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Satoshi KOIKE, Ta Lê LOI,
Laurentiu PAUNESCU & Masahiro SHIOTA

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DIRECTIONAL PROPERTIES OF SETS DEFINABLE IN O-MINIMAL STRUCTURES

by Satoshi KOIKE, Ta Lê LOI,
Laurentiu PAUNESCU & Masahiro SHIOTA (*)

ABSTRACT. — In a previous paper by Koike and Paunescu, it was introduced the notion of direction set for a subset of a Euclidean space, and it was shown that the dimension of the common direction set of two subanalytic subsets, called *the directional dimension*, is preserved by a bi-Lipschitz homeomorphism, provided that their images are also subanalytic. In this paper we give a generalisation of the above result to sets definable in an o-minimal structure on an arbitrary real closed field. More precisely, we first prove our main theorem and discuss in detail directional properties in the case of an Archimedean real closed field, and in §7 we give a proof in the case of a general real closed field. In addition, related to our main result, we show the existence of special polyhedra in some Euclidean space, illustrating that the bi-Lipschitz equivalence does not always imply the existence of a definable one.

RÉSUMÉ. — Dans un article précédent par Koike et Paunescu, la notion d'ensemble de directions pour un sous-ensemble d'un espace euclidien a été introduite, et les auteurs ont montré que la dimension de l'ensemble des directions communes de deux sous-ensembles sous-analytiques, nommée *la dimension directionnelle*, est préservée par un homéomorphisme bi-Lipschitz, à condition que leurs images sont également sous-analytiques. Dans cet article, nous donnons une généralisation de ce résultat à des ensembles définissables dans une structure o-minimale sur un corps réel clos quelconque. Plus précisément, nous prouvons d'abord le théorème principal et nous discussons en détail les propriétés directionnelles dans le cas d'un corps archimèdien réel clos, et dans §7, nous donnons une preuve dans le cas d'un corps général fermé réel. En outre, en relation avec notre résultat principal, nous montrons l'existence des polyèdres spéciaux dans un espace euclidien, ce qui montre que l'équivalence bi-Lipschitz n'implique pas toujours l'existence d'une équivalence définissable.

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1. Introduction

We first recall the notions of direction set and real tangent cone in $\mathbb{R}^n$.

Definition 1.1. — *Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. We define the* direction set $D(A)$ *of A at $0 \in \mathbb{R}^n$ by*

$$D(A) := \{a \in S^{n-1} \mid \exists \{x_i\} \subset A \setminus \{0\},\ x_i \to 0 \in \mathbb{R}^n \text{ s.t. } \frac{x_i}{\|x_i\|} \to a,\ i \to \infty\}.$$

Here S^{n-1} denotes the unit sphere centred at $0 \in \mathbb{R}^n$.

We denote by $LD(A)$ a half-cone of $D(A)$ with the origin $0 \in \mathbb{R}^n$ as the vertex:

$$LD(A) := \{ta \in \mathbb{R}^n \mid a \in D(A),\ t \geqslant 0\},$$

and call it the real tangent cone *of A at $0 \in \mathbb{R}^n$.*

Let us examine an example.

Example 1.2. — Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be a semialgebraic homeomorphism defined by $h(x, y, z) = (x, y, z^3)$, and let $V = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2-z^6 = 0\}$. Then V and $h(V) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$ are algebraic sets. It is easy to see that $\dim D(A) = 0$ and $\dim D(h(A)) = 1$. Therefore the dimension of direction sets is not a homeomorphic invariant.

We next investigate whether the dimension of direction sets is a Lipschitz invariant. There are many singular examples of bi-Lipschitz homeomorphisms. In [12] it was given an example of a “quick spiral bi-Lipschitz homeomorphism" and of a “zigzag bi-Lipschitz homeomorphism". Here we give a different one.

Example 1.3. — (Oscillation). Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a mapping defined by $h(x, y) = (x, y + f(x))$, where $f(x) = x \sin(\ln |x|)$, and let $A = \mathbb{R} \times 0$. Then we can see that h is a bi-Lipschitz homeomorphism and $h(A)$ is the graph of f. In addition, we have $\dim D(A) = 0$ and $\dim(D(h(A))) = 1$. Consequently the dimension of direction sets is not a bi-Lipschitz invariant either.

Note that in the above example $h(A)$ is not a subanalytic set. Therefore we may ask whether the dimension of direction sets is a bi-Lipschitz invariant in the case when the image $h(A)$ is also subanalytic. In fact, the following result is shown in [12].

Theorem 1.4. — *(Main Theorem in* [12]*) Let A, $B \subset \mathbb{R}^n$ be subanalytic set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$*

be a bi-Lipschitz homeomorphism. Suppose that $h(A)$, $h(B)$ *are also subanalytic. Then we have*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

See H. Hironaka [9] for subanalyticity.

An ordered field R is called *Archimedean* if for any positive elements a and b of R there exists a natural number n such that $an > b$. We regard an Archimedean field as a subfield of $\mathbb{R}$ since an ordered field R is Archimedean if and only if R is isomorphic to a subfield of $\mathbb{R}$ and the isomorphism is unique.

An ordered field R is called *real closed* if its complexification $R[t]/(1+t^2)R[t]$ is algebraically closed. The field of real numbers $\mathbb{R}$ is an Archimedean real closed field. Another well-known example of Archimedean real closed field is the field of real algebraic numbers. We refer the readers to [2] for properties of the real closed field.

In this paper we give a generalization of Theorem 1.4 to the case of sets definable in an arbitrary o-minimal structure on an arbitrary real closed field (Theorem 7.1). We first show the main theorem for the case of an Archimedean real closed field. Namely, we show the following:

THEOREM 1.5. — *Let R be an Archimedean real closed field, and let A, B be definable set-germs at 0 in R^n in an o-minimal structure on R such that $0 \in \overline{A} \cap \overline{B}$. Let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(A)$, $h(B)$ are also definable. Then we have*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

See the next section for the definition of a definable set and of the direction set $D(A)$ of a set A in R^n.

Theorem 1.4 was shown using essentially the following ingredients:

(1) Sea-tangle properties;
(2) Sequence selection properties;
(3) Volume arguments.

In §2 we describe the notion of o-minimal structure and point out some of its properties. We give in §3 an important example concerning the relationship between bi-Lipschitz equivalence and definable equivalence. Then we introduce an adapted notion of sea-tangle neighbourhood, using ordered definable functions and describe several of its properties in §4. In §5 we discuss sequence selection properties, and we give the proof of our main theorem (Theorem 1.5) using volume arguments in §6. In §§4 - 6 we develop the arguments in any o-minimal structure on any Archimedean real closed field. In §7 we generalise the main theorem to any real closed field and give a proof. This proof is rather resembling proofs in Logic Theory. Essentially, this shows that in our proof of the main theorem we do not use all special properties of the real number field (e.g. local compactness).

Concerning the preservation of the dimension of a tangent cone under a bi-Lipschitz map there is an easy case, namely the case where the dimension of the tangent cone is the dimension of the set germ under consideration.

Actually, denoting by Θ the local density (the localization of the $\dim(A)$-volume in $\mathbb{R}^n$), we can bound from above and behind $\Theta(h(A))$ by some multiples of $\Theta(A)$; the multiplicative constants depending on the Lipschitz constants of h and being non zero (see [3]).

On the other hand, we know how the local density can be computed in terms of the volume of the tangent cone, we may cite for instance the so-called real Thie formula proved by K. Kurdyka and G. Raby, see [13], or for instance the local Crofton formula proved by G. Comte (see [4]).

It follows from these two remarks that the $\dim(A)$-volume of the tangent cone $T(A)$ of A is non zero if and only if the $\dim(A)$-volume of the tangent cone of $h(A)$ is non zero, that is: $\dim(T(A)) = \dim(T(h(A)))$ when $\dim(A) = \dim(T(A))$.

Accordingly the difficult case is when $\dim(T(A)) < \dim(A)$. The proof of the main result of Koike-Paunescu [12], Theorem 1.4 in this paper, can somehow be regarded as using the neighbourhoods ST_θ as a refinement of the above arguments on the local density (asymptotically responsible for cancellation of localization of volume).

