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CLOSED UNIVERSAL SUBSPACES OF SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

by Stéphane CHARPENTIER, Quentin MENET & Augustin MOUZE

Abstract. — We exhibit the first examples of Fréchet spaces which contain a closed infinite dimensional subspace of universal series, but no restricted universal series. We consider classical Fréchet spaces of infinitely differentiable functions which do not admit a continuous norm. Furthermore, this leads us to establish some more general results for sequences of operators acting on Fréchet spaces with or without a continuous norm. Additionally, we give a characterization of the existence of a closed subspace of universal series in the Fréchet space $\mathbb{K}^{\mathbb{N}}$.

Résumé. — On exhibe les premiers exemples d'espaces de Fréchet contenant un sous-espace fermé de dimension infinie de séries universelles, mais ne contenant aucune série universelle restreinte. Pour cela, on considère les espaces de Fréchet classiques de fonctions indéfiniment dérivables qui n'admettent pas de norme continue. On établit alors des résultats plus généraux pour des suites d'opérateurs qui agissent sur des espaces de Fréchet avec ou sans norme continue. Enfin, on caractérise complètement l'existence de sous-espaces fermés de séries universelles dans l'espace de Fréchet $\mathbb{K}^{\mathbb{N}}$.

1. Introduction

In approximation theory, very strange behaviors occur. For instance, Fekete proved [17] that there exists a sequence $(a_n)_{n\geqslant 1}$ in $\mathbb{R}$ such that, for every continuous function h on $[-1,1]$, with $h(0)=0$, there exists an increasing sequence $(\lambda_n)_{n\geqslant 0}$ of positive integers such that

$$\sup_{x\in[-1,1]}\left|\sum_{j=1}^{\lambda_n} a_j x^j - h(x)\right| \to 0, \text{ as } n\to+\infty.$$

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In the same spirit, Seleznev showed [18] the existence of a complex sequence $(c_n)_{n\geqslant 0}$ such that, for every entire function $h \in H(\mathbb{C})$ and for every compact set $K \subset \mathbb{C}$, $0 \notin K$, with connected complement, there exists an increasing sequence $(\lambda_n)_{n\geqslant 0}$ of positive integers such that

$$\sup_{z\in K}\left|\sum_{j=0}^{\lambda_n} c_j z^j - h(z)\right| \to 0, \text{ as } n \to +\infty.$$

Fekete's series or Seleznev's series are called *universal series*. Since these examples, many of such results were given. The theory of universal series is a part of universality, which is a very active branch of analysis. We refer the reader to the nice survey of Grosse-Erdmann [9]. Let us explain the basic definitions of universal series. Let X be a metrizable vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ endowed with a translation-invariant metric. In the sequel X will be a Fréchet space. Let us denote $(x_n)_{n\geqslant 1}$ a fixed sequence of elements in X.

Definition 1.1. — A sequence $a = (a_1, a_2, \dots)$ defines an *universal series* (with respect to $(x_n)_{n\geqslant 1}$) if the sequence $\sum_{j=1}^{n} a_j x_j$ is dense in X. We denote by $\mathcal{U}(X)$ the set of such series.

We fix a subspace A of $\mathbb{K}^{\mathbb{N}}$ which carries a complete metrizable vector space topology induced by a translation-invariant metric d. We assume that the coordinate projections $A \to \mathbb{K}$, $a \mapsto a_m$ are continuous for all $m \in \mathbb{N}^*$, and that the set of polynomials $\{a = (a_n)_{n\geqslant 1} \in \mathbb{K}^{\mathbb{N}} : \{n;\ a_n \neq 0\}$ is finite$\}$ is contained and dense in A. As usual we denote by $(e_n)_{n\geqslant 1}$ the canonical basis of $\mathbb{K}^{\mathbb{N}}$.

Definition 1.2. — A sequence $a = (a_1, a_2, \dots) \in A$ defines a *restricted universal series* (with respect to $(x_n)_{n\geqslant 1}$) if, for every $x \in X$, there exists an increasing sequence $(\lambda_n)_{n\geqslant 0}$ in $\mathbb{N}^*$ such that

$$\sum_{j=1}^{\lambda_n} a_j x_j \to x \text{ as } n \to +\infty \text{ and } \sum_{j=1}^{\lambda_n} a_j e_j \to a \text{ as } n \to +\infty.$$

We denote by $\mathcal{U}_A(X)$ the set of such series.

Clearly we have $\mathcal{U}_A(X) \subset \mathcal{U}(X) \cap A$. Obviously if for every $a = (a_1, a_2, \dots) \in A$ we have $\sum_{j=1}^{n} a_j e_j \to a$, as $n \to +\infty$, then $\mathcal{U}_A(X) = \mathcal{U}(X) \cap A$ (this is the case if the sequence $(e_n)_{n\geqslant 1}$ is a Schauder basis of A, e.g. $A = \mathbb{K}^{\mathbb{N}}$ endowed with its cartesian topology). Bayart, Grosse-Erdmann, Nestoridis and Papadimitropoulos established a nice abstract theory of universal series, where they gave a necessary and sufficient condition for the existence

of universal elements in terms of polynomial approximation [3]. The existence of such series is always surprising. In fact there are many such series in the sense that the existence is equivalent to the topological genericity. It is natural to ask if there are large vector spaces of such series. It is proved that if there exists a restricted universal series, then the set of restricted universal series is *densely lineable*, i.e. $\mathcal{U}_A(X)$ contains a dense vector space except $\{0\}$ [3]. Then it is natural to wonder whether the set of universal series (resp. restricted universal series) is *spaceable*, i.e. if $\mathcal{U}(X) \cap A$ (resp. $\mathcal{U}_A(X)$) contains a closed infinite dimensional vector subspace of A, except $\{0\}$.

We know several examples of sets of strange functions which are spaceable : for instance the set of continuous and nowhere differentiable functions in the set of continuous real-valued functions on $[0, 1]$, the set of entire functions f whose set of translates is dense in $H(\mathbb{C})$ [1]. In 2005, Bayart proved that the set of universal Taylor series of holomorphic functions on the unit disk is spaceable [2]. Later Charpentier proved that $\mathcal{U}_A(X)$ is spaceable if A is a Banach space and $\mathcal{U}_A(X) \neq \emptyset$ [8]. In the same paper, Charpentier studied the case where A is a Fréchet space and, under the hypotheses that A admits a continuous norm and $\mathcal{U}_A(X) \neq \emptyset$, the author obtains a weaker conclusion: $\mathcal{U}(X) \cap A$ contains a closed infinite dimensional vector space except $\{0\}$. Let us recall that (A, d) admits a continuous norm means that one can find a norm $\|\cdot\|$ on A which is continuous in the topology defined by d. It is of course equivalent to the fact that the topology of (A, d) can be defined by a family of norms. We mention that the existence of a continuous norm is a standard assumption in such related topics of universality (see [4], [5], [15] and the references therein). Therefore several natural questions arise in the Fréchet case:

- Can we obtain the conclusion $\mathcal{U}_A(X)$ is spaceable ?
- Replacing $\mathcal{U}_A(X) \neq \emptyset$ by $\mathcal{U}(X) \cap A \neq \emptyset$ can we obtain $\mathcal{U}(X) \cap A$ spaceable ?
- Removing the condition of continuous norm can we obtain examples where $\mathcal{U}(X) \cap A$ is spaceable again ?

Recently Menet gave a positive answer to the first question under the hypothesis that A admits a continuous norm using the same techniques as Charpentier together with the theory of basic sequences in Fréchet spaces with a continuous norm [12]. Besides, Charpentier gave an example where A does not admit a continuous norm and $\mathcal{U}_A(X)$ is not spaceable using Fekete universal series [8, Theorem 5.9]. The author takes $A = \mathbb{R}^{\mathbb{N}}$ and argues by contradiction. The particular structure of $\mathbb{R}^{\mathbb{N}}$ seems to play an essential

role in the proof as well as the fact that for every element $(a_n)_{n\geqslant 1}$ in $\mathbb{R}^{\mathbb{N}}$ the series $\sum_{j\geqslant 1} a_j e_j$ is convergent. With similar arguments, we can prove that there does not exist a closed infinite dimensional subspace without 0 in $\mathbb{C}^{\mathbb{N}}$ consisting of universal Seleznev series in the complex plane.

This somehow justifies the study of universal series in abstract Fréchet spaces which are not necessarily sequence spaces. Through a convenient continuous linear map from a Fréchet space E into a subspace of $\mathbb{K}^{\mathbb{N}}$, we can define the elements of E which are universal as those for which the image under this map is an universal series. Restricted universality also makes sense. More precisely, let $A \subset \mathbb{K}^{\mathbb{N}}$ be a Fréchet space of sequences, let $(f_n)_{n\geqslant 1}$ be a sequence in E and let $T_0 : E \to A$ be a continuous linear map such that $T_0(f_n) = e_n$ for any $n \geqslant 1$. Then universal elements are defined in the following way.

Definition 1.3. — We keep the above assumptions.

(1) An element $f \in E$, with $T_0(f) = (a_n)_{n\geqslant 1} \in A$, is universal (with respect to T_0 and $(x_n)_{n\geqslant 1}$) if, for every $x \in X$, there exists an increasing sequence $(\lambda_n)_{n\geqslant 0}$ in $\mathbb{N}^*$ such that

$$\sum_{j=1}^{\lambda_n} a_j x_j \to x \text{ as } n \to +\infty. \tag{1.1}$$

We denote by $\mathcal{U}(X) \cap E$ the set of such universal elements.

(2) An element $f \in E$, with $T_0(f) = (a_n)_{n\geqslant 1} \in A$, is restrictively universal (with respect to T_0, $(f_n)_{n\geqslant 1}$ and $(x_n)_{n\geqslant 1}$) if, for every $x \in X$, there exists an increasing sequence $(\lambda_n)_{n\geqslant 0}$ in $\mathbb{N}^*$ such that (1.1) holds and such that

$$\sum_{j=1}^{\lambda_n} a_j f_j \to f \text{ in } E, \text{ as } n \to +\infty.$$

We denote by $\mathcal{U}_E(X)$ the set of such restricted universal elements.

Actually, the structure of the sets of universal elements is deeply connected to the space in which the universal elements under consideration live. Anyway, such elements are often called universal series, keeping in mind that this term referred to the approximation in X which is realized by partial sums of a series. This more general setting has been considered in [3, 8, 12]. In particular, Charpentier and Menet's results still hold and the two last questions above remain relevant.

