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RATIONAL APPROXIMATION TO REAL POINTS ON CONICS

by Damien ROY (*)

Abstract. — A point \((\xi_1, \xi_2)\) with coordinates in a subfield of \(\mathbb{R}\) of transcendence degree one over \(\mathbb{Q}\), with \(1, \xi_1, \xi_2\) linearly independent over \(\mathbb{Q}\), may have a uniform exponent of approximation by elements of \(\mathbb{Q}^2\) that is strictly larger than the lower bound \(1/2\) given by Dirichlet’s box principle. This appeared as a surprise, in connection to work of Davenport and Schmidt, for points of the parabola \(\{ (\xi, \xi^2) ; \xi \in \mathbb{R} \}\). The goal of this paper is to show that this phenomenon extends to all real conics defined over \(\mathbb{Q}\), and that the largest exponent of approximation achieved by points of these curves satisfying the above condition of linear independence is always the same, independently of the curve, namely \(1/\gamma \cong 0.618\) where \(\gamma\) denotes the golden ratio.

Résumé. — Un point \((\xi_1, \xi_2)\) à coordonnées dans un sous-corps de \(\mathbb{R}\) de degré de transcendance un sur \(\mathbb{Q}\), avec \(1, \xi_1, \xi_2\) linéairement indépendants sur \(\mathbb{Q}\), peut admettre un exposant d’approximation uniforme par les éléments de \(\mathbb{Q}^2\) qui soit strictement plus grand que la borne inférieure \(1/2\) que garantit le principe des tiroirs de Dirichlet. Ce fait inattendu est apparu, en lien avec des travaux de Davenport et Schmidt, pour les points de la parabole \(\{ (\xi, \xi^2) ; \xi \in \mathbb{R} \}\). Le but de cet article est de montrer que ce phénomène s’étend à toutes les coniques réelles définies sur \(\mathbb{Q}\) et que le plus grand exposant d’approximation atteint par les points de ces courbes, sujets à la condition d’indépendance linéaire mentionnée plus tôt, est toujours le même, indépendamment de la courbe, à savoir \(1/\gamma \cong 0.618\) où \(\gamma\) désigne le nombre d’or.

1. Introduction

Let \(n\) be a positive integer and let \(\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\). The uniform exponent of approximation to \(\xi\) by rational points, denoted \(\lambda(\xi)\), is defined as the supremum of all real numbers \(\lambda\) for which the system of inequalities

\[
|x_0| \leq X, \quad \max_{1 \leq i \leq n} |x_0 \xi_i - x_i| \leq X^{-\lambda}
\]

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admits a non-zero solution \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1} \) for each sufficiently large real number \( X > 1 \). It is one of the classical ways of measuring how well \( \xi \) can be approximated by elements of \( \mathbb{Q}^n \), because each solution of (1.1) with \( x_0 \neq 0 \) provides a rational point \( r = (x_1/x_0, \ldots, x_n/x_0) \) with denominator dividing \( x_0 \) such that \( \| \xi - r \| \leq |x_0|^{-\lambda - 1} \), where the symbol \( \| \| \) stands for the maximum norm. We call it a “uniform exponent” following the terminology of Y. Bugeaud and M. Laurent in [2, §1] because we require a solution of (1.1) for each sufficiently large \( X \) (but note that our notation is slightly different as they denote it \( \hat{\lambda}(\xi) \)). This exponent depends only on the \( \mathbb{Q} \)-vector subspace of \( \mathbb{R} \) spanned by \( 1, \xi_1, \ldots, \xi_n \) and so, by a result of Dirichlet [12, Chapter II, Theorem 1A], it satisfies \( \lambda(\xi) \geq 1/(s - 1) \) where \( s \geq 1 \) denotes the dimension of that subspace. In particular we have \( \lambda(\xi) = \infty \) when \( \xi \in \mathbb{Q}^n \), while it is easily shown that \( \lambda(\xi) \leq 1 \) when \( \xi \notin \mathbb{Q}^n \) (see for example [2, Prop. 2.1]).

In their seminal work [3], H. Davenport and W. M. Schmidt determine an upper bound \( \lambda_n \), depending only on \( n \), for \( \lambda(\xi, \xi_2, \ldots, \xi_n) \) where \( \xi \) runs through all real numbers such that \( 1, \xi_1, \ldots, \xi_n \) are linearly independent over \( \mathbb{Q} \), a condition which amounts to asking that \( \xi \) is not algebraic over \( \mathbb{Q} \) of degree \( n \) or less. Using geometry of numbers, they deduce from this a result of approximation to such \( \xi \) by algebraic integers of degree at most \( n + 1 \). In particular they prove that \( \lambda(\xi, \xi^2) \leq \lambda_2 := 1/\gamma \cong 0.618 \) for each non-quadratic irrational real number \( \xi \), where \( \gamma = (1 + \sqrt{5})/2 \) denotes the golden ratio. It is shown in [7, 9] that this upper bound is best possible and, in [8], that the corresponding result of approximation by algebraic integers of degree at most 3 is also best possible. For \( n \geq 3 \), no optimal value is known for \( \lambda_n \). At present the best known upper bounds are \( \lambda_3 \leq (1 + 2\gamma - \sqrt{1 + 4\gamma^2})/2 \cong 0.4245 \) (see [11]) and \( \lambda_n \leq 1/\lceil n/2 \rceil \) for \( n \geq 4 \) (see [5]).

As a matter of approaching this problem from a different angle, we propose to extend it to the following setting.

**Definition 1.1.** — Let \( \mathcal{C} \) be a closed algebraic subset of \( \mathbb{R}^n \) of dimension 1 defined over \( \mathbb{Q} \), irreducible over \( \mathbb{Q} \), and not contained in any proper affine linear subspace of \( \mathbb{R}^n \) defined over \( \mathbb{Q} \). Then, we put \( \lambda(\mathcal{C}) = \sup \{ \lambda(\xi) ; \xi \in \mathcal{C}^{li} \} \) where \( \mathcal{C}^{li} \) denotes the set of points \( \xi = (\xi_1, \ldots, \xi_n) \in \mathcal{C} \) such that \( 1, \xi_1, \ldots, \xi_n \) are linearly independent over \( \mathbb{Q} \).

Equivalently, such a curve may be described as the Zariski closure over \( \mathbb{Q} \) in \( \mathbb{R}^n \) of a point \( \xi \in \mathbb{R}^n \) whose coordinates \( \xi_1, \ldots, \xi_n \) together with 1 are linearly independent over \( \mathbb{Q} \) and generate over \( \mathbb{Q} \) a subfield of \( \mathbb{R} \).
of transcendence degree one. In particular $C^{li}$ is not empty as it contains that point. From the point of view of metrical number theory the situation is simple since, for the relative Lebesgue measure, almost all points $\xi$ of $C$ have $\lambda(\xi) = 1/n$ (see [4]). Of special interest is the curve $C_n := \{(\xi, \xi^2, \ldots, \xi^n) ; \xi \in \mathbb{R}\}$ for any $n \geq 2$. As mentioned above, we have $\lambda(C_2) = 1/\gamma$ and the problem remains to compute $\lambda(C_n)$ for $n \geq 3$. In this paper, we look at the case of conics in $\mathbb{R}^2$ and prove the following result.

**Theorem 1.2.** — Let $C$ be a closed algebraic subset of $\mathbb{R}^2$ of dimension 1 and degree 2. Suppose that $C$ is defined over $\mathbb{Q}$ and irreducible over $\mathbb{Q}$. Then, we have $\lambda(C) = 1/\gamma$. Moreover, the set of points $\xi \in C^{li}$ with $\lambda(\xi) = 1/\gamma$ is countably infinite.