However this way of thinking is not working for a general real closed field (there is no definite notion of volume on a general real closed field). Nevertheless we have managed to introduce a notion of volume in this paper, and in the case of the field of Puiseux series, we could only prove the following:

Let h be a bi-Lipschitz homeomorphism, and let A be a definable set such that $h(A)$ is definable. Then, for $0 < q < p < 1$, each point of A has a neighbourhood U such that $Vol(U) \leqslant Vol(h(U))^p \leqslant Vol(U)^q$.

2. o-minimal structure

Throughout this paper, except §7, R denotes an Archimedean real closed field.

Concerning the direction set, let us analyse the following example:

Example 2.1. — Let R be the field of real algebraic numbers, and let $\{a_m\}$ be the sequence of points of R^2 defined by

$$a_m = (\frac{1}{m}, \frac{1}{m}(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{m!})).$$

Then a_m tends to $0 \in R^2$, and $\frac{a_m}{\|a_m\|}$ tends to a pair of transcendental numbers $(\frac{1}{\sqrt{1+e^2}}, \frac{e}{\sqrt{1+e^2}})$ which is not an element of R^2. Similarly we can see that for any $p \in S^1 \subset \mathbb{R}^2$, there is a sequence of points $\{a_m\}$ of R^2 tending to $0 \in R^2$ such that $\frac{a_m}{\|a_m\|}$ tends to p.

Let $\{b_m\}$ be the sequence of points of R^2 defined by

$$b_m = (0, \frac{1}{m}(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{m!})).$$

Then we can see that there is a bi-Lipschitz homeomorphism $h : (R^2, 0) \to (R^2, 0)$ such that for all m, $h(a_m) = b_m$. On the other hand we have

$$\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \notin R^2, \quad \lim_{m\to\infty} \frac{b_m}{\|b_m\|} \in R^2.$$

To avoid any confusion when we consider the limit, we give the precise definition of the direction set over an Archimedean real closed field R.

Definition 2.2. — *Let A be a set-germ at $0 \in R^n$ such that $0 \in \overline{A}$. We define the* direction set $D(A)$ *of A at $0 \in R^n$ by*

$$D(A) := \{a \in S^{n-1} \mid \exists\{x_i\} \subset A\setminus\{0\},\ x_i \to 0 \in R^n \text{ s.t. } \frac{x_i}{\|x_i\|} \to a,\ i \to \infty\}.$$

Here $S^{n-1} \subset R^n$ denotes the unit sphere centred at $0 \in R^n$.

We denote by $LD(A)$ a half-cone of $D(A)$ with the origin $0 \in R^n$ as the vertex:

$$LD(A) := \{ta \in R^n \mid a \in D(A),\ t \geqslant 0\}.$$

Let us recall the definition of an o-minimal structure on a real closed field R.

Definition 2.3. — *Let $\mathcal{D}$ be a sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$, $\mathcal{D}_n$ is a family of subsets of R^n. We say that $\mathcal{D}$ is an* o-minimal structure *on R if:*

(D1) $\mathcal{D}_n$ is a boolean algebra.

(D2) If $A \in \mathcal{D}_n$, then $A \times R$ and $R \times A \in \mathcal{D}_{n+1}$.

(D3) If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi : R^{n+1} \to R^n$ is the projection on the first n coordinates.

(D4) $\mathcal{D}_n$ contains $\{x \in R^n : P(x) = 0\}$ for every polynomial $P \in R[X_1, \cdots, X_n]$.

(D5) Each set in $\mathcal{D}_1$ is a finite union of intervals and points.

A subset A of R^n belonging to $\mathcal{D}_n$ is called *definable* in $\mathcal{D}$. A map $f : A \to R^m$ is *definable* in $\mathcal{D}$, if its graph is a definable subset of $R^n \times R^m$ in $\mathcal{D}$.

The class of semi-algebraic sets and the class of global sub-analytic sets are examples of o-minimal structures on the field of real numbers $(\mathbb{R}, +, \cdot)$. We refer the readers to [7], [8] and [5] for the basic properties of o-minimal structures. In particular, the following results and properties are frequently used in our paper.

(1) The dimension of definable sets is well-defined (this follows from the Cell Decomposition Theorem ([7], Chapter 3 (2.11)));
(2) Monotonicity ([7], Chapter 3 (1.2));
(3) Curve Selection Lemma ([7], Chapter 6 (1.5)).

We mention one more fact. The Lojasiewicz inequalities for definable sets in an o-minimal structure are discussed in [8] and [14]. The Lojasiewicz inequalities in the sense of [15] hold in *polynomially bounded* o-minimal structures.

Before ending this section, we introduce a notation which we will often use in this paper, namely we denote by Φ the set of all odd, strictly increasing, continuous definable germs from $(R, 0)$ to $(R, 0)$. Note that, by Monotonicity, Φ is ordered by the following relation:

$$\theta_1 \leqslant \theta_2 \text{ iff } \theta_1(t) \leqslant \theta_2(t), \text{ for all } t > 0 \text{ sufficiently small.}$$

3. Bi-Lipschitz equivalence does not always imply definable one

In our main theorem (Theorem (1.5)) we did not assume the definability of the Lipschitz homeomorphism $h : (R^n, 0) \to (R^n, 0)$. In the case when h is definable, we can easily show the theorem as follows:

Since $h : (R^n, 0) \to (R^n, 0)$ is a bi-Lipschitz homeomorphism, there are positive numbers $K_1,\ K_2 \in R$ with $0 < K_1 \leqslant K_2$, called *Lipschitz constants*, such that

$$K_1\|x_1 - x_2\| \leqslant \|h(x_1) - h(x_2)\| \leqslant K_2\|x_1 - x_2\|$$

in a small neighbourhood of $0 \in R^n$. Let $\overline{h} : S^{n-1} \to S^{n-1}$, $S^{n-1} \subset R^n$, be a mapping defined by

$$\overline{h}(a) = \lim_{t \to 0} \frac{h(ta)}{\|h(ta)\|}.$$

By Monotonicity, we can see that $\bar{h}$ is well-defined. In addition, it is easy to see that $\bar{h}$ is definable and bijective with $(\bar{h})^{-1} = \overline{h^{-1}}$. Let $a, b \in S^{n-1}$. Then for sufficiently small, arbitrary $t > 0$, we have

$$\begin{aligned}\left\|\frac{h(ta)}{\|h(ta)\|} - \frac{h(tb)}{\|h(tb)\|}\right\| &\leqslant \frac{\|h(ta) - h(tb)\|}{\min(\|h(ta)\|, \|h(tb)\|)} \\ &\leqslant \frac{K_2\|ta - tb\|}{\min(K_1\|ta\|, K_1\|tb\|)} \leqslant \frac{K_2}{K_1}\|a - b\|.\end{aligned}$$

Taking the limit as $t \to 0^+$, we have $\|\bar{h}(a) - \bar{h}(b)\| \leqslant \frac{K_2}{K_1}\|a - b\|$. Therefore it follows that $\bar{h}$ is a definable bi-Lipschitz homeomorphism. This property allows us to claim that $\bar{h}(D(A)) = D(h(A)))$.

Related to the above fact, it may be natural to ask if we can replace a bi-Lipschitz homeomorphism h with a definable bi-Lipschitz homeomorphism h'. That is to say, whether the existence of h implies the existence of a definable bi-Lipschitz homeomorphism h' with $h'(A) = h(A)$ and $h'(B) = h(B)$. If the answer were positive, we would have a different proof of our main theorem, without using the main properties mentioned in the introduction. Nevertheless, the answer to this question is negative. More precisely, bi-Lipschitz equivalence does not always guarantee the existence of a definable one.

Theorem 3.1. — *There exist $n \in \mathbb{N}$ and compact polyhedra A_1 and A_2 in $\mathbb{R}^n$, such that the germs of $(\mathbb{R}^n, A_1)$ and $(\mathbb{R}^n, A_2)$ at $0 \in \mathbb{R}^n$ are bi-Lipschitz homeomorphic but not definable homeomorphic in any o-minimal structure on $\mathbb{R}$.*

Proof. — We first recall the following result of R. C. Kirby and L. C. Siebenmann [11].

