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ON THE GEOMETRY OF POLYNOMIAL MAPPINGS AT INFINITY

by Anna VALETTE & Guillaume VALETTE (*)

ABSTRACT. — We associate to a given polynomial map from $\mathbb{C}^2$ to itself with nonvanishing Jacobian a variety whose homology or intersection homology describes the geometry of singularities at infinity of this map.

RÉSUMÉ. — On associe à une application polynomiale de $\mathbb{C}^2$ dans lui-même à Jacobien constant non nul, une variété dont l'homologie ou l'homologie d'intersection décrit la géométrie à l'infini de cette application.

0. Introduction

In the study of geometrical or topological properties of polynomial mappings, the set of points at which those maps fail to be proper plays an important role. As an example, recall the famous Jacobian conjecture, formulated in 1939 by O. H. Keller [8] and asserting that any polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with nowhere vanishing Jacobian is a polynomial automorphism. The problem remains open today. We call the smallest set A such that the map $F : \mathbb{C}^n \setminus F^{-1}(A) \to \mathbb{C}^n \setminus A$ is proper, the asymptotic set of F. The Jacobian conjecture reduces to show that the asymptotic set of a complex polynomial mapping with nonzero constant Jacobian is empty. It is therefore natural to study the topology of the asymptotic set of polynomial maps. In the 90's of the previous century Z. Jelonek studied the properties of this set and obtained very important results. We briefly recall some of them in the next section. The starting point of his study is the simple observation that the asymptotic set of a given map $F : \mathbb{C}^n \to \mathbb{C}^n$ is the image

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under canonical projection of the $\overline{\text{graph}(F)} \setminus \text{graph}(F)$, where the closure is taken in the $\mathbb{P}^n \times \mathbb{C}^n$. This gives many important consequences. But if one wishes to keep the assumption of constancy of the Jacobian, compactifications have to be avoided. For instance a compactification drastically affects the volume of subsets of $\mathbb{C}^n$. If the Jacobian is constant then the mapping F preserves the volume locally. Thus, most of the information is lost in the compactification.

Our aim is to give a new approach of the Jacobian conjecture. In general, singularities of mappings are much more difficult to handle than singularities of sets. Our theorem reduces the study of a mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ to the study of a singular semi-algebraic set. We construct a pseudomanifold N_F associated to a given polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$. In the case $n = 2$, we will then prove that the map F with non-vanishing Jacobian is not proper iff the homology or the intersection homology of N_F is nontrivial.

Our approach is of metric nature. The significant advantage of a metric approach is to be able to investigate the singularities, without looking at singularities themselves, but just at the regular points nearby. For instance, it was proved in [11] that the L^∞ cohomology of the regular locus of a given subanalytic pseudomanifold (which may be proved to be a subanalytic bi-Lipschitz invariant of the regular locus) is a topological invariant of the pseudomanifold. Hence, being able to work beside the singularities could prove to be useful to investigate singularities at infinity (where we hardly can go). In this paper we shall relate the L^∞ cohomology of the regular locus of the constructed pseudomanifold N_F to the behavior of F at infinity (Corollary 3.4). This result relies on the de Rham theorem proved in [11], which yields an isomorphism between L^∞ cohomology and intersection cohomology in the maximal perversity.

The idea is to transfer the problem to differential geometry so as to reformulate the problem in terms of differential equations later in order to make it possible to take advantage of the fact that the Jacobian is not only nonzero but constant.

It seems indeed more useful to find intersection homology than homology. The reason is that we are also able to prove that the set N_F is a stratified pseudomanifold with only even codimensional strata. The main feature of intersection homology is that M. Goresky and R. MacPherson were able to show in their fundamental paper that their theory satisfies Poincaré duality for stratified pseudomanifolds and this duality is particularly nice in the case of the middle perversity if the considered pseudomanifold can be stratified by only even codimensional strata. The case of the maximal perversity is also of importance since, as we said above, it makes it possible to relate the geometry of F at infinity to the L^∞ cohomology of N_F. Our main theorem (Theorem 3.2) relates the intersection homology of N_F to the behavior of F at infinity.

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1. Preliminaries and basic definitions

In this section we set-up our framework. We begin by recalling the concept of intersection homology and then define the L^∞ cohomology.

1.1. Notations and conventions.

Given a semialgebraic set X, the singular simplices of X will be the semi-algebraic continuous mappings $\sigma : T_i \to X$, where T_i is the standard simplex of $\mathbb{R}^i$. We denote by $C_i(X)$ the group of i-dimensional singular chains with coefficients in $\mathbb{R}$; if c is an element of $C_i(X)$, we denote by $|c|$ its support. By $Reg(X)$ and $Sing(X)$ we denote respectively the regular and singular locus of the set X. Given $A \subset \mathbb{R}^n$, $\overline{A}$ will stand for the topological closure of A. Given a point $x \in \mathbb{R}^n$ and $\alpha > 0$, we write $S(x, \alpha)$ for the sphere of radius α centered at x and $B(x, \alpha)$ for the corresponding ball.

1.2. Intersection homology.

We briefly recall the definition of intersection homology; for details, we refer to the fundamental work of Goresky and MacPherson [2]. This requires to first introduce the notion of stratification.

