^{\ell} \wedge T := \lim_{j} dd^c \left( v_j \wedge (dd^c \log |\xi|)^{\ell-1} \wedge T \right), \]
where \( v_j \) is a decreasing sequence of smooth psh functions converging pointwise to \( \log |\xi| \). Because of the dimension principle it is not necessary to insert \( 1_X \setminus \{x\} \) in this definition, cf., Section 4. See Remark 5.1 below for another possible definition of \( M_{n-k}^\xi \wedge T \). Clearly \( M_{n-k}^\xi \wedge T \) is an \((n,n)\)-current with support at \( x \), and it is in fact equal to \( \alpha[x] \), where \( \alpha \) is the Lelong number of \( T \) at \( x \), see, e.g., [4, Lemma 2.1].

Remark 5.1. — As is pointed out in [4, Section 5] one can define \( M^\xi \wedge T \) as the value at \( \lambda = 0 \) of the current-valued analytic function
\[ \lambda \mapsto \frac{\bar{\partial}|\xi|^{2\lambda} \wedge \partial|\xi|^2}{2\pi i|\xi|^2} \wedge (dd^c \log |\xi|)^{n-k-1} \wedge T. \]

6. Proof of Theorem 1.1

We will prove the slightly more general formulation of Theorem 1.1 stated in Remark 1.2, i.e., we let \( G \) be any psh function of the form (1.1).

We still assume that \( \pi: \tilde{X} \to X \) is a smooth modification and use the notation from the proof of Proposition 4.1. Notice that \( L_D \) has a Hermitian metric such that \( |f^0|_{L_D} = |\pi^* f| \). By the Poincaré-Lelong formula,
\[ dd^c \log |\pi^* f| = [D] + \omega_f, \]
where \( \omega_f \) is the first Chern form for \( L_D^{-1} \).

Let us fix a local holomorphic frame so that \( \log |f'| \) is a well-defined function as above. Since
\[ \log |\pi^* f| = \log |f^0| + \log |f'|, \]
from (6.1) we have that
\[ \omega_f = dd^c \log |f'|. \]

Let \( b \) be the psh bounded function outside \( |D| \) defined in (4.4). If we choose another local frame for \( L_D \), then \( \log |f'| \) is changed to \( \log |f'| + \alpha \) where \( \alpha \) is pluriharmonic, and \( b \) is thus changed to \( \tilde{b} := b + \alpha \). Moreover \( \tilde{B} := B + \alpha \)
is the unique psh extension of $\tilde{b}$ across $|D|$, cf. the proof of Proposition 4.1. It follows that $A$, locally defined as

$$A := B - \log |f'|,$$

is a global upper semicontinuous extension of $\pi^*h$ across $|D|$. Notice also that $A(ddcB)^\ell$ is well-defined on $\tilde{\mathcal{X}}$ and, in light of (6.2) and (6.3), that

$$(ddcB)^{k-1} - \omega_f^{k-1} = ddc \left( A \sum_{\ell=0}^{k-2} (ddcB)^\ell \land \omega_f^{k-2-\ell} \right).$$

Assume now that $Y \subset \tilde{\mathcal{X}}$ is a smooth submanifold and that $i: Y \to \tilde{\mathcal{X}}$ is the natural inclusion. Then $i^*B$ is psh and bounded, $i^* \log |f'|$ is smooth, and, in the same way as above, $i^*A$ is a global upper semi-continuous function on $Y$ and

$$(ddc^iB)^{k-1} - i^*\omega_f^{k-1} = ddc \left( i^* A \sum_{\ell=0}^{k-2} (ddc^iB)^\ell \land i^* \omega_f^{k-2-\ell} \right).$$

In view of Lemma 3.4, (6.4) implies that

$$[Y] \land \left( (ddcB)^{k-1} - \omega_f^{k-1} \right) = ddc^i \left( i^* A \sum_{\ell=0}^{k-2} (ddc^iB)^\ell \land i^* \omega_f^{k-2-\ell} \right).$$

The currents $(ddc \log |f|)^k$ and $M^f_k$ are defined in a completely analogous way as $(ddcG)^k$ and $M^G_k$, just replacing $G$ by $\log |f|$, cf., the introduction and the end of Section 2 and also [4]. Arguing as in the proof of Proposition 4.1, we get, cf., (4.11), that

$$M^f_k = \pi_*([D] \land \omega_f^{k-1}), \quad 1_{X \backslash Z}(ddc \log |f|)^k = \pi_* \omega_f^k.$$

**Lemma 6.1.** — The currents $M^G_k$ and $M^f_k$ have the same Lelong number at each point $x \in X$. Moreover, the currents $1_{X \backslash Z}(ddcG)^k$ and $1_{X \backslash Z}(ddc \log |f|)^k$ have the same Lelong number at each point $x \in X$.