Here the degree of $C$ simply refers to the degree of the irreducible polynomial of $\mathbb{Q}[x_1, x_2]$ defining it. The curve $C_2$ is the parabola of equation $x_2 - x_1^2 = 0$ but, as we will see, other curves are easier to deal with, for example the curve defined by $x_1^2 - 2 = 0$ which consists of the pair of vertical lines $\{ \pm \sqrt{2} \} \times \mathbb{R}$. Note that, for the latter curve, Theorem 1.2 simply says that any $\xi \in \mathbb{R} \setminus \mathbb{Q}(\sqrt{2})$ has $\lambda(\sqrt{2}, \xi) \leq 1/\gamma$, with equality defining a denumerable subset of $\mathbb{R} \setminus \mathbb{Q}(\sqrt{2})$. Our main result in the next section provides a slightly finer result.

In [6], it is shown that the cubic $C$ defined by $x_2 - x_1^3 = 0$ has $\lambda(C) \leq 2(9 + \sqrt{11})/35 \approx 0.7038$, but the case of the line $\sqrt{2} \times \mathbb{R}$ should be simpler to solve and could give ideas to determine the precise value of $\lambda(C)$ for that cubic $C$. Similarly, looking at lines $(\omega_2, \ldots, \omega_n) \times \mathbb{R}$ where $(1, \omega_2, \ldots, \omega_n)$ is a basis over $\mathbb{Q}$ of a number field of degree $n$ could provide new ideas to compute $\lambda(C_n)$.

This paper is organized as follows. In the next section, we state a slightly stronger result in projective setting and note that, for curves $C$ which are irreducible over $\mathbb{R}$ and contain at least one rational point, the proof simply reduces to the known case of the parabola $C_2$. In Section 3, we prove the inequality $\lambda(C) \leq 1/\gamma$ for the remaining curves $C$ by an adaptation of the original argument of Davenport and Schmidt in [3, § 3]. However, the fact that these curves have at most one rational point brings a notable simplification in the proof. In Section 4, we adapt the arguments of [9, § 5] to establish a certain rigidity property for the sequence of minimal points attached to points $\xi \in C^{li}$ with $\lambda(\xi) = 1/\gamma$, and deduce from it that the set of these points $\xi$ is at most countable. We conclude in Section 5, with the most delicate part, namely the existence of infinitely many points $\xi \in C^{li}$ having exponent $1/\gamma$. 

TOME 63 (2013), FASCICULE 6
2. The main result in projective framework

For each $n \geq 2$, we endow $\mathbb{R}^n$ with the maximum norm, and identify its exterior square $\bigwedge^2 \mathbb{R}^n$ with $\mathbb{R}^{n(n-1)/2}$ via an ordering of the Plücker coordinates. In particular, when $n = 3$, we define the wedge product of two vectors in $\mathbb{R}^3$ as their usual cross-product. We first introduce finer notions of Diophantine approximation in the projective context.

Let $\Xi \in \mathbb{P}^n(\mathbb{R})$ and let $\Xi = (\xi_0, \ldots, \xi_n)$ be a representative of $\Xi$ in $\mathbb{R}^{n+1}$. We say that a real number $\lambda \geq 0$ is an exponent of approximation to $\Xi$ if there exists a constant $c = c_1(\Xi)$ such that the conditions

$$\|x\| \leq X \quad \text{and} \quad \|x \wedge \Xi\| \leq cX^{-\lambda}$$

admit a non-zero solution $x \in \mathbb{Z}^{n+1}$ for each sufficiently large real number $X$. We say that $\lambda$ is a strict exponent of approximation to $\Xi$ if moreover there exists a constant $c = c_2(\Xi) > 0$ such that the same conditions admit no non-zero solution $x \in \mathbb{Z}^{n+1}$ for arbitrarily large values of $X$. Both properties are independent of the choice of the representative $\Xi$, and we define $\lambda(\Xi)$ as the supremum of all exponents of approximations to $\Xi$. Clearly, when $\lambda$ is a strict exponent of approximation to $\Xi$, we have $\lambda(\Xi) = \lambda$.

Let $T : \mathbb{Q}^{n+1} \to \mathbb{Q}^{n+1}$ be an invertible $\mathbb{Q}$-linear map. It extends uniquely to a $\mathbb{R}$-linear automorphism of $\mathbb{R}^{n+1}$ and then to an automorphism of $\mathbb{P}^n(\mathbb{R})$. Moreover, upon choosing an integer $m \geq 1$ such that $mT(\mathbb{Z}^{n+1}) \subseteq \mathbb{Z}^{n+1}$, any non-zero point $x \in \mathbb{Z}^{n+1}$ gives rise to a non-zero point $y = mT(x) \in \mathbb{Z}^{n+1}$ satisfying

$$\|y\| \leq cT\|x\| \quad \text{and} \quad \|y \wedge T(\Xi)\| \leq cT\|x \wedge \Xi\|$$

for a constant $cT > 0$ depending only on $T$. Combined with the above definitions, this yields the following invariance property.

**Lemma 2.1.** — Let $\Xi \in \mathbb{P}^n(\mathbb{R})$ and $T \in \text{GL}_{n+1}(\mathbb{Q})$. Then we have $\lambda(\Xi) = \lambda(T(\Xi))$. More precisely a real number $\lambda \geq 0$ is an exponent of approximation to $\Xi$, respectively a strict exponent of approximation to $\Xi$, if and only if it is an exponent of approximation to $T(\Xi)$, respectively a strict exponent of approximation to $T(\Xi)$.

We also have a natural embedding of $\mathbb{R}^n$ into $\mathbb{P}^n(\mathbb{R})$, sending a point $\xi = (\xi_1, \ldots, \xi_n)$ to $(1 : \xi) := (1 : \xi_1 : \cdots : \xi_n)$. Identifying $\mathbb{R}^n$ with its image in $\mathbb{P}^n(\mathbb{R})$, the above notions of exponent of approximation and strict exponent of approximation carry back to points of $\mathbb{R}^n$. The next lemma, whose proof is left to the reader, shows how they translate in this context and shows moreover that $\lambda(\xi) = \lambda(1 : \xi)$, thus leaving no ambiguity as to the value of $\lambda(\xi)$. 
Lemma 2.2. — Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

(i) A real number $\lambda \geq 0$ is an exponent of approximation to $(1 : \xi)$ if and only if there exists a constant $c = c_1(\xi)$ such that the conditions

$$|x_0| \leq X \quad \text{and} \quad \max_{1 \leq i \leq n} |x_0\xi_i - x_i| \leq cX^{-\lambda}$$

admit a non-zero solution $x = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ for each sufficiently large $X$.

(ii) It is a strict exponent of approximation to $(1 : \xi)$ if and only if there also exists a constant $c = c_2(\xi) > 0$ such that the above conditions admit no non-zero integer solution for arbitrarily large values of $X$.

Finally, we have $\lambda(\xi) = \lambda(1 : \xi)$.

Our main result is the following strengthening of Theorem 1.2.

Theorem 2.3. — Let $\varphi$ be a homogeneous polynomial of degree 2 in $\mathbb{Q}[x_0, x_1, x_2]$. Suppose that $\varphi$ is irreducible over $\mathbb{Q}$ and that its set of zeros $\mathcal{C}$ in $\mathbb{P}^2(\mathbb{R})$ consists of at least two points.

(i) For each point $\Xi \in \mathcal{C}$ having $\mathbb{Q}$-linearly independent homogeneous coordinates, the number $1/\gamma$ is at best a strict exponent of approximation to $\Xi$: if it is an exponent of approximation to $\Xi$, it is a strict one.

(ii) There are infinitely many points $\Xi \in \mathcal{C}$ which have $\mathbb{Q}$-linearly independent homogeneous coordinates and for which $1/\gamma$ is an exponent of approximation.