In this paper we exhibit concrete examples of Fréchet spaces E of infinitely differentiable functions which lead to positive answer to these two questions. These classes of universal C^∞ elements have been studied in [14]

and they are examples where we have $\mathcal{U}_E(X) = \emptyset$ and $\mathcal{U}(X) \cap E \neq \emptyset$. In fact, we show that all the pathological cases also occur:

- $\mathcal{U}_E(X) = \emptyset$, $\mathcal{U}(X) \cap E \neq \emptyset$ and $\mathcal{U}(X) \cap E$ is spaceable,
- $\mathcal{U}_E(X) = \emptyset$, $\mathcal{U}(X) \cap E \neq \emptyset$, E does not admit a continuous norm and $\mathcal{U}(X) \cap E$ is spaceable.

More precisely, let E be the Fréchet space of all $\mathcal{C}^\infty$ functions on $\mathbb{R}$ vanishing at 0, $X = \mathcal{C}_0(\mathbb{R})$ the Fréchet space of all continuous functions on $\mathbb{R}$ vanishing at 0, and the fixed sequence $x_n = x^n$, $n \geqslant 1$, in X. It appears that moving from the point of view of sequences in $\mathbb{R}^{\mathbb{N}}$ to the point of view of infinitely differentiable functions changes the spaceability of the set of universal elements. Indeed, we prove the following result (see Theorem 2.9).

Theorem A. — *The set of $\mathcal{C}^\infty$ functions f vanishing at 0 such that the series $\left(\sum_{n\geqslant 1} \frac{f^{(n)}(0)}{n!} x^n\right)$ is universal for $\mathcal{C}_0(\mathbb{R})$ is spaceable.*

In passing, we would like to mention that as far as we know, there was only one example of a universal closed infinite dimensional subspace in a Fréchet space with no continuous norm (see [7]). Furthermore, we show that the spaceability of Fekete universal smooth functions even extends to a set of smooth functions which satisfies the universal Fekete property with respect to a denumerable set of isolated points. The same results are valid in other specific classes of C^∞ functions as well-known Gevrey spaces. Let us remark that all the results are written for functions defined on $\mathbb{R}$ for convenience of the reader, but a simple generalization shows that the theorems are valid in $\mathbb{R}^n$, $n \geqslant 1$, too (see [14] for details).

More generally, our study of Fekete spaceability in $\mathcal{C}^\infty(\mathbb{R})$ provides a way to produce examples of sequences of operators, acting on Fréchet spaces with or without a continuous norm, which admit universal closed subspaces. In some extend, we adopt an abstract point of view which aims to include universal series into the theory of universality for sequences of operators. For example, we show how general criterion for spaceability of sequences of operators can be applied to deduce spaceability in the context of universal series. In particular, if E is a separable Fréchet space and $T_0 : E \to A$ a continuous, linear and surjective map, then we have the following result (see Theorem 3.12).

Theorem B. — *Suppose that E admits a continuous norm. If $\mathcal{U}_A(X) \neq \emptyset$, then $\mathcal{U}(X) \cap E$ is spaceable.*

We finally recover previous results about spaceability of universal series (including that of the present paper that we mentioned above) without

using the tools strictly specific to universal series. The only restriction is to work with certain type of Fréchet spaces (but quite general even so).

Besides, the question of the spaceability in $\mathbb{K}^{\mathbb{N}}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ was far from being settled by the negative result obtained in [8] for Fekete universal series. Another quick look at the proof of this result points out that when the sequence $(x_k)_{k\geqslant 1} \subset X$ - from which the universal series is defined - is linearly independent, the set of associated universal series cannot be spaceable. Actually, thanks to a subtle refinement of the ideas of the proof of Charpentier's result, we characterize the sequences $(x_k)_{k\geqslant 1} \subset X$ for which the set of associated universal series in $\mathbb{K}^{\mathbb{N}}$ is spaceable. We obtain the following result (see Theorem 4.1).

Theorem C. — *Let X be any metrizable vector space over $\mathbb{K}$ with a continuous norm and let (x_n) be a fixed sequence in X. Then the following assertions are equivalent:*

(1) *the set $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$ is spaceable,*
(2) *for every $n \geqslant 1$, the set $\bigcup_{m\geqslant n}(\mathrm{span}\{x_k;\ n \leqslant k \leqslant m\} \cap \mathrm{span}\{x_k; k \geqslant m+1\})$ is dense in X.*

This characterization involves a "degree of linear independence" of the sequence $(x_k)_{k\geqslant 1} \subset X$. Surprisingly, it appears that, up to modify the sequence $\left(x^k\right)_{k\geqslant 1} \subset X$, we can recover Fekete universal closed subspaces in $\mathbb{R}^{\mathbb{N}}$.

The paper is organized as follows: Section 2 is devoted to spaceability for universal series in spaces of infinitely differentiable functions; in Section 3, we generalize the results of the previous section to arbitrary sequences of operators acting on Fréchet spaces with or without continuous norm. The last section deals with the characterization of those universal series living in $\mathbb{K}^{\mathbb{N}}$ whose set is spaceable.

Notation. — Given a sequence $(x_k)_{k\geqslant 1} \subset X$, we will denote by S_n, $n \geqslant 1$, the map which sends the formal series $\sum_{k\geqslant 1} a_k x_k$, $(a_k)_{k\geqslant 1} \subset \mathbb{K}$, to the n-th partial sum $\sum_{k=1}^{n} a_k x_k$.

2. Fekete universal closed subspaces of $\mathcal{C}^\infty$ functions

2.1. Framework

Let $K_n = [-(n+1), n+1]$, $n \in \mathbb{N}$, be an exhaustion of compact sets in $\mathbb{R}$. Notice that $0 \in K_0$. We also fix a connected compact set $K \subset \mathbb{R}$

so that 0 is in the interior of K. Let $\mathcal{C}^\infty(K)$ be the Fréchet space of $\mathcal{C}^\infty$ functions on K (i.e. the functions f which are $\mathcal{C}^\infty$ in the interior of K and such that every derivative of any order of f admits a limit at the boundary of K), endowed with the topology defined by the family of norms given by $\|f\|_n = \sum_{j=0}^n \sup_{x\in K} |f^{(j)}(x)|$, for any $f \in \mathcal{C}^\infty(K)$ and any $n \in \mathbb{N}$. We denote by d_K the associated translation-invariant metric. Let us also define the Fréchet space $\mathcal{C}^\infty(\mathbb{R})$ of infinitely differentiable functions on $\mathbb{R}$, whose topology is given by the seminorms

$$p_n := \sum_{j=0}^{n} \sup_{x\in K_n} \left| f^{(j)}(x) \right|.$$

We denote by $d_{\mathcal{C}^\infty}$ the associated metric. To simplify the notations, we shall use d instead of d_K or $d_{\mathcal{C}^\infty}$ when there will be no ambiguity. Finally let $\mathcal{C}(\mathbb{R})$ denote the Fréchet space of continuous functions on $\mathbb{R}$, whose topology is given by the seminorms $\|f\|^n = \sup_{x\in K_n} |f(x)|$.

Let T_0 denote the Borel map from $\mathcal{C}^\infty(\mathbb{R})$ to $\mathbb{R}^\mathbb{N}$ defined by

$$T_0(f) = \left(\frac{f^{(j)}(0)}{j!} \right)_{j\geqslant 0}.$$

By Borel theorem [6], T_0 is a continuous, surjective, hence open map from $\mathcal{C}^\infty(\mathbb{R})$ onto $\mathbb{R}^\mathbb{N}$. Since every restriction to K of $f \in \mathcal{C}^\infty(\mathbb{R})$ is an element of $\mathcal{C}^\infty(K)$, T_0 is also an open surjective continuous map from $\mathcal{C}^\infty(K)$ onto $\mathbb{R}^\mathbb{N}$. For greater convenience, we will sometimes identify T_0 with the map which sends $f \in \mathcal{C}^\infty(K)$ to the formal series $\sum_{j\geqslant 0} a_j x^j$ where $a_j = \dfrac{f^{(j)}(0)}{j!}$. Let observe that if $(f_n)_{n\geqslant 0}$ converges to f in $\mathcal{C}^\infty(K)$, then $(T_0(f_n))_{n\geqslant 0}$ converges to $T_0(f)$ in $\mathbb{R}^\mathbb{N}$, at least for the cartesian topology.

If $f \in \mathcal{C}^\infty(\mathbb{R})$ is such that $T_0(f) = (a_i)_{i\geqslant 0}$ is a polynomial (i.e. a finite linear combination of the e_i's), then we define the *valuation* $v(f)$ (resp. the *degree* $d(f)$) of f as the smallest (resp. the biggest) index $n \in \mathbb{N}$ such that $a_n \neq 0$. Observe that if f is itself a polynomial in $\mathcal{C}^\infty(\mathbb{R})$, then the valuation and the degree of f coincide with the usual ones.

2.2. Main results

We recall that a combination of Fekete's theorem with Borel's one yields to the existence of infinitely differentiable functions $f \in \mathcal{C}^\infty(\mathbb{R})$, with $f(0) = 0$, which are universal in the following sense: for every continuous

functions h, with $h(0) = 0$, and for every compact set $L \subset \mathbb{R}$, there exists an increasing sequence $(\lambda_n)_{n\geqslant 1}$ of integers such that

$$\sup_{x\in L}\left|\sum_{j=1}^{\lambda_n}\left(\frac{f^{(j)}(0)}{j!}\right)x^j - h(x)\right| \to 0, \text{ as } n \to +\infty.$$

Clearly the universal elements cannot satisfy, in addition to the universal approximation, the convergence

$$\sum_{j=1}^{\lambda_n}\left(\frac{f^{(j)}(0)}{j!}\right)x^j \to f(x), \text{ as } n \to +\infty,$$

in the topology of $\mathcal{C}^\infty(\mathbb{R})$. With the notations of Definition 1.3, this corresponds to take $E = \mathcal{C}^\infty(\mathbb{R})$, $A = \mathbb{R}^{\mathbb{N}}$, T_0 equal to the Borel map introduced in Section 2.1, $X = \mathcal{C}_0 = \{h \in \mathcal{C}(\mathbb{R}) : h(0) = 0\}$ and $(f_n)_{n\geqslant 1} = (x_n)_{n\geqslant 1} = (x^n)_{n\geqslant 1}$. Therefore, we deduce that $\mathcal{U}_{\mathcal{C}^\infty}(\mathcal{C}_0) = \emptyset$ whereas we have $\mathcal{U}(\mathcal{C}_0) \cap \mathcal{C}^\infty(\mathbb{R}) \neq \emptyset$.

We are interested in the spaceability. Let us remark that each step of the construction of universal closed infinite dimensional subspaces given in [8, 12] intensively used the building of *convenient* basic sequences in Banach or Fréchet spaces with continuous norm respectively and the following approximation result.

