ANNALES

DE

L'INSTITUT FOURIER

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Tome 64, nº 4 (2014), p. 1781-1805.

<http://aif.cedram.org/item?id=AIF_2014__64_4_1781_0>

Ann. Inst. Fourier, Grenoble
64, 4 (2014) 1781-1805

FOLIATIONS BY CURVES WITH CURVES AS SINGULARITIES

by M. CORRÊA Jr, A. FERNÁNDEZ-PÉREZ,
G. NONATO COSTA & R. VIDAL MARTINS

Dedicated to Márcio Gomes Soares, for his 60th birthday.

Abstract. — Let $\mathcal{F}$ be a holomorphic one-dimensional foliation on $\mathbb{P}^n$ such that the components of its singular locus Σ are curves C_i and points p_j. We determine the number of p_j, counted with multiplicities, in terms of invariants of $\mathcal{F}$ and C_i, assuming that $\mathcal{F}$ is special along the C_i. Allowing just one nonzero dimensional component on Σ, we also prove results on when the foliation happens to be determined by its singular locus.

Résumé. — Soit $\mathcal{F}$ un feuilletage holomorphe unidimensionnel sur $\mathbb{P}^n$, dont les composantes du lieu singulier Σ sont des courbes C_i et des points p_j. On exprime le nombre de tels points p_j, comptés avec leurs multiplicités, en termes des invariants de $\mathcal{F}$ et C_i, en supposant que $\mathcal{F}$ est spécial le long des courbes C_i. En supposant qu'il n'y a qu'une seule composante de Σ de dimension non nulle, on obtient aussi des résultats lorsque le feuilletage est déterminé par ses lieux singuliers.

1. Introduction

Let $\mathcal{F}$ be a foliation on a smooth projective scheme Y, and X a projective subscheme of Y. Let $\widetilde{Y}$ be the blowup of Y along X, and $\pi : \widetilde{Y} \to Y$ the blowup morphism with exceptional divisor $E := \pi^{-1}(X)$. The foliation $\mathcal{F}$ will be called *special* along X if the strict transform $\widetilde{\mathcal{F}}$ has E as an invariant set, and $\mathrm{Sing}(\widetilde{\mathcal{F}})$ meets E at isolated singularities at most. With this in mind, we prove the following.

Theorem 1.1. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on $\mathbb{P}^n$, $n \geqslant 3$, of degree k, such that its singular locus is the disjoint union of irreducible curves $C_1, \ldots, C_r$ and points $p_1, \ldots, p_s$. Assume each C_i is either*

Keywords: holomorphic foliations, non-isolated singularities.
Math. classification: 32S65, 58K45.

smooth, or a singular set theoretic complete intersection; assume also that $\mathcal{F}$ is special along each C_i, for $1 \leqslant i \leqslant r$. Then

$$\sum_{i=1}^{s} \mu(\mathcal{F}, p_i) = 1 + k + k^2 + \ldots + k^n + \sum_{i=1}^{r} \nu(\mathcal{F}, C_i)$$

where $\mu(\mathcal{F}, p_i)$ is the multiplicity of $\mathcal{F}$ at p_i, and where for any curve $C \subset \mathbb{P}^n$ of arithmetic genus g, degree d, with singular points, if any, $q_1, \ldots, q_l$, and along which $\mathcal{F}$ is special, we set

$$\nu(\mathcal{F}, C) := (\ell+1)^{n-2}\Bigg(\Big(2g - 2 - \sum_{i=1}^{l}(b_i - 1)\Big)(\ell^2 + \ell + 1) + (n+1)d\ell^2 - (k-1)d(n\ell + 1)\Bigg)$$

with b_i the number of branches of q_i, and $\ell := m_C(\mathcal{F})$ the multiplicity of $\mathcal{F}$ at C.

The above result generalizes a formula by the third named author (*cf.* [7]) which counts the number of isolated singularities of a foliation by curves on $\mathbb{P}^3$ admiting regular curves as singularities. We just relaxed the hypothesys basically allowing singular curves on the singular locus, as long as complete intersections, and consider the foliation on a generic projective space $\mathbb{P}^n$.

Our second task concerns stablishing conditions for when the singular locus happens to determine a foliation. In order to do so, we need another definition. Let X be a projective subscheme of $\mathbb{P}^n$ given by an ideal sheaf $\mathcal{I}_X$. We define the *generating degree of* X, denoted $\mathrm{gd}(X)$, as the least integer $d > 0$ such that $\mathcal{I}_X(d)$ is globally generated, *i.e.*, for which X is a set theoretic intersection of hypersurfaces of degree at most d.

Theorem 1.2. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on $\mathbb{P}^n$, $n \geqslant 3$, of degree k, such that its singular locus has just one nonzero dimensional component, which is an integral and smooth curve C. Assume also that $\mathcal{F}$ is special along C. Let $\pi : \widetilde{\mathbb{P}}^n \to \mathbb{P}^n$ be the blowup of $\mathbb{P}^n$ along C and E the exceptional divisor. If $\mathcal{F}'$ is another foliation of degree k on $\mathbb{P}^n$, with $k > \mathrm{gd}(C)$, and also $\mathrm{Sing}(\mathcal{F}) \subset \mathrm{Sing}(\mathcal{F}')$ and $\mathrm{Sing}(\widetilde{\mathcal{F}}|_E) \subset \mathrm{Sing}(\widetilde{\mathcal{F}'}|_E)$, then $\mathcal{F}' = \mathcal{F}$.*

The above result can be compared to A. Campillo and J. Olivares' [4, Cor. 3.2], the proof of which, along with X. Gomez-Mont and G. Kempf's [5], motivated the one here. In the very case of three dimensional ambient space, with additional requirements on the curve of singularities, a stronger sentence can be proved.

THEOREM 1.3. — *If $\mathcal{F}$ and $\mathcal{F}'$ are holomorphic foliations by curves on $\mathbb{P}^3$, of same degree, such that* $\operatorname{Sing}(\mathcal{F}') \supset \operatorname{Sing}(\mathcal{F}) = C \cup \{p_1, \dots, p_s\}$, *where C is a nondegenerated integral smooth set theoretic complete intersection curve, and $\mathcal{F}$ is special along C, then $\mathcal{F} = \mathcal{F}'$.*

So in this case we have the desired statement on determination. This unicity problem has been studied by many authors (see [1] and the references therein) and, in this sense, the above result is a step forward on dealing with the subject when the singular locus has unexpected codimension.

Acknowledgments. The second named author is partially supported by Fapemig-Brasil and PRPq-UFMG; the fourth named author is partially supported by the CNPq grant number 304919/2009-8.

2. Preliminaries

2.1. Multiplicities along subschemes

Let $\mathcal{F}$ be a foliation by curves on an n-dimensional smooth projective variety Y over $\mathbb{C}$. It is determined by an injective sheaf map $\varphi : \mathcal{L}_{\mathcal{F}} \hookrightarrow \mathcal{T}_Y$, where $\mathcal{L}_{\mathcal{F}}$ is an invertible sheaf and $\mathcal{T}_Y$ is the tangent bundle, such that $\mathcal{T}_Y/\mathcal{L}_{\mathcal{F}}$ is torsion free. The *singular locus of $\mathcal{F}$* is the closed subscheme Σ of Y defined by the ideal sheaf

$$\mathcal{I}_\Sigma := F_{n-1}(\mathcal{T}_Y/\mathcal{L}_{\mathcal{F}})$$

where F_{n-1} stands for the fitting ideal. So one may denote and write

$$\operatorname{Sing}(\mathcal{F}) := \Sigma = \operatorname{Spec} \mathcal{O}_Y/\mathcal{I}_\Sigma.$$

Now let $P \in Y$ be any point. The local ring $\mathcal{O}_{Y,P}$ may not be a discrete valuation ring, but one can at least consider the $\mathfrak{m}_P$-adic valuation, which we denote by v_P, where $\mathfrak{m}_P$ is the maximal ideal. So one defines the *multiplicity of $\mathcal{F}$ at P* as

$$m_P(\mathcal{F}) := \min_{f \in \mathcal{I}_{\Sigma,P}} \{v_P(f)\}$$

The multiplicity of $\mathcal{F}$ at an irreducible subscheme of Y will be the multiplicity of the foliation at its generic point. So multiplicities are well defined for irreducible components of the singular locus as well.