Definition 1.1. — *Let X be a semi-algebraic set of dimension m. A* **semi-algebraic stratification of** X *is a finite semi-algebraic filtration*

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_m = X,$$

such that for every i, $X_i \setminus X_{i-1}$ is either empty or a topological manifold of dimension i. A connected component of $X_i \setminus X_{i-1}$ is called **a stratum** *of X.*

We denote by cL the open cone on the space L, $c\emptyset$ being a point. Observe that if L is a stratified set then cL is stratified by the cones over the strata. We now define inductively on the dimension the topologically trivial stratifications. For $m = 0$, every stratification is topologically trivial.

A stratification of X is said to be **locally topologically trivial** *if for every $x \in X_i \setminus X_{i-1}$, $i \geqslant 0$, there is a semi-algebraic homeomorphism*

$$h : U_x \to (0;1)^i \times cL,$$

with U_x neighborhood of x in X and $L \subset X$ compact set having a locally topologically trivial semi-algebraic stratification such that h maps the strata of U_x (induced stratification) onto the strata of $(0;1)^i \times cL$ (product stratification).

Definitions 1.2. — *A* **pseudomanifold** *is a semi-algebraic subset $X \subset \mathbb{R}^n$ whose singular locus is of codimension at least 2 in X and whose regular locus is dense in X.*

A **stratified pseudomanifold** *(of dimension m) is the data of an m-dimensional pseudomanifold X together with a semi-algebraic filtration:*

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_m = X,$$

with $X_{m-1} = X_{m-2}$, which constitutes a locally topologically trivial stratification of X.

Definition 1.3. — *A* **stratified pseudomanifold with boundary** *is a semi-algebraic couple $(X, \partial X)$ together with a semi-algebraic filtration*

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{m-2} = X_{m-1} \subset X_m = X,$$

such that:

(1) *$X \setminus \partial X$ is an m-dimensional stratified pseudomanifold (with the filtration $X_j \setminus \partial X$),*
(2) *∂X is a stratified pseudomanifold (with the filtration $X'_j := X_{j+1} \cap \partial X$),*
(3) *∂X has a* **stratified collared neighborhood***: there exist a neighborhood U of ∂X in X and a semi-algebraic homeomorphism $h : \partial X \times [0,1] \to U$ such that $h(X'_{j-1} \times [0,1]) = U \cap X_j$ and $h(\partial X \times \{0\}) = \partial X$.*

Definition 1.4. — *A* **perversity** *is a $(m-1)$-uple of integers $\bar{p} = (p_2, p_3, \ldots, p_m)$ such that $p_2 = 0$ and $p_{k+1} \in \{p_k, p_k + 1\}$.*

Traditionally we denote the zero perversity by $\overline{0} = (0, \dots, 0)$, *the maximal perversity by* $\overline{t} = (0, 1, \dots, m-2)$, *and the middle perversities by* $\overline{m} = (0, 0, 1, 1, \dots, [\frac{m-2}{2}])$ *(lower middle) and* $\overline{n} = (0, 1, 1, 2, 2, \dots, [\frac{m-1}{2}])$ *(upper middle). We say that the perversities* $\overline{p}$ *and* $\overline{q}$ *are* **complementary** *if* $\overline{p} + \overline{q} = \overline{t}$.

Given a stratified pseudomanifold X, *we say that a semi-algebraic subset* $Y \subset X$ *is* $(\bar{p}, i)$**-allowable** *if* $\dim(Y \cap X_{m-k}) \leqslant i - k + p_k$ *for all* $k \geqslant 2$.

In particular, a subset $Y \subset X$ *is* $(\overline{t}, i)$*-allowable if* $\dim(Y \cap Sing(X)) < i - 1$.

Define $IC_i^{\overline{p}}(X)$ *to be the* $\mathbb{R}$*-vector subspace of* $C_i(X)$ *consisting of those chains* ξ *such that* $|\xi|$ *is* $(\overline{p}, i)$*-allowable and* $|\partial \xi|$ *is* $(\overline{p}, i-1)$*-allowable.*

Definition 1.5. — *The* i^{th} *intersection homology group of perversity* $\overline{p}$, *denoted by* $IH_i^{\overline{p}}(X)$, *is the* i^{th} *homology group of the chain complex* $IC_\bullet^{\overline{p}}(X)$.

Goresky and MacPherson proved that these groups are independent of the choice of the stratification and are finitely generated [2, 3].

Recall also the remarkable

Theorem 1.6 (Goresky, MacPherson [2]). — *For any orientable compact stratified semi-algebraic pseudomanifold* X, *generalized Poincaré duality holds:*

$$IH_k^{\overline{p}}(X) \simeq IH_{m-k}^{\overline{q}}(X), \tag{1.1}$$

where $\overline{p}$ *and* $\overline{q}$ *are complementary perversities.*

In the non-compact case the above isomorphism holds for Borel-Moore homology:

$$IH_k^{\overline{p}}(X) \simeq IH_{m-k,BM}^{\overline{q}}(X), \tag{1.2}$$

where $IH_{\bullet,BM}$ denotes the intersection homology with respect to Borel-Moore chains [3, 9]. A relative version is also true in the case where X has boundary.

1.3. L^∞ cohomology

Let $M \subset \mathbb{R}^n$ be a smooth submanifold.