(iii) There exists a positive $\epsilon$, independent of $\varphi$, such that the set of points $\Xi \in \mathcal{C}$ with $\lambda(\Xi) > 1/\gamma - \epsilon$ is countable.

To show that this implies Theorem 1.2, let $\mathcal{C}$ be as in latter statement. Then, the Zariski closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ in $\mathbb{P}^2(\mathbb{R})$ is infinite and is the zero set of an irreducible homogeneous polynomial of degree 2 in $\mathbb{Q}[x_0, x_1, x_2]$. Moreover, $\overline{\mathcal{C}}^\text{li}$ identifies with the set of elements of $\overline{\mathcal{C}}$ with $\mathbb{Q}$-linearly independent homogeneous coordinates. So, if we admit the above theorem, then, in view of Lemma 2.2, Part (i) implies that $\lambda(\mathcal{C}) \leq 1/\gamma$, Part (ii) shows that there are infinitely many $\xi \in \overline{\mathcal{C}}^\text{li}$ with $\lambda(\xi) = 1/\gamma$, and Part (iii) shows that the set of points $\xi \in \mathcal{C}$ with $\lambda(\xi) > 1/\gamma - \epsilon$ is countable. Altogether, this proves Theorem 1.2.

The proof of Part (iii) in Section 4 will show that one can take $\epsilon = 0.005$ but the optimal value for $\epsilon$ is probably much larger. In connection to (iii), we also note that the set of elements of $\mathcal{C}$ with $\mathbb{Q}$-linearly dependent homogeneous coordinates is at most countable because each such point belongs to a proper linear subspace of $\mathbb{P}^2(\mathbb{R})$ defined over $\mathbb{Q}$, there are
countably many such subspaces, and each of them meets $\mathcal{C}$ in at most two points. So, in order to prove (iii), we may restrict to the points of $\mathcal{C}$ with \( \mathbb{Q} \)-linearly independent homogeneous coordinates.

Lemma 2.1 implies that, if Theorem 2.3 holds true for a form $\varphi$, then it also holds for $\mu(\varphi \circ T)$ for any $T \in \text{GL}_3(\mathbb{Q})$ and any $\mu \in \mathbb{Q}^*$. Thus the next lemma reduces the proof of the theorem to forms of special types.

**Lemma 2.4.** — Let $\varphi$ be an irreducible homogeneous polynomial of $\mathbb{Q}[x_0, x_1, x_2]$ of degree 2 which admits at least two zeros in $\mathbb{P}^2(\mathbb{R})$.

(i) If $\varphi$ is irreducible over $\mathbb{R}$ and admits at least one zero in $\mathbb{P}^2(\mathbb{Q})$, then there exist $\mu \in \mathbb{Q}^*$ and $T \in \text{GL}_3(\mathbb{Q})$ such that $\mu(\varphi \circ T)(x_0, x_1, x_2) = x_0x_2 - x_1^2$.

(ii) If $\varphi$ is not irreducible over $\mathbb{R}$, then it admits exactly one zero in $\mathbb{P}^2(\mathbb{Q})$ and there exist $\mu \in \mathbb{Q}^*$ and $T \in \text{GL}_3(\mathbb{Q})$ such that we have $\mu(\varphi \circ T)(x_0, x_1, x_2) = x_0^2 - bx_1^2$ for some square-free integer $b > 1$.

(iii) If $\varphi$ has no zero in $\mathbb{P}^2(\mathbb{Q})$, then there exist $\mu \in \mathbb{Q}^*$ and $T \in \text{GL}_3(\mathbb{Q})$ such that $\mu(\varphi \circ T)(x_0, x_1, x_2) = x_0^2 - bx_1^2 - cx_2^2$ for some square-free integers $b > 1$ and $c > 1$.

**Proof.** — We view $(\mathbb{Q}^3, \varphi)$ as a quadratic space. We denote by $K$ its kernel, and by $\Phi$ the unique symmetric bilinear form such that $\Phi(x, x) = 2\varphi(x)$.

Suppose first that $K \neq \{0\}$. Then, by a change of variables over $\mathbb{Q}$, we can bring $\varphi$ to a diagonal form $rx_0^2 + sx_1^2$ with $r, s \in \mathbb{Q}$. We have $rs \neq 0$ since $\varphi$ is irreducible over $\mathbb{Q}$, and furthermore $rs < 0$ since otherwise the point $(0 : 0 : 1)$ would be the only zero of $\varphi$ in $\mathbb{P}^2(\mathbb{R})$. Thus, $\varphi$ is not irreducible over $\mathbb{R}$, and $\dim_{\mathbb{Q}} K = 1$.

In the case (i), the above observation shows that $\mathbb{Q}^3$ is non-degenerate. Then, since $\varphi$ has a zero in $\mathbb{P}^2(\mathbb{Q})$, the space $\mathbb{Q}^3$ decomposes as the orthogonal direct sum of a hyperbolic plane $H$ and a non-degenerate line $P$. We choose bases $\{v_0, v_2\}$ for $H$ and $\{v_1\}$ for $P$ such that $\varphi(v_0) = \varphi(v_2) = 0$ and $\Phi(v_0, v_2) = -\varphi(v_1)$. Then $\mu = -1/\varphi(v_1)$ and the linear map $T \colon \mathbb{Q}^3 \to \mathbb{Q}^3$ sending the canonical basis of $\mathbb{Q}^3$ to $(v_0, v_1, v_2)$ have the property stated in (i).

In the case (iii), we have $K = \{0\}$ and so we can write $\mathbb{Q}^3$ as an orthogonal direct sum of one-dimensional non-degenerate subspaces $P_0, P_1$ and $P_2$. We order them so that the non-zero values of $\varphi$ on $P_0$ have opposite sign to those on $P_1$ and $P_2$. This is possible since $\varphi$ is indefinite. Let $\{v_0\}$ be a basis of $P_0$ and put $\mu = 1/\varphi(v_0)$. For $i = 1, 2$, we can choose a basis $\{v_i\}$ of $P_i$ such that $\mu \varphi(v_i)$ is a square-free integer. Then $\mu$ and the linear map
$T : \mathbb{Q}^3 \to \mathbb{Q}^3$ sending the canonical basis of $\mathbb{Q}^3$ to $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ have the property stated in (iii).

In the case (ii), the form $\varphi$ factors over a quadratic extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$ as a product $\varphi(x) = \rho L(x)\bar{L}(x)$ where $L$ is a linear form, $\bar{L}$ its conjugate over $\mathbb{Q}$, and $\rho \in \mathbb{Q}^\ast$. As $\varphi$ is irreducible over $\mathbb{Q}$, the linear forms $L$ and $\bar{L}$ are not multiple of each other. Moreover, for a point $a \in \mathbb{Q}^3$, we have

$$\varphi(a) = 0 \iff L(a) = \bar{L}(a) = 0 \iff (L + \bar{L})(a) = \sqrt{d}(L - \bar{L})(a) = 0.$$

Since $L + \bar{L}$ and $\sqrt{d}(L - \bar{L})$ are linearly independent forms with coefficients in $\mathbb{Q}$, this means that the zero set of $\varphi$ in $\mathbb{Q}^3$ is a line, and so $\varphi$ has a unique zero in $\mathbb{P}^2(\mathbb{Q})$. As $\Phi(x, y) = \rho L(x)\bar{L}(y) + \rho L(x)L(y)$, this line is contained in the kernel $K$ of $\varphi$, and so is equal to $K$. By an earlier observation, this means that, by a change of variables over $\mathbb{Q}$, we may bring $\varphi$ to a diagonal form $rx_0^2 + sx_1^2$ with $r, s \in \mathbb{Q}$, $rs < 0$. We may further choose $r$ and $s$ so that $-s/r$ is a square-free integer $b > 0$. Then, the same change of variables brings $r^{-1}\varphi$ to $x_0^2 - bx_1^2$. Finally, we have $b \neq 1$ since $\varphi$ is irreducible over $\mathbb{Q}$. \hfill \Box