For a PL manifold X_1 of dimension $\geqslant 5$ with $H^3(X_1; Z_2) \neq 0$, there exists a PL manifold X_2 which is homeomorphic but not PL homeomorphic to X_1.

Let X_1 and X_2 be such compact manifolds contained in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively, and let $h : X_1 \to X_2$ be a homeomorphism. On the other hand, by D. Sullivan [18], the Lipschitz manifold structure on a topological manifold of dimension $\neq 4$ is unique up to bi-Lipschitz homeomorphisms. Therefore we can choose h as a bi-Lipschitz homeomorphism since a PL manifold is a Lipschitz manifold. For a point x in $\mathbb{R}^n$ and a subset X of $\mathbb{R}^n$, let $x * X$ denote the cone with vertex x and base X, and let X_x be the germ of X at x. Set

$$\begin{aligned} Y_i &:= 0 * (X_i \times \{1\}) \subset \mathbb{R}^{m_i} \times \mathbb{R} \quad \text{for } i = 1, 2, \\ A_1 &:= Y_1 \times \{0\} \subset \mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1}, \\ A_2 &:= \{0\} \times Y_2 \subset \mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1}, \\ n &:= m_1 + m_2 + 2. \end{aligned}$$

We first show the following claim.

Claim 1. The following germs at 0, $(\mathbb{R}^n, A_1)_0$ and $(\mathbb{R}^n, A_2)_0$, are bi-Lipschitz homeomorphic.

Proof. — The idea of our proof comes from the proof of Proposition 10.4 in R. J. Daverman [6]. First we extend $h \times \mathrm{id} : X_1 \times \{1\} \to X_2 \times \{1\}$ to a bi-Lipschitz homeomorphism $h^* : Y_1 \to Y_2$ by cone extension. To be precise, set $h^*(0) := 0$ and

$$h^*(tx, t) := (th(x), t) \quad \text{for } (t, x) \in (0,\, 1) \times X_1.$$

Then h^* is bijective and $(h^*)^{-1}(tx, t) = (th^{-1}(x), t)$. Moreover, we can see that h^* is Lipschitz as follows. Let $(t, x), (t', x') \in [0,\, 1] \times X_1$. Then we have

$$\|h^*(tx, t) - h^*(t'x', t')\| \leqslant \|th(x) - t'h(x')\| + \|t - t'\|,$$

$$\begin{aligned} \|th(x) - t'h(x')\| &\leqslant \|th(x) - t'h(x)\| + \|t'h(x) - t'h(x')\| \\ &\leqslant c\|t - t'\| + ct'\|x - x'\|, \end{aligned}$$

$$\begin{aligned} t'\|x - x'\| &\leqslant \|tx - t'x\| + \|tx - t'x'\| \\ &\leqslant c\|t - t'\| + \|tx - t'x'\| \end{aligned}$$

for some constant real number $c > 0$. Hence we have

$$\|h^*(tx, t) - h^*(t'x', t')\| \leqslant c'\|t - t'\| + c'\|tx - t'x'\|$$

for some constant real number $c' > 0$. In the same way we can see that $(h^*)^{-1}$ is Lipschitz. Thus h^* is bi-Lipschitz.

Secondly we will extend h^* to a Lipschitz (not bi-Lipschitz) map $\tilde{h} : \mathbb{R}^{m_1+1} \to \mathbb{R}^{m_2+1}$. Let K be a simplicial decomposition of $\mathbb{R}^{m_1+1}$ such that Y_1 is the underlying polyhedron of a full subcomplex K_1 of K. K_1 is called *full* in K if each simplex in K with all its vertices in K_1 is necessarily contained in K_1. (When K_1 is not full in K, we replace K and K_1 with their barycentric subdivisions K' and K_1'. Then K_1' is full in K'. See C. P. Rourke and B. J. Sanderson [16] for details.) Let K^r denote the r-skeleton of K, namely, the simplexes in K of dimension $\leqslant r$. We define $\tilde{h}$ on the underlying polyhedron $|K^r|$ of K^r by induction on r. If $r = 0$, set $\tilde{h} := 0$ on $|K^0| - Y_1$. Assume that $\tilde{h}$ is already defined on $|K^{r-1}|$ for some $r > 0$. For each $\sigma \in K^r - K^{r-1}$ with $\sigma \not\subset Y_1$, let $v_0, \cdots, v_r$ be the vertices of σ such that $v_0 \notin Y_1$, which exists by the fullness of K_1. Note that $v_0, v_1 * \cdots * v_r \in K^{r-1}$. Set

$$\tilde{h}(\sum_{i=0}^{r} t_i v_i) := \sum_{k=1}^{r} t_k \tilde{h}(\sum_{i=1}^{r} t_i v_i / \sum_{j=1}^{r} t_j)$$
$$\text{for } (t_0, ..., t_r) \in [0,\, 1]^{r+1} \text{ with } \sum_{i=0}^{r} t_i = 1 \text{ and } \sum_{i=1}^{r} t_i \neq 0,$$

which is well-defined because $\sum_{i=1}^r t_i v_i / \sum_{j=1}^r t_j \in v_1 * \cdots * v_r$. Then $\tilde{h}$ is a map from $|K^r|$ to $\mathbb{R}^{m_2+1}$ and we claim that it is Lipschitz. In order to see this, it suffices to show that $\tilde{h}|_\sigma$ is Lipschitz for the above σ, because $\tilde{h} = 0$ outside of a compact neighbourhood of Y_1 in $\mathbb{R}^{m_1+1}$. By the above definition of $\tilde{h}|_\sigma$, $\tilde{h}|_\sigma(v_0) = 0$ and $\tilde{h}|_\sigma$ is the cone extension of $\tilde{h}|_{v_1 * \cdots * v_r}$. Hence, as shown above, $\tilde{h}|_\sigma$ is Lipschitz since so is $\tilde{h}|_{v_1 * \cdots * v_r}$.

Set $A_3 :=$ graph $h^* \subset \mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1}$. We shall prove that $(\mathbb{R}^n, A_3)_0$ is bi-Lipschitz homeomorphic to $(\mathbb{R}^n, A_1)_0$ and $(\mathbb{R}^n, A_2)_0$. Consider $(\mathbb{R}^n, A_3)_0$ and $(\mathbb{R}^n, A_1)_0$. Set

$$\phi(x, y) := (x, y - \tilde{h}(x)) \quad \text{for } (x, y) \in \mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1}.$$

Then ϕ is a homeomorphism of $\mathbb{R}^n$, $\phi^{-1}(A_1) = A_3$, $\phi(0) = 0$, and ϕ is Lipschitz since so is $\tilde{h}$. In addition, ϕ has its inverse given by

$$\mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1} \ni (x, y) \to (x, y + \tilde{h}(x)) \in \mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1},$$

which is also Lipschitz. Thus ϕ is a bi-Lipschitz homeomorphism from $(\mathbb{R}^n, A_3)_0$ to $(\mathbb{R}^n, A_1)_0$. In the same way, by extending $(h^*)^{-1}$ to a Lipschitz map from $\mathbb{R}^{m_2+1}$ to $\mathbb{R}^{m_1+1}$, we see that $(\mathbb{R}^n, A_3)_0$ and $(\mathbb{R}^n, A_2)_0$ are bi-Lipschitz homeomorphic. □

We next show the following claim.

Claim 2. $(\mathbb{R}^n, A_1)_0$ and $(\mathbb{R}^n, A_2)_0$ are not definable homeomorphic.

Proof. — Assume that they are definable homeomorphic. Then $(A_1)_0$ and $(A_2)_0$ are definable homeomorphic. Hence shrinking A_1 we have a definable embedding $f : A_1 \to A_2$ such that $f(0) = 0$ and $f(A_1)$ is a neighbourhood of 0 in A_2.

In order to complete the proof we recall the following facts from [17].

