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SPHERICAL VARIETIES AND WAHL'S CONJECTURE

by Nicolas PERRIN

Abstract. — Using the theory of spherical varieties, we give a type independent very short proof of Wahl's conjecture for cominuscule homogeneous varieties for all primes different from 2.

Résumé. — En utilisant les variétés sphériques, nous donnons, en toute caractéristique impaire, une preuve courte et uniforme de la conjecture de Wahl pour les variétés homogènes cominuscules.

1. Introduction

Let V be a smooth projective variety and let L and M be two line bundles on V. It is natural to consider the so called *Gaussian map*:

$$H^0\big(V \times V, \mathcal{I}_\Delta \otimes L \boxtimes M\big) \longrightarrow H^0\big(V, \Omega^1_V \otimes L \otimes M\big),$$

where $\mathcal{I}_\Delta$ is the ideal of the diagonal in $V \times V$, where $L \boxtimes M$ is the external product on $V \times V$ and the map is induced by the restriction map $\mathcal{I}_\Delta \to \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \simeq \Omega^1_V$. Wahl studied this map in detail. In particular in [20] he conjectured that the Gaussian map should be surjective for V a projective rational homogeneous variety and L and M any ample line bundles. This conjecture was proved by Kumar in characteristic 0 in [10]. Lakshmibai, Mehta and Parameswaran [11] considered the situation in positive characteristic and proved that the following conjecture (now called *LMP-conjecture*) implies Wahl's conjecture in positive characteristic. From now on in the introduction, the base field k is algebraically closed of positive characteristic p.

Conjecture 1.1. — *Let V be a rational projective homogeneous variety, let $X = V \times V$ and let $\widetilde{X}$ be the blowing-up of the diagonal Δ in X. Then $\widetilde{X}$ is Frobenius split compatibly with the exceptional divisor E.*

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This conjecture is equivalent to the existence of a splitting on $V \times V$ with maximal multiplicity along the diagonal (see [11] for more on this). This conjecture has been considered by several authors (see for example [15], [12], [5], [13], [18]). In particular Brown and Lakshmibai in [5] proved this conjecture for minuscule homogeneous varieties using Representation Theoretic techniques and a case by case analysis.

In this paper we give a new proof of LMP-conjecture and therefore of Wahl's conjecture for cominuscule homogeneous varieties (see Definition 4.1) using the theory of spherical varieties. Let V be a cominuscule homogeneous variety and let $\widetilde{X}$ be the blow-up of the diagonal in $X = V \times V$.

Theorem 1.2. — *Assume that $p \neq 2$, then $\widetilde{X}$ is Frobenius split compatibly with the exceptional divisor.*

Since any minuscule homogeneous variety is cominuscule for some other group this also implies the result in the minuscule case. The advantages of this proof is that it is mainly geometric, it completely avoids the case by case analysis in [5] and it is very short.

One of the main arguments is to use a consequence of results of Littelmann: If V is cominuscule, then $X = V \times V$ is spherical. Using this and a result of Brion and Inamdar [3], a very simple proof of Theorem 1.2 is given in Section 2 for large primes and in particular in characteristic 0. This result might have been known to Inamdar as a remark at the end of the paper [15] seems to indicate. We thank Niels Lauritzen for pointing this to us.

To obtain the result for all odd primes, we need to do a parabolic induction from a symmetric variety and use a result of de Concini and Springer [6].

Acknowledgement. — I want to thank Michel Brion for useful email exchanges on the subject in particular for the reference [6] and Martí Lahoz Vilalta for useful discussions. I also thank the referee for valuable comments especially corrections and simplifications for the proofs of Corollaries 3.2 and 3.3.

2. Very simple proof for large primes

In this section we give a very short proof of Theorem 1.2 for large primes. Here we assume that the base field k is of characteristic 0. Write $V = G/P$ with G semisimple and P a parabolic subgroup. Write $X = V \times V$, write

$\widetilde{X}$ for the blow-up of the diagonal and E for the exceptional divisor. The following is a consequence of results of Littelmann [14].

Theorem 2.1. — *If P is cominuscule, then X and $\widetilde{X}$ are spherical for the diagonal action of G.*

Actually $G/P \times G/P$ is G-spherical if and only if P is minuscule or cominuscule, see [17]. We may thus apply the following result (see [3, Theorem 1]) to $\widetilde{X}$ and E.