Lemma 2.1. — [3, Theorem 1] *With the notations of Definition 1.3, the following assertions are equivalent:*

(1) $\mathcal{U}_E(X) \neq \emptyset$

(2) *For every* $p \in \mathbb{N}^*$, $x \in X$ *and* $\varepsilon > 0$, *there exists* $n \geqslant p$ *and* $a_p, \dots, a_n \in \mathbb{K}$ *such that*

$$d_X\left(\sum_{j=p}^{n} a_j x_j, x\right) < \varepsilon \quad \text{and} \quad d_E\left(\sum_{j=p}^{n} a_j f_j, 0\right) < \varepsilon.$$

Remark 2.2. — Actually, only (1) $\Rightarrow$ (2) is needed in [8, 12]. As we already said, Theorem 1 in [3] also states that under assumption (1) or (2) of the preceding lemma, $\mathcal{U}_E(X)$ is a dense G_δ and contains a dense subspace (except 0).

Since $\mathcal{U}_{\mathcal{C}^\infty}(\mathcal{C}_0) = \emptyset$, Lemma 2.1 is useless in the present context. In the recent abstract theory of universal series, the authors gave another approximation lemma under the assumption $\mathcal{U}(X) \cap A \neq \emptyset$ [3, Theorem 30]. Yet, as mentioned in [8], this result cannot be used to improve the construction in an abstract way under assumption $\mathcal{U}(X) \cap A \neq \emptyset$. Nevertheless,

we have the following approximation lemma, which is a refinement of that mentioned above.

Lemma 2.3. — *For any $L \subset \mathbb{R}$ compact set, any continuous function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$, for any $p \in \mathbb{N}^*$ and for any $\varepsilon > 0$, there exist $n \in \mathbb{N}$, $n \geqslant p$, $a_p, \ldots, a_n$ in $\mathbb{R}$ and $f \in \mathcal{C}^\infty(\mathbb{R})$ such that*

$$\sup_{x \in L} \left| \sum_{j=p}^{n} a_j x^j - h(x) \right| < \varepsilon \text{ and } d(f, 0) < \varepsilon \text{ with } T_0(f) = \sum_{j=p}^{n} a_j x^j,$$

where d is the distance in the Fréchet space $C^\infty(\mathbb{R})$.

Proof. — Since there exist restricted universal series in $\mathbb{R}^\mathbb{N}$ by Fekete's theorem, the abstract theory of universal series [3] ensures that, for any $\eta > 0$, there exist $n \geqslant p$ and $a_p, \ldots, a_n$ in $\mathbb{R}$ such that

$$\sup_{x \in L} \left| \sum_{j=p}^{n} a_j x^j - h(x) \right| < \eta \text{ and } d_{\mathbb{R}^\mathbb{N}}(a, 0) < \eta,$$

where $a = \sum_{j=p}^{n} a_j x^j$. As we already said, the linear Borel map $T_0 : \mathcal{C}^\infty(\mathbb{R}) \to \mathbb{R}^\mathbb{N}$ is open. Hence with a good choice of η, $\eta < \varepsilon$, there is a function $f \in \mathcal{C}^\infty(\mathbb{R})$ such that $T_0 f = \sum_{j=p}^{n} a_j x^j$ and $d_{\mathcal{C}^\infty}(f, 0) < \varepsilon$. This finishes the proof. □

We recall that a basic sequence in a Fréchet space is a sequence $(u_n)_{n \geqslant 0}$ whose closed linear hull is given by all convergent series $\sum_{n \geqslant 0} a_n u_n$ and where the expansion is unique. As mentioned above, the construction of universal closed subspaces given in [8, 12] lays on the possibility to build *convenient* basic sequences in Banach or Fréchet spaces with a continuous norm. However, it is well-known that $\mathcal{C}^\infty(\mathbb{R})$ does not admit a continuous norm (see [15] for example). For this reason, we first work in $\mathcal{C}^\infty(K)$. Then Lemma 1 in [15] or Lemma 2.2 in [12] directly yields the following result.

Lemma 2.4. — *Let $(u_0, \ldots, u_n) \subset \mathcal{C}^\infty(K)$ be a finite family and let F be an infinite dimensional subspace of $\mathcal{C}^\infty(K)$. For every real number $\varepsilon > 0$, there exists $u_{n+1} \in F$ with $\|u_{n+1}\|_\infty = 1$ and such that*

$$\left\| \sum_{k=0}^{n} \lambda_k u_k \right\|_j \leqslant (1 + \varepsilon) \left\| \sum_{k=0}^{n+1} \lambda_k u_k \right\|_j \tag{2.1}$$

for every scalar λ_k, $0 \leqslant k \leqslant n+1$ and every $0 \leqslant j \leqslant n$.

In particular, we can construct by induction a sequence of polynomials $(u_n)_{n \geqslant 0}$, where each u_n is built as above with F the infinite dimensional

subspace of $\mathcal{C}^\infty(K)$ consisting of polynomials of valuation greater than a chosen sequence $(l_n)_{n\geqslant 0}$ (assuming that $v(0)=+\infty$), such that

(1) *$(u_n)_{n\geqslant 0}$ is a basic sequence in the Fréchet space $\mathcal{C}^\infty(K)$;*
(2) *$\|u_n\|_\infty = 1$ for any $n\in\mathbb{N}$;*
(3) *$v(u_n)\geqslant l_n$ for any $n\in\mathbb{N}$.*

Now, a combination of Lemma 2.3 and Lemma 2.4 together with the construction of universal closed subspaces in [8, 12] yields the following theorem. The proof, quite similar to that given in [8, 12], is omitted.

Theorem 2.5. — *The set $\mathcal{U}(\mathcal{C}_0)\cap\mathcal{C}^\infty(K)$ is spaceable.*

Remark 2.6. — Theorem 2.5 is the first result about spaceability for universal series when $\mathcal{U}_E(X)=\emptyset$ and $\mathcal{U}(X)\cap E\neq\emptyset$. Moreover it is a quite natural one.

Actually, a refinement of Lemma 2.4 shows that even $\mathcal{C}^\infty(\mathbb{R})$ admits a convenient basic sequence. To see this, we need the following result.

Lemma 2.7. — *Let $L_n:\mathcal{C}^\infty(\mathbb{R})\to\mathcal{C}^\infty([-(n+1),n+1])$, $n\geqslant 0$, be the restriction mapping defined by $L_n(u)=u_{|[-(n+1),n+1]}$, and M be a subspace of $\mathcal{C}^\infty(\mathbb{R})$ such that $L_0(M)$ is infinite dimensional. Then there exists a sequence $(u_k)_{k\geqslant 0}$ in M such that:*

(1) *For every $k\geqslant 0$, $\|L_0(u_k)\|_\infty=1$;*
(2) *For every $n\geqslant 0$, the sequence $(L_n(u_k))_{k\geqslant n}$ is basic in $\mathcal{C}^\infty([-(n+1),n+1])$.*

Proof. — Let $(\varepsilon_k)_{k\geqslant 0}$ be a sequence of positive real numbers such that $\prod_{k=0}^{+\infty}(1+\varepsilon_k)$ converges. We consider a subspace $N\subset M$ such that $L_0(N)$ is infinite dimensional and, for every non-zero $u\in N$, $L_0(u)$ is non-zero. Note that such a subspace exists, by considering the linear span of any extension to $\mathbb{R}$ of any infinite free sequence of functions in $L_0(M)$. The sequence $(u_k)_{k\geqslant 0}$ is built by induction. First we take u_0 in N of supremum norm 1 on $[-1,1]$. We turn to the inductive step and we assume that $u_0,\dots,u_k$, $k\geqslant 0$, have been built. For any $0\leqslant i\leqslant k$ and any $0\leqslant l\leqslant k$, let $(z_{i,j,l})_{j=0}^{m_{k,i,l}}$ be an $\dfrac{\varepsilon_k}{1+\varepsilon_k}$-net of the unit sphere of $\operatorname{span}\left(u_{0|[-(i+1),i+1]},\dots,u_{k|[-(i+1),i+1]}\right)$ in $\left(\widehat{\mathcal{C}^\infty([-(i+1),i+1])},\|\cdot\|_{i,l}\right)$, where $\|\cdot\|_{i,l}$ stands for the l-th norm in $\mathcal{C}^\infty([-(i+1),i+1])$. Let $(\varphi_{i,j,l})_{j=0}^{m_{k,i,l}}$ be continuous linear functionals of norm 1 on $\left(\widehat{\mathcal{C}^\infty([-(i+1),i+1])},\|\cdot\|_{i,l}\right)$ such that, for any $0\leqslant i\leqslant k$, any $0\leqslant l\leqslant k$ and any $0\leqslant j\leqslant m_{k,i,l}$, we

have

$$\varphi_{i,j,l}\left(z_{i,j,l}\right)=1.$$

Now, for every $0 \leqslant i \leqslant k$, let E_i be the subspace of $\mathcal{C}^\infty([-(i+1), i+1])$ defined by

$$E_i := \bigcap_{j=0}^{m_{k,i,l}} \bigcap_{l=0}^{k} \ker\left(\varphi_{i,j,l}\right).$$

Since every L_i is a continuous linear map, $i \geqslant 0$, $L_i^{-1}\left(E_i\right)$ has finite codimension and, by definition of N, we can choose u_{k+1} non-zero in $N \cap \bigcap_{i=0}^{k} L_i^{-1}\left(E_i\right)$ such that $\left\|u_{k+1|[-1,1]}\right\|_\infty = 1$.
To finish, we can conclude as at the end of the proof of [15, Lemma 1] to show that $\left(u_k\right)_{k\geqslant 0}$ satisfies the assertions (1) and (2) in the statement of the lemma. □

We now show how the previous lemma actually produces a basic sequence in $\mathcal{C}^\infty(\mathbb{R})$.