Definable Triangulation Theorem. (Theorem II.2.1 in [17]) *Any compact definable set X is definable homeomorphic to some polyhedron X'.*

Remark 3.2. — (Remark II.2.3 in [17]) If the above X is contained in the underlying polyhedron of a finite simplicial complex K, then we can choose X' in $|K|$ and a definable homeomorphism $g : X \to X'$ so that $g(X \cap |\sigma|) \subset |\sigma|$ for each $\sigma \in K$.

Definable Hauptvermutung. (Corollary III.1.4 in [17]) *Any two compact polyhedra are PL homeomorphic if they are definable homeomorphic.*

Applying the definable triangulation theorem and Remark 3.2 to $f(A_1)$ and 0, we can assume that $f(A_1)$ is a polyhedron. Therefore we may assume $f(A_1) = A_2$ from the beginning. The links of A_1 and A_2 at 0 are X_1 and X_2 respectively. By our choice of X_1, they are not PL homeomorphic to a sphere or to a ball, hence A_1 and A_2 are not PL manifolds at 0 (0 is a singular point). On the other hand the links at the other points are all PL homeomorphic to a sphere or to a ball. By the definable Hauptvermutung, A_1 and A_2 are PL homeomorphic as polyhedra. Moreover, because the origin 0 is the only singular point of A_1 and A_2, the PL homeomorphism has to carry 0 to 0. Thus $(A_1)_0$ and $(A_2)_0$ are PL homeomorphic, which is a contradiction because of our our choice of X_1 and X_2. □

This completes the proof of the theorem.

□

Remark 3.3. — In the proof of Claim 1 we constructed a Lipschitz extension of h^* to $\tilde{h}$ using the cone structure. However, in general, to extend a Lipschitz map is not difficult. Indeed, for a Lipschitz function with constant L, $f : A \to \mathbb{R}$, $A \subset X$, A endowed with the induced metric from (X, d), we have an extension formula (see S. Banach [1]):

$$\alpha(x) := \inf_{a \in A} (f(a) + Ld(x, a)).$$

Similarly one can extend it by

$$\beta(x) := \sup_{a \in A} (f(a) - Ld(x, a)).$$

Note that $\beta(x) \leqslant \alpha(x)$. Any convex combination $t\alpha(x) + (1-t)\beta(x)$, $0 \leqslant t \leqslant 1$, also gives a Lipschitz extension.

This construction can be used to extend Lipschitz maps as well, however, without preserving the Lipschitz constant.

4. Sea-Tangle Properties in o-minimal Structures

We recall the notion of sea-tangle neighbourhood for a subset of $\mathbb{R}^n$, originated from the classical notion of *horn-neighbourhood* for an analytic set or more generally a subanalytic set in $\mathbb{R}^n$.

Definition 4.1. — *Let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $d,\ C > 0$. The* sea-tangle neighbourhood $ST_d(A;C)$ of A, of degree d and width C, *is defined by:*

$$ST_d(A;C) := \{x \in \mathbb{R}^n \mid dist(x,A) \leqslant C\|x\|^d\}.$$

See §4 of [12] for some sea-tangle properties. For instance, the following is shown.

Proposition 4.2. — *([12] Proposition 4.7) Let A be a subanalytic set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then there is $d_1 > 1$ such that $A \subset ST_d(LD(A);C)$ as set-germs at $0 \in \mathbb{R}^n$ for any d with $1 < d < d_1$ and $C > 0$.*

Both Hironaka's selection lemma for a subanalytic set and a Lojasiewicz inequality have played an important role in the proof of the above result. As mentioned in §2, it is known that the curve selection lemma holds also for definable sets in an o-minimal structure; on the other hand, the usual Lojasiewicz inequality does not always hold in an o-minimal structure. Accordingly, Proposition 4.2 might be false for a definable set in some o-minimal structure. Indeed, the following example confirms this.

Example 4.3. — Let $\pi : \mathcal{M}_2 \to \mathbb{R}^2$ be a blowing-up at $(0,0) \in \mathbb{R}^2$, and let $a = (0,1) \in S^1$. We denote by $L(a)$ the half line in $\mathbb{R}^2$ with the origin as the starting point passing through a and by $\hat{L}(a)$ the strict transform of $L(a)$ in $\mathcal{M}_2$ by π. In a suitable coordinate neighbourhood, $\pi : \mathbb{R}^2_{(X,Y)} \to \mathbb{R}^2$ can be expressed as $\pi(X,Y) = (XY,Y)$. Here $(0,0) \in \mathbb{R}^2_{(X,Y)}$ is the intersection of $\hat{L}(a)$ and the exceptional divisor $E = \pi^{-1}(0,0)$.

Let $B := \{(X,Y) \in \mathbb{R}^2_{(X,Y)} \mid Y = e^{-\frac{1}{|X|^2}},\ X \geqslant 0\}$. Then the curve B is not contained in $\{(X,Y) \in \mathbb{R}^2_{(X,Y)} \mid |Y| \geqslant C'|X|^{d'}\}$ as germs at $(0,0) \in \mathbb{R}^2_{(X,Y)}$, for any $d' > 0$, $C' > 0$.

Let $\mathbb{R}_{exp}$ be Wilkie's exponential field ([19]), and let $\mathcal{D}$ be the o-minimal structure on it. Set $A := \pi(B)$. Then we can see that $A \in \mathcal{D}$ and $LD(A) = L(a)$, but A is not contained in any sea-tangle neighbourhood $ST_d(L(a);C)$ as germs at $(0,0) \in \mathbb{R}^2$, for $d > 1$, $C > 0$. Therefore Proposition 4.2 does not hold for this definable set A in the o-minimal structure $\mathcal{D}$.

Taking into account the above fact, in order to develop sea-tangle properties in an o-minimal structure on an Archimedean real closed field R, the definition of sea-tangle neighbourhood has to be modified. Note that R^n has an induced metric from $\mathbb{R}^n$.

From now on let us fix an o-minimal structure on R. Here we recall that Φ is the set of all odd, strictly increasing, continuous definable germs from $(R,0)$ to $(R,0)$. Then we define the notion of sea-tangle neighbourhood of a definable set as follows:

Definition 4.4. — *Let $A \subset R^n$ such that $0 \in \overline{A}$, and let $\theta \in \Phi$. The* sea-tangle neighbourhood $ST_\theta(A)$ of A with respect to θ *is defined by:*

$$ST_\theta(A) := \{x \in R^n \mid d(x,A) \leqslant \theta(\|x\|)\|x\|\}.$$

Remark 4.5. — (1) Let $x \in R^n$ and $A \subset R^n$. In general, $d(x,A) = \inf_{a\in A} d(x,a)$ does not always belong to R; nonetheless it is always a non-negative real number.

(2) If A is definable, then $D(A)$, $LD(A)$ and $ST_\theta(A)$ are also definable.

Let $\mathcal{S}$ be the set of set-germs $A \subset R^n$ at $0 \in R^n$ such that $0 \in \overline{A}$.

Definition 4.6. — *Let A, $B \in \mathcal{S}$. We say that A and B are* ST-equivalent, *if there are $\theta_1,\ \theta_2 \in \Phi$ such that $B \subset ST_{\theta_1}(A)$ and $A \subset ST_{\theta_2}(B)$ as germs at $0 \in R^n$. We write $A \underset{st}{\sim} B$.*

Remark 4.7. — ST-equivalence $\underset{\mathrm{st}}{\sim}$ is an equivalence relation in $\mathcal{S}$.

We first describe several sea-tangle properties for general subsets of R^n.

Let $h : (R^n,0) \to (R^n,0)$ be a bi-Lipschitz homeomorphism with Lipschitz constants $K_1,\ K_2 \in R$ such that $0 < K_1 \leqslant K_2$, that is

$$K_1\|x_1 - x_2\| \leqslant \|h(x_1) - h(x_2)\| \leqslant K_2\|x_1 - x_2\|$$

in a small neighbourhood of $0 \in R^n$. Conversely, we have

$$\frac{1}{K_2}\|y_1 - y_2\| \leqslant \|h^{-1}(y_1) - h^{-1}(y_2)\| \leqslant \frac{1}{K_1}\|y_1 - y_2\|$$

in a small neighbourhood of $0 \in R^n$. With these Lipschitz constants we can formulate the following Sandwich Lemma.