Theorem 2.2. — *For Z a G-spherical variety, its reduction Z_p modulo a prime p is Frobenius split compatibly with all closed G-stable subvarieties for all but finitely many p.*

Corollary 2.3. — *The variety $\widetilde{X}_p$ is Frobenius split compatibly with E_p for all but finitely many p.*

3. Frobenius splitting of symmetric varieties

In this section we extend results of de Concini and Springer [6] on Frobenius splitting of wonderful compactifications to any symmetric varieties. The results we obtain are probably well known to the experts but we could not find a reference for them. We assume from now on that the base field k has positive characteristic $p \neq 2$.

Recall that for L a connected reductive group, a closed subgroup K is called spherical if L/K has a dense orbit of a Borel subgroup B_L of L. An embedding of L/K is a normal L-variety equivariantly containing L/K as an open orbit. An embedding is called simple if it contains a unique closed L-orbit and toroidal if the B_L-stable divisors containing an L-orbit are L-stable. We refer to [9] for further results on spherical embeddings.

From now on in this section, we fix L a connected semisimple algebraic group of adjoint type and θ a group involution on L. Let L^θ be the subgroup of θ-invariant elements. We have $N_L(L^\theta) = L^\theta$ (see [8, Corollary 1.3]). Write $(L^\theta)^0$ for the connected component containing the identity element in L^θ and let K be a subgroup such that $(L^\theta)^0 \subset K \subset L^\theta$. Then L/L^θ is a homogeneous symmetric variety and therefore a spherical variety (see [19]) thus L/K is also spherical. It is actually symmetric for the universal covering $\tilde{L}$ of L (see Remark 3.5). We want to extend Frobenius splitting results for embeddings of L/L^θ obtained by de Concini and Springer to embeddings of L/K. We fix some notation. Let T_L be a maximal torus of L containing a split maximal torus S (*i.e.* a maximal torus such that $\theta|_S$

acts as the inverse) and let B_L be a Borel subgroup of L containing T_L. Recall from [4, Theorem 4.1.15], that for any parabolic subgroup Q^- of L, there is a unique B_L-canonical splitting $\tau_{L/Q^-} \in H^0(L/Q^-, \omega_{L/Q^-}^{1-p})$. We recall the following results from [6].

Proposition 3.1. — *Let L be of adjoint type and θ be a group involution of L.*

(i) There exists a unique simple smooth projective toroidal embedding $\mathbf{Y}$ of L/L^θ.

(ii) There exist a parabolic subgroup Q of L containing B_L, an open affine subset $\mathbf{Y}_0$ of $\mathbf{Y}$ which meets all the L-orbits and $\mathbf{Z}$ a closed subvariety of $\mathbf{Y}_0$ such that:

- *The Levi subgroup $L(Q)$ of Q containing T_L acts on $\mathbf{Z}$ and its derived subgroup $D(L(Q))$ acts trivially on $\mathbf{Z}$ so that $\mathbf{Z}$ is a toric variety for a quotient of $L(Q)/D(L(Q))$;*
- *The multiplication map $R_u(Q) \times \mathbf{Z} \to \mathbf{Y_0}$ is an isomorphism.*

(iii) Let Q^- be the parabolic subgroup opposite to Q with respect to T_L. Then the unique closed orbit in $\mathbf{Y}$ is isomorphic to L/Q^- and the pull-back map $\mathrm{Pic}(\mathbf{Y}) \to \mathrm{Pic}(L/Q^-)$ is injective.

(iv) The irreducible L-stable divisors $(\mathbf{Y}_i)_{i\in\mathbf{I}}$ in $\mathbf{Y}$ are smooth with normal crossing and any L-orbit closure is the intersection of a unique subfamily $(\mathbf{Y}_i)_{i\in\mathbf{J}}$ with $\mathbf{J} \subset \mathbf{I}$ of irreducible L-stable divisors.

(v) Write $\partial\mathbf{Y}$ for the union of the divisors $(\mathbf{Y}_i)_{i\in\mathbf{I}}$, we have the formula $\omega_{\mathbf{Y}}(\partial\mathbf{Y})|_{L/Q^-} = \omega_{L/Q^-}$.

(vi) The B_L-canonical splitting $\tau_{L/Q^-} \in H^0(L/Q^-, \omega_{L/Q^-}^{1-p})$ can be lifted through the restriction map to a B_L-semiinvariant $\tau_{\mathbf{Y}} \in H^0(\mathbf{Y}, \omega_{\mathbf{Y}}(\partial\mathbf{Y})^{1-p})$.