3. Proof of the first part of the main theorem

Let $\varphi$ and $C$ be as in the statement of Theorem 2.3. Suppose first that $\varphi$ is irreducible over $\mathbb{R}$ and that $C \cap \mathbb{P}^2(\mathbb{Q}) \neq \emptyset$. Then, by Lemma 2.4, there exists $T \in \text{GL}_3(\mathbb{Q})$ such that $T^{-1}(C)$ is the zero-set in $\mathbb{P}^2(\mathbb{R})$ of the polynomial $x_0x_2 - x_1^2$. Let $\Xi$ be a point of $C$ with $\mathbb{Q}$-linearly independent homogeneous coordinates. Its image $T^{-1}(\Xi)$ has homogeneous coordinates $(1 : \xi : \xi^2)$, for some irrational non-quadratic $\xi \in \mathbb{R}$. Then, by [3, Theorem 1a], the number $1/\gamma$ is at best a strict exponent of approximation to $T^{-1}(\Xi)$, and, by Lemma 2.1, the same applies to $\Xi$. This proves Part (i) of the theorem in that case.

Otherwise, Lemma 2.4 shows that $\varphi$ has at most one zero in $\mathbb{P}^2(\mathbb{Q})$. Taking advantage of the major simplification that this entails, we proceed as Davenport and Schmidt in [3, §3]. We fix a point $\Xi \in C$ with $\mathbb{Q}$-linearly independent homogeneous coordinates $(1 : \xi_1 : \xi_2)$ and an exponent of approximation $\lambda \geq 1/2$ for $\Xi$. Then, by Lemma 2.2, there exists a constant $c > 0$ such that, for each sufficiently large $X$, the system

$$(3.1) \quad |x_0| \leq X, \quad L(x) := \max \{|x_0\xi_1 - x_1|, |x_0\xi_2 - x_2|\} \leq cX^{-\lambda}$$

has a non-zero solution $x = (x_0, x_1, x_2) \in \mathbb{Z}^3$. To prove Part (i) of Theorem 2.3, we simply need to show that $\lambda \leq 1/\gamma$ and that, when $\lambda = 1/\gamma$, the constant $c$ cannot be chosen arbitrarily small.
To this end, we first note that there exists a sequence of points \((x_i)_{i \geq 1}\) in \(\mathbb{Z}^3\) such that

(a) their first coordinates \(X_i\) form an increasing sequence \(1 \leq X_1 < X_2 < X_3 < \cdots\),

(b) the quantities \(L_i := L(x_i)\) form a decreasing sequence \(1 > L_1 > L_2 > L_3 > \cdots\),

(c) for each \(x = (x_0, x_1, x_2) \in \mathbb{Z}^3\) and each \(i \geq 1\) with \(|x_0| < X_{i+1}\), we have \(L(x) \geq L_i\).

Then, each \(x_i\) is a primitive point of \(\mathbb{Z}^3\), by which we mean that the gcd of its coordinates is 1. Moreover, the hypothesis that (3.1) has a solution for each large enough \(X\) implies that

\[
L_i \leq cX_i^{-\lambda}
\]

for each sufficiently large \(i\), say for all \(i \geq i_0\). Since \(\varphi\) has at most one zero in \(\mathbb{P}^2(\mathbb{Q})\), we may further assume that \(\varphi(x_i) \neq 0\) for each \(i \geq i_0\). Then, upon normalizing \(\varphi\) so that it has integer coefficients, we conclude that \(|\varphi(x_i)| \geq 1\) for the same values of \(i\).

Put \(\Xi = (1, \xi_1, \xi_2) \in \mathbb{Q}^3\), and let \(\Phi\) denote the symmetric bilinear form for which \(\Phi(x, x) = 2\varphi(x)\). Then, upon writing \(x_i = X_i\Xi + \Delta_i\) and noting that \(\varphi(\Xi) = 0\), we find

\[
\varphi(x_i) = X_i\Phi(\Xi, \Delta_i) + \varphi(\Delta_i).
\]

As \(\|\Delta_i\| = L_i\), this yields \(|\varphi(x_i)| \leq c_1 X_i L_i\) for a constant \(c_1 = c_1(\varphi, \Xi) > 0\). Using (3.2), we conclude that, for each \(i \geq i_0\), we have \(1 \leq |\varphi(x_i)| \leq cc_1 X_i X_i^{-\lambda}\), and so

\[
X_i^{\lambda} \leq cc_1 X_i.
\]

We also note that there are infinitely many values of \(i > i_0\) for which \(x_{i-1}, x_i\) and \(x_{i+1}\) are linearly independent. For otherwise, all points \(x_i\) with \(i\) large enough would lie in a two dimensional subspace \(V\) of \(\mathbb{R}^3\) defined over \(\mathbb{Q}\). As the products \(X_i^{-\lambda} x_i\) converge to \(\Xi\) when \(i \to \infty\), this would imply that \(\Xi \in V\), in contradiction with the hypothesis that \(\Xi\) has \(\mathbb{Q}\)-linearly independent coordinates. Let \(I\) denote the set of these indices \(i\).

For \(i \in I\), the integer \(\det(x_{i-1}, x_i, x_{i+1})\) is non-zero and [3, Lemma 4] yields

\[
1 \leq |\det(x_{i-1}, x_i, x_{i+1})| \leq 6X_{i+1}L_i L_{i-1} \leq 6c^2 X_i^{1-\lambda} X_i^{-\lambda},
\]

thus \(X_i^{\lambda} \leq 6c^2 X_i^{1-\lambda}\). Combining this with (3.4), we deduce that \(X_i^{\lambda} \leq (6c^2)^{\lambda/(cc_1 X_i)^{1-\lambda}}\) for each \(i \in I\), thus \(\lambda^2 \leq 1 - \lambda\) and so \(\lambda \leq 1/\gamma\). Moreover,
if $\lambda = 1/\gamma$, this yields $1 \leq 6c^2(c_1)^{1/\gamma}$, and so $c$ is bounded below by a positive constant depending only on $\varphi$ and $\Xi$.

4. Proof of the third part of the main theorem

The arguments in [9, §5] can easily be adapted to show that, for some $\epsilon > 0$ there are at most countably many irrational non-quadratic $\xi \in \mathbb{R}$ with $\lambda(1 : \xi : \xi^2) \geq 1/\gamma - \epsilon$. This is, originally, an observation of S. Fischler who, in unpublished work, also computed an explicit value for $\epsilon$. The question was later revisited by D. Zelo who showed in [13, Cor. 1.4.7] that one can take $\epsilon = 3.48 \times 10^{-3}$, and who also proved a $p$-adic analog of this result. More recently, the existence of such $\epsilon$ was established by P. Bel, in a larger context where $\mathbb{Q}$ is replaced by a number field $K$, and $\mathbb{R}$ by a completion of $K$ at some place [1, Theorem 1.3]. By Lemmas 2.1 and 2.4 (i), this proves Theorem 2.3 (iii) when $\varphi$ is irreducible over $\mathbb{R}$ and has a non-trivial zero in $\mathbb{P}^2(\mathbb{Q})$.

We now consider the complementary case. Using the notation and results of the previous section, we need to show that, when $\lambda$ is sufficiently close to $1/\gamma$, the point $\Xi$ lies in a countable subset of $\mathbb{C}$. For this purpose, we may assume that $\lambda > 1/2$. The next two lemmas introduce a polynomial $\psi(x, y)$ with both algebraic and numerical properties analog to that of the operator $[x, x, y]$ from [9, §2] (cf. Lemmas 2.1 and 3.1(iii) of [9]).