If $\mathcal{F}$ is given by a vector field which, in a neighbourhood of P, is written by

$$\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F},P} = f_1 \frac{\partial}{\partial z_1} + \ldots + f_n \frac{\partial}{\partial z_n} \tag{2.1}$$

then we have

$$m_P(\mathcal{F}) := \min\{v_P(f_1), \ldots, v_P(f_n)\}.$$

If $p \in Y$ is a closed point then

$$m_p(\mathcal{F}) = \operatorname{length}_{\mathcal{O}_{Y,p}} \frac{\mathcal{O}_{Y,p}}{(f_1, \ldots, f_n)}$$

which agrees with the classical Milnor number. So in this case we adopt the standard notation

$$\mu(\mathcal{F}, p) = m_p(\mathcal{F}).$$

For later use, we now describe the multiplicity of $\mathcal{F}$ at an irreducible curve C which is a component of $\operatorname{Sing}(\mathcal{F})$. By a holomorphic change of coordinates, C can be locally given as $z_1 = \ldots = z_{n-1} = 0$. Therefore, one may write the local sections in (2.1) as

$$f_i(z) = \sum_{|a|=m_i} z_1^{a_1} \cdots z_{n-1}^{a_{n-1}} f_{i,a}(z) \tag{2.2}$$

where $a := (a_1, \ldots, a_{n-1})$ with $|a| := a_1 + \ldots + a_{n-1}$, and at least one among the $f_{i,a}(z)$ does not vanish in the z_n-axis. One rapidly sees that the number m_i in (2.2) agrees with $v_C(f_i)$ so

$$m_C(\mathcal{F}) = \min\{m_1, \ldots, m_n\}. \tag{2.3}$$

We may change coordinates and assume for the remainder that

$$m_{n-1} \leqslant \ldots \leqslant m_1.$$

Now we blowup Y along C and describe the behavior of $\mathcal{F}$ under this transformation. Just in order to fix notation we recall the blowup procedure in this specific case. If Δ is an n-dimensional polydisc with holomorphic coordinates $z_1, \ldots, z_n$ and $\Gamma \subset \Delta$ is the locus $z_1 = \ldots = z_{n-1} = 0$, take $[y_1, \ldots, y_{n-1}]$ to be homogeneous coordinates on $\mathbb{P}^{n-2}$. The blowup of Δ along Γ is the smooth variety

$$\widetilde{\Delta} = \{(z, [y]) \in \Delta \times \mathbb{P}^{n-2} \mid z_i y_j = z_j y_i \text{ for } 1 \leqslant i, j \leqslant n-1\}.$$

The projection $\pi : \widetilde{\Delta} \to \Delta$ on the first factor is an isomorphism away from Γ, while the inverse image of a point $z \in \Gamma$ is a projective space $\mathbb{P}^{n-2}$. The inverse image $E = \pi^{-1}(\Gamma)$ is the exceptional divisor of the blowup.

The standard open cover $U_j = \{[y_1, \dots, y_{n-1}] \mid y_j \neq 0\}$, with $1 \leqslant j \leqslant n-1$, of $\mathbb{P}^{n-2}$ yields a cover of $\widetilde{\Delta}$ where each open set, for $1 \leqslant j \leqslant n-1$, is defined by

$$\widetilde{U_j} = \{(z, [y]) \in \widetilde{\Delta} \ \mid \ [y] \in U_j\} \tag{2.4}$$

with holomorphic coordinates $\sigma(u_1, \dots, u_n) = (z_1, \dots, z_n)$ given by

$$z_i = \begin{cases} u_i & \text{if } i = j \text{ or } i = n \\ u_i u_j & \text{if } i = 1, \dots, \widehat{j}, \dots, n-1. \end{cases}$$

The coordinates $u \in \mathbb{C}^n$ are affine coordinates on each fiber $\pi^{-1}(p) \cong \mathbb{P}^{n-2}$ of E.

Now consider the curve $C \subset Y$. Let $\{\phi_\lambda, U_\lambda\}$ be a collection of local charts covering C and $\phi_\lambda : U_\lambda \to \Delta_\lambda$, where Δ_λ is an n-dimensional polydisc. One may suppose that $\Gamma_\lambda = \phi_\lambda(C \cap U_\lambda)$ is given by $z_1 = \ldots = z_{n-1} = 0$. Let $\pi_\lambda : \widetilde{\Delta}_\lambda \to \Delta_\lambda$ be the blowup of Δ_λ along Γ_λ. One can patch the π_λ together and use the chart maps ϕ_λ to get a blowup $\widetilde{Y}$ of Y along C and a blowup morphism $\pi : \widetilde{Y} \to Y$. The exceptional divisor E is a fibre bundle over C with fiber $\mathbb{P}^{n-2}$ which is naturally identified with the projectivization $\mathbb{P}(\mathcal{N}_{C/Y})$ of the normal bundle $\mathcal{N}_{C/Y}$.

In the open set $\widetilde{U}_1$, as in (2.4), we have

$$\sigma(u) = (u_1, u_1 u_2, \dots, u_1 u_{n-1}, u_n) = (z_1, \dots, z_n).$$

If $i = 1$ or $i = n$, since $u_i = z_i$ we get

$$\begin{aligned} \dot{u}_i &= \sum_{|a|=m_i} u_1^{a_1} (u_1 u_2)^{a_2} \cdots (u_1 u_{n-1})^{a_{n-1}} f_{i,a}(\sigma(u)) \\ &= u_1^{m_i} \sum_{|a|=m_i} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} f_{i,a}(\sigma(u)) \end{aligned}$$

but we may write $f_{i,a}(\sigma(u)) = f_{i,a}(0, \dots, 0, u_n) + u_1 \widetilde{f}_{i,a}(u) = p_{i,a}(u_n) + u_1 \widetilde{f}_{i,a}(u)$ and hence

$$\dot{u}_i = u_1^{m_i} \left(\sum_{|a|=m_i} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} p_{i,a}(u_n) + u_1 \widetilde{f}_i(u) \right) \tag{2.5}$$

for some functions $\widetilde{f}_i(u)$, with $i = 1$ or $i = n$.

If $2 \leqslant i \leqslant n-1$, since $z_i = u_1 u_i$, we have that $\dot{z}_i = \dot{u}_1 u_i + u_1 \dot{u}_i$ and thus

$$\dot{u}_i = u_1^{m_i-1}\Bigg(\sum_{|a|=m_i} u_2^{a_2} \dots u_{n-1}^{a_{n-1}} p_{i,a}(u_n) - u_1^{m_1-m_i} u_i \sum_{|a|=m_1} u_2^{a_2} \dots u_{n-1}^{a_{n-1}} p_{1,a}(u_n) + u_1 \widetilde{f}_i(u)\Bigg) \tag{2.6}$$

for some functions $\widetilde{f}_i(u)$, with $2 \leqslant i \leqslant n-1$.

Combining (2.5) and (2.6) we have that $\pi^*\mathcal{F}$ is described by the vector field

$$\mathcal{D}_{\pi^*\mathcal{F}} = u_1^{m_1}\Big(g_1(u) + u_1\widetilde{f}_1(u)\Big)\frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1} u_1^{m_i-1}\Big(h_i(u) + u_1\widetilde{f}_i(u)\Big)\frac{\partial}{\partial u_i} + u_1^{m_n}\Big(g_n(u) + u_1\widetilde{f}_n(u)\Big)\frac{\partial}{\partial u_n} \tag{2.7}$$

where

$$g_i(u) := \sum_{|a|=m_i} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} p_{i,a}(u_n) \quad \text{and} \quad h_i(u) := g_i(u) - u_1^{m_1-m_i} u_i g_1(u).$$

Now all points of E, given by $u_1 = 0$, are singularities of $f^*\mathcal{F}$. We have some ways of desingularizing, according to the possible values of m_i. Furthermore, if $m_1 = m_i$ for some i, we must verify whether

$$r_i(u) := g_i(u) - u_i g_1(u) \tag{2.8}$$

is identically zero or not. In this way, we may divide it in two cases, dicritical or nondicrital curves of singularities, according to if the exceptional divisor is or is not invariant by the induced foliation $\widetilde{\mathcal{F}}$.

- Non-dicritical curve of singularities.

(i) $m_n + 1 = m_{n-1} = \ldots = m_2$ with $r_i \not\equiv 0$ for all $2 \leqslant i \leqslant n-1$ if $m_{n-1} = m_1$.

Dividing (2.7) by $u_1^{m_n}$ we get the vector field defining $\widetilde{\mathcal{F}}$ which is

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = u_1^{m_1-m_n}(g_1(u)+u_1\widetilde{f}_1(u))\frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1} \widetilde{h}_i(u)\frac{\partial}{\partial u_i} + (g_n(u)+u_1\widetilde{f}_n(u))\frac{\partial}{\partial u_n} \tag{2.9}$$

where

$$\widetilde{h}_i(u) := g_i(u) - u_1^{m_1-m_i} u_i g_1(u) + u_1 \widetilde{f}_i(u).$$

The singularities on E are given by the roots of the system

$$\widetilde{h}_2(u) = \widetilde{h}_3(u) = \cdots = \widetilde{h}_{n-1}(u) = g_n(u) = 0$$

and (i) implies that they should be isolated, *i.e.*, $\mathcal{F}$ is special along C.

(ii) $m_n+1 \leqslant m_{n-1}$ with $r_{i_0} \equiv 0$ for some $2 \leqslant i_0 \leqslant n-1$ if $m_n+1 = m_1$. Dividing (2.7) by $u_1^{m_n}$ we get

(2.10)

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = u_1^{m_1-m_n}\big(g_1(u)+u_1\widetilde{f}_1(u)\big)\frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1} u_1^{m_i-m_n-1}\big(h_i(u)+u_1\widetilde{f}_i(u)\big)\frac{\partial}{\partial u_i} + \big(g_n(u)+u_1\widetilde{f}_n(u)\big)\frac{\partial}{\partial u_n}$$

and we see that the exceptional divisor is also invariant by the foliation $\widetilde{\mathcal{F}}$, but this turn, the singularities are always nonisolated. Furthermore, the leaves of (2.10) when restricted to E are contained in the hyperplane given by $u_i = c_i$ for those i such that $m_i - 1 > m_n$ or $r_i \equiv 0$, where c_i is a constant.