Proof.

(i) and (iv) are proved in [6, Theorem 3.9]. (ii) is proved in [6, Proposition 3.8]. (iii) is proved in [6, Theorem 3.9 and Theorem 4.2]. (v) is simply adjunction formula. For (vi) the result is not stated in [6] but follows from their results. Write $M = H^0(\mathbf{Y}, \omega_{\mathbf{Y}}(\partial\mathbf{Y})^{1-p})$ and $M'' = H^0(L/Q^-, \omega_{L/Q^-}^{1-p})$. The representation M'' trivially has a good filtration (see [4, Definition 4.2.4] for the definition of good filtrations). By [6, Proposition 5.7] and since p is odd, the restriction map $M \to M''$ is surjective. We need to prove that this map is again surjective on B_L-invariants. By [6, Theorem 5.10], the restriction map $M \to M''$ is the first step of a good filtration of M and the kernel M' of this map also admits a good filtration. By properties of good filtrations (see [4, Definition 4.2.4 and

Exercise 4.2.E.5]) we have for μ the highest weight of M'' the equality $\dim M_\mu^{(B_L)} = \dim M_\mu'^{(B_L)} + \dim M_\mu''^{(B_L)}$ where $M_\mu^{(B_L)}$ denotes the B_L-semiinvariants of weight μ in M (note that this also follows from [7, Part (iv) in Proposition p. 121]). This proves the result. $\square$

Corollary 3.2. — *Let Y be an embedding of L/K.*

(i) There exists a toroidal embedding Y' of L/K and equivariant morphisms $Y \xleftarrow{p} Y' \xrightarrow{\pi} \mathbf{Y}$ with p projective.

(ii) Let $\pi\colon Y' \to \mathbf{Y}$ as in (i). There exist an open affine subset Y_0' of Y' which meets all the L-orbits and Z' a closed subvariety of Y_0' such that:

- *The Levi subgroup $L(Q)$ of Q containing T_L acts on Z' and its derived subgroup $D(L(Q))$ acts trivially on Z' so that Z' is a toric variety for a quotient of $L(Q)/D(L(Q))$;*
- *The multiplication map $R_u(Q) \times Z' \to Y_0'$ is an isomorphism.*

(iii) There exists morphisms $\pi\colon Y' \to \mathbf{Y}$ and $p\colon Y' \to Y$ as in (i) with Y' smooth. If Y is complete, we may choose Y' complete as well.

(iv) Assume Y complete and let $\pi\colon Y' \to \mathbf{Y}$ as in (i) with Y' complete. Any closed orbit $\mathcal{O}$ in Y' is isomorphic to L/Q^-.

(v) Let $\pi\colon Y' \to \mathbf{Y}$ as in (i) and assume that Y' is smooth. Then the irreducible L-stable divisors $(Y_i')_{i\in I}$ in Y' are smooth with normal crossing and any L-orbit closure is the intersection of a unique subfamily $(Y_i')_{i\in J'}$ with $J' \subset I'$ of irreducible L-stable divisors.

(vi) Let $\pi\colon Y' \to \mathbf{Y}$ as in (i) and assume that Y' is smooth. Write $\partial Y'$ for the union of the divisors $(Y_i')_{i\in I'}$, we have the formula $\omega_{Y'}(\partial Y') = \pi^*\omega_{\mathbf{Y}}(\partial \mathbf{Y})$.

Proof.