**Proof.** — Let us fix a point $x \in X$ and let $\xi$ be a tuple of functions that defines the maximal ideal $m_x$ at $x$. We can choose the modification $\pi: \tilde{\mathcal{X}} \to X$ so that also $\pi^*m_x$ is principal, i.e., $\pi^*\xi = \xi^0 \xi'$, where $\xi^0$ is a section of a line bundle $L_E$ that defines the exceptional divisor $E$, and $\xi'$ is a non-vanishing tuple of sections of $L_E^{-1}$. Let us assume that

$$E = \sum_{\kappa} \beta_{\kappa} E_{\kappa},$$

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where $E_\kappa$ are irreducible with simple normal crossings and $\beta_\kappa$ are integers. We may also assume that, for each $j$, cf., (4.3), either $D_j \subset |E|$ or all $E_\kappa$ intersect $D_j$ properly and that

$$E^{D_j}_\kappa := E_\kappa \cap D_j$$

are smooth. Let $\omega_\xi$ be the first Chern form of $L^{-1}_{E^1}$ with respect to the metric induced by $\xi$, so that

$$\omega_\xi = dd^c \log |\xi'|,$$

cf., (6.2), and

$$dd^c \log |\pi^* \xi| = [E] + \omega_\xi.$$  

Let $i_j: D_j \to \tilde{X}$ be the injection of $D_j$ as a submanifold of $\tilde{X}$. It follows from (4.3), (4.11) and Lemma 3.4 that

$$M^J_k = \sum_j \alpha_j \pi_*(i_j)^*((dd^c i_j)^*B)^{k-1}).$$

In order to prove the first part of the lemma, it is enough to consider one single term in (6.6) and verify that

$$T^J_k := \pi_* i_*((dd^c i^* B)^{k-1})$$

and

$$T^f_k := \pi_* i_* (i^* \omega_j^{k-1})$$

have the same Lelong numbers, where we write $D = D_j$ and $i = i_j$ for simplicity.

Let us first assume that $k = n$. If $D \subset |E|$, then $T^J_n$ and $T^f_n$ both have support at $x$. In view of (6.4), with $Y = D$, we have that

$$T^J_k - T^f_k = dd^c \pi_* i_* (i^* A \sum_{\ell=1}^{k-2} (dd^c i^* B)^\ell \wedge i^* \omega_j^{k-2-\ell}) =: dW,$$

where $W$ has support at $x$. By Stokes’ theorem thus

$$\int (T^J_n - T^f_n) = \int dW = 0,$$

which means that $T^J_n$ and $T^f_n$ have the same Lelong number at $x$. If $D$ is not contained in $E$, then $i^{-1}E$ has positive codimension in $D$ and therefore,

$$1_{\{x\}} T^J_n = \pi_* i_* (1_{|i^{-1}E|}(dd^c i^* B)^{n-1}) = 0$$

by Corollary 3.3. In the same way we see that $1_{\{x\}} T^f_n = 0$.

Let us now assume that $k < n$. If $D \subset |E|$, then $T^J_k$ and $T^f_k$ are positive closed $(k, k)$-currents with support at $x$, so by the dimension principle they both vanish. We can therefore assume that $i^* \pi^* \xi$ does not vanish identically.
on $D$; by assumption it then defines a smooth divisor $E^D$ on $D$. Locally on $D$,

$$\log |i^*\pi^*\xi| = \log |i^*\xi^0| + \log |i^*\xi'|,$$

and thus

(6.7) \hspace{1cm} dd^c \log |i^*\pi^*\xi| = [E^D] + i^*\omega_\xi,

where $[E^D]$ is the Lelong current on $D$ associated to $E^D$. If $v_j$ are as in Section 5, then

$$dd^c i^*\pi^*v_j \to [E^D] + i^*\omega_\xi.$$

Now

$$dd^c(v_jT^J_k) = \pi_* i_*(dd^c i^*\pi^*v_j \wedge (dd^c i^*B)^{k-1})$$

so that

$$dd^c \log |\xi| \wedge T^J_k = \pi_* i_*(([E^D] + i^*\omega_\xi) \wedge (dd^c i^*B)^{k-1})$$

by Proposition 3.2 and (6.7). Moreover, since $\pi_* i_*(([E^D] \wedge (dd^c i^*B)^{k-1})$ has support at $x$, by the dimension principle,

$$dd^c \log |\xi| \wedge T^J_k = \pi_* i_*(i^*\omega_\xi \wedge (dd^c i^*B)^{k-1}).$$

By induction we get

$$(dd^c \log |\xi|)^{n-k} \wedge T^J_k = \pi_* i_*/ [E^D] + i^*\omega_\xi) \wedge i^*\omega_\xi^{n-k-1} \wedge (dd^c i^*B)^{k-1}).$$