(iii) $m_n \geqslant m_{n-1}$ with $r_{i_0} \not\equiv 0$ for some $2 \leqslant i_0 \leqslant n-1$ if $m_n = m_{n-1} = m_1$. Dividing (2.7) by $u_1^{m_{n-1}-1}$ we get

(2.11)

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = u_1^{m_1-m_{n-1}+1}\Big(g_1(u)+u_1\widetilde{f}_1(u)\Big)\frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1} u_1^{m_i-m_{n-1}}\widetilde{h}_i(u)\frac{\partial}{\partial u_i} + u_1^{m_n-m_{n-1}+1}\Big(g_n(u)+u_1\widetilde{f}_n(u)\Big)\frac{\partial}{\partial u_n}$$

and the exceptional divisor is also invariant by the foliation $\widetilde{\mathcal{F}}$, but again with nonisolated singularities on $\widetilde{C}$. The leaves of $\widetilde{\mathcal{F}}$ on E are contained in the hyperplane $u_n = c$ for a constant c.

- Dicritical curve of singularities.

(i) $m_1 = \cdots = m_n$ and $r_i \equiv 0$ for all $2 \leqslant i \leqslant n-1$.

Dividing (2.7) by $u_1^{m_n}$ we get

(2.12)

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = \Big(g_1(u)+u_1\widetilde{f}_1(u)\Big)\frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1}\widetilde{f}_i(u)\frac{\partial}{\partial u_i} + \Big(g_n(u)+u_1\widetilde{f}_n(u)\Big)\frac{\partial}{\partial u_n}.$$

Combining this with the corresponding expression in the other coordinate systems, we get defining equations for a foliation $\widetilde{\mathcal{F}}$ which coincides with $f^*\mathcal{F}$ outside E but this time the exceptional divisor is not an invariant set. The foliation $\widetilde{\mathcal{F}}$ is transverse to E except at the hypersurface locally given by $g_1(u) = 0$ which may or may not consist of singularities of $\widetilde{\mathcal{F}}$.

(ii) $m_{n-1} = \ldots = m_1 < m_n$ and $r_i \equiv 0$ for all $2 \leqslant i \leqslant n-1$.

Dividing (2.7) by $u_1^{m_1}$ we get

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = \left(g_1(u) + u_1\widetilde{f}_1(u)\right)\frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1}\widetilde{f}_i(u)\frac{\partial}{\partial u_i} + u_1^{m_n-m_1}\left(g_n(u) + u_1\widetilde{f}_n(u)\right)\frac{\partial}{\partial u_n} \tag{2.13}$$

and the exceptional divisor is not invariant by the foliation $\widetilde{\mathcal{F}}$, but the last component of the vector field (2.13) vanishes on it.

Keeping the notation above, for later use we sketch what we get as follows.

Lemma 2.1. — *The following hold:*

(i) $m_C(\mathcal{F}) = \min\{m_1, \dots, m_n\}$;

(ii) *if ℓ is the integer such that*

$$\mathcal{L}_{\widetilde{\mathcal{F}}} \cong \pi^*\mathcal{L}_{\mathcal{F}} \otimes \mathcal{O}_{\widetilde{Y}}(\ell E)$$

then

$$\ell = \begin{cases} \min\{m_1, m_2 - 1, \dots, m_{n-1} - 1, m_n\} & \text{if } C \text{ is nondicritical} \\ \min\{m_1, \dots, m_n\} & \text{if } C \text{ is dicritical} \end{cases}$$

(iii) *$\mathcal{F}$ is special along C if and only if $m_n + 1 = m_{n-1} = \dots = m_2$ with $r_i \not\equiv 0$ for all $2 \leqslant i \leqslant n-1$ if $m_{n-1} = m_1$. In particular, $\ell = m_C(\mathcal{F})$ in this case.*

2.2. Chern classes

Now we relate cohomology groups of schemes and blowups. Let $\pi : \widetilde{\mathbb{P}}^n \to \mathbb{P}^n$, $n \geqslant 3$, be the blowup of $\mathbb{P}^n$ along a regular curve C, with exceptional divisor E. Set $\mathcal{N} := \mathcal{N}_{C/\mathbb{P}^n}$ and $\rho := \pi|_E$. Since $E \cong \mathbb{P}(\mathcal{N})$, recall that $A(E)$ is generated as an $A(C)$-algebra by the Chern class

$$\zeta := c_1(\mathcal{O}_{\mathcal{N}}(-1))$$

with the single relation

$$\zeta^{n-1} - \rho^*c_1(\mathcal{N})\zeta^{n-2} + \dots + (-1)^{n-1}\rho^*c_{n-2}(\mathcal{N})\zeta + (-1)^{n-1}\rho^*c_{n-1}(\mathcal{N}) = 0. \tag{2.14}$$

The normal bundle $\mathcal{N}_{E/\widetilde{\mathbb{P}}^n}$ agrees with the tautological bundle $\mathcal{O}_{\mathcal{N}}(-1)$, and hence

$$\zeta = c_1(\mathcal{N}_{E/\widetilde{\mathbb{P}}^n}). \tag{2.15}$$

If $\iota : E \hookrightarrow \widetilde{\mathbb{P}}^n$ is the inclusion map, we also get

$$\iota_*(\zeta^i) = (-1)^i E^{i+1}. \tag{2.16}$$

Given that

$$\int_E \rho^* c_i(\mathcal{N})\zeta^{n-i-1} = (-1)^{n-i-1} \int_C c_i(\mathcal{N}) = 0$$

for $i \geqslant 2$, we have

(2.17)
$$\begin{aligned}\int_E \zeta^{n-1} &= \int_E \rho^* c_1(\mathcal{N})\zeta^{n-2} = (-1)^n \int_C c_1(\mathcal{N})\\ &= (-1)^n \int_C c_1(\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{O}_C) - c_1(C) = (-1)^n\big((n+1)d - 2 + 2g\big)\end{aligned}$$

where g is the genus and d is the degree of C_{red}.

From Porteous Theorem (see [8]), it holds that

$$c(\widetilde{\mathbb{P}}^n) - \pi^* c(\mathbb{P}^n) = \iota_*(\rho^* c(C)\alpha) \tag{2.18}$$

where

$$\alpha = \frac{1}{\zeta} \sum_{i=0}^{n-1} \big(1 - (1-\zeta)(1+\zeta)^i\big)\rho^* c_{n-1-i}(\mathcal{N}). \tag{2.19}$$

We may rewrite (2.19) taking $(1+\zeta)^i = \sum_{l=0}^{i} \binom{i}{l}\zeta^l$ and setting $j := n-1-i$ as

$$\alpha = \sum_{j=0}^{n-1} \sum_{l=0}^{n-1-j} \left(\binom{n-1-j}{l} - \binom{n-1-j}{l+1}\right)\zeta^l \rho^* c_j(\mathcal{N}) \tag{2.20}$$

with the convention, also for the remainder, that $\binom{p}{q} := 0$ whenever $q > p$. Since i does not appear in (2.20) we reset $i := j + l$ and write

$$\alpha = \sum_{i=0}^{n-1} \alpha_i$$

where

$$\alpha_i = \sum_{j=0}^{i} \left(\binom{n-1-j}{i-j} - \binom{n-1-j}{i-j+1}\right)\zeta^{i-j} \rho^* c_j(\mathcal{N}).$$

Consequently

$$c(\widetilde{\mathbb{P}}^n) - \pi^* c(\mathbb{P}^n) = \iota_*(\rho^* c(C)\alpha) = \iota_*\left(\sum_{i=0}^{n-1} \beta_i\right)$$

where

$$\beta_i = \sum_{j=0}^{i} \alpha_j \rho^* c_{i-j}(C).$$

Then $\beta_0 = \alpha_0 = -(n-2)$ and $\beta_i = \alpha_i + \alpha_{i-1}\rho^* c_1(C)$ for $i \geqslant 1$. Now, in order to calculate the Chern class $c(\widetilde{\mathbb{P}}^n)$ we have to compare the terms of (2.18) of same degree. Therefore

$$c_i(\widetilde{\mathbb{P}}^n) - \pi^* c_i(\mathbb{P}^n) = \iota_*(\beta_{i-1})$$

which yields

$$c_1(\widetilde{\mathbb{P}}^n) - \pi^* c_1(\mathbb{P}^n) = \iota_*(\beta_0) = -(n-2)E \tag{2.21}$$

and for $i \geqslant 2$,

(2.22)

$$c_i(\widetilde{\mathbb{P}}^n) = \pi^* c_i(\mathbb{P}^n) + \sum_{j=0}^{i-1} \left(\binom{n-1-j}{i-1-j} - \binom{n-1-j}{i-j} \right) (-1)^{i-1-j} \rho^* c_j(\mathcal{N}) E^{i-j}$$
$$+ \sum_{j=0}^{i-2} \left(\binom{n-1-j}{i-2-j} - \binom{n-1-j}{i-j-1} \right) (-1)^{i-2-j} \rho^* c_j(\mathcal{N}) \rho^* c_1(C) E^{i-1-j}.$$

3. Special Foliations along Regular Curves

In this section, $\mathcal{F}$ is always a holomorphic foliation by curves on $\mathbb{P}^n$, $n \geqslant 3$, with

$$\operatorname{Sing}(\mathcal{F}) = C \cup \{p_1, \dots, p_s\}, \tag{3.1}$$

where the union is disjoint, C is an irreducible smooth projective curve, the p_i are isolated closed points, and $\mathcal{F}$ is special along C. This means that for the blowup $\pi : \widetilde{\mathbb{P}}^n \to \mathbb{P}^n$ along C, we obtain a foliation $\widetilde{\mathcal{F}}$ on $\widetilde{\mathbb{P}}^n$ which has only isolated singularities, and the exceptional divisor E is an invariant set of $\widetilde{\mathcal{F}}$.