(i) Take for Y' the normalisation of the closure of the diagonal embedding of L/K in $Y \times \mathbf{Y}$. Note that if Y is projective, so is Y'. (ii) Set $y' = K/K$ and $\mathbf{y} = L^\theta/L^\theta$. Let $Y_0' = \pi^{-1}(\mathbf{Y}_0)$ which is therefore Q-stable. Let $Z' = \pi^{-1}(\mathbf{Z})$ and let $Z'' = \overline{T_L \cdot y'}$ be the closure of the T_L-orbit in Y_0'. The map π being L-equivariant we get from the isomorphism $R_u(Q) \times \mathbf{Z} \to \mathbf{Y}_0$ an isomorphism $R_u(Q) \times Z' \to Y_0'$. The equivariance of π implies the equality $L \cdot Y_0' = Y'$. Furthermore, since $\pi(y') = \mathbf{y}$, the orbit $T_L \cdot y'$ is contained in Z' which is closed in Y_0' so we have the inclusion $Z'' \subset Z'$. But the isomorphism $R_u(Q) \times Z' \to Y_0'$ and the fact that Y_0' is reduced, irreducible and normal imply that Z' is reduced, irreducible and normal. In particular, since Z'' and Z' have the same dimension we deduce $Z'' = Z'$. Therefore Z' is also a toric variety as claimed. (iii) Since any toroidal variety with a morphism $Y' \to \mathbf{Y}$ as in (i), has locally the structure of a toric variety by (ii), this follows from classical results on toric varieties (see for example [16, Section 1.5]). (iv) This comes from (ii). (v) The L-orbit structure is the one of a smooth toric variety via the isomorphism given in (ii). The result follows. (vi) We first prove the following result: Every non trivial element in the group L^θ/K has order 2. In particular the order of L^θ/K is prime to the characteristic. We thank the referee for the statement and the proof of this fact. We may assume that $K = (L^\theta)^0$ since L^θ/K is a quotient of $L/(L^\theta)^0$. Let S be a maximal θ-split torus in L, *i.e.* a torus S such that $\theta(s) = s^{-1}$ for all $s \in S$ and maximal for this property. By a result of Vust [19, Proposition 7], we have $L^\theta = KS^\theta$. The group morphism $S^\theta \to L^\theta/K$ is therefore surjective. But since $S^\theta = \{s \in S \mid s^2 = 1\}$, the result follows.

We now proceed with the proof of (vi). The varieties $\mathbf{Z}$ and Z' are toric varieties for two tori $\mathbf{T}$ and T'. These groups are quotients of T of the same dimension and we have a finite morphism $T' \to \mathbf{T}$. Let $\mathbf{N}$ and N' be the groups of cocharacters of $\mathbf{T}$ and T'. Induced by $T' \to \mathbf{T}$, we have a morphism $N' \to \mathbf{N}$ whose cokernel is torsion of order $|L^\theta/K|$. By [16, Chapter 3], the natural morphisms $\mathcal{O}_{\mathbf{Z}} \otimes_{\mathbb{Z}} \mathbf{N} \to T_{\mathbf{Z}}$ and $\mathcal{O}_{Z'} \otimes_{\mathbb{Z}} N' \to T_{Z'}$ induded by the action of $\mathbf{T}$ and T' induce isomorphisms onto their images: $\mathcal{O}_{\mathbf{Z}} \otimes_{\mathbb{Z}} \mathbf{N} \simeq T_{\mathbf{Z}}(-\log \partial\mathbf{Z})$ and $\mathcal{O}_{Z'} \otimes_{\mathbb{Z}} N' \simeq T_{Z'}(-\log \partial Z)$ where $\partial\mathbf{Z}$ and $\partial Z'$ are the unions of $\mathbf{T}$-divisors and T'-divisors respectively and where $T_{\mathbf{Z}}(-\log \partial\mathbf{Z})$ and $T_{Z'}(-\log \partial Z)$ denote the sheaves of logarithmic vector fields *i.e.* derivations of $\mathcal{O}_{\mathbf{Z}}$ (resp. $\mathcal{O}_{Z'}$) preserving the ideal sheaf of $\partial\mathbf{Z}$ (resp. $\partial Z'$). This induces a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{Z'} \otimes_{\mathbb{Z}} N' \arrow[r, "\sim"] \arrow[d] & T_{Z'}(-\log \partial Z') \arrow[d] \\
\mathcal{O}_{Z'} \otimes_{\mathbb{Z}} \mathbf{N} \arrow[r, "\sim"] & \pi^*(T_{\mathbf{Z}}(-\log \partial\mathbf{Z})).
\end{tikzcd}$$

Taking the top exterior power and because $|L^\theta/K|$ is prime to the characteristic, we get an isomorphism $\omega_{Z'}(\partial Z') \simeq \pi^*\omega_{\mathbf{Z}}(\partial\mathbf{Z})$. Thanks to the isomorphisms $Y_0' \simeq R_u(Q) \times Z'$ and $\mathbf{Y}_0 \simeq R_u(Q) \times \mathbf{Z}$ we get an isomorphism $\omega_{Y_0'}(\partial Y_0') \simeq \pi^*\omega_{\mathbf{Y}_0}(\partial\mathbf{Y}_0)$ and by L-invariance this implies the result. □

Corollary 3.3. — *Any complete embedding of L/K admits a B_L-canonical splitting compatible with all closed L-stable subvarieties.*

Proof. — Let Y be such an embedding and let Y' be a smooth and complete embedding with morphisms $Y \xleftarrow{p} Y' \xrightarrow{\pi} \mathbf{Y}$ as in Corollary 3.2 (iii). By [4, Lemma 1.1.8]) it is enough to prove this result for Y'.