Lemma 4.8. — *Let $A \subset R^n$ such that $0 \in \overline{A}$. Then, for $\theta \in \Phi$,*

(i) $h(ST_\theta(A)) \subset ST_{\theta_1}(h(A))$ where $\theta_1(t) = \frac{K_2}{K_1}\theta(\frac{t}{K_1}) \in \Phi$. and

(ii) $ST_{\theta_2}(h(A)) \subset h(ST_\theta(A))$ where $\theta_2(t) = \frac{K_1}{K_2}\theta(\frac{t}{K_2}) \in \Phi$

in a small neighbourhood of $0 \in R^n$.

Remark 4.9. — Lemma 4.8 (i) shows that if for some $\theta \in \Phi$, $A \subset ST_\theta(B)$ and h is bi-Lipschitz, then there exists $\theta_1 \in \Phi$ so that $h(A) \subset ST_{\theta_1}(h(B))$.

The next proposition follows from the above Sandwich Lemma.

PROPOSITION 4.10. — *ST-equivalence is preserved by a bi-Lipschitz homeomorphism.*

In [12] there are mentioned some directional properties for the original notion of sea-tangle neighbourhood $ST_d(A;C)$. The same properties hold also for our sea-tangle neighbourhood $ST_\theta(A)$. Throughout this section, let A, $B \subset R^n$ be set-germs at $0 \in R^n$ such that $0 \in \overline{A} \cap \overline{B}$, namely A, $B \in \mathcal{S}$, and let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism.

In the case of sequences of real points $a_m, b_m \in \mathbb{R}^n$ we use the following notation

$$\|a_m\| \ll \ \|b_m\|,$$

as a substitute for

$$\lim_{m\to\infty} \frac{\|a_m\|}{\|b_m\|} = 0.$$

LEMMA 4.11. — *Suppose that there is $\theta \in \Phi$ such that $A \subset ST_\theta(B)$ as set-germs at $0 \in R^n$. Then we have $D(h(A)) \subset D(h(B))$. In addition, we have $D(ST_{\theta'}(h(A))) \subset D(h(B))$ for any $\theta' \in \Phi$.*

Proof. — Let $a \in D(h(A))$. Then there is a sequence of points $\{a_m\} \subset A$ tending to $0 \in R^n$ such that $\lim_{m\to\infty} \frac{h(a_m)}{\|h(a_m)\|} = a$. By assumption, $A \subset ST_\theta(B)$. Therefore, for each m we can take $b_m \in B$ such that

$$\|a_m - b_m\| \ll \|a_m\|, \ \|b_m\|.$$

Since h is a bi-Lipschitz homeomorphism,

$$\|h(a_m) - h(b_m)\| \ll \|h(a_m)\|, \ \|h(b_m)\|.$$

Thus we have

$$a = \lim_{m\to\infty} \frac{h(a_m)}{\|h(a_m)\|} = \lim_{m\to\infty} \frac{h(b_m)}{\|h(b_m)\|} \in D(h(B)).$$

The second statement follows from the first one. □

Remark 4.12. — For h identity in the above lemma we get the following implication :

$$A \subset ST_\theta(B) \Rightarrow D(A) \subset D(B).$$

The converse is not always true, however if both A and B are definable the converse is true (see Corollary 4.18).

We have the following corollary of Lemma 4.11.

Corollary 4.13. — (1) *$D(ST_\theta(A)) = D(A)$ for any $\theta \in \Phi$.*

(2) $D(ST_\theta(h(A))) = D(h(A))$ for any $\theta \in \Phi$.

(3) If A and B are ST-equivalent, then we have $D(A) = D(B)$ and $D(h(A)) = D(h(B))$.

Remark 4.14. — The first equality in the above corollary clearly implies the following inclusion:

$$D(ST_{\theta_1}(A) \cap ST_{\theta_2}(B)) \subset D(A) \cap D(B)$$

for any θ_1, $\theta_2 \in \Phi$. The equality holds for some θ_1, $\theta_2 \in \Phi$, when both A and B are definable.

For a definable set, we have more specific sea-tangle properties. Proposition 4.2 is modified to the following.

Proposition 4.15. — *Let A be a definable set-germ at $0 \in R^n$ such that $0 \in \overline{A}$. Then there is $\theta \in \Phi$ such that $A \subset ST_\theta(LD(A))$ as set-germs at $0 \in R^n$.*

Proof. — Let $g(x) = \frac{d(x, LD(A))}{\|x\|}$ for $x \in \overline{A}$, $x \neq 0$.

Claim. $\lim_{x \to 0} g(x) = 0$.

Proof. — If the claim does not hold, then by the Curve selection lemma, there exist $c > 0$ and a definable curve $\gamma : (0,1) \to \overline{A} \setminus \{0\}$ such that $\lim_{t \to 0} \gamma(t) = 0$ and $\frac{d(\gamma(t), LD(A))}{\|\gamma(t)\|} \geqslant c$ for sufficiently small $t > 0$. By Monotonicity, $\lim_{t \to 0} \frac{\gamma(t)}{\|\gamma(t)\|} = a \in D(A)$. Therefore we have

$$\frac{d(\gamma(t), LD(A))}{\|\gamma(t)\|} \leqslant \frac{d(\gamma(t), L(a))}{\|\gamma(t)\|} \to 0$$

when $t \to 0$. This is a contradiction. □

By this claim, g can be naturally extended to a continuous definable function $g : \overline{A} \to R$ with $g(0) = 0$. By Lojasiewicz inequality ([8]), there exists $\theta \in \Phi$ such that $g(x) \leqslant \theta(\|x\|)$ for $x \in A$ near $0 \in R^n$. This means that $A \subset ST_\theta(LD(A))$ as germs at $0 \in R^n$. □

Proposition 4.16. — *Suppose that A is definable. Then, for any $\theta_1 \in \Phi$, there is $\theta_2 \in \Phi$ such that $ST_{\theta_1}(LD(A)) \subset ST_\theta(A)$ as germs at $0 \in R^n$, for any $\theta \in \Phi$ with $\theta \geqslant \theta_2$.*

Proof. — Let $\theta_1 \in \Phi$. Then, by Corollary 4.13 (1), we have

$$D(ST_{\theta_1}(LD(A)) = D(LD(A)) = D(A).$$

Using the same arguments for $g(x) = \frac{d(x,A)}{\|x\|}$ $(x \in ST_{\theta_1}(LD(A)), x \neq 0)$ as in the proof of Proposition 4.15, we can find $\theta_2 \in \Phi$ such that $ST_{\theta_1}(LD(A)) \subset ST_{\theta_2}(A)$. Hence $LD(A) \subset ST_{\theta_2}(A)$ as germs at $0 \in R^n$. □

By Propositions 4.15, 4.16, we have

THEOREM 4.17. — *If A is definable, then A is ST-equivalent to $LD(A)$.*

Using Propositions 4.15, 4.16 and Lemma 4.8, we can show the following result in a similar way to Corollary 4.15 in [12]:

COROLLARY 4.18. — *Suppose that $h(A)$, $h(B)$ are definable. If $D(h(A)) \subset D(h(B))$, then there is $\theta \in \Phi$ such that $A \subset ST_\theta(B)$ as germs at $0 \in R^n$.*

By Corollary 4.18 and Lemma 4.11, we have

THEOREM 4.19. — *Suppose that $h(A)$, $h(B)$ are definable. Then the following conditions are equivalent.*

(1) $D(h(A)) \subset D(h(B))$.

(2) There is $\theta \in \Phi$ such that $A \subset ST_\theta(B)$ as germs at $0 \in R^n$.

5. Sequence Selection Property

In this section we discuss directional properties of sets with the sequence selection property, denoted by (SSP) for short, over an Archimedean real closed field R. The set $h(LD(A))$ takes a very important role in the proof of our main theorem. In [12] it is shown that over the field of real numbers $\mathbb{R}$, $h(LD(A))$ satisfies condition (SSP) provided that A and $h(A)$ are both subanalytic. We give an improvement of this result here.

Let us recall the notion of the sequence selection property.