**Lemma 4.1.** — For any $x, y \in \mathbb{Z}^3$, we define

$$\psi(x, y) := \Phi(x, y)x - \varphi(x)y \in \mathbb{Z}^3.$$  

Then, $z = \psi(x, y)$ satisfies $\varphi(z) = \varphi(x)^2\varphi(y)$ and $\psi(x, z) = \varphi(x)^2y$.

**Proof.** — For any $a, b \in \mathbb{Q}$, we have $\varphi(ax + by) = a^2\varphi(x) + ab\Phi(x, y) + b^2\varphi(y)$. Substituting $a = \Phi(x, y)$ and $b = -\varphi(x)$ in this equality yields $\varphi(z) = \varphi(x)^2\varphi(y)$. The formula for $\psi(x, z)$ follows from the linearity of $\psi$ in its second argument. \(\square\)

**Lemma 4.2.** — Let $i, j \in \mathbb{Z}$ with $i_0 \leq i < j$. Then, the point $w = \psi(x_i, x_j) \in \mathbb{Z}^3$ is non-zero and satisfies

$$\|w\| \ll X_i^2L_j + X_jL_i^2 \quad \text{and} \quad L(w) \ll X_jL_i^2.$$  

Here and for the rest of this section, the implied constants depend only on $\Xi, \varphi, \lambda$ and $c$.  

TOME 63 (2013), FASCICULE 6
Proof. — Since \( x_i \) and \( x_j \) are distinct primitive elements of \( \mathbb{Z}^3 \), they are linearly independent over \( \mathbb{Q} \). As \( \varphi(x_i) \neq 0 \), this implies that \( w = \Phi(x_i, x_j)x_i - \varphi(x_i)x_j \neq 0 \). By (3.3), we have

\[
\varphi(x_i) = X_i \Phi(\Xi, \Delta_i) + \mathcal{O}(L_i^2)
\]

where \( \Delta_i = x_i - x_i \Xi \). Similarly, for \( \Delta_j = x_j - x_j \Xi \), we find

\[
\Phi(x_i, x_j) = X_j \Phi(\Xi, \Delta_i) + X_i \Phi(\Xi, \Delta_j) + \Phi(\Delta_i, \Delta_j) = X_j \Phi(\Xi, \Delta_i) + \mathcal{O}(X_i L_j).
\]

Substituting these expressions in the formula for \( w = \psi(x_i, x_j) \), we obtain

\[
w = (X_j \Phi(\Xi, \Delta_i) + \mathcal{O}(X_i L_j))(x_i \Xi + \Delta_i)
\]

\[
- (X_i \Phi(\Xi, \Delta_i) + \mathcal{O}(L_i^2))(x_j \Xi + \Delta_j)
\]

\[
= \mathcal{O}(X_j^2 L_j + X_j L_i^2) \Xi + \mathcal{O}(X_j L_i^2),
\]

and the conclusion follows. \( \square \)

We will also need the following result, where the set \( I \) (defined in Section 3) is endowed with its natural ordering as a subset of \( \mathbb{N} \).

**Lemma 4.3.** — For each triple of consecutive elements \( i < j < k \) in \( I \), the points \( x_i, x_j \) and \( x_k \) are linearly independent. We have

\[
X_j^\alpha \ll X_i \ll X_j^\theta \quad \text{and} \quad L_i \ll X_j^{-\alpha} \quad \text{where} \quad \alpha = \frac{2\lambda - 1}{1 - \lambda} \quad \text{and} \quad \theta = \frac{1 - \lambda}{\lambda}.
\]

Proof. — The fact that \( i \) and \( j \) are consecutive elements of \( I \) implies that \( x_i, x_{i+1}, \ldots, x_j \) belong to the same 2-dimensional subspace \( V_i = \langle x_i, x_{i+1} \rangle_\mathbb{R} \) of \( \mathbb{R}^3 \). Similarly, \( x_j, x_{j+1}, \ldots, x_k \) belong to \( V_j = \langle x_j, x_{j+1} \rangle_\mathbb{R} \). Thus \( x_i, x_j \) and \( x_k \) span \( V_i + V_j = \langle x_{j-1}, x_j, x_{j+1} \rangle_\mathbb{R} = \mathbb{R}^3 \), and so they are linearly independent. Then, the normal vectors \( x_i \wedge x_{i+1} \) to \( V_i \) and \( x_j \wedge x_{j+1} \) to \( V_j \) are non-parallel and both orthogonal to \( x_j \). So, their cross-product is a non-zero multiple of \( x_j \). Since \( x_j \) is a primitive point of \( \mathbb{Z}^3 \) and since these normal vectors have integer coordinates, their cross-product is more precisely a non-zero integer multiple of \( x_j \). This yields

\[
X_j \ll \|x_j\| \ll \|x_i \wedge x_{i+1}\| \|x_j \wedge x_{j+1}\| \ll (X_{i+1} L_i) (X_{j+1} L_j)
\]

\[
\ll (X_i+1 X_{j+1})^{1-\lambda}.
\]

If we use the trivial upper bounds \( X_{i+1} \ll X_j \) and \( X_{j+1} \ll X_k \) to eliminate \( X_{i+1} \) and \( X_{j+1} \) from the above estimate, we obtain \( X_j \ll X_k^\theta \). If instead we use the upper bounds \( X_{i+1} \ll X_j^{1/\lambda} \) and \( X_{j+1} \ll X_j^{1/\lambda} \) coming from (3.4), we find instead \( X_j^\alpha \ll X_i \). Finally, if we only eliminate \( X_{j+1} \) using \( X_{j+1} \ll X_j^{1/\lambda} \), we obtain \( X_j^{1/\lambda} \ll X_{i+1} \) and thus \( L_i \ll X_{i+1}^{-\lambda} \ll X_{j}^{-\alpha} \). \( \square \)
Proposition 4.4. — Suppose that \( \lambda \geq 0.613 \). For each integer \( k \geq 1 \), put \( y_k = x_{ik} \) where \( i_k \) is the \( k \)-th element of \( I \). Then, for each sufficiently large \( k \), the point \( y_{k+1} \) is a rational multiple of \( \psi(y_k, y_{k-2}) \).

Proof. — For each integer \( k \geq 1 \), let \( Y_k \) denote the first coordinate of \( y_k \). Then, according to Lemma 4.3, we have \( Y_{k+1}^\alpha < Y_k < Y_{k+1}^\theta \) and \( L(y_k) < Y_{k+1}^{-\alpha} \), with \( \alpha \geq 0.5839 \) and \( \theta \leq 0.6314 \). Put \( w_k = \psi(y_k, y_{k+1}) \). By Lemma 4.2, the point \( w_k \) is non-zero, and the above estimates yield
\[
L(w_k) < Y_{k+1} L(y_k)^2 < Y_{k+1}^{1-2\alpha} \quad \text{and} \quad \|w_k\| < Y_{k+1}^2 L(y_k+1) < Y_{k+1}^{1+2\alpha}
\]
we dropped the term \( Y_{k+1} L(y_k)^2 \) in the upper bound for \( \|w_k\| \) because it tends to 0 as \( k \to \infty \) while \( \|w_k\| \geq 1 \). Using these estimates, we find
\[
|\det(y_{k-2}, y_{k-1}, w_k)| \ll \|w_k\| L(y_{k-2}) L(y_{k-1}) + \|y_{k-1}\| L(y_{k-2}) L(w_k) \ll Y_{k+1}^{1-2\alpha - \alpha} + Y_{k+1}^{1-\alpha} Y_{k+1}^{1-2\alpha},
\]
\[
|\det(y_{k-3}, y_{k-2}, w_k)| \ll \|w_k\| L(y_{k-3}) L(y_{k-2}) + \|y_{k-2}\| L(y_{k-3}) L(w_k) \ll Y_{k+1}^{1-2\alpha - \alpha} + Y_{k+1}^{1-\alpha} Y_{k+1}^{1-2\alpha},
\]
Thus both determinants tend to 0 as \( k \to \infty \) and so, for each sufficiently large \( k \), they vanish. Since, by Lemma 4.3, \( y_{k-3}, y_{k-2}, y_{k-1} \) are linearly independent, this implies that, for those \( k \), the point \( w_k \) is a rational multiple of \( y_{k-2} \). As Lemma 4.1 gives \( \psi(y_k, w_k) = \varphi(y_k)^2 y_{k+1} \), we conclude that \( y_{k+1} \) is a rational multiple of \( \psi(y_k, y_{k-2}) \) for each large enough \( k \). □

We end this section with two corollaries. The first one gathers properties of the sequence \( (y_k)_{k \geq 1} \) when \( \lambda = 1/\gamma \). The second completes the proof of Theorem 2.3(iii).