Let $\tau_{L/Q^-} \in H^0(L/Q^-, \omega_{L/Q^-}^{1-p})$ be the unique B_L-semi-invariant section. By Proposition 3.1 (vi) this element lifts to a B_L-semiinvariant $\tau_{\mathbf{Y}}$ in $H^0(\mathbf{Y}, \omega_{\mathbf{Y}}(\partial\mathbf{Y})^{1-p})$. But by Corollary 3.2 (vi) we have $\omega_{Y'}(\partial Y') = \pi^*\omega_{\mathbf{Y}}(\partial\mathbf{Y})$ therefore if $\mathcal{O} \simeq L/Q^-$ is a closed orbit of Y', we have a commutative diagram

$$\begin{tikzcd} H^0(\mathbf{Y}, \omega_{\mathbf{Y}}(\partial\mathbf{Y})^{1-p}) \arrow[r] \arrow[d] & H^0(L/Q^-, \omega_{L/Q^-}^{1-p}) \arrow[d, equal] \\ H^0(Y', \omega_{Y'}(\partial Y')^{1-p}) \arrow[r] & H^0(\mathcal{O}, \omega_{\mathcal{O}}^{1-p}). \end{tikzcd}$$

The image τ'_Y of $\tau_{\mathbf{Y}}$ is a B_L-semiinvariant element in $H^0(Y', \omega_{Y'}(\partial Y')^{1-p})$ with $\tau'_Y|_{\mathcal{O}} = \tau_{L/Q^-}$. If $(Y_i)_{i\in I}$ are the irreducible L-stable divisors and if σ_i is the canonical section of $\mathcal{O}_Y(Y_i)$, multiplying with $\prod_{i\in I}\sigma_i^{p-1}$ yields an element in $H^0(Y', \omega_{Y'}^{1-p})$. By recursive application of [4, Exercise 1.3.E.4] and the fact that a closed L-orbit $\mathcal{O}$ in Y' is isomorphic to L/Q^- and is split by $\tau'_Y|_{\mathcal{O}} = \tau_{L/Q^-}$ the result follows. □

Corollary 3.4. — *Let L' be a connected reductive group, let θ' be an involution on L' and let K' be a subgroup such that $(L'^{\theta'})^0 \subset K' \subset N_{L'}(L'^{\theta'})$. Then any embedding of L'/K' is $B_{L'}$-canonically Frobenius split compatibly with all closed L'-subvarieties with $B_{L'}$ a Borel subgroup of L'.*

Proof. — Let L be the adjoint group of L'. We have a quotient $\sigma\colon L' \to L$ by the center and θ' descends to an involution θ on L. Write $K = \sigma(K')$, then $\sigma((L'^{\theta'})^0) = (L^\theta)^0$ therefore we have $(L^\theta)^0 \subset K \subset N_L(L^\theta) = L^\theta$. The last inclusion as well as the last equality follows, since L is adjoint, from [8, Corollary 1.3]: For G connected reductive with an involution θ and for $g \in G$ there is an equivalence between

- $g\theta(g)^{-1} \in Z(G)$
- $g\theta(g)^{-1} \in Z_G(G^\theta)$
- $g \in N_G(G^\theta)$
- $g \in N_G((G^\theta)^0)$.

Since $L'/K' \simeq L/K$ the result follows from Corollary 3.3. □

Remark 3.5. — Let L be of adjoint type, θ an involution and K a subgroup such that $(L^\theta)^0 \subset K \subset L^\theta$. Let $\sigma\colon \tilde{L} \to L$ be the universal cover of L. Then any involution θ on L lifts to an involution $\tilde{\theta}$ in $\tilde{L}$ and $\tilde{L}^{\tilde{\theta}}$ is connected while $\sigma^{-1}(L^\theta) = N_{\tilde{L}}(\tilde{L}^{\tilde{\theta}})$ (see the above equivalence or [8, Corollary 1.3]). Let $\tilde{K} = \sigma^{-1}(K)$, then we have $\tilde{L}^{\tilde{\theta}} \subset \tilde{K} \subset N_{\tilde{L}}(\tilde{L}^{\tilde{\theta}})$ and $L/K \simeq \tilde{L}/\tilde{K}$ so that L/K is a symmetric variety for $\tilde{L}$.