DEFINITION 5.1. — *Let $A \subset R^n$ be a set-germ at $0 \in R^n$ such that $0 \in \overline{A}$. We say that A satisfies* condition (SSP), *if for any sequence of points $\{a_m\}$ of R^n tending to $0 \in R^n$ such that $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$, there is a sequence of points $\{b_m\} \subset A$ such that*

$$\|a_m - b_m\| \ll \|a_m\|,\ \|b_m\|.$$

We have some remarks on (SSP) (cf. [12]).

Remark 5.2. — Condition (SSP) is C^1 invariant, but not bi-Lipschitz invariant.

Remark 5.3. — Let $A \subset R^n$ be a set-germ at $0 \in R^n$ such that $0 \in \overline{A}$.

(1) The cone $LD(A)$ satisfies condition (SSP).

(2) If $A \subset \mathbb{R}^n$ is subanalytic, then it satisfies condition (SSP).

(3) If $A \subset R^n$ is definable in an o-minimal structure, then it satisfies condition (SSP).

Remark 5.4. — We can describe condition (SSP) without using the convergence of a sequence of points as follows:

for any $\epsilon, \delta \in R$ and $x \in R^n$ with $\epsilon > 0,\ \delta > 0,\ 0 < \|x\| \leqslant \delta$ and $dist(\frac{x}{\|x\|}, D(A)) \leqslant \epsilon$ there exists $y \in A$ such that $\|x - y\| \leqslant \epsilon\|x\|$.

We recall the following lemma.

Lemma 5.5. — *(Lemma 5.6 in* [12] *) Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$. Then $D(h(A)) \subset D(h(LD(A)))$. If A satisfies condition (SSP), the equality holds.*

In order to show the above lemma, the following property was used in [12] :

Let $\{a_m\}$ be a sequence of points of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$. Then there is a subsequence of points $\{a_k\}$ of $\{a_m\}$ such that

$$\lim_{k\to\infty} \frac{a_k}{\|a_k\|} \in S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

This property does not always hold on an Archimedean real closed field R. In fact, Lemma 5.5 is false for the real closed field of algebraic numbers. Let us recall Example 2.1. Let $A = \{a_m\}$. Then $h(A) = \{b_m\}$. Since $D(A) = \emptyset$ we have $D(h(LD(A))) = \emptyset$. Nevertheless $D(h(A)) = \{(0, 1)\}$.

Over an Archimedean real closed field, we can show the following weaker result, which is enough to show our main theorem.

Lemma 5.6. — *Let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset R^n$ such that $0 \in \overline{A}$. Suppose that A is definable. Then we have $LD(h(A)) = LD(h(LD(A)))$.*

Proof. — We first show the inclusion $LD(h(A)) \subset LD(h(LD(A)))$. By Proposition 4.15. there is $\theta_1 \in \Phi$ such that $A \subset ST_{\theta_1}(LD(A))$ as set-germs at $0 \in R^n$. Then, by the sandwich lemma, we have

$$h(A) \subset h(ST_{\theta_1}(LD(A))) \subset ST_{\theta_2}(h(LD(A)))$$

for some $\theta_2 \in \Phi$. Therefore it follows from Corollary 4.13 (2) that

$$LD(h(A)) \subset LD(ST_{\theta_2}(h(LD(A)))) = LD(h(LD(A))).$$

The opposite inclusion $LD(h(A)) \supset LD(h(LD(A)))$ follows from a similar argument, replacing Proposition 4.16 with Proposition 4.15. $\square$

As a corollary of Lemma 5.6 and Remark 4.5 we have the following lemma:

LEMMA 5.7. — *Let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset R^n$ such that $0 \in \overline{A}$. If A and $h(A)$ are definable, then $LD(h(LD(A))) = LD(h(A))$ is definable.*

We give one more example having condition (SSP). Using a similar argument to Proposition 6.4 in [12], we can show the following:

PROPOSITION 5.8. — *Let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism, and let A, $h(A) \subset R^n$ be definable set-germs at $0 \in R^n$ such that $0 \in \overline{A}$. Then the set $h(LD(A))$ satisfies condition (SSP).*

Let us discuss more on the sequence selection property over the field of real numbers $\mathbb{R}$. In this note we consider also the notion of weak sequence selection property, denoted by $(WSSP)$ for short.

DEFINITION 5.9. — *Let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. We say that A satisfies* condition $(WSSP)$, *if for any sequence of points $\{a_m\}$ of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$ such that $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$, there exist a subsequence $\{m_j\}$ and $\{b_{m_j}\} \subset A$ such that*

$$\|a_{m_j} - b_{m_j}\| \ll \|a_{m_j}\|,\ \|b_{m_j}\|.$$

We have the following characterisation of condition (SSP).

LEMMA 5.10. — *Let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. If A satisfies condition $(WSSP)$, then it satisfies condition (SSP). Namely, conditions (SSP) and $(WSSP)$ are equivalent.*

Proof. — We show that $(WSSP)$ implies (SSP). Assume that A does not satisfy condition (SSP). Then there is a sequence of points $\{a_m\}$ tending to $0 \in \mathbb{R}^n$ with $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$ such that for any sequence of points $\{b_m\} \subset A$, the following is not satisfied:

$$\|a_m - b_m\| \ll \|a_m\|.$$

Therefore there is a subsequence of points $\{a_{m_j}\}$ of $\{a_m\}$ such that $\lim_{m_j\to\infty} \frac{d(a_{m_j}, A)}{\|a_{m_j}\|} = \alpha > 0$, where $d(a_{m_j}, A)$ denotes the distance between a_{m_j} and A. Taking this $\{a_{m_j}\}$ as the first $\{a_m\}$, we can assume from the beginning that $\lim_{m\to\infty} \frac{d(a_m, A)}{\|a_m\|} = \alpha > 0$. This implies that there does not exist a sequence of points $\{b_{m_j}\} \subset A$ such that $\|a_{m_j} - b_{m_j}\| \ll \|a_{m_j}\|$. Therefore A does not satisfy condition $(WSSP)$. $\square$

Using Lemmas 5.5 and 5.10 we can improve Proposition 6.4 in [12] as follows:

Theorem 5.11. — *Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$. Assume that A satisfies condition (SSP). Then $h(A)$ satisfies condition (SSP), if and only if, $h(LD(A))$ satisfies condition (SSP).*

Proof. — We first show the "only if" part. By assumption, A satisfies condition (SSP). Therefore it follows from Lemma 5.5 that $D(h(LD(A))) = D(h(A))$. Let $\{y_m\}$ be an arbitrary sequence of points of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$ such that

$$\lim_{m\to\infty} \frac{y_m}{\|y_m\|} \in D(h(LD(A))) = D(h(A)).$$

Let $y_m = h(x_m)$ for each m. Since $h(A)$ satisfies condition (SSP), there is a sequence of points $\{z_m\} \subset A$ such that

$$\|h(x_m) - h(z_m)\| \ll \|h(x_m)\|,\ \|h(z_m)\|.$$

On the other hand, there is a subsequence $\{z_{m_j}\}$ of $\{z_m\}$ such that $\lim_{m_j\to\infty} \frac{z_{m_j}}{\|z_{m_j}\|} \in D(A)$. Since $LD(A)$ satisfies condition (SSP), there is a sequence of points $\{\theta_{m_j}\} \subset LD(A)$ such that

$$\|z_{m_j} - \theta_{m_j}\| \ll \|z_{m_j}\|,\ \|\theta_{m_j}\|.$$

It follows from the bi-Lipschitz of h that

$$\|h(z_{m_j}) - h(\theta_{m_j})\| \ll \|h(z_{m_j})\|,\ \|h(\theta_{m_j})\|.$$

Then we have

$$\|h(x_{m_j}) - h(\theta_{m_j})\| \leqslant \|h(x_{m_j}) - h(z_{m_j})\| + \|h(z_{m_j}) - h(\theta_{m_j})\| \ll \|h(z_{m_j})\|.$$

Therefore we have

$$\|h(x_{m_j}) - h(\theta_{m_j})\| \ll \|h(x_{m_j})\|,\ \|h(\theta_{m_j})\|.$$

Thus $h(LD(A))$ satisfies condition $(WSSP)$, and also condition (SSP) by Lemma 5.10.