Definition 1.7. — *We say that a differential form* ω *on* M *is* L^∞ *if there exists a constant* C *such that for any* $x \in M$:

$$|\omega(x)| \leqslant C.$$

We denote by $\Omega^j_\infty(M)$ *the cochain complex constituted by all the* j*-forms* ω *such that* ω *and* $d\omega$ *are both* L^∞*.*

The cohomology groups of this cochain complex are called the **L^∞-cohomology groups of** M *and will be denoted by* $H^\bullet_\infty(M)$*.*

Recently the second author showed that the L^∞ cohomology of a pseudomanifold coincides with its intersection cohomology in the maximal perversity ([11], Theorem 1.2.2):

Theorem 1.8. — *Let X be a compact subanalytic pseudomanifold (possibly with boundary). Then, for any j:*

$$H^j_\infty(Reg(X)) \simeq IH^{\bar{t}}_j(X).$$

Furthermore, the isomorphism is induced by the natural map provided by integration on allowable simplices.

The theorem presented in the latter paper was devoted to pseudomanifolds without boundary but actually still applies when the boundary is nonempty (as mentioned in the introduction of the cited article).

2. The variety N_F

We will consider polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ as real ones $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. By $Sing(F)$ we mean the singular locus of F, which is the zero set of its Jacobian determinant and we denote by $K_0(F)$ the set of critical values of F, i.e. the set $F(Sing(F))$.

We denote by ρ the Euclidean Riemannian metric of $\mathbb{R}^{2n}$. We can pull it back in a natural way:

$$F^*\rho_x(u, v) := \rho(d_xF(u), d_xF(v)).$$

This metric is non degenerate outside the singular locus of F.

Define the Riemannian manifold $M_F := (\mathbb{C}^n \setminus Sing(F), F^*\rho)$ and observe that the map F induces a local isometry nearby any point of M_F.

2.1. The Jelonek set.

For a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^m$, we denote by J_F the set of points at which the map F is not proper, i.e.

$$J_F = \{y \in \mathbb{C}^m \ \textit{such that}\ \exists \{x_k\} \subset \mathbb{C}^n,\ |x_k| \to \infty,\ F(x_k) \to y\},$$

and call it the **asymptotic variety** or the **Jelonek set** of F. The geometry of this set was studied by Jelonek in series of papers [5, 6, 7]. Jelonek obtained a nice description of this set and gave an upper bound for the degree of this set. For the details and applications of these results we refer to the works of Jelonek. In our paper, we will need the following powerful theorem.

THEOREM 2.1 (Jelonek [6]). — *If $F : \mathbb{C}^n \to \mathbb{C}^m$ is a generically finite polynomial map then J_F is either an $(n-1)$ pure dimensional $\mathbb{C}$-unirulled algebraic variety or the empty set.*

Here, by $\mathbb{C}$-unirulled variety we mean that through any point of this variety passes a rational curve included in this variety (X is $\mathbb{C}$-unirulled if for all $x \in X$ there exists a non-constant polynomial map $\varphi_x : \mathbb{C} \to X$ such that $\varphi_x(0) = x$).

In the real case, the Jelonek set is an $\mathbb{R}$-unirulled semi-algebraic set but there is not such a severe restriction on its dimension: it can be any number between 1 and $(n-1)$ (if J_F is nonempty) [7].

2.2. Construction of an embedding

The first important step is to embed the manifold M_F into an affine space. To do this, we can make use of F. The only problem is that F is not necessarily one-to-one but just locally one-to-one. We begin by proving the following lemma. We will implicitly assume below that F is generically finite since otherwise M_F is reduced to the empty set and all the results of this section are clear.

LEMMA 2.2. — *There exists a finite covering of M_F by open semi-algebraic subsets such that on every element of this covering, the mapping F induces a diffeomorphism onto its image.*

Proof. — Let $\Gamma_F \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ be the set constituted by the elements (y, x) in $\mathbb{R}^{2n} \times M_F$ such that $y = F(x)$. Consider a Nash cell decomposition $\mathcal{C}$ of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ compatible with the set Γ_F (see [1]). Let π_1 and π_2 be the mappings respectively defined by $\pi_1(y, x) := y$ and $\pi_2(y, x) := x$, if $(y, x) \in \Gamma_F$. The mapping π_2 is a homeomorphism onto its image and the mapping π_1 is locally invertible near every point of Γ_F. Moreover, by definition of cells, if the restriction of π_1 to a cell C of $\mathcal{C}$ which is included in Γ_F is not injective then it cannot be finite-to-one. But, as F is finite-to-one, so is π_1. Consequently, π_1 induces a one-to-one map on every cell of $\mathcal{C}$ which is a subset of Γ_F.

Now, fix a cell $C \in \mathcal{C}$ included in Γ_F. Since C is a Nash manifold, it admits a tubular neighborhood U_C (see [1]). There exists a strong deformation of U_C onto C. Hence, as the restriction of π_1 to C is one-to-one and since the mapping π_1 is locally invertible near every point, the mapping π_1 must be injective on the whole of U_C (since U_C retracts by deformation on C). It means that the mapping F induces an injective map on V_C, if $V_C := \pi_2(U_C)$. As the family constituted by the U_C, $C \in \mathcal{C}$, $C \subset \Gamma_F$, covers Γ_F, the subsets V_C cover the whole of M_F. On every V_C, F is injective and, since it is a local diffeomorphism at every point of M_F, it induces a diffeomorphism onto its image. □

The next proposition will enable us to transfer the geometry at infinity of a given polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ to the constructed set. Namely, the intersection homology of the set N_F provided by the following proposition determines the geometry of F at infinity as we will see in the main theorem.