4. Structure of the open orbit

Let T be a maximal torus of a connected semisimple group G and let W be the associated Weyl group. Recall that if $\varpi^\vee\colon \mathbb{G}_m \to G$ is a cocharacter of T, we may define a parabolic subgroup $P_{\varpi^\vee}$ of G as follows:

$$P_{\varpi^\vee} = \Big\{ g \in G \;/\; \lim_{t\to 0} \varpi^\vee(t) g \varpi^\vee(t)^{-1} \text{ exists} \Big\}.$$

Note that $P_{\varpi^\vee}$ contains T. Any parabolic subgroup containing T can be defined this way and there exists a unique minimal cocharacter $\varpi_P^\vee$ such that $P = P_{\varpi_P^\vee}$.

Definition 4.1. — *A parabolic subgroup containing T is cominuscule*[1] *if its associated cocharacter $\varpi_P^\vee$ satisfies $|\langle \varpi_P^\vee, \alpha\rangle| \leqslant 1$ for any root α.*

Let P be a parabolic subgroup, let w_0 be the longest element in W and set $H = P \cap P^{w_0}$.

Definition 4.2.

(i) Let $\varpi^\vee = \varpi_P^\vee + w_0(\varpi_P^\vee)$ and let $R = P_{\varpi^\vee}$ be the parabolic subgroup associated to $\varpi^\vee$.

(ii) Let L be the Levi subgroup of R containing T and let U_R be the unipotent radical of R.

(iii) Define $(\overline{P}, \overline{P}') = (L \cap P, L \cap P^{w_0})$.

Remark 4.3. — The parabolic subgroups $\overline{P}$ and $\overline{P}'$ are cominuscule and opposite in L. This was the main motivation for the above definition.

Proposition 4.4.

(i) The parabolic subgroup R contains H.

(ii) The cocharacter $\varpi_{\overline{P}}^\vee + \varpi_{\overline{P}'}^\vee$ is orthogonal to all roots of L.

Proof.

(i) This is obvious by definition.

(ii) Note that the roots of L are the roots α such that $\langle \varpi_P^\vee, \alpha\rangle = -\langle \varpi_{P^{w_0}}^\vee, \alpha\rangle$. The result follows. □

Set $p\colon G/H \to G/R$ and let $K = \overline{P} \cap \overline{P}'$. We assume from now on that P is cominuscule.

Proposition 4.5.

(i) The morphism p is locally trivial with fibre L/K.

(ii) There is an involution θ of L such that $K = (L^\theta)^0$ and L^θ/K is of order at most 2.

[1] The notion of minuscule weight is introduced in [2, Chapter VI.1 Exercice 24], the notion of cominuscule weight is the dual notion for the duality of root systems.

Proof.

(i) The map is clearly locally trivial. Its fiber is R/H. We need to prove that the map $R \to L = R/U_R$ induces an isomorphism $R/H \simeq L/K$. For this it is enough to check that the roots in U_R are in H. Such a root α satisfies, $\langle \varpi^\vee, \alpha\rangle > 0$. On the other hand, we have $\langle w_0(\varpi_P^\vee), \alpha\rangle \leqslant 1$ (since P is cominuscule) therefore $\langle \varpi_P^\vee, \alpha\rangle = \langle \varpi^\vee, \alpha\rangle - \langle w_0(\varpi_P^\vee), \alpha\rangle \geqslant 0$. Thus α is a root of P and by the same argument it is a root of P^{w_0} and thus of H.