The "if" part can be proved in a similar way. □

6. Proof of Main Theorem

Our main theorem is proved in the same way as Theorem 1.4 in [12]. Since the reduction arguments in §6 of [12] work also for definable sets, it only suffices to show that under our conditions the Lemma 6.8 and Proposition 6.1

hold true (basically showing that $\dim D(A) = \dim D(h(A))$). Indeed one can calculate $\dim(D(A) \cap D(B)) = \dim(D(ST_{\theta_1}(A) \cap ST_{\theta_2}(B))$ (following Remark 4.14), and this last number behaves well under bi-Lipschitz homeomorphisms (apply Lemma 6.8 and Proposition 6.1), allowing us to switch to $\dim(D(h(A)) \cap D(h(B)))$ and claim the desired equality. In the proofs below, we adjust the details of the proof of the main theorem in [12] to the new context of o-minimality.

Proposition 6.1. — *Let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset R^n$ such that $0 \in \overline{A}$. Suppose that A, $h(A)$ are definable. Then*

$$\dim h(LD(A)) \geqslant \dim LD(h(LD(A))).$$

As mentioned in the introduction, Theorem 1.4 was proved using sea-tangle properties, sequence selection properties and volume arguments. In this section we discuss volume arguments and give a proof of our main theorem. In order to avoid considering the volume on a general Archimedean real closed field R, we take the closure of a subset of R^n in $\mathbb{R}^n$. Note that the completion of R is $\mathbb{R}$ and that R^n is dense in $\mathbb{R}^n$.

For a subset A of R^n ($\subset \mathbb{R}^n$), let $\overline{A}^{\mathbb{R}}$ denote the closure of A in $\mathbb{R}^n$ (not in R^n), and let $B_\epsilon(0)$ denote a closed ϵ ball in $\mathbb{R}^n$ centred at $0 \in \mathbb{R}^n$ for $\epsilon > 0$, $\epsilon \in \mathbb{R}$.

Let $f,\ g : [0, \delta) \to \mathbb{R}$, $\delta > 0$, be non-negative functions, where $[0, \delta)$ is a half open interval of $\mathbb{R}$. If there are real numbers $K > 0$, $0 < \delta_1 \leqslant \delta$ such that

$$f(\epsilon) \leqslant K g(\epsilon) \quad \text{for} \quad 0 \leqslant \epsilon \leqslant \delta_1,$$

then we write $f \precsim g$ (or $g \succsim f$). If $f \precsim g$ and $f \succsim g$, we write $f \approx g$.

We can easily see the following property on volumes:

Lemma 6.2. — *Let $A \subset R^n$ such that $0 \in \overline{A}$. Then*

$$Vol(\overline{ST_{c\theta}(A)}^{\mathbb{R}} \cap B_\varepsilon(0)) \approx Vol(\overline{ST_\theta(A)}^{\mathbb{R}} \cap B_\varepsilon(0))$$

for $\theta \in \Phi$ and $c > 0$.

Using a similar argument as in Lemma 7.1 of [12], we can show the following lemma.

Lemma 6.3. — *Let α, β be linear subspaces of R^n. Suppose that $\dim \alpha < \dim \beta$. Then, for $\theta \in \Phi$,*

$$\lim_{\epsilon \to 0} \frac{Vol(\overline{ST_\theta(\alpha)}^{\mathbb{R}} \cap B_\epsilon(0))}{Vol(\overline{ST_\theta(\beta)}^{\mathbb{R}} \cap B_\epsilon(0))} = 0.$$

We have the following volume properties analogous to those in [12].

Proposition 6.4. — *Let α, $\beta \subset R^n$ be definable cones at $0 \in R^n$. Suppose that* $\dim \alpha < \dim \beta$. *Then, for $\theta \in \Phi$,*

$$\lim_{\epsilon \to 0} \frac{Vol(\overline{ST_\theta(\alpha)}^{\mathbb{R}} \cap B_\epsilon(0))}{Vol(\overline{ST_\theta(\beta)}^{\mathbb{R}} \cap B_\epsilon(0))} = 0.$$

Proof. — Let γ be a definable cone at $0 \in R^n$ of dimension r, and let M be an r-dimensional linear subspace of R^n. Then the proposition follows easily from Lemma 6.3 and the fact that

$$Vol(\overline{ST_\theta(\gamma)}^{\mathbb{R}} \cap B_\epsilon(0)) \approx Vol(\overline{ST_\theta(M)}^{\mathbb{R}} \cap B_\epsilon(0)) \tag{6.1}$$

for $\theta \in \Phi$.

Let us show (6.1). We may assume that γ is equidimensional. Then there exist a finite partition of γ into r-dimensional definable cones $\gamma_1, \cdots, \gamma_s$ with $0 \in R^n$ as a vertex, and r-dimensional linear subspaces $M_1, \cdots, M_s$ of R^n such that for each orthogonal projection $\Pi_i : R^n \to M_i$, $1 \leqslant i \leqslant s$, γ_i is expressed as the graph of a definable map from $\Pi_i(\gamma_i) \subset M_i$ and the diameter of $ST_\theta(\gamma) \cap \Pi_i^{-1}(u)$, $u \in \Pi_i(\gamma_i)$, is less than or equal to 4θ. Then we can see that

$$Vol(\overline{ST_\theta(\gamma)}^{\mathbb{R}} \cap B_\epsilon(0)) \precsim Vol(\overline{ST_{4\theta}(M)}^{\mathbb{R}} \cap B_\epsilon(0)). \tag{6.2}$$

On the other hand, we may assume that one of $\Pi_i(\gamma_i)$'s is a closed cone in M_i, taking a finite subdivision of γ_i's if necessary. Then we can see that M_i is covered with a finite number of r-dimensional closed cones isometric to $\Pi_i(\gamma_i)$. It follows that

$$\begin{aligned} Vol(\overline{ST_\theta(M_i)}^{\mathbb{R}} \cap B_\epsilon(0)) &\precsim Vol(\overline{ST_\theta(M_i) \cap \Pi_i^{-1}(\Pi_i(\gamma_i))}^{\mathbb{R}} \cap B_\epsilon(0)) \\ &\precsim Vol(\overline{ST_\theta(\gamma_i)}^{\mathbb{R}} \cap B_\epsilon(0)). \end{aligned}$$

Therefore we have

$$Vol(\overline{ST_\theta(\gamma)}^{\mathbb{R}} \cap B_\epsilon(0)) \succsim Vol(\overline{ST_\theta(M)}^{\mathbb{R}} \cap B_\epsilon(0)). \tag{6.3}$$

Then (6.1) follows from (6.2), (6.3) and Lemma 6.2. □

The next lemma follows in the same way as Lemma 7.2 in [12]:

Lemma 6.5. — *Let $A \subset R^n$ be a definable set-germ at $0 \in R^n$ such that $0 \in \overline{A}$. Then we have* $\dim LD(A) \leqslant \dim A$.

Proposition 6.6. — *Let A, B be set-germs at 0 in R^n such that $0 \in \overline{A} \cap \overline{B}$. Suppose that A and B are ST-equivalent. Then there is $\theta_1 \in \Phi$ such that*

$$Vol(\overline{ST_\theta(A)}^{\mathbb{R}} \cap B_\epsilon(0)) \approx Vol(\overline{ST_\theta(B)}^{\mathbb{R}} \cap B_\epsilon(0))$$

for any $\theta \in \Phi$ with $\theta \geqslant \theta_1$.

Proof. — Let $\theta_3,\ \theta_4 \in \Phi$ such that $A \subset ST_{\frac{\theta_3}{2}}(B),\ B \subset ST_{\frac{\theta_4}{2}}(A)$. Take $\theta_1 \in \Phi$ so that $\theta_1(t) \geqslant 2\max(\theta_3(2t), \theta_4(2t))$ for $0 < t < 1$.