Corollary 2.8. — *Under the above notations, the sequence $\left(u_k\right)_{k\geqslant 0}$ is a basic sequence in $\mathcal{C}^\infty(\mathbb{R})$.*

Proof. — Let $F := \overline{\operatorname{span}\left\{u_k,\, k \geqslant 0\right\}}$, with $\left(u_k\right)_{k\geqslant 0}$ given by Lemma 2.7. Let us fix $x \in F$. There exists a sequence $\left(\alpha_{k,l}\right)_{k,l\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\sum_{k=0}^{m_l} \alpha_{k,l} u_k \to x, \text{ as } l \to +\infty.$$

Since L_0 is a continuous map and $\left(u_{k|[-1,1]}\right)_{k\geqslant 0}$ is a basic sequence in $\mathcal{C}^\infty\left([-1,1]\right)$ (see Lemma 2.7 for the definition of L_0), there exists a sequence $\left(\alpha_k\right)_{k\geqslant 0}$ of real numbers such that

$$\sum_{k=0}^{m_l} \alpha_{k,l} u_{k|[-1,1]} \to \sum_{k=0}^{+\infty} \alpha_k u_{k|[-1,1]} = x_{|[-1,1]}, \text{ as } l \to +\infty.$$

Because the linear coordinate functionals which take a convergent series $\sum_{k=0} \gamma_k u_{k|[-1,1]}$ to γ_k are continuous for every $k \geqslant 0$, we must have $\alpha_{k,l} \to \alpha_k$, as $l \to +\infty$, for every $k \geqslant 0$.
Now for every $n \geqslant 1$ and l large enough (up to take $\alpha_{k,l} = 0$ for k large, we may and shall assume that $m_l \to +\infty$, as $l \to +\infty$), we write

$$\sum_{k=0}^{m_l} \alpha_{k,l} u_k = \alpha_{0,l} u_0 + \ldots + \alpha_{n-1,l} u_{n-1} + \sum_{k=n}^{m_l} \alpha_{k,l} u_k.$$

Letting l tends to $+\infty$, we get that $\lim_{l\to+\infty} \sum_{k=n}^{m_l} \alpha_{k,l} u_k$ exists in $\mathcal{C}^\infty(\mathbb{R})$. Therefore, taking the restriction to $[-n, n]$ and using that $\left(u_{k|[-n,n]}\right)_{k\geqslant n}$ is

a basic sequence in $\mathcal{C}^\infty([-n,n])$, if follows that

$$x_{|[-n,n]} = \alpha_0 u_{0|[-n,n]} + \ldots + \alpha_{n-1} u_{n-1|[-n,n]} + \sum_{k\geqslant n} \beta_k u_{k|[-n,n]}$$

for some sequence $(\beta_k)_{k\geqslant n}$ of real numbers. But, up to take the restriction to $[-1,1]$, we can still use the uniqueness of the development along a basic sequence to see that each β_k, $k \geqslant n$, must be equal to α_k.
Finally, we have shown that for every $n \geqslant 1$, $x_{|[-n,n]} = \sum_{k\geqslant 0} \alpha_k u_{k|[-n,n]}$, where the convergence of the series holds in $\mathcal{C}^\infty([-n,n])$. It means that $x = \sum_{k\geqslant 0} \alpha_k u_k$ by definition of the topology of $\mathcal{C}^\infty(\mathbb{R})$. Moreover this development is unique since its restriction $x_{|[-1,1]}$ to $[-1,1]$ admits a unique development along the sequence $\left(u_{k|[-1,1]}\right)_{k\geqslant 0}$ (because this latter is basic in $\mathcal{C}^\infty([-1,1])$).

□

We now prove that Theorem 2.5 extends to $\mathcal{C}^\infty(\mathbb{R})$.

Theorem 2.9. — *The set $\mathcal{U}(\mathcal{C}_0) \cap \mathcal{C}^\infty(\mathbb{R})$ is spaceable.*

Proof. — By Theorem 2.5, there exists a universal closed infinite dimensional subspace $H \subset \mathcal{C}^\infty([-1,1])$. Let $M = L_0^{-1}(H)$ and let then $(u_k)_{k\geqslant 0} \subset M$ be given by Lemma 2.7. We prove that the space

$$F := \overline{\operatorname{span}\{u_k,\, k \geqslant 0\}}$$

is a (Fekete) universal closed infinite dimensional subspace of $\mathcal{C}^\infty(\mathbb{R})$ (except 0). We have to show that every non-zero element of F is universal. By Corollary 2.8, if $h \in F \setminus \{0\}$, then $h = \sum_{k\geqslant 0} \alpha_k u_k$ with $\alpha_k \neq 0$ for some $k \geqslant 0$. By construction in Lemma 2.7, $h_{|[-1,1]} = \sum_{k\geqslant 0} \alpha_k u_{k|[-1,1]}$ is in particular a non-zero element of H and so it is universal. □

2.3. Further extension

Let $(b_n)_{n\geqslant 0}$ be a sequence of distinct real numbers without accumulation point and let $(c_n)_{n\geqslant 0}$ be an arbitrary sequence of real numbers. Mouze and Nestoridis proved that there exists $f \in \mathcal{C}^\infty(\mathbb{R})$ with $f(b_n) = c_n$, $n \in \mathbb{N}$, such that for every selection $h_n : \mathbb{R} \to \mathbb{R}$ of continuous functions with $h_n(b_n) = 0$, $n \in \mathbb{N}$, there exists a subsequence $(\lambda_j)_{j\geqslant 0}$ of integers such that, for all $n \in \mathbb{N}$, $\sum_{k=0}^{\lambda_j} \frac{f^{(k)}(b_n)}{k!}(x-b_n)^k$ converges to $c_n + h_n(x)$ uniformly on compact subsets of $\mathbb{R}$, as $j \to +\infty$ [14, Theorem 4.2]. Observe that we realize the universal approximation property for the same subsequence of partial sums. For convenience, let us denote by $\mathcal{U}_\infty((b_n),(c_n))$ the set of

such universal functions. Clearly if $f, g \in \mathcal{U}_\infty((b_n),(c_n))$, then $f+g$ does not belong to $\mathcal{U}_\infty((b_n),(c_n))$ except if $c_n = 0$ for all n. So $\mathcal{U}_\infty((b_n),(c_n)) \cap \mathcal{C}^\infty$ cannot be spaceable when (c_n) is a non-zero sequence. But we intend to prove that $\mathcal{U}_\infty((b_n),0) \cap \mathcal{C}^\infty$ contains a closed infinite dimensional subspace.

THEOREM 2.10. — *With the above notations, $\mathcal{U}_\infty((b_n),0) \cap \mathcal{C}^\infty$ is spaceable.*

The main ingredient of the proof is the following lemma, which is a refinement of Lemma 2.3. For any $n \in \mathbb{N}$, we define $T_0^n(f) = \left(\frac{f^{(k)}(b_n)}{k!}\right)_{k \geqslant 0}$.

LEMMA 2.11. — *Let $(b_n)_{n \geqslant 0}$ be a sequence of distinct real numbers without accumulation point such that the sequence $(|b_n|)_{n \geqslant 0}$ is strictly increasing. For any $N \in \mathbb{N}$, any compact set $K' \subset \mathbb{R}$, any continuous functions $h_0, \ldots, h_N : \mathbb{R} \to \mathbb{R}$, with $h_i(b_i) = 0$, $i \in \{0, \ldots, N\}$, any $p > 0$ and any $\varepsilon > 0$, there exist an integer $n \geqslant p$, real numbers $a_p^i, \ldots, a_n^i$, $i \in \{0, \ldots, N\}$, and $f \in \mathcal{C}^\infty(\mathbb{R})$, such that*

$$\sup_{x \in K'} \left| \sum_{k=p}^{n} a_k^i (x - b_i)^k - h_i(x) \right| < \varepsilon \text{ and } d(f,0) < \varepsilon,$$

with $T_0^i(f) = \left(0, \ldots, 0, a_p^i, \ldots, a_n^i, 0 \ldots\right)$, $i = 0, \ldots, N$, and $T_0^i(f) = 0$, for $i \geqslant N+1$.

Proof. — Using that the sequence $(|b_n|)_{n \geqslant 0}$ is strictly increasing and has no accumulation points, we fix a function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ supported by $[-|b_{N+1}|, |b_{N+1}|]$ and equal to 1 on $[-|b_N|, |b_N|]$. By continuity of the multiplication by φ in $\mathcal{C}^\infty(\mathbb{R})$, we can choose $\delta > 0$ such that for any $g \in \mathcal{C}^\infty$, $d(g,0) < \delta$ implies $d(\varphi g, 0) < \varepsilon$. Then, we apply [14, Proposition 4.1] and [19, Theorem 2.6] to find $a_p^i, \ldots, a_n^i$, $i \in \{1, \ldots, N\}$, in $\mathbb{R}$ so that

$$\sup_{x \in K'} \left| \sum_{k=p}^{n} a_k^i (x - b_i)^k - h_i(x) \right| < \varepsilon \text{ and } d_{(\mathbb{R}^\mathbb{N})^N}(a,0) < \eta,$$

where $a = ((a^1), \ldots, (a^N))$ with $a^i = (0, \ldots, 0, a_p^i, \ldots, a_n^i, 0, \ldots)$, $i = 1, \ldots, N$. Now, it is easy to check that the linear map from $\mathcal{C}^\infty(\mathbb{R})$ into $\left(\mathbb{R}^\mathbb{N}\right)^N$ which takes a function f to $(T_0^0(f), \ldots, T_0^N(f))$ is continuous, onto between two Fréchet spaces, hence the open mapping theorem ensures that the image of every δ-neighborhood of 0 in $\mathcal{C}^\infty(\mathbb{R})$ contains some η-neighborhood of 0 in $\left(\mathbb{R}^\mathbb{N}\right)^N$ (endowed with the product topology). Therefore, for any $\delta > 0$, there exists $g \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $d(g,0) < \delta$ and

$T_0^i(g) = a^i$, $i = 0, \dots, N$. Observe that the function $f = \varphi g$ satisfies the lemma. □

According to Corollary 2.8, there exist basic sequences in $\mathcal{C}^\infty(\mathbb{R})$. Our proof of Theorem 2.9 did not need this basic sequence to consist of polynomials of convenient valuation, as it is usually the case in order to build universal closed subspaces ([8, 12] or for e.g. Theorem 2.5). To prove Theorem 2.10, we need to work again with basic sequences of polynomials of arbitrary valuations. In fact, an improvement of Lemma 2.7 gives the following.

Lemma 2.12. — *Let $(b_n)_{n\geqslant 0}$ be a sequence of real numbers with no accumulation points. With the above notations, there exists a basic sequence $(u_k)_{k\geqslant 0}$ in $\mathcal{C}^\infty(\mathbb{R})$ of polynomials such that, for every $k \geqslant 0$, we have $v(T_0^i(u_k)) \geqslant l_k$, $i = 0, \dots, k$, where $(l_k)_{k\geqslant 0}$ is a chosen sequence of natural numbers.*

Proof. — It is a slight modification of the proof of Lemma 2.7. Considering F_n the subspace of polynomial functions whose valuation with respect to the centers b_i, $i = 0, \dots, n$, is greater than l_n (with the usual convention $v(0) = +\infty$), the sequence $(u_k)_{k\geqslant 0}$ is built by induction exactly as in the proof of Lemma 2.7, except that at each step k, the infinite dimensional subspace M is replaced by F_k. □

We turn to the proof of the main result of this subsection.

Proof of Theorem 2.10. — It is still a modification of the standard construction of universal closed infinite dimensional subspace given in [8, 12]. We make a sketch of it. Without loss of generality we may assume that, for every $n \geqslant 1$, the interval $[-n, n]$ contains exactly the n first terms of the sequence $(b_i)_{i\geqslant 0}$ (since we assumed that $(b_i)_{i\geqslant 0}$ has no accumulation point). Therefore we have Lemma 2.11 and Lemma 2.12 at our disposal.