(ii) Take θ to be the involution defined by conjugating with $\varpi_{\overline{P}}^\vee(-1)$. We compute L^θ. For $w \in W_L$, we choose a representative $n_w \in N_L(T_L)$ of w. Let $g \in L^\theta$, by the Bruhat decomposition in L, there exist $w \in W_L$, $t \in T$, scalars x_α for each positive root α of L and scalars y_α for each positive root α of L with $w^{-1}(\alpha)$ negative such that

$$g = \prod_{\alpha>0, w^{-1}(\alpha)<0} \mathbf{u}_\alpha(y_\alpha) t n_w \prod_{\alpha>0} \mathbf{u}_\alpha(x_\alpha),$$

where $\mathbf{u}_\alpha \colon \mathbb{G}_a \to U_\alpha$ is an isomorphism such that $z\mathbf{u}_\alpha(x)z^{-1} = \mathbf{u}_\alpha(\alpha(z)x)$ for all $z \in T$ and $x \in \mathbb{G}_a$. The elements w, t, x_α, y_α are uniquely determined by g. Applying θ we get

$$\theta(g) = \prod_{\alpha>0, w^{-1}(\alpha)<0} \mathbf{u}_\alpha((-1)^{\langle \varpi_{\overline{P}}^\vee, \alpha\rangle} y_\alpha) t \varpi_{\overline{P}}^\vee(-1)(\varpi_{w(\overline{P})}^\vee(-1))^{-1} n_w \prod_{\alpha>0} \mathbf{u}_\alpha((-1)^{\langle \varpi_{\overline{P}}^\vee, \alpha\rangle} x_\alpha).$$

In particular $\theta(g) = g$ if and only if $x_\alpha = y_\alpha = 0$ for all α such that $(-1)^{\langle \varpi_{\overline{P}}^\vee, \alpha\rangle} = -1$ and $\varpi_{w(\overline{P})}^\vee(-1) = \varpi_{\overline{P}}^\vee(-1)$. This is equivalent to $x_\alpha = y_\alpha = 0$ for α not a root of $\overline{P}$ and $\varpi_{w(\overline{P})}^\vee(-1) = \varpi_{\overline{P}}^\vee(-1)$. The latter is equivalent to

$$\begin{aligned}(-1)^{\langle \varpi_{\overline{P}}^\vee, w^{-1}(\alpha_i)\rangle} = (-1)^{\langle \varpi_{w(\overline{P})}^\vee, \alpha_i\rangle} &= \alpha_i(\varpi_{w(\overline{P})}^\vee(-1)) \\ &= \alpha_i(\varpi_{\overline{P}}^\vee(-1)) = (-1)^{\langle \varpi_{\overline{P}}^\vee, \alpha_i\rangle}\end{aligned}$$

for any simple root α_i of L. Note that this equality is satisfied for any $w \in W_{\overline{P}}$ so that $K \subset L^\theta$.

We check that, modulo $W_{\overline{P}}$, there is at most one element $w \in W_L$ satisfying $(-1)^{\langle \varpi_{\overline{P}}^\vee, w^{-1}(\alpha_i)\rangle} = (-1)^{\langle \varpi_{\overline{P}}^\vee, \alpha_i\rangle}$ for any simple root α_i of L. Since we work modulo $W_{\overline{P}}$, we may assume that w lies in $W^{\overline{P}}$ the set of minimal length representatives of $W_L/W_{\overline{P}}$. Let $R_{\overline{P}}$ be the root system of $\overline{P}$. The condition is equivalent to

$$w^{-1}(\alpha_i) \in R_{\overline{P}} \Leftrightarrow \alpha_i \in R_{\overline{P}}.$$

We prove that this condition implies $w=1$ or $w=w^{\overline{P}}$ the maximal element in $W^{\overline{P}}$. Indeed, assume that $w\neq 1$. Then there exists a simple root α_i with $s_{\alpha_i}w\in W^{\overline{P}}$ and $s_{\alpha_i}w<w$. In particular, we have $(s_{\alpha_i}w)^{-1}(\alpha_i)\notin R_{\overline{P}}$ *i.e.* $w^{-1}(\alpha_i)\notin R_{\overline{P}}$. This implies that $\alpha_i\notin R_{\overline{P}}$. Assume furthermore that $w\neq w^{\overline{P}}$, then there exists a simple root α_j with $s_{\alpha_j}w\in W^{\overline{P}}$ and $s_{\alpha_j}w>w$. In particular, we have $w^{-1}(\alpha_j)\notin R_{\overline{P}}$ *i.e.* $\alpha_j\notin R_{\overline{P}}$. But since $\overline{P}$ is maximal there is a unique simple root not in $R_{\overline{P}}$. This implies $\alpha_i=\alpha_j$ a contradiction. Note that we have the additional condition $w^{\overline{P}}(\overline{P})=\overline{P}'$ for the element $w^{\overline{P}}$ to appear in L^θ.