Proposition 2.3. — *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. There exists a real semi-algebraic pseudomanifold $N_F \subset \mathbb{R}^\nu$, for some $\nu \geqslant 2n$, such that*

$$Sing(N_F) \subset (J_F \cup K_0(F)) \times \{0\} \tag{2.1}$$

and there exists a semi-algebraic bi-Lipschitz map:

$$h_F : M_F \to Reg(N_F),$$

where N_F is equipped with the Riemannian metric induced by $\mathbb{R}^\nu$.

Proof. — Let $U_1, \dots, U_p$ be the open sets provided by Lemma 2.2. We may find some semi-algebraic closed subsets of M_F, $V_i \subset U_i$, which cover M_F as well.

Thanks to Mostowski's separation theorem [10], see also Lemma 8.8.8 in [1], there exist Nash functions $\psi_i : M_F \to \mathbb{R}$, $i = 1, \dots, p$, such that for each i, ψ_i is positive on V_i and negative on $M_F \setminus U_i$. Define:

$$h_F := (F, \psi_1, \dots, \psi_p) \text{ and } N_F := \overline{h_F(M_F)}. \tag{2.2}$$

We first check that the mapping h_F is injective on M_F. Take $x \neq x'$ with $h_F(x) = h_F(x')$, and let i be such that $x \in V_i$. As $F(x') = F(x)$, the injectivity of F on U_i entails that $x' \notin U_i$. But this means that $\psi_i(x) > 0$ and $\psi_i(x') < 0$. A contradiction.

We claim that if the functions ψ_i are chosen sufficiently small then the mapping h_F is bi-Lipschitz. To show this, we will use the famous Łojasiewicz inequality as well as the following less usual form of this inequality:

Proposition 2.4. — *Let $A \subset \mathbb{R}^n$ be a closed semi-algebraic set and $f : A \to \mathbb{R}$ a continuous semi-algebraic function. There exist $c \in \mathbb{R}$ and $p \in \mathbb{N}$ such that for any $x \in A$ we have*

$$|f(x)| \leqslant c(1+|x|^2)^p.$$

For the proof see [1], Proposition 2.6.2.

Fix an open set U_l. Define, for $y \in F(U_l)$, the following functions:

$$\tilde{\psi}_i(y) := \psi_i \circ (F_{|U_l})^{-1}(y), \tag{2.3}$$

for $i = 1, \dots, p$, and

$$\hat{\psi}(y) := (y, \tilde{\psi}_1(y), \dots, \tilde{\psi}_p(y)). \tag{2.4}$$

We then have the formula for $x \in U_l$:

$$h_F(x) = (F(x), \tilde{\psi}_1(F(x)), \dots, \tilde{\psi}_p(F(x))) = \hat{\psi}(F(x)). \tag{2.5}$$

As the map $F : (U_l, F^*\rho) \to F(U_l)$ is bi-Lipschitz, it is enough to show that $\hat{\psi} : F(U_l) \to \mathbb{R}^{2n+p}$ is bi-Lipschitz. This amounts to prove that $\tilde{\psi}_i$ has bounded derivatives for any $i = 1, \dots, p$. By Łojasiewicz inequality (see Proposition 2.4), there exist a positive constant ε and integers μ and N such that

$$\inf_{u \in S^{2n-1}} |d_x F(u)| \geqslant \varepsilon \frac{jac(F)(x)^{2\mu}}{(1+|x|^2)^N},$$

for any $x \in M_F$ (here $jac(F)$ stands for the Jacobian determinant of F). Since for $y = F(x)$ we have $\sup_{u \in S^{2n-1}} |d_y F^{-1}(u)| = 1/\inf_{u \in S^{2n-1}} |d_x F(u)|$, this implies that we have:

$$|d_y F^{-1}| \leqslant C(1+|x|^2)^N \cdot jac(F)(x)^{-2\mu},$$

where $C = \frac{1}{\varepsilon}$. By Łojasiewicz inequality, possibly multiplying the ψ_i's by a huge power of $jac(F)^2$, we may assume that they extend to C^1 functions on $\mathbb{R}^{2n}$. Hence, by Proposition 2.4, the ψ_i's satisfy for some constant C' and some integer N':

$$|\psi_i(x)| + |d_x \psi_i| \leqslant C'(1+|x|^2)^{N'},$$

and a similar inequality also holds for $jac(F)$.

The partial derivatives of $\tilde{\psi}_i$ are combinations of the $\frac{\partial F^{-1}_{|U_l}}{\partial y_j}(y)$'s and the $\frac{\partial \psi_i}{\partial x_j}(F^{-1}_{|U_l}(y))$'s. Therefore, the two above inequalities show that, multiplying ψ_i by a huge power of $jac(F)^2$ and then a power of $\frac{1}{1+|x|^2}$ (which are Nash functions) if necessary, we can assume that the first order partial derivatives of $\tilde{\psi}_i$ are bounded, as required. It establishes that h_F is bi-Lipschitz provided the ψ_i's are decreasing fast enough.