Let us also consider an enumeration $(\varphi(n), \psi(n))_{n\geqslant 1}$ of pairs (σ, l) where σ is a finite sequence with coordinates in $\mathbb{N}$ and $l \in \mathbb{N}$, such that, for every integer l, there are infinitely many integers k_n, $n \geqslant 1$, such that $\psi(k_n) = l$ for any n. As a notation, $\varphi(n)(i)$ will stand for the i-th coordinate of the sequence $\varphi(n)$. Let also $(Q_n)_{n\geqslant 0}$ be an enumeration of real polynomials with coefficients in $\mathbb{Q}$. Let us fix smooth functions χ_n^i so that $\chi_n^i \equiv 0$ on $]b_i - 1/2(n+1), b_i + 1/2(n+1)[$, $\chi_n^i \equiv 1$ on $\mathbb{R}\backslash]b_i - 1/(n+1), b_i + 1/(n+1)[$ and $|\chi_n^i| \leqslant 1$.

Let us fix $u_0 = x$. We recall that the notation $\|\cdot\|^n$ stands for the n-th seminorm in $\mathcal{C}(\mathbb{R})$ (see Section 2.1). Proceeding as in [8, Theorem 3.1] or in [12], we use Lemma 2.11 and Lemma 2.12 to build sequences $(u_k)_{k\geqslant 0}$,

$(g_{n,k})_{n\geqslant k\geqslant 0}$ and $(f_{n,k})_{n\geqslant k\geqslant 0}$ in $\mathcal{C}^\infty(\mathbb{R})$, with convenient valuations, such that:

(1) For any $i\in[0,n]$, $\left\|T_0^i(g_{n,k})-\chi_n^i Q_{\varphi(n)(i)}\right\|^{\psi(n)}<\eta_n$;
(2) For any $i\in[0,n]$, $\left\|T_0^i(f_{n,k})\right\|^{\psi(n+1)}<\eta_n$;
(3) For any $i\geqslant n+1$, $T_0^i(f_{n,k})=0$;
(4) $\max\left(\|f_{n+1,k}-f_{n,k}\|_k, d(f_{n+1,k},f_{n,k})\right)<\eta_n$;
(5) $\max\left(\|f_{k,k}-u_k\|_k, d(f_{k,k},u_k)\right)<\eta_k$;
(6) $\left\|u_{k|[-1,1]}\right\|_\infty=1$,

where $(\eta_n)_{n\geqslant 0}$ is a well chosen sequence which decreases to 0 fast enough. We can define in $\mathcal{C}^\infty(\mathbb{R})$ the sequence $(f_k)_{k\geqslant 0}$ where each f_k is given by

$$f_k=\sum_{n\geqslant k}(f_{n+1,k}-f_{n,k})+f_{k,k}.$$

The sequence $\left(u_{k|[-n,n]}\right)_{k\geqslant n}$ is a basic sequence, to which $\left(f_{k|[-n,n]}\right)_{k\geqslant n}$ is equivalent, hence the sequence $\left(f_{k|[-n,n]}\right)_{k\geqslant n}$ is also basic. It follows that $(f_k)_{k\geqslant 0}$ is basic in $\mathcal{C}^\infty(\mathbb{R})$ (as in Corollary 2.8). We consider the infinite dimensional closed subspace

$$F=\left\{\sum_{k\geqslant 0}\alpha_k f_k,\ \sum_{k\geqslant 0}\alpha_k f_k \text{ converges in } \mathcal{C}^\infty(\mathbb{R})\right\}.$$

It remains to show that for every $h\in F\setminus\{0\}$, every selection $h_i:\mathbb{R}\to\mathbb{R}$ of continuous functions with $h_i(b_i)=0$, $i\in\mathbb{N}$, and every $n\geqslant 1$, there exists a sequence $(N_j)_{j\geqslant 0}$ such that

$$\sup_{x\in[-n,n]}\left|S_{N_j}\left(\sum_{k\geqslant 0}\alpha_k T_0^i(f_k)\right)-h_i\right|\to 0,\ \text{as } j\to+\infty,$$

for any $i\in\mathbb{N}$ and where $h=\sum_{k\geqslant 0}\alpha_k f_k$. Let k_0 be the smallest integer k such that $\alpha_k\neq 0$, let $(v_j)_{j\geqslant 0}\subset\mathbb{N}$ be a strictly increasing sequence such that $\psi(v_j+k_0)=n$ and $Q_{\varphi(v_j+k_0)(i)}$ converges to h_i, uniformly on every compact set, as $j\to+\infty$. Combining the continuity of h_i and the equality $h_i(b_i)=0$, with the properties of functions χ_n^i, it is easy to check that $\chi_{v_j+k_0}^i Q_{\varphi(v_j+k_0)(i)}$ converges to h_i, uniformly on $[-n,n]$, as $j\to+\infty$. Let $N_j=\max\limits_{i\leqslant v_j+k_0}\left\{d\left(T_0^i(g_{v_j+k_0})\right)\right\}$. We may suppose that $\alpha_{k_0}=1$. We fix an integer $i\geqslant 0$. Now, for j large enough, we have $i\in[0,v_j+k_0-1]$. The rest

of the proof is standard, we show that

$$\left\|S_{N_j}\left(T_0^i(h)\right)-\chi^i_{v_j+k_0}Q_{\varphi(v_j+k_0)(i)}\right\|^n$$
$$=\left\|S_{N_j}\left(\sum_{k=k_0}^{v_j+k_0}\alpha_k T_0^i(f_k)\right)-\chi^i_{v_j+k_0}Q_{\varphi(v_j+k_0)(i)}\right\|^{\psi(v_j+k_0)}\to 0,\ \text{as } j\to+\infty,$$

as in [8] or [12]. □

2.4. Fekete universal closed subspaces of ultradifferentiable functions

Let $(M_n)_{n\geqslant 0}$ be an increasing sequence of real numbers satisfying the following conditions:

(1) The sequence $\left(\dfrac{M_{n+1}}{M_n}\right)_{n\geqslant 0}$ is increasing;

(2) There is a constant $C>0$ such that $\displaystyle\sum_{n\geqslant p}\frac{M_n}{(n+1)\,M_{n+1}}\leqslant C\frac{M_p}{M_{p+1}}$, $p\geqslant 0$.

Let us introduce the classical Beurling space $\mathcal{C}^\infty_{(M)}(\mathbb{R})$ as the subspace of $\mathcal{C}^\infty(\mathbb{R})$ consisting of those functions f such that for any compact set K in $\mathbb{R}$, $\sup_{j\in\mathbb{N},\,x\in K}\frac{\left|f^{(j)}(x)\right|}{j!M_jC^j}<\infty$ for any $C>0$ [10]. We endowed this space with the Fréchet topology defined by the family of seminorms

$$\|f\|_{(M),n}:=\sup_{j\in\mathbb{N},\,x\in K_n}\frac{\left|f^{(j)}(x)\right|}{j!M_jC_n^j}$$

where $(C_n)_{n\geqslant 0}$ is a sequence of positive real numbers, decreasing to 0, with $C_0=1$. Let T_0 denote the Borel map from $\mathcal{C}^\infty(\mathbb{R})$ to $\mathbb{R}^{\mathbb{N}}$ defined by $T_0(f)=\left(\frac{f^{(j)}(0)}{j!}\right)_{j\geqslant 0}$. A refinement of Borel theorem due to Petzsche [16] ensures, under condition (2) above, that $T_{0|\mathcal{C}^\infty_{(M)}(\mathbb{R})}$ is also a surjective, continuous and open map from $\mathcal{C}^\infty_{(M)}(\mathbb{R})$ onto the subspace $\mathcal{F}_{(M)}$ of $\mathbb{R}^{\mathbb{N}}$ consisting of those sequences $(a_j)_{j\geqslant 0}$ such that

$$\sup_{j\in\mathbb{N}}\frac{|a_j|}{M_jC^j}<\infty$$

for any $C>0$.

This allows us to define both universal series in $\mathcal{C}^\infty_{(M)}(\mathbb{R})$ and restricted universal series in $\mathcal{F}_{(M)}$ in the sense of Fekete. As for $\mathcal{C}^\infty(\mathbb{R})$, we have

$\mathcal{U}_{\mathcal{C}^\infty_{(M)}}(\mathcal{C}_0) = \emptyset$ and $\mathcal{U}(\mathcal{C}_0) \cap \mathcal{C}^\infty_{(M)}(\mathbb{R}) \neq \emptyset$ [14]. Moreover the Beurling space does not admit a continuous norm. Yet everything can be done exactly as in the previous section for $\mathcal{C}^\infty(\mathbb{R})$. We can define $\mathcal{C}^\infty_{(M)}(K)$ for K a compact set in $\mathbb{R}$. By Petzsche's theorem and because $\mathcal{U}_{\mathcal{F}_{(M)}}(\mathcal{C}_0) \neq \emptyset$ [14], the approximation Lemma 2.3 is valid in $\mathcal{C}^\infty_{(M)}(\mathbb{R})$ and an analogue of Lemma 2.4 holds for $\mathcal{C}^\infty_{(M)}(K)$. Therefore we can prove that $\mathcal{U}(\mathcal{C}_0) \cap \mathcal{C}^\infty_{(M)}(K)$ is spaceable (see Theorem 2.5). Using the same ideas as that of the previous section, we can prove that $\mathcal{U}(\mathcal{C}_0) \cap \mathcal{C}^\infty_{(M)}(\mathbb{R})$ is also spaceable. In the same spirit, Theorem 2.10 remains true in $\mathcal{C}^\infty_{(M)}(\mathbb{R})$. To see this, it suffices again to observe that $\mathcal{U}_\infty((b_n),(c_n)) \cap \mathcal{C}^\infty_{(M)} \neq \emptyset$ [14] and to write the analogue of Lemma 2.11 in Beurling spaces using Petzsche's theorem and the existence of cut functions.

We end this section with the following remark.