We proved that if $w^{\overline{P}}(\overline{P})=\overline{P}'$, then $L^\theta=w^{\overline{P}}K\cup K$ and $L^\theta=K$ otherwise. Therefore L^θ/K is of order at most 2. Since L^θ/K is finite and K is connected we have $K=(L^\theta)^0$. □

Remark 4.6.

(i) Note that this proposition implies that G/H and thus X are spherical.

(ii) In [1], we extend this construction to study B-orbit closures in products $G/P\times G/Q$ with P and Q cominuscule parabolic subgroups.

5. Proof of the main result

Let Y be a L-equivariant embedding of L/K and define $X_Y=G\times^R Y$. Choose B a Borel subgroup of G such that BR is dense in G.

Proposition 5.1. — *The variety X_Y is B-canonically split compatibly with its closed G-stable subvarieties.*

Proof. — The intersection $B_L:=B\cap L$ has a dense orbit in Y. Furthermore, if T' is a maximal torus of G contained in $B\cap R$ and if R^- is the parabolic subgroup opposite to R with respect to T' we have the inclusion $B\subset R^-$. We deduce the inclusion $B^-\subset R$ where B^- is the Borel subgroup opposite to B with respect to T'.

By Corollary 3.3, there exists a B_L-canonical splitting φ of Y compatibly splitting all the L-stable closed subvarieties. By [4, Proposition 4.1.10] the splitting φ is also a B_L^--canonical splitting (where $B_L^-=B^-\cap L$ is also the Borel subgroup in L, opposite to B_L with respect to $T'_L=T'\cap L$). Since B_L^- is also the quotient of B^- by the unipotent radical U_R of R the action of B_L^- on Y induces an action of B^- on Y and the splitting φ is also B^--canonical (see for example [4, Lemma 4.1.6]). We may therefore induce this splitting to get a B^--canonical splitting ψ of $G\times^{B^-}Y$ which splits compatibly the subvarieties $G\times^{B^-}Y'$ where Y' is a closed L-stable subvariety in Y

(see [4, Proposition 4.1.17 and Exercise 4.1.E.4]). Consider the morphism $q\colon G\times^{B^-}Y\to G\times^R Y$ obtained by base change from $G/B^-\to G/R$. We have $q_*\mathcal{O}_{G\times^{B^-}Y}=\mathcal{O}_{G\times^R Y}$ therefore ψ induces (see [4, Lemma 1.1.8]) a B^--canonical splitting compatibly splitting the varieties $G\times^R Y'$. This splitting is also a B-canonical splitting by [4, Proposition 4.1.10] again.

The fact that the G-stable closed subvarieties in X_Y are of the form $G\times^R Y'$ with Y' any L-stable closed subvariety in Y concludes the proof. □

Proposition 5.2. — *Any toroidal embedding of G/H is of the form $G\times^R Y$ where Y is a toroidal embedding of L/K.*

Proof. — Let X be a toroidal embedding of G/H. We prove that the morphism $p\colon G/H\to G/R$ extends to a morphism $q\colon X\to G/R$. This morphism is induced by a linear system containing the B-stable divisor D obtained as the pull-back by p of the union of the Schubert B-stable divisors in G/R. Since p is G-equivariant, this linear system also contains the G translates of D. But X is toroidal so that the closure of D contains no G-orbit and the linear system has no base point.

The fiber Y of $q\colon X\to G/R$ over R/R is a toroidal embedding of L/K and $X\simeq G\times^R Y$. □

Corollary 5.3. — *The variety $\widetilde{X}$ is Frobenius split compatibly with the exceptional divisor E.*

Proof. — By Proposition 4.5, the variety X is spherical therefore $\widetilde{X}$ is also spherical. Furthermore, any spherical variety $\widetilde{X}$ admits a projective birational morphism $\widehat{X}\to\widetilde{X}$ with $\widehat{X}$ toroidal (take the normalisation of the graph of a birational transformation $\widetilde{X}\dashrightarrow X'$ where X' is a projective toroidal embedding of G/H — see [9, Lemma 5.2] for the existence of a projective toroidal embedding of G/H). Note that this gives back Corollary 3.2.(1). By the former two Propositions, the variety $\widehat{X}$ is B-canonically Frobenius split compatibly with its closed G-stable subvarieties, in particular compatibly with the proper transform of E. By [4, Lemma 1.1.8] and because $\widetilde{X}$ is normal, we deduce that the variety $\widetilde{X}$ is B-canonically Frobenius split compatibly with E. □

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