Our goal is to compute the number of isolated singularities of $\mathcal{F}$, counted with multiplicities. We assume (5.1) for simplicity, but the general case where $\operatorname{Sing}(\mathcal{F})$ has more than one curve as a component is straight forward from this one. The case where C is a singular set theoretic complete intersection is left to the following section.

We start by calculating the Chern class of the invertible sheaf $\mathcal{L}_{\widetilde{\mathcal{F}}}$, the tangent bundle of the foliation $\widetilde{\mathcal{F}}$. From Lemma 2.1, it follows that

$$\mathcal{L}_{\widetilde{\mathcal{F}}} \cong \pi^* \mathcal{L}_{\mathcal{F}} \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^n}(\ell E)$$

where $\ell = m_C(\mathcal{F})$. Therefore

$$c_1(\mathcal{L}_{\widetilde{\mathcal{F}}}) = \pi^* c_1(\mathcal{L}_{\mathcal{F}}) + \ell E. \tag{3.2}$$

The result below is the first step to get Theorem 1.1, announced in the Introduction.

THEOREM 3.1. — *Let $\mathcal{F}$ has degree k, and multiplicity ℓ at C; let C has genus g and degree d; and let* $\mathrm{Sing}(\widetilde{\mathcal{F}}|_E) = \{\widetilde{q}_1, \ldots, \widetilde{q}_t\}$. *Then*

$$\sum_{i=1}^{t} \mu(\widetilde{\mathcal{F}}|_E, \widetilde{q}_i) = (2-2g)\Big(1 + (\ell+1) + (\ell+1)^2 + \ldots + (\ell+1)^{n-3}\Big) \\ + (\ell+1)^{n-2}\Big((2-2g)(\ell+1) - (n+1)d\ell + (k-1)d(n-1)\Big).$$

Proof. — By Baum-Bott's formula [2], we have that

$$\sum_{i=1}^{t} \mu(\widetilde{\mathcal{F}}|_E, \widetilde{q}_i) = \int_E c_{n-1}(\mathcal{T}_E \otimes \mathcal{L}^*_{\widetilde{\mathcal{F}}})$$

with

$$c_{n-1}(\mathcal{T}_E \otimes \mathcal{L}^*_{\widetilde{\mathcal{F}}}) = \sum_{i=0}^{n-1} c_i(E) \cdot c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i-1}.$$

On the one hand,

$$c_i(E) = c_i(\mathcal{T}_{\widetilde{\mathbb{P}^n}} \otimes \mathcal{O}_E) - c_{i-1}(E)\zeta \tag{3.3}$$

and reapplying (3.3) recursively we obtain

$$c_i(E) = \sum_{j=0}^{i} (-1)^j c_{i-j}(\mathcal{T}_{\widetilde{\mathbb{P}^n}} \otimes \mathcal{O}_E)\zeta^j.$$

Set $\rho := \pi|_E$ and $\mathcal{N} := \mathcal{N}_{C/\mathbb{P}^n}$. Then, using also (2.22), for $i \geqslant 1$ we get

$$c_i(E) = \sum_{j=0}^{i-1} (-1)^j \pi^* c_{i-j}(\mathbb{P}^n)\zeta^j + (-1)^i \binom{n-1}{i}\zeta^i \\ + \sum_{j=1}^{i-1} (-1)^{i-j-1}\left(1 - \binom{n-j-1}{i-j}\right)\rho^* c_j(\mathcal{N})\zeta^{i-j} \\ + \sum_{j=0}^{i-2} (-1)^{i-j}\left(1 - \binom{n-j-1}{i-j-1}\right)\rho^* c_j(\mathcal{N})\rho^* c_1(C)\zeta^{i-j-1}.$$

On the other hand, as $c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}}) = \pi^* c_1(\mathcal{L}^*_{\mathcal{F}}) - \ell E$, we have

$$c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i-1} = \sum_{l=0}^{n-i-1} \binom{n-i-1}{l} \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l (-\ell E)^{n-i-l-1}.$$

Passing from E to C one rapidly sees that

$$\begin{aligned}\int_E \pi^* c_{i-j}(\mathbb{P}^n)\pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l \zeta^{n-i+j-l-1} &= 0 \quad \text{for } j \leqslant i-2 \text{ or } l \geqslant 1\\ \int_E \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l \zeta^{n-l-1} &= 0 \quad \text{for } l \geqslant 2\\ \int_E \rho^* c_j(\mathcal{N})\pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l \zeta^{n-j-l-1} &= 0 \quad \text{for } j \geqslant 2 \text{ or } l \geqslant 1\\ \int_E \rho^* c_j(\mathcal{N})\rho^* c_1(C)\pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l \zeta^{n-j-l-2} &= 0 \quad \text{for } j \geqslant 1 \text{ or } l \geqslant 1\end{aligned}$$

and we obtain for $i \geqslant 1$,

$$\begin{aligned}\int_E c_i(E)\cdot c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i-1} = {} & (-1)^n \ell^{n-i-1}\int_E \pi^* c_1(\mathbb{P}^n)\zeta^{n-2}\\ & + (-1)^{n-1}\ell^{n-i-1}\binom{n-1}{i}\int_E \zeta^{n-1}\\ & + (-1)^n \ell^{n-i-2}\binom{n-1}{i}\binom{n-i-1}{1}\int_E \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})\zeta^{n-2}\\ & + (-1)^{n-1}\ell^{n-i-1}\left(1-\binom{n-2}{i-1}\right)\int_E \rho^* c_1(\mathcal{N})\zeta^{n-2}\\ & + (-1)^{n-1}\ell^{n-i-1}\left(1-\binom{n-1}{i-1}\right)\int_E \rho^* c_1(C)\zeta^{n-2}.\end{aligned}$$

Finally,

$$\int_E c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-1} = (-1)^{n-1}\ell^{n-1}\int_E \zeta^{n-1} + (-1)^n \ell^{n-2}\binom{n-1}{1}\int_E \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})\zeta^{n-2}.$$

Using (2.17), it follows that

$$\begin{aligned}\sum_{i=1}^{t}\mu(\widetilde{\mathcal{F}}|_E,\widetilde{q}_i) &= (n+1)d\sum_{i=1}^{n-1}\ell^{n-i-1}-((n+1)d-2+2g)\sum_{i=0}^{n-1}\ell^{n-i-1}\binom{n-1}{i}\\ &\quad+(k-1)d\sum_{i=0}^{n-1}\ell^{n-i-2}\binom{n-i-1}{1}\binom{n-1}{i}\\ &\quad+((n+1)d-2+2g)\sum_{i=1}^{n-1}\ell^{n-i-1}\left(\binom{n-2}{i-1}-1\right)\\ &\quad+(2-2g)\sum_{i=1}^{n-1}\ell^{n-i-1}\left(\binom{n-1}{i-1}-1\right)\\ &= -((n+1)d-2+2g)\sum_{i=0}^{n-1}\ell^{n-i-1}\binom{n-1}{i}\\ &\quad+(k-1)d\sum_{i=0}^{n-1}\ell^{n-i-2}\binom{n-i-1}{1}\binom{n-1}{i}\\ &\quad+((n+1)d-2+2g)\sum_{i=1}^{n-1}\ell^{n-i-1}\binom{n-2}{i-1}\\ &\quad+(2-2g)\sum_{i=1}^{n-1}\ell^{n-i-1}\binom{n-1}{i-1}\\ &= -((n+1)d-2+2g)(\ell+1)^{n-1}+(k-1)d(n-1)(\ell+1)^{n-2}\\ &\quad+((n+1)d-2+2g)(\ell+1)^{n-2}+(2-2g)\sum_{i=0}^{n-2}(\ell+1)^i\end{aligned}$$

and it is straight forward obtaining the formula stated in the theorem. □

Example 3.2. — Let $\mathcal{F}$ be a holomorphic foliation by curves of degree $k \geqslant 2$ on $\mathbb{P}^n$, induced on the affine open set $U_0 = \{[x_0,\dots,x_n]\in\mathbb{P}^n \mid x_0 \neq 0\}$ by the vector field

$$\mathcal{D}_{\mathcal{F}} = \sum_{i=1}^{n-1}\Bigg(\sum_{|a|=k} c_{i,a} z_1^{a_1}\cdots z_{n-1}^{a_{n-1}}\Bigg)\frac{\partial}{\partial z_i} + \Bigg(\sum_{|a|=k-1} z_1^{a_1}\cdots z_{n-1}^{a_{n-1}} h_a(z)\Bigg)\frac{\partial}{\partial z_n}$$

where $z_i = x_i/x_0$, $a=(a_1,\dots,a_{n-1})$ is a multi-index with $|a| = \sum_{i=1}^{n-1} a_i$, $a_i \geqslant 0$, $c_{i,a}$ are constants and $h_a(z) = c'_{0,a} + c'_{1,a}z_1 + \ldots + c'_{n,a}z_n$ a linear function. We also consider the $f_i(z) = \sum_{|a|=k} c_{i,a}z^a$ linearly independent over $\mathbb{C}$.