We may also assume that the functions ψ_i tend to zero at infinity and near $Sing(F)$. Consequently, the set $N_F \setminus h_F(M_F)$ is contained in $(J_F \cup K_0(F)) \times \{0\}$, which is of codimension 2 in $\mathbb{R}^{2n}$ thanks to Theorem 2.1, so that the set N_F is a pseudomanifold. □

Remark 2.5. — From the proof we get immediately that the image of h_F is $N_F \setminus \mathbb{R}^{2n} \times \{0\}$ (since the ψ_i's tend to zero at the points of $Sing(F)$ and at infinity). We thus have the following commutative diagram:

$$\begin{array}{ccc} & & N_F \setminus \mathbb{R}^{2n} \times \{0\} \\ & \overset{h_F}{\nearrow} & \big\downarrow \pi_F \\ M_F & \xrightarrow{\quad F \quad} & \mathbb{R}^{2n} \end{array}$$

Where π_F is induced by the canonical projection and is locally bi-Lipschitz.

Lemma 2.6. — *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map. There exists a natural stratification of the set N_F, with only strata of even (real) dimension, which is locally topologically trivial along the strata.*

Proof. — For simplicity set $D := J_F \cup K_0(F)$ and consider the set

$$A := \pi_F^{-1}(D) \setminus \mathbb{R}^4 \times \{0_{\mathbb{R}^{\nu-4}}\}.$$

We first check the following fact:

Claim. The set $B := \overline{A} \cap \mathbb{R}^4 \times \{0_{\mathbb{R}^{\nu-4}}\}$ is finite.

Let $x \in B$. There exists a (real analytic) curve α in A tending to x. This curve lies in the image of h_F (see Remark 2.5). Clearly, $\pi_F(\alpha(s)) \in D$ entails $h_F^{-1}(\alpha(s)) \in F^{-1}(D)$. As α ends at point which does not belong to the image of h_F, the preimage of α by h_F must go either to infinity or to a point of $Sing(F) \cap \overline{F^{-1}(D) \setminus Sing(F)}$.

If $h_F^{-1}(\alpha)$ goes to infinity then $\pi_F(x)$ is an asymptotical value of $F_{|F^{-1}(J_F)}$. As $F^{-1}(D)$ is a complex algebraic curve and F is a polynomial, the set of asymptotical values of $F_{|F^{-1}(D)}$ is finite. We are thus done in this case. It is therefore enough to make sure that $Sing(F) \cap \overline{F^{-1}(D) \setminus Sing(F)}$ is finite. Both $Sing(F)$ and $F^{-1}(D)$ are complex algebraic curves. The set $\overline{F^{-1}(D) \setminus Sing(F)}$ is therefore constituted by the branches of $F^{-1}(D)$ which are not branches of $Sing(F)$. The intersection with $Sing(F)$ is therefore finite because it is the intersection of distinct branches of complex algebraic curves. This yields the claim.

Let now

$$N_0 = B \cup Sing(J_F \cup K_0(F)) \times \{0_{\mathbb{R}^{\nu-4}}\},$$

and take $x_0 \in D \times \{0_{\mathbb{R}^{\nu-4}}\} \setminus N_0$.

We are going to show that x_0 has a neighborhood in N_F which is locally topologically trivial. As we will also show that this topological trivialization preserves $D \times \{0_{\mathbb{R}^{\nu-4}}\}$, this will show that the stratification given by $(N_0, D \times \{0_{\mathbb{R}^{\nu-4}}\})$ is locally topologically trivial along the strata.

By definition of N_0, x_0 is a nonsingular point of $D \times \{0_{\mathbb{R}^{\nu-4}}\}$. Since the problem is purely local we may identify $D \times \{0_{\mathbb{R}^{\nu-4}}\}$ with $\mathbb{R}^2 \times \{0_{\mathbb{R}^{\nu-4}}\}$ and work nearby the origin. Consider now the following local isotopies

$$\mu_i : U \times [-\varepsilon, \varepsilon] \to U,$$

where U is a neighborhood of 0 in $\mathbb{R}^4$ and $\mu_i(x,t) = x + te_i$, $i = 1,2$. Choosing U small enough, we can assume $U \times \{0_{\mathbb{R}^{\nu-4}}\} \cap B = \emptyset$. Above the complement of $D \cap U$, these isotopies may be lifted to local isotopies in $\pi_F^{-1}(U)$ for π_F induces a covering map.

We denote this lifting by $\tilde{\mu}_i$. The obtained isotopies may not fall into $D \times \{0_{\mathbb{R}^{\nu-4}}\}$ since μ_i preserves the complement of D and U does not meet the set B. On $U \cap D \times \{0_{\mathbb{R}^{\nu-4}}\}$, $\pi_F(x,0) = x$, so that each $\tilde{\mu}_i$ extends continuously. □

Remark 2.7. — The set M_F is indeed well defined for any semi-algebraic map $F : \mathbb{R}^n \to \mathbb{R}^n$. The proof of Proposition 2.3 is still valid in this framework. Nevertheless, the obtained set N_F is no longer a pseudomanifold since $K_0(F)$ and J_F have no reason to be of codimension 2. It is just a semi-algebraic stratified set.