Remark 2.13. — Let us introduce the quotients space $\mathcal{E}$ and $\mathcal{E}_{(M)}$ given by $\mathcal{C}^\infty(\mathbb{R})/\ker(T_0)$ and $\mathcal{C}^\infty_{(M)}(\mathbb{R})/\ker(T_0)$ respectively. Both $\mathcal{E}$ and $\mathcal{E}_{(M)}$ are Fréchet spaces since T_0 is continuous. Moreover they are isomorphic to $\mathbb{R}^\mathbb{N}$ and $\mathcal{F}_{(M)}$ respectively. In particular, $\mathcal{E}$ has no continuous norm but $\mathcal{E}_{(M)}$ has a continuous norm. By definition, two elements in the same class in $\mathcal{E}$ or $\mathcal{E}_{(M)}$ have the same Taylor expansion at 0. In particular, we can define universal and restricted universal elements $\dot{f}$ in $\mathcal{E}$ (resp. $\mathcal{E}_{(M)}$) as those for which $T_0(f)$ is a universal or a restricted universal series, where f is any element in the class $\dot{f}$. Now it follows that:

(1) $\mathcal{U}_\mathcal{E}(\mathcal{C}_0) = \mathcal{U}(\mathcal{C}_0) \cap \mathcal{E} \neq \emptyset$ is not spaceable ([8, Theorem 5.9]);
(2) $\mathcal{U}_{\mathcal{E}_{(M)}}(\mathcal{C}_0) \neq \emptyset$ is spaceable (by Menet's result [12]).

Furthermore, this section highlights that all the previous ideas can be extended to a much more general context. This is the purpose of the next section.

3. Abstract theory

3.1. General framework

Let Y be a separable Fréchet space and X be a separable topological vector space whose topology is given by a sequence of seminorms. Let us consider a sequence $T_n : Y \to X$ of continuous linear mappings.

Definition 3.1. — The sequence (T_n) is *mixing* if for any open sets $U \in Y$, $V \in X$, with $U \neq \emptyset$ and $V \neq \emptyset$, there exists $N \in \mathbb{N}$ such that one has $T_n(U) \cap V \neq \emptyset$ for every $n \geqslant N$.

Definition 3.2. — We say that the sequence (T_n) *satisfies condition* (C) if there exist an increasing sequence of positive integers (n_k) and a dense subset $Y_0 \subset Y$ such that

- $T_{n_k} y \to 0$, for all $y \in Y_0$, as $k \to +\infty$;
- $\cup_{k \geqslant 0} T_{n_k}(\{y \in Y \ : \ p(y) < 1\})$ is dense in X, for every continuous seminorms p on Y.

Definition 3.3. — We say that the sequence (T_n) is *topologically transitive* if for any open sets $U \in Y$, $V \in X$, with $U \neq \emptyset$ and $V \neq \emptyset$, there exists $n \in \mathbb{N}$ such that one has $T_n(U) \cap V \neq \emptyset$.

Definition 3.4. — We say that the sequence (T_n) is *universal* if there exists $y \in Y$ such that the set $\{T_n y;\ n \geqslant 1\}$ is dense in X.

Let us remark that condition (C) has been introduced in [11] in the case of sequences of operators between Banach spaces and it has been extended in Fréchet spaces in [13]. The characterization of condition (C) given for Banach spaces by León and Müller [11, Theorem 4] is still verified for Fréchet spaces:

Proposition 3.5. — *The sequence (T_n) satisfies condition (C) if and only if for any $j \geqslant 1$, for any non-empty open sets $U_0, \dots, U_j \subset Y$, with $0 \in U_0$, and for any non-empty open sets $V_0, V \subset X$, with $0 \in V_0$, there exists $n \in \mathbb{N}$ such that for $1 \leqslant i \leqslant j$, $T_n(U_i) \cap V_0 \neq \emptyset$ and $T_n(U_0) \cap V \neq \emptyset$.*

Proof. — The proof is similar that of Theorem 4 in [11].
($\Rightarrow$) This implication directly follows from the fact that if for every continuous seminorm p, $\cup_{k \geqslant 0} T_{n_k}(\{y \in Y : p(y) < 1\})$ is dense in X, then for every continuous seminorm p, for every $\varepsilon > 0$ and for every $N \geqslant 1$, the set $\bigcup_{k \geqslant N} T_{n_k}(\{y \in Y : p(y) < \varepsilon\})$ is still dense in X.
($\Leftarrow$) Let (y_n) be a dense sequence in Y, (x_n) a dense sequence in X, (p_n) an increasing sequence of seminorms defining the topology of Y and (q_n) an increasing sequence of seminorms defining the topology of X. By induction on k we can construct a family $(u_{n,k})_{k \geqslant n} \subset Y$, a sequence $(v_k) \subset X$ and an increasing sequence of integers (n_k) such that for all $j, n < k$,

$$u_{k,k} = y_k, \quad q_k\left(T_{n_k} u_{n,k}\right) < \frac{1}{2^k}, \quad p_k(u_{n,k} - u_{n,k-1}) < \frac{1}{2^k},$$
$$q_k\left(T_{n_j}(u_{n,k} - u_{n,k-1})\right) < \frac{1}{2^k}, \quad p_k(v_k) < 1 \quad \text{and} \quad q_k\left(T_{n_k} v_k - x_k\right) < \frac{1}{2^k}.$$

For all $n \geqslant 1$, the sequence $(u_{n,k})_{k \geqslant n}$ is Cauchy and thus converges to a vector $u_n \in Y$. Therefore we have

$$\begin{aligned} p_n(y_n - u_n) &\leqslant \sum_{k=n+1}^{\infty} p_n(u_{n,k} - u_{n,k-1}) \leqslant \sum_{k=n+1}^{\infty} p_k(u_{n,k} - u_{n,k-1}) \\ &\leqslant \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}. \end{aligned}$$

The sequence (u_n) is thus dense in Y and we have for all $j \geqslant n$,

$$\begin{aligned} q_n\left(T_{n_j} u_n\right) &\leqslant q_j\left(T_{n_j} u_{n,j}\right) + \sum_{k=j+1}^{\infty} q_k\left(T_{n_j}(u_{n,k} - u_{n,k-1})\right) \\ &\leqslant \frac{1}{2^j} + \frac{1}{2^j} \xrightarrow[j \to \infty]{} 0. \end{aligned}$$

We conclude by remarking that as $(T_{n_k} v_k)$ is dense and (p_n) is an increasing sequence of seminorms defining the topology of Y, the sets $\cup_{k \geqslant 0} T_{n_k}(\{y \in Y \ : \ p(y) < 1\})$ are dense in X, for every continuous seminorm p on Y. $\square$

Corollary 3.6. — *If (T_n) satisfies condition (C) then (T_n) is topologically transitive.*

Proof. — Let U, V be non-empty open sets in Y, X respectively. There exist U_0, U_1 non-empty open sets in Y such that $U_0 + U_1 \subset U$ and $0 \in U_0$, and there exist V_0, V_1 non-empty open sets in X such that $V_0 + V_1 \subset V$ and $0 \in V_0$. By Proposition 3.5, we then know that there exists $n \in \mathbb{N}$ such that

$$T_n(U_0) \cap V_1 \neq \emptyset \quad \text{and} \quad T_n(U_1) \cap V_0 \neq \emptyset.$$

We conclude that $T_n(U) \cap V \neq \emptyset$. $\square$

We deduce that we have the following implications:

$$(T_n) \text{ mixing} \Rightarrow (T_n) \text{ satisfies condition } (C) \Rightarrow (T_n) \text{ topologically transitive} \Rightarrow (T_n) \text{ universal.}$$

In [11], the authors give conditions using condition (C) which imply the existence of a universal closed infinite dimensional subspace for the sequence (T_n), in the setting of Banach spaces. In particular Theorem 20 in [11], useful in the context of universal series, has been extended in [13] to Fréchet spaces. A direct consequence of this result is the following:

Proposition 3.7. — *Let Y be a Fréchet space and X be a separable topological vector space whose topology is given by a sequence of seminorms. Let us consider a sequence $T_n : Y \to X$ of continuous linear mappings. Assume that Y admits a continuous norm, that the sequence (T_n) satisfies condition (C) and that, for any $n \geqslant 1$, $\cap_{k=1}^n \ker T_k$ is infinite dimensional. Then the sequence (T_n) admits a universal closed infinite dimensional subspace.*

Let us consider another Fréchet space E and another map $T_0 : E \to Y$. We have the following result.

Lemma 3.8. — *We use the previous notations and we suppose that T_0 is a continuous linear surjective map. Then we have the following assertions:*

(1) *If the sequence (T_k) satisfies condition (C), then the sequence $(T_k \circ T_0)$ satisfies condition (C);*
(2) *For any $n \geqslant 1$, if $\cap_{k=1}^n \ker T_k$ is an infinite dimensional space, then $\cap_{k=1}^n \ker(T_k \circ T_0)$ is an infinite dimensional space.*

Proof. — T_0 is a continuous, surjective map between two Fréchet spaces, hence T_0 is open.
(1) It suffices to combine the equivalence of Proposition 3.5 together with the fact that T_0 is an open map to conclude.
(2) The space $\cap_{k=1}^n \ker(T_k \circ T_0)$ contains $T_0^{-1}(\cap_{k=1}^n \ker T_k)$. Hence if $\cap_{k=1}^n \ker T_k$ is an infinite dimensional space, then the surjectivity of T_0 ensures that $\cap_{k=1}^n \ker(T_k \circ T_0)$ is infinite dimensional. □

Now the following statement gives a condition of spaceability when E admits a continuous norm.

Theorem 3.9. — *Let E, Y be Fréchet spaces and X be a separable topological vector space whose topology is given by a sequence of seminorms. Let $T_k : Y \to X$ be a sequence of continuous linear maps ($k \geqslant 1$) and let T_0 be a surjective continuous linear map from E onto Y. Assume that E admits a continuous norm, that the sequence $(T_k)_{k\geqslant 1}$ satisfies condition (C) and that, for any $n \geqslant 1$, $\cap_{k=1}^n \ker T_k$ is an infinite dimensional space. Then the sequence $(T_k \circ T_0)_{k\geqslant 1}$ admits a universal closed infinite dimensional subspace.*

Proof. — Applying Lemma 3.8 we know that the sequence $(T_k \circ T_0)_{k\geqslant 1}$ satisfies condition (C) and that $\cap_{k=1}^n \ker(T_k \circ T_0)$ is an infinite dimensional space. Since E admits a continuous norm, it suffices to apply Proposition 3.7 to have the desired conclusion. □

Applications. — The results of section 3.1 apply to universal series. We use the notations of the introduction. First we can easily deduce from [3, Theorem 1] the following proposition.

Proposition 3.10. — *If $\mathcal{U}_A(X) \neq \emptyset$, then the sequence of partial sums $(S_n)_{n\geqslant 1}$ is a mixing sequence.*

Let us consider a separable Fréchet space E and $T_0 : E \to A$ a continuous, linear and surjective map. We recall the following definition.

Definition 3.11. — We say that $f \in E$ is a universal series if the sequence $T_0(f)$ defines a universal series. We denote by $\mathcal{U}(X) \cap E$ the set of such universal series.

Combining Proposition 3.10 together with Theorem 3.9, we obtain the following statement.