Let C be the curve defined by $x_i = 0$ for $i = 1, \dots, n-1$. It is a curve of singularities of $\mathcal{F}$ and we blowup $\mathbb{P}^n$ along it. In the open set $\widetilde{U}_1$ with coordinates $u \in \mathbb{C}^n$, we have the relations

$$\sigma(u) = (u_1, u_1u_2, \dots, u_1u_{n-1}, u_n) = z \in \mathbb{C}^n.$$

Therefore, $\pi^*\mathcal{F}$ is generated by the vector field

$$\mathcal{D}_{\pi^*\mathcal{F}} = u_1^k \sum_{|a|=k} c_{1,a} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} \frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1} u_1^{k-1} g_{i,a}(u) \frac{\partial}{\partial u_i} + u_1^{k-1} \sum_{|a|=k-1} h_a(\sigma(u)) u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} \frac{\partial}{\partial u_n},$$

where

$$g_{i,a}(u) = \sum_{|a|=k} c_{i,a} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} - u_i \sum_{|a|=k} c_{1,a} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}}.$$

Since $m_C(f_i) = m_C(f_n) + 1 = k$, for $i = 1, \dots, n-1$, we have that the multiplicity $\ell := m_C(\mathcal{F}) = \operatorname{tang}(\pi^*\mathcal{F}, E) = k - 1$. In this way, the foliation $\widetilde{\mathcal{F}}$ induced via π is given in $\widetilde{U}_1$ by the vector field

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = u_1 \sum_{|a|=k} c_{1,a} u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} \frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1} g_{i,a}(u) \frac{\partial}{\partial u_i} + \sum_{|a|=k-1} h_a(\sigma(z)) u_2^{a_2} \cdots u_{n-1}^{a_{n-1}} \frac{\partial}{\partial u_n}.$$

It is easily seeing that on the affine open set, $u_n \in \mathbb{C}$, the foliation $\widetilde{\mathcal{F}}$, when restricted to the exceptional divisor E, which is given by $u_1 = 0$, defines a holomorphic foliation on $\mathbb{P}^{n-2}$ of degree k and with infinite hyperplane noninvariant. Consequently, there are $\sum_{i=0}^{n-2} k^i$ isolated singularities on E because for each $(u_2, \dots, u_{n-1})$ vanishing the $n-2$ first terms of $\mathcal{D}_{\widetilde{\mathcal{F}}|E}$ there is a unique u_n vanishing the last term of $\mathcal{D}_{\widetilde{\mathcal{F}}}$, namely, $\sum_{|a|=k-1} c'_{0,a} u_2^{a_2} \dots u_{n-1}^{a_{n-1}} + u_n \sum_{|a|=k-1} c'_{n,a} u_2^{a_2} \dots u_{n-1}^{a_{n-1}}$. Furthermore, at the fiber $\pi^{-1}[0 : 0 : \cdots : 0 : 1]$ the foliation $\widetilde{\mathcal{F}}$ has $\sum_{i=0}^{n-2} k^i$ additional singularities. Therefore, $\widetilde{\mathcal{F}}$ when restricted to E has $2\sum_{i=0}^{n-2} k^i$ singularities, which agrees with the number obtained by Theorem 3.1 taking $\ell = k-1$, $g = 0$ and $d = 1$.

Theorem 3.3. — *Let $\mathcal{F}$ has degree k, and multiplicity ℓ at C; let C has genus g and degree d; and let $\operatorname{Sing}(\widetilde{\mathcal{F}}) = \{\widetilde{p}_1, \ldots, \widetilde{p}_r\}$. Then*

$$\sum_{i=1}^{r} \mu(\widetilde{\mathcal{F}}, \widetilde{p}_i) = 1+k+k^2+\ldots+k^n+(2-2g)\Big(1+(\ell+1)+\ldots+(\ell+1)^{n-3}\Big) + (\ell+1)^{n-2}\Big((n+1)d(\ell^2-\ell) - (2-2g)\ell^2 - (k-1)d(n\ell-n+2)\Big).$$

Proof. — By Baum-Bott's formula, we have that

$$\sum_{i=1}^{r} \mu(\widetilde{\mathcal{F}}, \widetilde{p}_i) = \int_{\widetilde{\mathbb{P}}^n} c_n(\mathcal{T}_{\widetilde{\mathbb{P}}^n} \otimes \mathcal{L}^*_{\widetilde{\mathcal{F}}})$$

with

$$c_n(\mathcal{T}_{\widetilde{\mathbb{P}}^n} \otimes \mathcal{L}^*_{\widetilde{\mathcal{F}}}) = \sum_{i=0}^{n} c_i(\widetilde{\mathbb{P}}^n) \cdot c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i}.$$

If $i \geqslant 2$, the factor $c_i(\widetilde{\mathbb{P}}^n)$ is expressed by (2.22). And from (3.2) we get

$$c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i} = \sum_{l=0}^{n-i} \binom{n-i}{l} \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l (-\ell E)^{n-i-l}.$$

Passing from $\widetilde{\mathbb{P}}^n$ to E and then to C, one sees that

$$\begin{aligned}
\int_{\widetilde{\mathbb{P}}^n} \pi^* c_i(\mathbb{P}^n) \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l E^{n-i-l} &= 0 \quad \text{for } l \neq n-i \\
\int_{\widetilde{\mathbb{P}}^n} \rho^* c_j(\mathcal{N}) \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l E^{n-j-l} &= 0 \quad \text{for } j \geqslant 2 \text{ or } l \geqslant 2 \\
\int_{\widetilde{\mathbb{P}}^n} \rho^* c_j(\mathcal{N}) \rho^* c_1(C) \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^l E^{n-j-l-1} &= 0 \quad \text{for } j \geqslant 1 \text{ or } l \geqslant 1
\end{aligned}$$

hence

$$\begin{aligned}
&\int_{\widetilde{\mathbb{P}}^n} c_i(\widetilde{\mathbb{P}}^n) c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i} \\
&= \int_{\widetilde{\mathbb{P}}^n} \pi^* c_i(\mathbb{P}^n) \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^{n-i} + (-1)^{n-1} \ell^{n-i} \left(\binom{n-1}{i-1} - \binom{n-1}{i} \right) \int_{\widetilde{\mathbb{P}}^n} E^n \\
&\quad + (-1)^n \ell^{n-i-1} \binom{n-i}{1} \left(\binom{n-1}{i-1} - \binom{n-1}{i} \right) \int_{\widetilde{\mathbb{P}}^n} \pi^* c_1(\mathcal{L}^*_{\mathcal{F}}) E^{n-1} \\
&\quad + (-1)^n \ell^{n-i} \left(\binom{n-2}{i-2} - \binom{n-2}{i-1} \right) \int_{\widetilde{\mathbb{P}}^n} \rho^* c_1(\mathcal{N}) E^{n-1} \\
&\quad + (-1)^n \ell^{n-i} \left(\binom{n-1}{i-2} - \binom{n-1}{i-1} \right) \int_{\widetilde{\mathbb{P}}^n} \rho^* c_1(C) E^{n-1}
\end{aligned}$$

which yields

$$\begin{aligned}
&\int_{\widetilde{\mathbb{P}}^n} c_i(\widetilde{\mathbb{P}}^n) c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i} \\
&= \int_{\mathbb{P}^n} c_i(\mathbb{P}^n) c_1(\mathcal{L}^*_{\mathcal{F}})^{n-i} \\
&\qquad + (-1)^{n-1}\ell^{n-i}\left(\binom{n-1}{i-1} - \binom{n-1}{i}\right)\int_E \zeta^{n-1} \\
&\qquad + (-1)^{n}\ell^{n-i-1}\binom{n-i}{1}\left(\binom{n-1}{i-1} - \binom{n-1}{i}\right)\int_E \pi^* c_1(\mathcal{L}^*_{\mathcal{F}})\zeta^{n-2} \\
&\qquad + (-1)^{n}\ell^{n-i}\left(\binom{n-2}{i-2} - \binom{n-2}{i-1}\right)\int_E \rho^* c_1(\mathcal{N})\zeta^{n-2} \\
&\qquad + (-1)^{n}\ell^{n-i}\left(\binom{n-1}{i-2} - \binom{n-1}{i-1}\right)\int_E \rho^* c_1(C)\zeta^{n-2}
\end{aligned}$$

and thus

$$\begin{aligned}
\int_{\widetilde{\mathbb{P}}^n} c_i(\widetilde{\mathbb{P}}^n) c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-i} &= (k-1)^{n-i}\binom{n+1}{i} \\
&\quad - \ell^{n-i}((n+1)d-2+2g)\left(\binom{n-1}{i-1} - \binom{n-1}{i}\right) \\
&\quad + \ell^{n-i-1}(k-1)d\binom{n-i}{1}\left(\binom{n-1}{i-1} - \binom{n-1}{i}\right) \\
&\quad + \ell^{n-i}((n+1)d-2+2g)\left(\binom{n-2}{i-2} - \binom{n-2}{i-1}\right) \\
&\quad + \ell^{n-i}(2-2g)\left(\binom{n-1}{i-2} - \binom{n-1}{i-1}\right).
\end{aligned}$$