Let us provide a concrete example by drawing the set N_F for a specific $F : \mathbb{R}^2 \to \mathbb{R}^2$.

Example 2.8. — Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x,y) = (x, \frac{xy^3}{1+y^6})$. First, a quick computation gives:

$$Sing(F) = \{(x,y) \in \mathbb{R}^2 : y = \pm 1 \quad \text{or} \quad xy = 0\},$$

so that

$$K_0(F) = \{(x,y) \in \mathbb{R}^2 : y = \pm \frac{x}{2} \quad \text{or} \quad y = 0\}.$$

Let us determine the set of asymptotical values. If x_i is a sequence of real numbers tending to infinity then $F(x_i, y)$ goes to infinity. If x_i is a sequence tending to some real number a and y_i is a sequence going to infinity then $F(x_i, y_i)$ tends to $(a, 0)$. This shows that the Jelonek set is the x-axis.

For $a \neq 0$, the equation $F(x,y) = (a,b)$ reduces to $x = a$ and

$$by^6 - ay^3 + b = 0,$$

which has two solutions if $|b| < \frac{|a|}{2}$, one if $b = \pm\frac{a}{2}$ or $b = 0$, and zero if $|b| > \frac{|a|}{2}$. Hence, the fiber of F at an element (a, b) with $a \neq 0$ is constituted by two points if $0 < |b| < \frac{|a|}{2}$. The fibers of π_F have the same cardinal since h_F is a bijection.

We thus can draw a picture of the set N_F in this case:

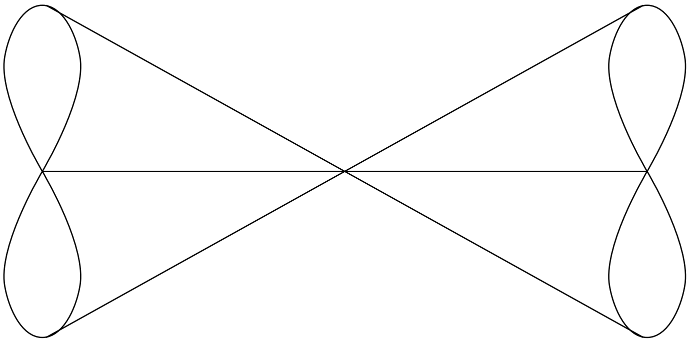

Figure 2.1. The set N_F for $F : \mathbb{R}^2 \to \mathbb{R}^2$, $(x; y) \mapsto (x; \frac{xy^3}{1+y^6})$.

The projection π_F maps N_F onto the subset $\{|y| \leqslant \frac{|x|}{2}\}$.

3. Non properness and vanishing cycles.

In the case $n = 2$, the singular homology as well as the intersection homology of the constructed set N_F captures the behavior at infinity of a nonsingular polynomial mapping F. Indeed, if F is proper then N_F is nothing but $\mathbb{R}^4$ and thus

$$H_2(N_F) = IH_2^{\bar{t}}(N_F) = 0.$$

We are going to show that the converse is also true (Theorem 3.2). For this purpose we need a preliminary lemma.

A **semi-algebraic family of subsets of $\mathbb{R}^m$ (parametrized by $\mathbb{R}^n$)** is a semi-algebraic set $A \subset \mathbb{R}^m \times \mathbb{R}^n$, the last n variables being considered as parameters.

In the lemma below, the subset A of $\mathbb{R}^m \times \mathbb{R}$ will be considered as a family parametrized by $t \in \mathbb{R}$. We write A_t, for "the fiber of A at t" i.e.:

$$A_t := \{x \in \mathbb{R}^m : (x, t) \in A\}.$$

LEMMA 3.1. — *Let β be a j-cycle and let $A \subset \mathbb{R}^m \times \mathbb{R}$ be a compact semi-algebraic family of sets with $|\beta| \subset A_t$ for any t. Assume that β bounds a $(j+1)$-chain in each A_t, $t > 0$ small enough. Then β bounds a chain in A_0.*

Proof. — Given a compact family of sets $A \subset \mathbb{R}^n \times [0, \varepsilon]$, there exists a semi-algebraic triangulation of A such that A_0 is a union of images of simplices (here A_0 denotes the zero fiber of the family A). Therefore, there exists a continuous strong deformation retraction of a neighborhood U of A_0 in A onto A_0. As $A_t \subset U$ for t small enough, the lemma ensues. □

3.1. Nonproper maps of the complex plane.

By Lemma 2.6, if $F : \mathbb{C}^2 \to \mathbb{C}^2$ denotes a polynomial map with nowhere vanishing Jacobian, the set N_F has a natural stratification. Indeed, in this case, we have shown that it is simply given by the filtration:

$$(Sing(J_F) \cup B) \times \{0\} \subset J_F \times \{0\} \subset N_F,$$

where B is given by all the asymptotical values at infinity of the restriction $F_{|F^{-1}(J_F)} : F^{-1}(J_F) \to J_F$. As the strata are all of even real dimension and since the dimension of N_F is 4, there are in fact basically three perversities which may provide different intersection homology groups: the top perversity $\overline{t}$, the $\overline{0}$ perversity and the middle perversity $\overline{m}$.