Theorem 3.12. — *Suppose that E admits a continuous norm. If $\mathcal{U}_A(X) \neq \emptyset$, then $\mathcal{U}(X) \cap E$ is spaceable.*

In particular, Theorem 3.12 allows to recover the spaceability of the sets $\mathcal{U}(\mathcal{C}_0) \cap \mathcal{C}^\infty(K)$ and $\mathcal{U}(\mathcal{C}_0) \cap \mathcal{C}^\infty_{(M)}(K)$.

3.2. Universal closed subspaces in projective limits

We keep some of the notations and assumptions of Section 2. Yet, we prefer to describe the framework entirely. Let A, A_n, E, E_n, X, $n \geqslant 1$, be Fréchet spaces endowed with translation-invariant metrics d_A, d_{A_n}, d_E, d_{E_n}, d_X respectively. For $n \geqslant 1$, let $(S_k)_k$ and $(S_k^n)_k$ be sequences of linear continuous operators from E to X and from E_n to X_n respectively. We make the following assumptions.

Assumptions. — *There exist continuous linear maps T_0, T_0^n, T_k, T_k^n, K_n, K_n^{n+1}, L_n and L_n^{n+1}, $n \geqslant 1$, such that for every $n \geqslant 2$, the following diagram and each of its sub-diagrams are commutative:*

$$\begin{tikzcd}
E \arrow[r, "T_0"] \arrow[d, "L_n"'] & A \arrow[d, "K_n"'] \arrow[dr, "T_k"] & \\
E_n \arrow[r, "T_0^n"] \arrow[d, "L_{n-1}^n"'] & A_n \arrow[r, "T_k^n"] \arrow[d, "K_{n-1}^n"'] & X \\
E_{n-1} \arrow[r, "T_0^{n-1}"] & A_{n-1} \arrow[ur, "T_k^{n-1}"'] &
\end{tikzcd}$$

We assume that all these spaces and maps satisfy the following assumptions:

(1) *The Fréchet spaces E_n, $n \geqslant 1$, admit a continuous norm;*
(2) *For any $n \geqslant 1$ and any $k \geqslant 0$, $S_k = T_k \circ T_0$ and $S_k^n = T_k^n \circ T_0^n$;*
(3) *The maps T_0, T_0^n, L_n^{n+1} and K_n^{n+1}, $n \geqslant 1$, are surjective;*
(4) *For any $n \geqslant 2$, $L_{n-1} = L_{n-1}^n \circ L_n$ and $K_{n-1} = K_{n-1}^n \circ K_n$;*
(5) *The Fréchet spaces E and A are the projective limit of the sequences $(E_n)_n$ and $(A_n)_n$ respectively, that is*

$$E = \left\{ (x_n)_{n\geqslant 1} \in \prod_{n\geqslant 1} E_n \,:\, L_n^{n+1} \circ \ldots \circ L_{m-1}^m (x_m) = x_n, \text{ whenever } m > n. \right\},$$

endowed with the product topology (A satisfies analogue equality), where L_n (resp. K_n) is the canonical mapping $E \to E_n$ (resp. $A \to A_n$). Note that, under Assumption (3), L_n and K_n are surjective maps.

Remark 3.13. — (1) Observe that $x \in E_n$ is universal for $(S_k^n)_k$ for some $n \geqslant 1$ if and only if $L_j^{j+1} \circ \ldots \circ L_{n-1}^n(x) \in E_j$ is universal for $\left(S_k^j\right)_k$ for some $1 \leqslant j \leqslant n$. Indeed, for any $k \geqslant 1$, we have $S_k^j \circ L_j^{j+1} \circ \ldots \circ L_{n-1}^n = S_k^n$. Similarly $x \in E$ is universal for $(S_k)_k$ if and only if $L_j(x) \in E_j$ is universal for $\left(S_k^j\right)_k$.
(2) Theorem 3.9 coincides with the case where there is only one row in the previous diagram.

We have the following general result.

Lemma 3.14. — *With the above notations, if there exists $n \geqslant 1$ such that $(S_k^n)_k$ admits a universal closed subspace in E_n, then $(S_k)_k$ admits a universal closed subspace in E.*

The following corollary is a direct application of this Lemma and Theorem 3.9 to the present context.

Corollary 3.15. — *Under the above assumptions, assume that there exists some $n \geqslant 1$ such that the sequence $(T_k^n)_{k\geqslant 1}$ satisfies condition (C) and that, for any $m \geqslant 1$, $\cap_{k=1}^m \ker T_k^n$ is an infinite dimensional space. Then $(S_k)_{k\geqslant 1}$ admits a universal closed infinite dimensional subspace.*

The proof of Lemma 3.14 relies on the following lemma. This is a generalization of Lemma 2.7.

Lemma 3.16. — *Let M be subspace of E such that $L_1(M)$ is infinite dimensional. Then there exists a sequence $(u_k)_{k\geqslant 0}$ in M such that:*

(1) *For every $k \geqslant 0$, $\|L_1(u_k)\|_1 = 1$, where $\|\cdot\|_1$ is a continuous norm of E_1;*
(2) *For every $n \geqslant 1$, the sequence $(L_n(u_k))_{k\geqslant n}$ is basic in E_n.*

The proof of this lemma is identical to that of Lemma 2.7, with the updated notations. By definition of the topology of projective limit of E, we directly deduce the following (see the proof of Corollary 2.8).

Proposition 3.17. — *With the above notations, the sequence $(u_k)_{k\geqslant 0}$ given by Lemma 3.16 is basic in E.*

We now turn to the proof of Lemma 3.14.

Proof of Lemma 3.14. — Without loss of generality, we may and shall assume that $n = 1$. By hypothesis, there exists a universal closed infinite dimensional subspace $H \subset E_1$. Let $M = L_1^{-1}(H)$ and let then $(u_k)_{k\geqslant 0} \in M$ be given by Lemma 3.16. We prove that the space

$$F := \overline{\operatorname{span}\{u_k,\, k \geqslant 0\}}$$

is an universal closed infinite dimensional subspace of E (except 0). We have to show that every non-zero element of F is universal. By Proposition 3.17, if $h \in F \setminus \{0\}$, then $h = \sum_{k\geqslant 0} \alpha_k u_k$ with $\alpha_k \neq 0$ for some $k \geqslant 0$. Hence, by construction, $L_1(h) = \sum_{k\geqslant 0} \alpha_k L_1(u_k)$ is in particular a non-zero element of H and so it is universal. □

Applications. — It is easy to check that Theorem 2.10 and its analogue for the Beurling space are straightforward applications of Lemma 3.14.

An interesting consequence of Lemma 3.14 is the following theorem.

Theorem 3.18. — *Let E and X be two separable Fréchet spaces, A a Fréchet space and $T_0 : E \to A$ a continuous surjective mapping. If there exists a continuous seminorm p on E satisfying $\ker p \subset \ker T_0$, then the condition $\mathcal{U}_A(X) \neq \emptyset$ implies that $\mathcal{U}(X) \cap E$ is spaceable.*

Proof. — We may assume that the topology of E is given by an increasing family of seminorms $(p_n)_n$ such that $p_1 = p$. Setting $E_n = E/\ker p_n$, it is not difficult to check that the five assumptions above are satisfied with $A_n = A$ for every $n \geqslant 1$. Then we can apply Theorem 3.12 (for E_1) and Lemma 3.14 to conclude. □

Remark 3.19. — To show that $\mathcal{U}(X) \cap E$ is spaceable, it thus suffices to *factorize* the map T_0 in a convenient way through Fréchet spaces with a continuous norm.

Observe that we exhibited two different manners to recover Theorem 2.10 (and its analogue for the Beurling space) from Lemma 3.14. The first one consists in *factorizing* by $E_n = \mathcal{C}^\infty([-n, n])$, while the second one consists in *factorizing* by $E_n = \mathcal{C}^\infty(\mathbb{R})/\ker p_n$ where p_n is the n-th seminorm on $\mathcal{C}^\infty(\mathbb{R})$ (see Section 2.1 for their definition).

4. Universal closed subspaces in $\mathbb{R}^{\mathbb{N}}$

Let us return in this section to the space $\mathbb{K}^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}}$ (or $\mathbb{C}^{\mathbb{N}}$) endowed with the cartesian topology. Let X be a metrizable vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with a continuous norm $\|\cdot\|$. Let us denote $(x_n)_{n\geqslant 1}$ a fixed sequence of elements in X. In this context we always have $\mathcal{U}_{\mathbb{K}^{\mathbb{N}}}(X) = \mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$. Notice that the set $\mathcal{U}(\mathcal{C}_0) \cap \mathbb{K}^{\mathbb{N}}$ of all Fekete sequences is not spaceable [8]. The particular structure of $\mathbb{K}^{\mathbb{N}}$ plays an essential role in the proof of this result. A natural question arises: is it possible to find a set $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$ which is spaceable? The following theorem gives a complete answer.

Theorem 4.1. — *Under the above assumptions, the following assertions are equivalent:*

(1) *the set $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$ is spaceable,*
(2) *for every $n \geqslant 1$, the set $\bigcup_{m\geqslant n}(\mathrm{span}\{x_k;\ n \leqslant k \leqslant m\} \cap \mathrm{span}\{x_k; k \geqslant m+1\})$ is dense in X.*

Proof. — Given a sequence $(x_k)_{k\geqslant 1} \subset X$, we recall that the notation S_n, $n \geqslant 1$, stands for the n-th partial sum of the formal series $\sum_{k\geqslant 1} a_k x_k$, $(a_k)_k \subset \mathbb{K}$.