Corollary 4.5. — Suppose that \( \lambda = 1/\gamma \). Then, the sequence \( (y_k)_{k \geq 1} \) consists of primitive points of \( \mathbb{Z}^3 \) such that \( \psi(y_k, y_{k-2}) \) is an integer multiple of \( y_{k+1} \) for each sufficiently large \( k \). Any three consecutive points of this sequence are linearly independent and, for each \( k \geq 1 \), we have \( \|y_{k+1}\| \asymp \|y_k\| \gamma, L(y_k) \asymp \|y_k\|^{-1} \) and \( |\varphi(y_k)| \asymp 1 \).

Proof. — The first assertion simply adds a precision on Proposition 4.4 based on the fact that \( y_{k+1} \) is a primitive integer point. Aside from the estimate for \( |\varphi(y_k)| \), the second assertion is a direct consequence of Lemma 4.3 since, for \( \lambda = 1/\gamma \), we have \( \alpha = \theta = 1/\gamma \). To complete the proof, we use
the estimate $|\varphi(x_i)| \ll X_i L_i$ established in the previous section as a consequence of (3.3). Since $\varphi(y_k)$ is a non-zero integer, it yields $1 \leq |\varphi(y_k)| \ll 1$.

**Corollary 4.6.** — Suppose that $\lambda \geq 0.613$. Then, $\Xi$ belongs to a countable subset of $\mathcal{C}$.

**Proof.** — Since each $y_k$ is a primitive point of $\mathbb{Z}^3$ with positive first coordinate, the proposition shows that the sequence $(y_k)_{k \geq 1}$ is uniquely determined by its first terms. As there are countably many finite sequences of elements of $\mathbb{Z}^3$ and as the image of $(y_k)_{k \geq 1}$ in $\mathbb{P}^2(\mathbb{R})$ converges to $\Xi$, the point $\Xi$ belongs to a countable subset of $\mathcal{C}$. □

5. Proof of the second part of the main theorem

By [9, Theorem 1.1], there exist countably many irrational non-quadratic real numbers $\xi$ for which $1/\gamma$ is an exponent of approximation to $(1 : \xi : \xi^2)$. Thus Part (ii) of Theorem 2.3 holds for $\varphi = x_0 x_2 - x_1^2$ and consequently, by Lemmas 2.1 and 2.4, it holds for any quadratic form $\varphi \in \mathbb{Q}[x_0, x_1, x_2]$ which is irreducible over $\mathbb{R}$ and admits at least one zero in $\mathbb{P}^2(\mathbb{Q})$. These lemmas also show that, in order to complete the proof of Theorem 2.3(ii), we may restrict to a diagonal form $\varphi = x_0^2 - bx_1^2 - cx_2^2$ where $b > 1$ is a square free integer and where $c$ is either 0 or a square-free integer with $c > 1$. In fact, this even covers the case of $\varphi = x_0 x_2 - x_1^2$ since $(x_0 + x_1 + x_2)(x_0 - x_1 - x_2) - (x_1 - x_2)^2 = x_0^2 - 2x_1^2 - 2x_2^2$.

We first establish four lemmas which apply to any quadratic form $\varphi \in \mathbb{Q}[x_0, x_1, x_2]$ and its associated symmetric bilinear form $\Phi$ with $\Phi(x, x) = 2\varphi(x)$. Our first goal is to construct sequences $(y_i)$ as in Corollary 4.5. On the algebraic side, we first make the following observation.

**Lemma 5.1.** — Suppose that $y_{-1}, y_0, y_1 \in \mathbb{Z}^3$ satisfy $\varphi(y_i) = 1$ for $i = -1, 0, 1$. We extend this triple to a sequence $(y_i)_{i \geq -1}$ in $\mathbb{Z}^3$ by defining recursively $y_{i+1} = \psi(y_i, y_{i-2})$ for each $i \geq 1$. We also define $t_i = \Phi(y_{i+1}, y_i) \in \mathbb{Z}$ for each $i \geq -1$. Then, for any integer $i \geq 1$, we have

(a) $\varphi(y_{i-2}) = 1$,
(b) $\det(y_i, y_{i-1}, y_{i-2}) = (-1)^{i-1} \det(y_1, y_0, y_{-1})$,
(c) $t_i = \Phi(y_{i+1}, y_i) = \Phi(y_i, y_{i-2})$,
(d) $y_{i+1} = t_i y_i - y_{i-2}$,
(e) $t_{i+1} = t_i t_{i-1} - t_{i-2}$.

In particular, $t_{-1} = \Phi(y_0, y_{-1})$, $t_0 = \Phi(y_1, y_0)$ and $t_1 = \Phi(y_1, y_{-1})$.
Proof. — By Lemma 4.1, we have \( \varphi(y_{i+1}) = \varphi(y_i)^2 \varphi(y_{i-2}) \) for each \( i \geq 1 \). This yields (a) by recurrence on \( i \). Then, by definition of \( \psi \), the recurrence formula for \( y_{i+1} \) simplifies to

\[
y_{i+1} = \Phi(y_i, y_{i-2})y_i - y_{i-2} \quad (i \geq 1),
\]

and so \( \det(y_{i+1}, y_i, y_{i-1}) = -\det(y_i, y_{i-1}, y_{i-2}) \) for each \( i \geq 1 \), by multilinearity of the determinant. This proves (b) by recurrence on \( i \). From (5.1), we deduce that

\[
t_i = \Phi(y_{i+1}, y_i) = \Phi(y_i, y_{i-2})\Phi(y_i, y_i) - \Phi(y_{i-2}, y_i) = \Phi(y_i, y_{i-2}) \quad (i \geq 1),
\]

which is (c). Then (d) is just a rewriting of (5.1). Combining (c) and (d), we find

\[
t_{i+1} = \Phi(y_{i+1}, y_{i-1}) = t_i\Phi(y_i, y_{i-1}) - \Phi(y_{i-2}, y_{i-1}) = t_it_{i-1} - t_{i-2} \quad (i \geq 1),
\]

which is (e). Finally, the formulas given for \( t_{-1} \) and \( t_0 \) are taken from the definition while the one for \( t_1 \) follows from (c).

The next lemma provides mild conditions under which the norm of \( y_i \) grows as expected.

**Lemma 5.2.** — With the notation of the previous lemma, suppose that \( 1 \leq t_{-1} < t_0 < t_1 \) and that \( 1 \leq ||y_{-1}|| < ||y_0|| < ||y_1|| \). Then, \( (t_i)_{i \geq -1} \) and \( (||y_i||)_{i \geq -1} \) are strictly increasing sequences of positive integers with \( t_{i+1} \asymp t_i^2 \) and \( ||y_{i+1}|| \asymp t_{i+2} \asymp ||y_i||^7 \).