From (2.21) and (3.2) we have that

$$\begin{aligned}
&\int_{\widetilde{\mathbb{P}}^n} c_1(\widetilde{\mathbb{P}}^n) c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^{n-1} \\
&= \int_{\widetilde{\mathbb{P}}^n} \pi^* c_1(\mathbb{P}^n)\pi^* c_1(\mathcal{L}^*_{\mathcal{F}})^{n-1} + (-\ell)^{n-1}\int_{\widetilde{\mathbb{P}}^n} \pi^* c_1(\mathbb{P}^n) E^{n-1} \\
&\qquad + (-1)^{n-1}\ell^{n}(n-1)(n-2)\int_{\widetilde{\mathbb{P}}^n} \pi^* c_1(\mathcal{L}^*_{\mathcal{F}}) E^{n-1} \\
&\qquad + (-1)^{n}\ell^{n-1}(n-2)\int_{\widetilde{\mathbb{P}}^n} E^n \\
&= (n+1)(k-1)^{n-1} + \ell^{n-1}(n-2)((n+1)d-2+2g) \\
&\qquad - \ell^{n-2}(n-1)(n-2)(k-1)d - \ell^{n-1}(n+1)d.
\end{aligned}$$

Finally, from (3.2) we get

$$\int_{\widetilde{\mathbb{P}^n}} c_1(\mathcal{L}^*_{\widetilde{\mathcal{F}}})^n = (k-1)^n + \ell^n((n+1)d - 2 + 2g) - \ell^{n-1}(k-1)nd.$$

Summing up the cases $i=0$, $i=1$ and $i \geqslant 2$, we obtain

$$\begin{aligned}
\sum_{i=1}^{r} \mu(\widetilde{\mathcal{F}}, \widetilde{p}_i) = & \sum_{i=0}^{n} (k-1)^{n-i} \binom{n+1}{i} - \ell^{n-1}(n+1)d \\
& - ((n+1)d - 2 + 2g) \sum_{i=0}^{n} \ell^{n-i} \left(\binom{n-1}{i-1} - \binom{n-1}{i} \right) \\
& + (k-1)d \sum_{i=0}^{n} \ell^{n-i-1} \binom{n-i}{1} \left(\binom{n-1}{i-1} - \binom{n-1}{i} \right) \\
& + ((n+1)d - 2 + 2g) \sum_{i=2}^{n} \ell^{n-i} \left(\binom{n-2}{i-2} - \binom{n-2}{i-1} \right) \\
& + (2-2g) \sum_{i=2}^{n} \ell^{n-i} \left(\binom{n-1}{i-2} - \binom{n-1}{i-1} \right) \\
= & \sum_{i=0}^{n} k^i - \ell^{n-1}(n+1)d \\
& + (2-2g) \left(\sum_{i=0}^{n-2} (\ell+1)^i - (\ell+1)^{n-1} + \ell^{n-1} \right) \\
& - ((n+1)d - 2 + 2g)(\ell+1)^{n-2}(\ell^2 - 1) \\
& - (k-1)d(\ell+1)^{n-2}(n\ell - n + 2) \\
& + ((n+1)d - 2 + 2g)((\ell+1)^{n-2}(1-\ell) + \ell^{n-1})
\end{aligned}$$

and the desired formula is straight forward. □

As a consequence we get Theorem 1.1 in the case we are dealing with here.

Corollary 3.4. — *Let $\mathcal{F}$ has degree k, and multiplicity ℓ at C; let C has genus g and degree d. Then*

$$\begin{aligned}
\sum_{i=1}^{s} \mu(\mathcal{F}, p_i) = & \, 1 + k + k^2 + \ldots + k^n \\
& + (\ell+1)^{n-2} \Big((2g-2)(\ell^2 + \ell + 1) + (n+1)d\ell^2 - (k-1)d(n\ell + 1) \Big).
\end{aligned}$$

Proof. — Just note that

$$\sum_{i=1}^{s}\mu(\mathcal{F},p_i)=\sum_{i=1}^{r}\mu(\widetilde{\mathcal{F}},\widetilde{p}_i)-\sum_{i=1}^{t}\mu(\widetilde{\mathcal{F}}|_E,\widetilde{q}_i)$$

and recall Theorems 3.1 and 3.3. □

Let $\mathcal{F}$ be as in the Example 3.2. It has no singularities in U_0 but the ones in $C\cap U_0$. The hyperplane $H_0:=\mathbb{P}^n\setminus U_0$ is isomorphic to $\mathbb{P}^{n-1}$ as well as invariant by $\mathcal{F}$. As the degree of $\mathcal{F}|_{H_0}$ remains k, the number of isolated singularities, counted with multiplicities, of $\mathcal{F}$ on H_0 is $\sum_{i=0}^{n-1}k^i$. Given that the singularity $q=[0:0:\cdots 0:1]\in C$ has Milnor number $\mu(\mathcal{F}|_{H_0},q)=k^{n-1}$, it follows that $\mathcal{F}$ has $\sum_{i=0}^{n-2}k^i$ isolated singularities in $\mathbb{P}^n$, counted with multiplicities, which agrees with the number obtained by Corolarry 3.4 taking $\ell=k-1$, $g=0$ and $d=1$.

4. Special Foliations along Complete Intersections

The aim of this section is proving Theorem 1.1 on its full generality. In order to do so we need a result on foliations admiting a complete intersection curve in its singular locus.

Lemma 4.1. — *Let $\mathcal{F}$ be a holomorphic foliation by curves on $\mathbb{P}^n$, $n\geqslant 3$, with*

$$\operatorname{Sing}(\mathcal{F})=C\cup\{p_1,\dots,p_s\},$$

where the union is disjoint, C is an irreducible singular projective curve, the p_i are isolated closed points, and $\mathcal{F}$ is special along C. Then there exists a one-parameter family of holomorphic foliations by curves on $\mathbb{P}^n$, given by $\{\mathcal{F}_t\}_{t\in D}$ where $D=\{t\in\mathbb{C}\,|\,|t|<\epsilon\}$ such that:

(i) $\mathcal{F}_0=\mathcal{F}$;
(ii) $\deg(\mathcal{F}_t)=\deg(\mathcal{F})$;
(iii) $\operatorname{Sing}(\mathcal{F}_t)=C_t\cup\{p_1^t,\dots,p_{s_t}^t\}$, *where C_t is a regular irreducible projective curve with $\deg(C_t)=\deg(C)$, and the p_i^t are closed points;*
(iv) *$\mathcal{F}_t$ is special along C_t and $m_{C_t}(\mathcal{F}_t)=m_C(\mathcal{F})$;*
(v) $\sum_{i=1}^{s_t}\mu(\mathcal{F}_t,p_i^t)=\sum_{i=1}^{s}\mu(\mathcal{F},p_i)$.

Proof. — Assume C is given, in an affine standard chart of $\mathbb{P}^n$, by the zeros of the polynomials $f_1,\dots,f_{n-1}$. Take polynomials $h_1,\dots,h_{n-1}$ and consider the holomorphic function

$$\begin{array}{cccc} F_t: & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & z=(z_1,\dots,z_n) & \longmapsto & \Big(f_1(z)+th_1(z),\dots,f_{n-1}(z)+th_{n-1}(z),z_n\Big). \end{array}$$

For each $t \in \mathbb{C}$ and any $z \in \mathbb{C}^n$, set $M_t(z)$ to be the first $(n-1) \times (n-1)$-minor of the jacobian matrix $D_z F_t$. Define $U_t := \mathbb{C}^n \setminus \{\det M_t = 0\}$ and let C_t be the projective closure of the common zeros locus of the $f_i + t h_i$. Note that $F_t|_{U_t}$ is a local biholomorphism onto an open set $V_t \subset \mathbb{C}^n$ and the image of $C_t \cap U_t$ by F_t is the w_n-axis restricted to V_t so one may fix coordinates $w = F_t(z)$. In particular, we may describe the pushforward $(F_0)_*\mathcal{F}$ in V_0, as in (2.2), by the vector field

$$\mathcal{D}_{(F_0)_*\mathcal{F}} = P_1 \frac{\partial}{\partial w_1} + \ldots + P_n \frac{\partial}{\partial w_n}$$

where

$$P_i(w) = \sum_{|a|=m_i} w_1^{a_1} \cdots w_{n-1}^{a_{n-1}} P_{i,a}(w) \tag{4.1}$$

with at least one $P_{i,a}(z)$ not vanishing in the w_n-axis.