(1) $\Rightarrow$ (2): First assume that there exists $N \geqslant 1$ such that the set $\bigcup_{m\geqslant N}(\mathrm{span}\{x_k;\ N \leqslant k \leqslant m\} \cap \mathrm{span}\{x_k; k \geqslant m+1\})$ is not dense in

X. Therefore there exists an open set $U \subset X$ satisfying the following property:

for every $x \in U$ and for every $m \geqslant N$,
$$x \notin \operatorname{span}\{x_k;\ N \leqslant k \leqslant m\} \cap \operatorname{span}\{x_k; k \geqslant m+1\}.$$

Suppose that there exists a closed infinite dimensional subspace F_0 in $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$. Clearly one can find a sequence $(u_n)_{n\geqslant 1}$ in $F_0 \setminus \{0\}$ such that $(v(u_n))_{n\geqslant 1}$ is strictly increasing and $v(u_1) \geqslant N$ ($v(u_n)$ is the valuation of $u_n \in \mathbb{K}^{\mathbb{N}}$ with respect to the canonical basis) [8, Lemma 5.1]. Since for every $n \geqslant 1$, u_n is an universal element, there exists $m_n \geqslant v(u_n)$ such that $S_{m_n}(u_n) \in U$. This property implies
$$S_{m_n}(u_n) \notin \operatorname{span}\{x_k; k \geqslant m_n + 1\}.$$

Thus we have $S_k(u_n) \neq 0$ for all $k > m_n$. Without loss of generality one may consider that the sequence $(m_n)_{n\geqslant 1}$ strictly increases and $m_1 < v(u_2)$. Using that $\sum_{n\geqslant 1} \alpha_n u_n$ converges in $\mathbb{K}^{\mathbb{N}}$ for every sequence $(\alpha_n)_{n\geqslant 1} \in \mathbb{K}^{\mathbb{N}}$ because $(v(u_n))_{n\geqslant 1}$ is strictly increasing, we will construct an element $u = \sum_{n\geqslant 1} \alpha_n u_n$ such that $\|S_k(u)\| > 1$ for every $k > m_1$, what will contradict the universality of $u \in F_0$. First let us choose α_1 so that we have
$$\|\alpha_1 S_k(u_1)\| > 1, \quad \text{for } m_1 < k \leqslant m_2.$$

By induction we choose α_n satisfying the following
$$\left\| S_k\left(\sum_{j=1}^{n-1} \alpha_j u_j\right) + \alpha_n S_k(u_n) \right\| > 1, \quad \text{for } v(u_n) \leqslant k \leqslant m_{n+1}.$$

Notice that we can always find α_n because for every $m_n < k \leqslant m_{n+1}$ we know that $S_k(u_n) \neq 0$ and for every $v(u_n) \leqslant k \leqslant m_n$ two cases occur: $S_k(u_n) \neq 0$, thus it suffices to choose α_n sufficiently large, or $S_k(u_n) = 0$ and we obtain the result by induction hypothesis. Therefore, for every $k > m_1$, if $v(u_n) \leqslant k < v(u_{n+1}) \leqslant m_{n+1}$, then we have $\|S_k(u)\| = \left\| S_k\left(\sum_{j=1}^{n} \alpha_j u_j\right)\right\| > 1$ and we are done.

(2) $\Rightarrow$ (1): Now assume that, for every $N \geqslant 1$, the set $\bigcup_{m\geqslant N}(\operatorname{span}\{x_k;\ N \leqslant k \leqslant m\} \cap \operatorname{span}\{x_k; k \geqslant m+1\})$ is dense in X. Therefore for any open set $U \subset X$ and for any $N \geqslant 1$, there exist $x \in U$ and $M \geqslant N$ such that the following holds
$$x \in \operatorname{span}\{x_k;\ N \leqslant k \leqslant M\} \cap \operatorname{span}\{x_k; k \geqslant M+1\}.$$

Let us consider a countable basis of open sets (U_k) in X (which is separable as hypothesis (2) shows). We want to construct a sequence $(u_k)_{k\geqslant 1}$ in $\mathbb{K}^{\mathbb{N}}$ such that the sequence of valuations $(v(u_k))_{k\geqslant 1}$ is strictly increasing and all the non-zero elements $\sum_{k\geqslant 1}\alpha_k u_k$ are universal. Using the hypothesis, for any $k\geqslant 1$, any $N\geqslant 1$, there exist $b\in\mathbb{K}^{\mathbb{N}}$, $M\geqslant N$ and $M'\geqslant M$ such that

$$N\leqslant v(b),\ S_M(b)\in U_k \text{ and } S_M(b)=\sum_{k=M+1}^{M'} b_k x_k.$$

Define the sequence $a=(0,\dots,0,b_N,\dots,b_M,-b_{M+1},\dots,-b_{M'},0,\dots)$ and $d(a)=M'$. Therefore we have

$$N\leqslant v(a),\ S_M(a)\in U_k \text{ and } S_{d(a)}(a)=0.$$

Let us choose a strict total order $\prec$ on the set of couples (i,j), $i,j\geqslant 1$. By induction we construct also polynomials $y_{i,j}$ and integers $n_{i,j}$ so that

$$S_{n_{i,j}}(y_{i,j})\in U_j,\ S_{d(y_{i,j})}(y_{i,j})=0 \text{ and } d(y_{i',j'})<v(y_{i,j}) \text{ for } (i',j')\prec(i,j).$$

In particular we have $S_k(y_{i,j})=0$ for any $k<v(y_{i,j})$, any $k\geqslant d(y_{i,j})$ and any (i,j). In addition, for any (i,j), the intervals $(v(y_{i,j}),\dots,d(y_{i,j}))$ are pairwise disjoint and contains $n_{i,j}$. Then we define, for any $k\geqslant 1$, $u_k=\sum_{j\geqslant 1}y_{k,j}$. Let $u=\sum_{k\geqslant 1}\alpha_k u_k$, with $\alpha_{k_0}=1$ for some k_0. We observe that

$$S_{n_{k_0,j}}(u)=S_{n_{k_0,j}}(\alpha_{k_0}y_{k_0,j})\in U_j.$$

This finishes the proof. □

Remark 4.2. — The proof of the implication (2) ⇒ (1) does not need the existence of a continuous norm in X.

We immediately deduce the following result.

Corollary 4.3. — *If $(x_n)_{n\geqslant 1}$ is a free family, then the set $\mathcal{U}(X)\cap\mathbb{K}^{\mathbb{N}}$ is not spaceable.*

Moreover it is well known that there exists an universal element in $\mathcal{U}(X)\cap\mathbb{K}^{\mathbb{N}}$ if and only if for every $n_0\in\mathbb{N}$, $\operatorname{span}\{x_k;\ k\geqslant n_0\}$ is dense in X (see [9, Proposition 7] or [3] for e.g.). A careful examination of the condition (2) of Theorem 4.1 shows that it is close to condition

$$(2')\qquad \bigcap_{n\geqslant 1}\operatorname{span}\{x_k;\ k\geqslant n\} \text{ is dense in } X.$$

Proposition 4.4. — *Condition* $(2')$ *implies condition* (2) *of Theorem 4.1 but the converse implication is false. In particular, if* $(2')$ *holds, then* $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$ *is spaceable.*

Proof. — $(2') \Rightarrow (2)$: Let $N \geqslant 1$ and let U be a non-empty open subset of X. Using $(2')$ one can find $y \in \cap_{n \geqslant N} \mathrm{span}\{x_k;\ k \geqslant n\} \cap U$. Observe that one can write $y = a_N x_N + \cdots + a_M x_M$ or $y = a_{M+1} x_{M+1} + \cdots + a_{M'} x_{M'}$. To prove that $(2) \Rightarrow (2')$ does not hold, we are going to build a sequence $(\tilde{x}_k)_{k \geqslant 1}$ which satisfies condition (2) but not $(2')$. Let us consider a free family $(x_n)_{n \geqslant 1}$ in X (endowed with the metric ρ) such that for every $n_0 \in \mathbb{N}$, $\mathrm{span}\{x_k;\ k \geqslant n_0\}$ is dense in X. Let also $(q_n)_{n \geqslant 1}$ be an enumeration of rational numbers and G_k be the set $\{\alpha_1 x_1 + \ldots + \alpha_k x_k,\ \alpha_j \in \{q_1, \ldots, q_k\},\ j = 1, \ldots, k\}$. By induction on $k \geqslant 1$, there exists $n_k \geqslant 1$ (with $n_{k-1} < n_k$, $n_0 = 0$) such that, for any $p_k \in G_k$, we have

$$\rho\left(p_k, \mathrm{span}\left(x_{k+1}, \ldots, x_{n_k}\right)\right) < \frac{1}{2^k}.$$

Let us consider the sequence $(\tilde{x}_n)_{n \geqslant 1} = (x_1, b_1, x_2, b_2, \ldots)$ where $b_k = (x_{k+1}, \ldots, x_{n_k})$. Clearly we have $\cap_{n \geqslant 1} \mathrm{span}\{\tilde{x}_k;\ k \geqslant n\} = \{0\}$ which implies that condition $(2')$ is not fulfilled. Combining the inclusion $\mathrm{span}(b_k) \subset \mathrm{span}(x_k, b_{k+1})$ together with the density of the family

$$\left\{\sum_{i=1}^{n} \alpha_i x_i,\ \alpha_1, \ldots, \alpha_n \in \mathbb{Q},\ n \in \mathbb{N}\right\},$$

we conclude that $(\tilde{x}_n)_{n \geqslant 1}$ satisfies condition (2). □

All the previous yields the following general result on the spaceability of $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$ when X is finite dimensional.

Theorem 4.5. — *Under the above assumptions, if* X *is a finite dimensional space, the following assertions are equivalent:*

(1) $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}} \neq \emptyset$,
(2) $\mathcal{U}(X) \cap \mathbb{K}^{\mathbb{N}}$ *is spaceable,*
(3) *for every* $n_0 \in \mathbb{N}$, $\mathrm{span}\{x_k;\ k \geqslant n_0\} = X$.

Example 4.6. — The simplest non-trivial couple $(X, (x_n)_{n \geqslant 1})$ satisfying Assumption (3) of Theorem 4.5 is $(\mathbb{R}, (x_n)_{n \geqslant 1} = (1))$. In this setting, Theorem 4.5 ensures that there exists a universal sequence $a = (a_n)_{n \geqslant 1}$ of real numbers such that the set of all partial sums $\sum_{j=1}^{n} a_j$ is dense in $\mathbb{R}$ and that the set of such sequences is spaceable.

Example 4.7. — Let us return to the universal series of Fekete type. Applying Corollary 4.3 with the couple $(X = \mathcal{C}([-1, 1]), (x_n)_{n \geqslant 1} = (x^n)_{n \geqslant 1})$ we

recover that the set of universal Fekete series is not spaceable again [8, Theorem 5.9]. Now let us consider a slight modification of the family $(x_n)_{n\geqslant 1}$. We set $y_k = x^i$, if $k = i + j(j-1)/2$, for $j \geqslant 1$ and $1 \leqslant i \leqslant j$. We obtain the couple $(X, (y_n)_{n\geqslant 1}) = (\mathcal{C}([-1,1], (x, x, x^2, x, x^2, x^3, x, x^2, x^3, x^4, \dots))$. It is easy to check that this new family satisfies the condition of Theorem 4.1 so that the set of associated Fekete universal series becomes spaceable.

More generally, if we have a couple $(X, (x_n)_{n\geqslant 1})$ such that for every $n_0 \in \mathbb{N}$, $\operatorname{span}\{x_k;\ k \geqslant n_0\}$ is dense in X, then there exist universal series. If $(x_n)_{n\geqslant 1}$ does not satisfy the condition of Theorem 4.1, this set of universal series is not spaceable. But one can obtain a set of universal series of the same type which becomes spaceable, by setting $(\tilde{x}_n)_{n\geqslant 1} = (x_1, x_1, x_2, x_1, x_2, x_3, x_1, x_2, x_3, x_4, \dots)$.

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