Here and below, the implied constants are simply meant to be independent of \( i \).

Proof. — Lemma 5.1(e) implies, by recurrence on \( i \), that the sequence \( (t_i)_{i \geq -1} \) is strictly increasing and, more precisely, that it satisfies

\[
(t_i - 1)t_{i-1} < t_{i+1} < t_it_{i-1} \quad (i \geq 1),
\]

which by [10, Lemma 5.2] implies that \( t_{i+1} \asymp t_i^7 \). In turn, Lemma 5.1(d) implies, by recurrence on \( i \), that the sequence \( (||y_i||)_{i \geq -1} \) is strictly increasing with

\[
(t_i - 1)||y_i|| < ||y_{i+1}|| < (t_i + 1)||y_i|| \quad (i \geq 1).
\]

Combining this with (5.2), we find that the ratios \( \rho_i = ||y_i||/t_{i+1} \) satisfy

\[
(1 - 1/t_i)\rho_i \leq \rho_{i+1} \leq \frac{1 + 1/t_i}{1 - 1/t_{i+1}} \rho_i \leq \frac{1}{(1 - 1/t_i)^2} \rho_i \quad (i \geq 1),
\]

and so \( \rho_1 c_1 \leq \rho_i \leq \rho_1/c_1^2 \) for each \( i \geq 1 \) where \( c_1 = \prod_{i \geq 1} (1 - 1/t_i) > 0 \) is a converging infinite product because \( t_i \) tends to infinity with \( i \) faster.

RATIONAL APPROXIMATION ON CONICS
than any geometric series. This means that \( \rho_i \asymp 1 \), thus \( \|y_i\| \asymp t_{i+1} \), and so \( \|y_{i+1}\| \asymp t_{i+2} \asymp \|y_i\|^\gamma \) because \( t_{i+2} \asymp t_i^\gamma \).

For any \( x, y \in \mathbb{R}^3 \), we denote by \( \langle x, y \rangle \) their standard scalar product. When \( x \neq 0 \) and \( y \neq 0 \), we also denote by \( [x], [y] \) their respective classes in \( \mathbb{P}^2(\mathbb{R}) \), and define the projective distance between these classes by

\[
\text{dist}([x], [y]) = \frac{\|x \wedge y\|}{\|x\| \|y\|}.
\]

It is not strictly speaking a distance on \( \mathbb{P}^2(\mathbb{R}) \) but it behaves almost like a distance since it satisfies

\[
\text{dist}([x], [z]) \leq \text{dist}([x], [y]) + 2 \text{dist}([y], [z])
\]

for any non-zero \( z \in \mathbb{R}^3 \) (see [10, § 2]). Moreover, the open balls for the projective distance form a basis of the usual topology on \( \mathbb{P}^2(\mathbb{R}) \). We can now prove the following result.

**Lemma 5.3.** — With the notation and hypotheses of Lemmas 5.1 and 5.2, suppose that \( y_{-1}, y_0 \) and \( y_1 \) are linearly independent. Then there exists a zero \( \Xi = (1, \xi_1, \xi_2) \) of \( \varphi \) in \( \mathbb{R}^3 \) with \( \mathbb{Q} \)-linearly independent coordinates such that \( \|\Xi \wedge y_i\| \asymp \|y_i\|^{-1} \) for each \( i \geq 1 \). Moreover, \( 1/\gamma \) is an exponent of approximation to the corresponding point \( \Xi = (1 : \xi_1 : \xi_2) \in \mathbb{P}^2(\mathbb{R}) \).

**Proof.** — Our first goal is to show that \( ([y_i])_{i \geq 1} \) is a Cauchy sequence in \( \mathbb{P}^2(\mathbb{R}) \) with respect to the projective distance. To this end, we use freely the estimates of the previous lemma and define \( z_i = y_i \wedge y_{i+1} \) for each \( i \geq 1 \). By Lemma 5.1(b), the points \( y_{i-1}, y_i \) and \( y_{i+1} \) are linearly independent for each \( i \geq 0 \). Thus, none of the products \( z_i \) vanish, and so their norm is at least 1. Moreover, Lemma 5.1(d) applied first to \( y_{i+1} \) and then to \( y_i \) yields

\[
\text{(5.4)} \quad z_i = y_{i-2} \wedge y_i = t_{i-1} y_{i-2} \wedge y_{i-1} - y_{i-2} \wedge y_{i-3} = t_{i-1} z_{i-2} + z_{i-3}.
\]

The above equality \( z_i = y_{i-2} \wedge y_i \) with \( i \) replaced by \( i-3 \) implies that

\[
\|z_{i-3}\| \leq 2\|y_{i-5}\| \|y_{i-3}\| \ll t_{i-4} t_{i-2} \asymp t_{i-1} t_{i-5}^{-1} \leq t_{i-1} t_{i-5}^{-1} \|z_{i-2}\|.
\]

In view of (5.4), this means that \( \|z_i\| = t_{i-1} (1 + O(t_{i-5}^{-1})) \|z_{i-2}\| \), and thus

\[
\frac{\|z_i\|}{t_i} = \frac{t_{i-1} t_{i-2}^{-1}}{t_i} (1 + O(t_{i-5}^{-1})) \frac{\|z_{i-2}\|}{t_{i-2}} = (1 + O(t_{i-5}^{-1})) \frac{\|z_{i-2}\|}{t_{i-2}}
\]

since, by Lemma 5.1(e), we have \( t_{i-1} t_{i-2} = t_i (1 + t_{i-3} t_i^{-1}) = t_i (1 + O(t_{i-5}^{-1})) \). As the series \( \sum_{i \geq 1} t_i^{-1} \) converges, the same is true of the infinite products.
\( \prod_{i \geq i_0} (1 + ct_i^{-1}) \) for any \( c \in \mathbb{R} \). Thus the above estimates implies that \( \|z_i\| \asymp t_i \), and so we find

\[
\text{dist}([y_i], [y_{i+1}]) = \frac{\|z_i\|}{\|y_i\| \|y_{i+1}\|} \asymp \frac{t_i}{t_{i+1}t_{i+2}} \asymp t_{i+2}^{-2} \asymp \|y_i\|^{-2}.
\]

As the series \( \sum_{i \geq 1} 2^i t_{i+1}^{-2} \) is convergent, we deduce that \( (y_i)_{i \geq 1} \) forms a Cauchy sequence in \( \mathbb{P}^2(\mathbb{R}) \), and that its limit \( \Xi \in \mathbb{P}^2(\mathbb{R}) \) satisfies \( \text{dist}([y_i], \Xi) \asymp \|y_i\|^{-2} \). In terms of a representative \( \Xi \) of \( \Xi \) in \( \mathbb{R}^3 \), this means that

\[
(5.5) \quad \|y_i \wedge \Xi\| \asymp \|y_i\|^{-1}.
\]

To prove that \( \Xi \) has \( \mathcal{Q} \)-linearly independent coordinates, we use the fact that

\[
\| \langle u, y_i \rangle \Xi - \langle u, \Xi \rangle y_i \| \leq 2\|u\| \|y_i \wedge \Xi\|
\]

for any \( u \in \mathbb{R}^3 \) [10, Lemma 2.2]. So, if \( \langle u, \Xi \rangle = 0 \) for some \( u \in \mathbb{Z}^3 \), then, by (5.5), we obtain \( |\langle u, y_i \rangle| \ll \|y_i\|^{-1} \) for all \( i \). Then, as \( \langle u, y_i \rangle \) is an integer, it vanishes for each sufficiently large \( i \), and so \( u = 0 \) because any three consecutive \( y_i \) span \( \mathbb{R}^3 \). This proves our claim. In particular, the first coordinate of \( \Xi \) is non-zero, and we may normalize \( \Xi \) so that it is 1. Then, as \( i \) goes to infinity, the points \( \|y_i\|^{-1} y_i \) converge to \( \|\Xi\|^{-1} \Xi \) in \( \mathbb{R}^3 \) and, since \( \varphi(\|y_i\|^{-1} y_i) = \|y_i\|^{-2} \) tends to 0, we deduce that \( \varphi(\Xi) = 0 \). Finally, \( 1/\gamma \) is an exponent of approximation to \( \Xi \) because, for each \( X \geq \|y_1\| \), there exists an index \( i \geq 1 \) such that \( \|y_i\| \leq X \leq \|y_{i+1}\| \) and then, by (5.5), the point \( x := y_i \) satisfies both

\[
\|x\| \leq X \quad \text{and} \quad \|x \wedge \Xi\| \asymp \|y_i\|^{-1} \asymp \|y_{i+1}\|^{-1/\gamma} \leq X^{-1/\gamma}.
\]

The last lemma below will enable us to show that the above process leads to infinitely many limit points \( \Xi \).