For each $t \in \mathbb{C}$, we define $\mathcal{F}_t$ by the vector field

$$\mathcal{D}_{\mathcal{F}_t} = Q_1^t \frac{\partial}{\partial z_1} + \ldots + Q_n^t \frac{\partial}{\partial z_n}$$

where the Q_i^t are obtained by the system

$$\begin{pmatrix} P_1 \circ F_t \\ \vdots \\ P_n \circ F_t \end{pmatrix} = DF_t \cdot \begin{pmatrix} Q_1^t \\ \vdots \\ Q_n^t \end{pmatrix} \tag{4.2}$$

and DF_t is the Jacobian matrix

$$\begin{pmatrix} \partial(f_1+th_1)/\partial z_1 & \ldots & \partial(f_1+th_1)/\partial z_{n-1} & \partial(f_1+th_1)/\partial z_n \\ \vdots & \ddots & \vdots & \vdots \\ \partial(f_{n-1}+th_{n-1})/\partial z_1 & \ldots & \partial(f_{n-1}+th_{n-1})/\partial z_{n-1} & \partial(f_{n-1}+th_{n-1})/\partial z_n \\ 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Solving the system by Cramer's rule, we have

$$Q_i^t = \frac{\det A_i^t}{\det M_t}$$

where one gets A_i^t replacing the ith column of DF_t by the column vector at the left hand side of the equality (4.2). In particular,

$$Q_n^t = \frac{P_n \circ F_t \cdot \det M_t}{\det M_t} = P_n \circ F_t.$$

Therefore, normalizing by the factor $\det M_t$, one may describe $\mathcal{F}$ in U_t by

$$\mathcal{D}_{\mathcal{F}_t} = \det A_1^t \frac{\partial}{\partial z_1} + \ldots + \det A_{n-1}^t \frac{\partial}{\partial z_{n-1}} + P_n \circ F_t \frac{\partial}{\partial z_n}. \tag{4.3}$$

As the components of $\mathcal{D}_{\mathcal{F}_t}$ are polynomials, using Hartogs Extension Theorem, we can consider $\mathcal{D}_{\mathcal{F}_t}$ defined in $\mathbb{C}^n$.

It is immediate that $\mathcal{F}_0 = \mathcal{F}$. Besides, the $\mathcal{F}_t$ were built targeting property (iv) above. In fact, the pushforwards $(F_t)_*\mathcal{F}_t$ agree with $(F_0)_*\mathcal{F}$ no matter is $t \in \mathbb{C}$, hence, by (4.1) and Lemma 2.1.(i),(iii), it follows that $m_{C_t}(\mathcal{F}_t) = m_C(\mathcal{F})$ and $\mathcal{F}_t$ special along C_t for every $t \in \mathbb{C}$, because $\mathcal{F}$ is so. By construction, F_t is a local biholomorphism, so $\mathrm{Sing}(\mathcal{F}_t)$ must be the disjoint union of C_t and points, with C_t irreducible since its image by F_t is a line. Assuming $\deg(h_i) \leqslant \deg(f_i)$ for $1 \leqslant i \leqslant n-1$, one assures that $\deg(C_t) = \deg(C)$, and also, by (4.3), that $\deg(\mathcal{F}_t) = \deg(\mathcal{F})$.

Now set $\mathbb{C}^{n+1} = \{(z,t) \in \mathbb{C}^n \times \mathbb{C}\}$. Note that the family $S := \cup_{t\in\mathbb{C}} (C_t \cap \mathbb{C}^n)$ is an algebraic surface in $\mathbb{C}^{n+1}$. On the other hand, singularity imposes n conditions by the vanishing of the $(n-1)\times(n-1)$-minors of the jacobian matrix $DF_t(z)$ and generically determines an algebraic curve in $\mathbb{C}^{n+1}$. If these two varieties happen to meet, which is the case since C is singular, they do generically at isolated closed points, so one may adjust the h_i and find $\epsilon > 0$ sufficiently small such that $C_t \cap \mathbb{C}^n$ is regular for $0 < |t| < \epsilon$. The whole family surface $\overline{S} := \cup_{t\in\mathbb{C}} C_t$ intersects $H := (\mathbb{P}^n \setminus \mathbb{C}^n) \times \mathbb{C}$ at a curve. Change coordinates (a priori) and assume C has no singular points at H. Since singularity is a closed algebraic condition, either finitely many points of $\overline{S} \cap H$ are singular points of their curves, or all of them are so. But this contradicts our assumption, so one may take ϵ smaller if necessary to get the whole C_t regular if $0 < t < \epsilon$. And one may shrink ϵ even more in order to have

$$\sum_{i=1}^{s_t} \mu(\mathcal{F}_t, p_i^t) = \sum_{i=1}^{s} \mu(\mathcal{F}, p_i)$$

as well. □

Now Theorem 1.1 is straight forward. Suppose $\mathrm{Sing}(\mathcal{F})$ has a unique nonzero dimensional component C. Use Lemma 4.1 to pick up any $\mathcal{F}'$ among the $\{\mathcal{F}_t\}_{t\in D\setminus 0}$. Apply Corollary 3.4 to $\mathcal{F}'$ and get a formula for $\sum_{i=1}^{s'} \mu(\mathcal{F}', p_i')$. By Lemma 4.1, this formula holds for $\sum_{i=1}^{s} \mu(\mathcal{F}, p_i)$ with same d, ℓ, k up to the factor $2g-2$. So let g' be the genus of C' and let $\overline{g}$ and δ be respectively the geometric genus and the cogenus of C. Since there is a continuos deformation from C to C', and C is a set theoretic complete intersection, we have

$$\begin{aligned} 2g' - 2 = -\chi(C') &= -\chi(C) + \sum_{i=1}^{l} \mu(C, q_i) \\ &= 2\overline{g} - 2 + 2\delta - \sum_{i=1}^{l} (b_i - 1) \\ &= 2(\overline{g} + \delta) - 2 - \sum_{i=1}^{l} (b_i - 1) \\ &= 2g - 2 - \sum_{i=1}^{l} (b_i - 1) \end{aligned}$$

and one gets Theorem 1.1 for $\mathcal{F}$ in the case there is just one C. If there are many, say, $C_1, \ldots, C_r$, set $Y_0 = \mathbb{P}^n$ and take a sequence of blowups $\pi_i : Y_i \to Y_{i-1}$ centered at C_i with exceptional divisor E_i. Then it is just a matter of slightly adjusting succesively the proof of Theorem 3.3 just noticing that $E_i \cdot E_j = 0$ if $i \neq j$ because the curves C_i and C_j are disjoint.

5. The Unicity Problem

In this section we prove Theorems 1.2 and 1.3. For the remainder, $\mathcal{F}$ is a foliation by curves on $\mathbb{P}^n$, $n \geqslant 3$, with

$$\operatorname{Sing}(\mathcal{F}) = C \cup \{p_1, \ldots, p_s\} \tag{5.1}$$

where the union is disjoint, C is an integral and smooth projective curve, the p_i are closed points, $\mathcal{F}$ is special along C, and $\pi : \widetilde{\mathbb{P}}^n \to \mathbb{P}^n$ is the blowup of $\mathbb{P}^n$ along C with exceptional divisor E.

We also recall from the Introduction that $\operatorname{gd}(C)$ is the *generating degree of* C, the least integer $d > 0$ such that $\mathcal{I}_C(d)$ is globally generated.

Theorem 5.1. — *Let $\mathcal{F}'$ be a holomorphic foliation by curves on $\mathbb{P}^n$ for which we have $\deg(\mathcal{F}') = \deg(\mathcal{F}) > \operatorname{gd}(C)$, and such that $\operatorname{Sing}(\mathcal{F}) \subset \operatorname{Sing}(\mathcal{F}')$, and also $\operatorname{Sing}(\widetilde{\mathcal{F}}|_E) \subset \operatorname{Sing}(\widetilde{\mathcal{F}}'|_E)$. Then $\mathcal{F}' = \mathcal{F}$.*

Proof. — Set $\deg(\mathcal{F}) = \deg(\mathcal{F}') = k$. Then $\mathcal{F}$ and $\mathcal{F}'$ are induced by sections

$$s_{\mathcal{F}}, s_{\mathcal{F}'} \in H^0(\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(k-1)).$$

Similarly, $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ are induced by

$$s_{\widetilde{\mathcal{F}}}, s_{\widetilde{\mathcal{F}}'} \in H^0(\pi^* \mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^n}(k-1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^n}(-E))$$

because $\mathcal{L}^*_{\widetilde{\mathcal{F}}} = \mathcal{L}^*_{\widetilde{\mathcal{F}'}} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(k-1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-\ell E)$ and $\ell = 1$ since C is integral and $\mathcal{F}$ is special along C.

We will prove that

$$s_{\widetilde{\mathcal{F}}} = \lambda \cdot s_{\widetilde{\mathcal{F}'}} \quad \text{iff} \quad \operatorname{Sing}(\widetilde{\mathcal{F}}) \subset \operatorname{Sing}(\widetilde{\mathcal{F}'}) \tag{5.2}$$

for some $\lambda \in \mathbb{C}^*$ and $k > \operatorname{gd}(C)$. If so, we get the statement of the theorem since $\operatorname{Sing}(\mathcal{F}) \subset \operatorname{Sing}(\mathcal{F}')$ and $\operatorname{Sing}(\widetilde{\mathcal{F}}|_E) \subset \operatorname{Sing}(\widetilde{\mathcal{F}'}|_E)$ imply that $\operatorname{Sing}(\widetilde{\mathcal{F}}) \subset \operatorname{Sing}(\widetilde{\mathcal{F}'})$, and, by projection, $s_{\widetilde{\mathcal{F}}} = \lambda \cdot s_{\widetilde{\mathcal{F}'}}$ implies $s_{\mathcal{F}} = \lambda \cdot s_{\mathcal{F}'}$, and the latter yields $\mathcal{F}' = \mathcal{F}$.