Therefore, by Corollary 3.3,

$$M^\xi_{n-k} \wedge T^J_k = 1_{\{x\}}(dd^c \log |\xi|)^{n-k} \wedge T^J_k$$

$$= \pi_* i_*/ [E^D] \wedge i^*\omega_\xi^{n-k-1} \wedge (dd^c i^*B)^{k-1}).$$

Let $\iota_\kappa : E^D_\kappa \to D$ be the natural injection. By (6.5) and Lemma 3.4 we have that

$$M^\xi_{n-k} \wedge T^J_k = \sum_\kappa \beta_\kappa \pi_* i_*(\iota_\kappa)_* ([E^D] + i^*\omega_\xi) \wedge i^*\omega_\xi^{n-k-1} \wedge (dd^c i^*B)^{k-1}).$$

By analogous arguments,

$$M^\xi_{n-k} \wedge T^f_k = \sum_\kappa \beta_\kappa \pi_* i_*(\iota_\kappa)_* ([E^D] + i^*\omega_\xi) \wedge i^*\omega_\xi f^{n-k-1} \wedge (dd^c i^*B)^{k-1}).$$

For simplicity in notation let us assume that $E^D$ has just one irreducible component and let $\iota : E^D \to D$ be the natural injection. By (6.4) applied to $E^D$ we have that

$$M^\xi_{n-k} \wedge T^J_k - M^\xi_{n-k} \wedge T^f_k =$$

$$dd^c \pi_* i_*(\iota^* i^* A \iota^* i^* \omega_\xi^{n-k-1} \wedge \sum_{\ell=0}^{k-2} (dd^c \iota^* B)^{\ell} \iota^* \omega_f^{k-1-\ell}) =: dW,$$

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where $W$ has support at $x$. It follows by Stokes’ theorem that the integral of this current is zero, and thus the Lelong numbers at $x$ of $T^j_k$ and $T^f_k$ coincide. Thus the first part of the lemma is proved.

By analogous arguments we get that $\pi_*(dd^c B)^k$ and $\pi_*(\omega_f)^k$ have the same Lelong number at $x$, which proves the second part of the lemma, cf. (4.11) and (6.5).

We can now conclude the proof of Theorem 1.1.

**Proof of Theorem 1.1.** — Let $D^j_\ell$ be the irreducible components of $D$ such that $\pi(D^j_\ell)$ have codimension $\ell$. Then

$$M^J_k = \pi_*(\pi(D) \wedge (dd^c B)^{k-1}) = \pi_*\left(\sum_{\ell \leq k} \sum_j ([D^j_\ell] \wedge (dd^c B)^{k-1})\right)$$

since terms with $\ell > k$ vanish because of the dimension principle. We claim that

$$M^J_k = \pi_*\left(\sum_j ([D^j_k] \wedge (dd^c B)^{k-1}) + \sum_{\ell < k} \sum_j ([D^j_\ell] \wedge (dd^c B)^{k-1})\right)$$

$$=: S^J_k + N^J_k$$

is the Siu decomposition of $M^J_k$. First notice that since $\pi_*(\pi(D^j_k) \wedge (dd^c B)^{k-1})$ is a $(k,k)$-current with support on the set $Z := \pi(D^j_k)$ of codimension $k$ it must be of the form $\alpha[Z]$ where $\alpha$ is a constant, see Lemma 3.1.

It is now enough to see that if $W$ is a subvariety of codimension $k$, then $1_W N^J_k = 0$, i.e.,

$$1_W \pi_*(\pi(D^j_k) \wedge (dd^c B)^{k-1}) = 0$$

if $\ell < k$. Let $i: D^j_\ell \to \tilde{X}$ be the natural injection. By Lemma 3.4 we have

$$1_W \pi_*(\pi(D^j_k) \wedge (dd^c B)^{k-1}) = 1_W (\pi_* i_* (dd^c i^* B)^{k-1})$$

$$= \pi_* i_* (1_{(\pi \circ i)^{-1}(W)} (dd^c i^* B)^{k-1}).$$

Notice that since $\pi(D^j_k)$ is irreducible and not contained in $W$ it follows that $\pi^{-1}(W) \cap D^j_k$ has positive codimension in $D^j_\ell$, and hence

$$1_{(\pi \circ i)^{-1}(W)} (dd^c i^* B)^{k-1} = 0$$

in view of Corollary 3.3.

Thus (6.8) is the Siu decomposition. Since $M^J_k$ and $M^f_k$ have the same Lelong number at each point by Lemma 6.1 and the set where $N^J_k$ and $N^f_k$ have positive Lelong number have codimension $> k$ we conclude that $S^J_k = S^f_k$, see Remark 2.1. Since also $1_X \wedge Z (dd^c G)^k$ and $1_X \wedge Z (dd^c \log |f|)^k$
have the same Lelong numbers at \( x \) by Lemma 6.1, Theorem 1.1 follows from the analogous result, Theorem 1.1, for \( M^f \) in [4]. □

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