**Lemma 5.4.** — Suppose that \( (y_i)_{i \geq -1} \) and \( (y'_i)_{i \geq -1} \) are constructed as in Lemma 5.1 and that both of them satisfy the hypotheses of the three preceding lemmas. Suppose moreover that their images in \( \mathbb{P}^2(\mathbb{R}) \) have the same limit \( \Xi \). Then there exists an integer \( a \) such that \( y'_i = \pm y_{i+a} \) for each \( i \geq \max\{-1, -1 - a\} \).

**Proof.** — Let \( \Xi = (1, \xi_1, \xi_2) \) be a representative of \( \Xi \) in \( \mathbb{R}^3 \), and for each \( x \in \mathbb{Z}^3 \) define \( L(x) \) as in (3.1). The estimates of Lemma 5.3 imply that \( L(y_i) \asymp \|y_i\|^{-1} \) and \( L(y'_i) \asymp \|y'_i\|^{-1} \). For each sufficiently large index \( j \), we can find an integer \( i \geq 2 \) such that \( \|y_{i-1}\|^{3/2} \leq \|y'_j\| \leq \|y_i\|^{3/2} \) and the
standard estimates yield
\[
\left| \det(\mathbf{y}_{i-1}, \mathbf{y}_i, \mathbf{y}_j') \right| \ll \|\mathbf{y}_j'\|L(\mathbf{y}_i)L(\mathbf{y}_{i-1}) + \|\mathbf{y}_i\|L(\mathbf{y}_{i-1})L(\mathbf{y}_j')
\ll \|\mathbf{y}_i\|^{3/2}\|\mathbf{y}_i\|^{-1}\|\mathbf{y}_{i-1}\|^{-1} + \|\mathbf{y}_i\| \|\mathbf{y}_{i-1}\|^{-1}\|\mathbf{y}_{i-1}\|^{-3/2}
\ll \|\mathbf{y}_i\|^{1/2-1/\gamma} = o(1),
\]
and similarly \( \left| \det(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_j') \right| \ll \|\mathbf{y}_i\|^{-1/(2\gamma)} = o(1). \) Thus, both determinants vanish when \( j \) is large enough and then \( \mathbf{y}_j' \) is a rational multiple of \( \mathbf{y}_i \). However, both points are primitive elements of \( \mathbb{Z}^3 \) since \( \varphi \) takes value 1 on each of them. So, we must have \( \mathbf{y}_j' = \pm \mathbf{y}_i \). Since the two sequences have the same type of growth, we conclude that there exist integers \( a \) and \( i_0 \geq \max\{-1, -1 - a\} \) such that \( \mathbf{y}_i' = \pm \mathbf{y}_{i+a} \) for each \( i \geq i_0 \). Choose \( i_0 \) smallest with this property. If \( i_0 \geq \max\{0, -a\} \), then, using Lemma 4.1, we obtain
\[
\mathbf{y}_0' = \varphi(\mathbf{y}_{i_0+1}', \mathbf{y}_{i_0+2}') = \varphi(\pm \mathbf{y}_{i_0+1+a}, \pm \mathbf{y}_{i_0+2+a}) = \pm \mathbf{y}_{i_0-1+a}
\]
in contradiction with the choice of \( i_0 \). Thus we must have \( i_0 = \max\{-1, -1 - a\} \).

In view of the remarks made at the beginning of this section, the last result below completes the proof of Theorem 2.3(ii).

**Proposition 5.5.** — Let \( b > 1 \) be a square-free integer and let \( c \) be either 0 or a square-free integer with \( c > 1 \). Then the quadratic form \( \varphi = x_0^2 - bx_1^2 - cx_2^2 \) admits infinitely many zeros in \( \mathbb{P}^2(\mathbb{R}) \) which have \( \mathbb{Q} \)-linearly independent homogeneous coordinates and for which \( 1/\gamma \) is an exponent of approximation.

**Proof.** — The Pell equation \( x_0^2 - bx_1^2 = 1 \) admits infinitely many solutions in positive integers. We choose one such solution \((x_0, x_1) = (m, n)\). For the other solutions \((m', n') \in (\mathbb{N}^*)^2 \), the quantity \( mm' - bnn' \) behaves asymptotically like \( m'(m + n\sqrt{b}) \) as \( m' \to \infty \) and thus, we have \( m < mm' - bnn' < m' \) as soon as \( m' \) is large enough. We fix such a solution \((m', n')\). We also choose a pair of integers \( r, t > 0 \) such that \( r^2 - ct^2 = 1 \). Then, the three points
\[
\mathbf{y}_{-1} = (1, 0, 0), \quad \mathbf{y}_0 = (m, n, 0) \quad \text{and} \quad \mathbf{y}_1 = (rm', rn', t)
\]
are \( \mathbb{Q} \)-linearly independent. They satisfy
\[
\|\mathbf{y}_{-1}\| < \|\mathbf{y}_0\| = m < rm' \leq \|\mathbf{y}_1\| \quad \text{and} \quad \varphi(\mathbf{y}_i) = 1 \quad (i = -1, 0, 1).
\]
For such a triple, consider the corresponding sequences \((t_i)_{i \geq -1}\) and \((\mathbf{y}_i)_{i \geq -1}\) as defined in Lemma 5.1. The symmetric bilinear form attached
to $\varphi$ being $\Phi = 2(x_0y_0 - bx_1y_1 - cx_2y_2)$, we find
\[ t_{-1} = 2m < t_0 = 2r(mm' - bnn') < t_1 = 2rm'. \]
Therefore the hypotheses of Lemmas 5.2 and 5.3 are fulfilled and so the sequence $([y_i])_{i \geq -1}$ converges in $\mathbb{P}^2(\mathbb{R})$ to a zero $\Xi$ of $\varphi$ which has $\mathbb{Q}$-linearly independent homogeneous coordinates and for which $1/\gamma$ is an exponent of approximation. To complete the proof and show that there are infinitely many such points, it suffices to prove that any other choice of $m, n, m', n', r, t$ as above leads to a different limit point. Clearly, it leads to a different sequence $([y'_i])_{i \geq -1}$. If $[y'_i]$ and $[y_i]$ converge to the same point $\Xi$ as $i \to \infty$, then by Lemma 5.4, there exists $a \in \mathbb{Z}$ such that $y'_i = \pm y_{i+a}$ for each $i \geq \max\{-1, -1-a\}$. But, in both sequences $([y_i])_{i \geq -1}$ and $([y'_i])_{i \geq -1}$, the first point is the only one of norm 1, and moreover the first three points have non-negative entries. So, we must have $a = 0$ and $y'_i = y_i$ for $i = -1, 0, 1$, a contradiction. $\Box$

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