In order to get (5.2), adjusting the proofs of [4, Thm. 2.2, Cor. 3.2], with $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(1)$ playing the role of an ample bundle, it suffices checking that $\pi^*\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-E)$ is simple and that

$$H^p(\pi^*\Omega^q_{\mathbb{P}^n} \otimes \pi^*\mathcal{T}_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((1-q)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}((q-1)E)) = 0 \tag{5.3}$$

for $2 \leqslant q \leqslant n$ and $p = q-2, q-1$.

Simplicity immediately follows from projection formula

$$H^0(\pi^*\Omega^1_{\mathbb{P}^n} \otimes \pi^*\mathcal{T}_{\mathbb{P}^n}) \simeq H^0(\Omega^1_{\mathbb{P}^n} \otimes \mathcal{T}_{\mathbb{P}^n}) \simeq \mathbb{C}$$

while (5.3), within the desired range, deserves more care.

From [3, Lem. 1.4] we know that if $0 \leqslant t \leqslant n-2$, then

$$H^i(\pi^*F \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(tE)) \simeq H^i(F) \tag{5.4}$$

for all $i \in \mathbb{N}$ and any locally free sheaf F. On the other hand, from [5],

$$H^p(\Omega^q_{\mathbb{P}^n} \otimes \mathcal{T}_{\mathbb{P}^n}((1-q)(k-1))) = 0 \tag{5.5}$$

for $k \geqslant 0$ and $p < q$, $2 \leqslant q \leqslant n$.

Therefore, from (5.4) and (5.5), we have for $2 \leqslant q \leqslant n-1$

$$\begin{aligned} H^p(\pi^*\Omega^q_{\mathbb{P}^n} \otimes \pi^*\mathcal{T}_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((1-q)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}((q-1)E)) \\ = H^p(\Omega^q_{\mathbb{P}^n} \otimes \mathcal{T}_{\mathbb{P}^n}((1-q)(k-1))) = 0. \end{aligned}$$

Now we analyze the case $q = n$, that is, the vanishing of

$$H^p(\pi^*\Omega^n_{\mathbb{P}^n} \otimes \pi^*\mathcal{T}_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((1-n)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}((n-1)E))$$

for $p = n-2, n-1$. Observe that above groups are

$$H^p(\pi^*\mathcal{T}_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((1-n)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(E) \otimes \omega_{\widetilde{\mathbb{P}^n}})$$

since the dualizing sheaf on $\widetilde{\mathbb{P}^n}$ is $\omega_{\widetilde{\mathbb{P}^n}} = \pi^*\Omega^n_{\mathbb{P}^n} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}((n-2)E)$. By Serre's duality

$$\begin{aligned} H^{n-i}(\pi^*\mathcal{T}_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((1-n)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(E) \otimes \omega_{\widetilde{\mathbb{P}^n}}) \\ \simeq H^i(\pi^*\Omega^1_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((n-1)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-E)). \end{aligned}$$

Then, we must have to prove that

$$H^i(\pi^*\Omega^1_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((n-1)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^n}(-E)) = 0$$

for $i = 1, 2$.

Since $\pi_*\mathcal{O}_{\widetilde{\mathbb{P}}^n}(-E) = \mathcal{I}_C$ and $\mathrm{R}^i\pi_*\mathcal{O}_{\widetilde{\mathbb{P}}^n}(-E) = 0$, it follows from the projection formula and Leray's spectral sequence that

$$H^i(\pi^*\Omega^1_{\mathbb{P}^n} \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}((n-1)(k-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^n}(-E)) \simeq$$

$$H^i(\Omega^1_{\mathbb{P}^n} \otimes \mathcal{I}_C((n-1)(k-1))) \tag{5.6}$$

so we just have to check the vanishing of (5.6) for $i = 1, 2$. In order to get this, by Mumford's regularity theorem, it suffices to show that $(n-1)(k-1) \geqslant m-1$ if $\Omega^1_{\mathbb{P}^n} \otimes \mathcal{I}_C$ is m-regular.

From Bott's formulae, $\Omega^1_{\mathbb{P}^n}$ is 2-regular, while $\mathcal{I}_C$ is $((n-1)\mathrm{gd}(C)-n+2)$-regular by [3]. Hence $\Omega^1_{\mathbb{P}^n} \otimes \mathcal{I}_C$ is $((n-1)\mathrm{gd}(C) - n + 4)$-regular owing to [6, Prp. 1.8.9].

But, by hypothesis,

$$(n-1)(k-1) \geqslant (n-1)\mathrm{gd}(C) \geqslant (n-1)\mathrm{gd}(C) - n + 3$$

for $n \geqslant 3$ and we are done. □

In the case of three dimensional projective space, we can also get the following.

Theorem 5.2. — *Let $\mathcal{F}'$ be a foliation on $\mathbb{P}^3$ with $\deg(\mathcal{F}') = \deg(\mathcal{F})$, such that $\mathrm{Sing}(\mathcal{F}) \subset \mathrm{Sing}(\mathcal{F}')$. If C is also nondegenerated and a set theoretic complete intersection in $\mathbb{P}^3$, then $\mathcal{F}' = \mathcal{F}$.*

Proof. — Set $\deg(\mathcal{F}) = \deg(\mathcal{F}') = k$. According to the prior result, we just have to prove that, in $\mathbb{P}^3$, we always have $k > \mathrm{gd}(C)$; and $\mathrm{Sing}(\widetilde{\mathcal{F}}|_E) \subset \mathrm{Sing}(\widetilde{\mathcal{F}}'|_E)$ always holds as well.

For the first, under the hypothesis on C, assume it is given by the intersection of surfaces of degree d_1 and d_2, with $d_2 \geqslant d_1 \geqslant 2$. It follows from [7, Lem. 3.6] $\deg(\mathcal{F}) \geqslant (m_C(\mathcal{F}) + 1)d_2 + d_1 - 2$. Since $\mathcal{F}$ is special along C, which is integral, we have

$$k \geqslant 2d_2 + d_1 - 2 \geqslant 2d_2 > d_2 + 1 \geqslant \mathrm{gd}(C) + 1$$

because, by definition, $d_2 \geqslant \mathrm{gd}(C)$.

For the second, just recall that E is naturally identified with the projectivization $\mathbb{P}(\mathcal{N}_{C/\mathbb{P}^3})$ of the normal bundle $\mathcal{N}_{C/\mathbb{P}^3}$. Thus, since the vector fields inducing the foliations $\mathcal{F}$ and $\mathcal{F}'$ vanish identically along C, the lifts $\widetilde{\mathcal{F}}|_E$ and $\widetilde{\mathcal{F}}'|_E$ must be tangent to the the fibers of $\mathcal{N}_{C/\mathbb{P}^3}$, and hence coincide. □

Now we show that the assumption on the curve to be nondegenerated is necessary. In fact, we build a family of foliations by curves on $\mathbb{P}^3$ with same singular locus consisting of a degenerated smooth curve and isolated points, and all of them special along this curve.

Let $\mathcal{F}_t$ be a holomorphic foliation by curves on $\mathbb{P}^3$, with $t \in \mathbb{C}$, induced on the affine open set $U_0 = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \,|\, x_0 \neq 0\}$ by the vector field

$$\mathcal{D}_{\mathcal{F}_t} = \big(a_0 z_1^2 + a_1 z_1 z_2 + a_2 z_2^2\big)\frac{\partial}{\partial z_1} + \big(b_0 z_1^2 + b_1 z_1 z_2 + b_2 z_2^2\big)\frac{\partial}{\partial z_2} + \Big(z_1\big(\alpha_0 + \alpha_1 z_1 + (\alpha_2 - t) z_2 + \alpha_3 z_3\big) + z_2\big(\beta_0 + t z_1 + \beta_2 z_2 + \beta_3 z_3\big)\Big)\frac{\partial}{\partial z_3}$$

with $z_i = x_i/x_0$.

Assume the polynomials $\sum_{j=0}^2 a_j \lambda^j$ and $\sum_{j=0}^2 b_j \lambda^j$ have no common roots and let λ_i, for $i = 1, 2, 3$, be the roots of

$$\sum_{j=0}^{2} b_j \lambda^j - \lambda \sum_{j=0}^{2} a_j \lambda^j.$$

One can check that $\mathrm{Sing}(\mathcal{F}_t)$ is the union of the curve $C := \{x_1 = x_2 = 0\}$ and the points

$$[0 : u_i : \lambda_i u_i : 1] \in \mathbb{P}^3$$

where

$$u_i = \frac{a_0 + a_1\lambda_i + a_2\lambda_i^2 - \alpha_3 - \beta_3\lambda_i}{\alpha_1 + \alpha_2\lambda_i + \beta_2\lambda_i^2}$$

so $\mathrm{Sing}(\mathcal{F}_t)$ does not depend on t. Note also that the foliation $\widetilde{\mathcal{F}_t}$ induced by $\mathcal{F}_t$ via the blowup of $\mathbb{P}^3$ along C, has the same singular locus.

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Manuscrit reçu le 28 septembre 2012,
accepté le 26 novembre 2013.

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