Below, a **semi-algebraic family of simplices** (parametrized by $\mathbb{R}^m$) will be a continuous semi-algebraic map $\sigma : T_i \times \mathbb{R}^m \to \mathbb{R}^n$. A **semi-algebraic family of chains** will be a linear combination of semi-algebraic families of simplices.

For every $t \in \mathbb{R}^m$, $\sigma_t := \sigma(x, t)$ is then a semi-algebraic simplex. The union of all the $|\sigma_t| \times \{t\}$ is a semi-algebraic family of sets.

THEOREM 3.2. — *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map with nowhere vanishing Jacobian. The following conditions are equivalent:*

(1) *F is non proper,*
(2) *$H_2(N_F) \neq 0$,*
(3) *$IH_2^{\overline{p}}(N_F) \neq 0$ for any perversity $\overline{p}$,*
(4) *$IH_2^{\overline{p}}(N_F) \neq 0$ for some perversity $\overline{p}$.*

Proof. — We first show that (1) implies (2). As the cycle that we will exhibit will be $(\bar{p}, 2)$-allowable for any perversity $\bar{p}$, this will also establish that (1) implies (3).

Assume that F is not proper. It means that there exists a complex Puiseux arc $\gamma : \mathbb{D}(0,\eta) \to \mathbb{R}^4$,

$$\gamma(z) = az^\alpha + \dots,$$

(with α negative and $a \in \mathbb{R}^4$ unit vector) tending to infinity in such a way that $F(\gamma)$ converges to an element $x_0 \in \mathbb{R}^4$, generic point of J_F (i.e. $(x_0, 0)$ is not in a zero dimensional stratum of N_F). Indeed, as the Jelonek set coincides with the image under the canonical projection of the set $\overline{\operatorname{graph}(F)} \setminus \operatorname{graph}(F)$, where the closure is taken in the $\mathbb{P}^2 \times \mathbb{C}^2$, this arc is provided by the famous Curve Selection Lemma.

Let C be an oriented triangle in $\mathbb{R}^4$ whose barycenter is the origin. Then, as the map $h_F \circ \gamma$ extends continuously at 0, it provides a singular 2-simplex in N_F that we will denote by c. This simplex is $(\overline{0}, 2)$-allowable since

$$J_F \cap |c| = \{x_0\}.$$

The support of ∂c lies in $N_F \setminus J_F \times \{0\}$. As, by definition of N_F, $N_F \setminus J_F \times \{0\} \simeq \mathbb{R}^4$, it means that ∂c bounds a singular chain $e \in C_2(N_F \setminus J_F \times \{0\})$. But then $\sigma = c - e$ is a $(\overline{0}, 2)$-allowable cycle of N_F. We claim that σ may not bound a 3-chain in N_F.

Assume otherwise, i.e. assume that there is a chain $\tau \in C_3(N_F)$, satisfying $\partial \tau = \sigma$.

Let $A := h_F^{-1}(|\sigma| \setminus J_F)$ and $B := h_F^{-1}(|\tau| \setminus J_F)$. For R large enough, the sphere $S(0,R)$ is transverse to A and B (at regular points). Therefore, after a triangulation, the intersection $\sigma_R := S(0,R) \cap A$ is a chain bounding the chain $\tau_R := S(0,R) \cap B$.

Given a subset $X \subset \mathbb{R}^4$ we define the "tangent cone at infinity" by:

$$C_\infty(X) := \{\lambda \in S^3 : \exists \gamma : (0,\varepsilon] \to X \text{ semi-algebraic, } \lim_{t\to 0} \gamma(t) = \infty,\ \lim_{t \to 0} \frac{\gamma(t)}{|\gamma(t)|} = \lambda\},$$

(this is rather the link of a cone than a cone but it will be more convenient to work with this set).

Let $\hat{F}_1$ and $\hat{F}_2$ be the respective initial forms of the components of $F := (F_1, F_2)$ and let $\alpha(t) := bt^m + \dots$ be any real Puiseux arc in $\mathbb{R}^4$, tending to infinity and such that $F(\alpha)$ does not tend to infinity. For any $i = 1, 2$, the Puiseux expansion of $F_i(\alpha(t))$ must start like:

$$F_i(\alpha(t)) = \hat{F}_i(b) t^{md_i} + \dots, \tag{3.1}$$

where d_i is the homogeneous degree of $\widehat{F}_i$. As the arc $F(\alpha)$ does not diverge to infinity as t tends to zero and m is negative, we see that

$$\hat{F}_i(b) = 0, \qquad\qquad i = 1, 2.$$

As a matter of fact, if a semi-algebraic set $X \subset \mathbb{R}^4$ is mapped onto a bounded set by F then $C_\infty(X)$ is included in the zero locus V of $\hat{F} := (\hat{F}_1, \hat{F}_2)$. Hence, $C_\infty(A)$ and $C_\infty(B)$ are both subsets of $V \cap S^3$.

Observe that, in a neighborhood of infinity, A coincides with the support of the Puiseux arc γ. Thus, $C_\infty(A)$ is nothing but $S^1 \cdot a$ (denoting the orbit of $a \in \mathbb{C}^2$ under the action of S^1 on $\mathbb{C}^2$, $(e^{i\eta}, z) \mapsto e^{i\eta} z$). Observe that, since $V \cap S^3$ is a union of circles, the homology class of $S^1 \cdot a$ in this set is nontrivial.

Consider a Nash strong deformation retraction $\rho : W \times [0,1] \to S^1 \cdot a$, of a neighborhood W of $S^1 \cdot a$ in S^3 onto $S^1 \cdot a$.

Let $\tilde{\sigma}_R := \frac{\sigma_R}{R}$, R positive real number. The image of the restriction of γ to a sufficiently small neighborhood of 0 in $\mathbb{C}$ entirely lies in the cone over W. Consequently, $|\tilde{\sigma}_R|$ is included in W, for R large enough.

Let σ'_R be the image of $\tilde{\sigma}_R$ under the retraction ρ_1 (where $\rho_1(x) := \rho(x,1)$). As, near infinity, $|\sigma_R|$ coincides with the intersection of the support of the arc γ with $S(0,R)$, for R large enough the class of $|\sigma'_R|$ in $S^1 \cdot a$ is nonzero.

Since the retraction ρ_1 is isotopic to the identity, there exists a chain $\theta_R \in C_2(S^3)$ such that:

$$\partial \theta_R = \sigma'_R - \tilde{\sigma}_R. \tag{3.2}$$

As A is semi-algebraic, the family σ_R is a semi-algebraic family of chains (considering R as a parameter, see the beginning of this section for the definition of families). The chain θ_R actually results from the composition of $\tilde{\sigma}_R$ and ρ (after a subdivision of $T_i \times [0,1]$). As ρ is semi-algebraic, the family $\theta_{1/r}$, $r > 0$, must constitute a semi-algebraic family of chains. Their supports thus constitute a semi-algebraic family of sets. Denote by $E \subset \mathbb{R}^4 \times \mathbb{R}$ its closure and set $E_0 := E \cap \mathbb{R}^4 \times \{0\}$. As the strong deformation retraction ρ is the identity on $S^1 \cdot a \times [0,1]$, we see that

$$E_0 \subset \rho(C_\infty(A) \times [0,1]) = S^1 \cdot a \subset V \cap S^3. \tag{3.3}$$

Let now $\tilde{\tau}_R = \dfrac{\tau_R}{R}$ and define a semi-algebraic family of chains by:

$$\theta'_R := \tilde{\tau}_R + \theta_R,$$

and denote by E' the semi-algebraic family of sets constituted by the supports of the semi-algebraic family of chains $\theta'_{\frac{1}{r}}$, $r>0$. Denote by E'_0 the fiber at 0, that is to say, the set $E'\cap\mathbb{R}^4\times\{0\}$. By (3.3) and the definition of θ'_R we have

$$E'_0\subset E_0\cup C_\infty(B)\subset V\cap S^3. \tag{3.4}$$

Moreover, we immediately derive from (3.2) and the definition of θ'_R:

$$\partial\theta'_R=\sigma'_R.$$

The class of σ'_R in $S^1\cdot a$ is, up to a product with a nonzero constant, equal to the generator of $S^1\cdot a$. Therefore, since σ'_R bounds the chain θ'_R, the cycle $S^1\cdot a$ must bound a chain in $|\theta'_R|$ as well. By Lemma 3.1 (applied to the family $|\theta'_{1/r}|$ and the cycle $S^1\cdot a$, $r>0$) this implies that $S^1\cdot a$ bounds a chain in E'_0 which is included in $V\cap S^3$. This is a contradiction since $S^1\cdot a$ is a nontrivial cycle of $V\cap S^3$. This establishes that (1) implies (2), (3), and (4).

If F is a proper map, then the set N_F is homeomorphic to $\mathbb{C}^2$ and consequently $H_2(N_F)=IH_2^{\bar{t}}(N_F)=0$. Thus (2), (3) and (4) all fail. □

Observation 3.3. — *The set $N_F^R:=N_F\cap\bar{B}(0,R)$ is a pseudomanifold with boundary, for large enough R.*

Proof. — We have to construct a collared neighborhood of $\partial N_F^R := S(0,R)\cap N_F$, R large enough. First of all, observe that if R is chosen large enough then R is not a critical value of the distance function to the origin on the set N_F and thus N_F^R is a smooth manifold with boundary at any point of $N_F\cap S(0,R)\setminus J_F$. Furthermore, by Hardt's Theorem [4], the level surfaces $N_F\cap S(0,R)$ are pseudomanifolds constituting a semi-algebraically topologically trivial family. The trivialization may be required to preserve J_F. Consequently, the couple $(N_F^R,\partial N_F^R)$ constitutes a pseudomanifold with boundary if R is chosen large enough. □

Now, thanks to the de Rham theorem for L^∞ forms (Theorem 1.8) we get the following immediate corollary.

Corollary 3.4. — *Let $F:\mathbb{C}^2\to\mathbb{C}^2$ be a polynomial map with nowhere vanishing Jacobian. The following conditions are equivalent:*

(1) *F is nonproper,*
(2) *$H^2_\infty(Reg(N_F^R))\neq 0$.*

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