Claim. For any $\theta \in \Phi$ with $\theta \geqslant \theta_1$, we have

$$ST_{\frac{\theta}{2}}(A) \subset ST_{\frac{\theta}{2}}(ST_{\frac{\theta_3}{2}}(B)) \subset ST_{2\theta}(B) \subset ST_{4\theta}(A)$$

as germs at $0 \in R^n$.

Proof. — To see the second inclusion, let $x \in ST_{\frac{\theta}{2}}(ST_{\frac{\theta_3}{2}}(B))$ with $\theta(\|x\|) \leqslant 1$. Then there exists $y \in R^n$ such that $d(y, B) \leqslant \frac{\theta_3}{2}(\|y\|)\|y\|$ and $d(x, y) \leqslant \theta(\|x\|)\|x\|$. We also have $\|y\| \leqslant \|x\| + \theta(\|x\|)\|x\| \leqslant 2\|x\|$. Take $z \in B$ such that $d(y, z) \leqslant \theta_3(\|y\|)\|y\|$. Then

$$\begin{aligned} d(x,z) &\leqslant d(x,y) + d(y,z) \\ &\leqslant \theta(\|x\|)\|x\| + \theta_3(\|y\|)\|y\| \\ &\leqslant \theta(\|x\|)\|x\| + \theta_3(2\|x\|)2\|x\| \\ &\leqslant \theta(\|x\|)\|x\| + \theta(\|x\|)\|x\| = 2\theta(\|x\|)\|x\|. \end{aligned}$$

This implies $x \in ST_{2\theta}(B)$, and hence $ST_{\frac{\theta}{2}}(ST_{\frac{\theta_3}{2}}(B)) \subset ST_{2\theta}(B)$.

Let $x \in ST_{2\theta}(ST_{\frac{\theta_4}{2}}(A))$ with $\theta(\|x\|) \leqslant \frac{1}{3}$. Similarly as above, we can show

$$ST_{2\theta}(B) \subset ST_{2\theta}(ST_{\frac{\theta_4}{2}}(A)) \subset ST_{4\theta}(A).$$

□

The statement of the proposition follows from this claim and Lemma 6.2. □

The following corollary is an obvious consequence of Theorem 4.17, Lemma 6.5 and Propositions 6.4 and 6.6.

Corollary 6.7. — *Let $\alpha \subset R^n$ be a definable set-germ at $0 \in R^n$ such that $0 \in \overline{\alpha}$, and let $\beta \subset R^n$ be a definable cone at $0 \in R^n$. Suppose that $\dim \alpha < \dim \beta$. Then there is $\theta_1 \in \Phi$ such that*

$$\lim_{\epsilon \to 0} \frac{Vol(\overline{ST_\theta(\alpha)}^{\mathbb{R}} \cap B_\epsilon(0))}{Vol(\overline{ST_\theta(\beta)}^{\mathbb{R}} \cap B_\epsilon(0))} = 0$$

for any $\theta \in \Phi$ with $\theta \geqslant \theta_1$.

We next describe a key lemma needed for proving our main theorem.

Lemma 6.8. — *Let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism, let $E \subset R^n$ be a definable set-germ at $0 \in R^n$ such that $0 \in \overline{E}$, and let $F := h(E)$. Suppose that F and $LD(F)$ are ST-equivalent and $LD(F)$ is definable. Then we have $\dim LD(F) \leqslant \dim E$.*

Proof. — Since $h : (R^n, 0) \to (R^n, 0)$ is a bi-Lipschitz homeomorphism and R^n is dense in $\mathbb{R}^n$, h has a natural extension to a bi-Lipschiz homeomorphism $\overline{h} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. Then we have

$$\overline{h(ST_\theta(E))}^{\mathbb{R}} = \overline{h}(\overline{ST_\theta(E)}^{\mathbb{R}}) \ \text{ for } \theta \in \Phi.$$

Therefore it follows from Lemmas 4.8 and 6.2 that

$$Vol(\overline{ST_\theta(F)}^{\mathbb{R}} \cap B_\epsilon(0)) \approx Vol(\overline{ST_\theta(E)}^{\mathbb{R}} \cap B_\epsilon(0)). \tag{6.4}$$

On the other hand, F and $LD(F)$ are ST-equivalent. By Proposition 6.6, there is $\theta_1 \in \Phi$ such that

$$Vol(\overline{ST_\theta(F)}^{\mathbb{R}} \cap B_\epsilon(0)) \approx Vol(\overline{ST_\theta(LD(F))}^{\mathbb{R}} \cap B_\epsilon(0)) \tag{6.5}$$

for any $\theta \in \Phi$ with $\theta \geqslant \theta_1$.

By (6.4) and (6.5), we have

$$1 \approx \frac{Vol(\overline{ST_\theta(F)}^{\mathbb{R}} \cap B_\epsilon(0))}{Vol(\overline{ST_\theta(LD(F))}^{\mathbb{R}} \cap B_\epsilon(0))} \approx \frac{Vol(\overline{ST_\theta(E)}^{\mathbb{R}} \cap B_\epsilon(0))}{Vol(\overline{ST_\theta(LD(F))}^{\mathbb{R}} \cap B_\epsilon(0))} \tag{6.6}$$

for any $\theta \in \Phi$ with $\theta \geqslant \theta_1$. Assume that $\dim LD(F) > \dim E$. Then, by Corollary 6.7, the RHS ratio in (6.6) tends to 0 as $\epsilon \to 0$, if θ is sufficiently big. This is a contradiction. Thus we have $\dim LD(F) \leqslant \dim E$. □

Let us show Proposition 6.1. The sets A and $h(A)$ are assumed definable. Therefore, by Lemma 5.7, $LD(h(A)) = LD(h(LD(A)))$ is definable, and it follows from Theorem 4.17 that $LD(A)$ is ST-equivalent to A. Then, by Proposition 4.10, $h(LD(A))$ is ST-equivalent to $h(A)$. In addition, it follows from the definability of $h(A)$, that $h(A)$ is ST-equivalent to $LD(h(A)) = LD((h(LD(A)))$. Since the ST-equivalence is an equivalence relation, $h(LD(A))$ is ST-equivalent to $LD(h(LD(A)))$. Thus the proposition follows from Lemma 6.8 with $E = LD(A)$ and $F = h(LD(A))$, since $\dim h(LD(A)) = \dim LD(A)$.

This completes the proof of our main theorem.

7. General real closed field case

In this section we formulate and prove our main theorem for an arbitrary real closed field. Let R denote a real closed field with an o-minimal structure and consider the topology on R given by the open intervals of R, analogous to that on $\mathbb{R}$. We have already proved our main theorem for an arbitrary Archimedean real closed field. An example of a non-Archimedean real closed field is the field of Puiseux series, where a *Puiseux series* is a power series of the form $\sum_{i=k}^{\infty} a_i t^{i/p}$ for $a_i \in \mathbb{R}$, $p > 0 \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $\sum_{i=\max(0,k)}^{\infty} a_i t^i$ is a formal (convergent) power series in one variable t. One reason for why we consider problems on a general real closed field R, is that some problems on $\mathbb{R}$ are solved by replacing $\mathbb{R}$ with R. Actually, the Hilbert 17th problem is a famous illustration of this.

In order to treat our main theorem for R, we need to modify the previous definitions. Let A and B always denote subsets of R^n. In the case of definable A and B, let $dist(A,B)$ denote the maximal number of $t \in R$ such that $t \leqslant \|a-b\|$ for any $a \in A$ and $b \in B$. Let $D(A)$ denote the subset of $S^{n-1} = \{x \in R^n \mid \|x\| = 1\}$ consisting of points a such that for any $\epsilon, \delta \in R$ with $\epsilon > 0$ and $\delta > 0$ there exists $x \in A - \{0\}$ with $\|x\| \leqslant \delta$ and $dist(a, \frac{x}{\|x\|}) < \epsilon$. In the Archimedean case, $D(A)$ coincides with that in Definition 2.2. It may be empty for general A, but if A is definable then it is definable and not empty. We define $LD(A)$ in the same way as before, which is definable if so is A. We define $\dim A$ to be the dimension of the topological space A as in the dimension theory (see [10]). There are no problems with this definition because $\dim A = \dim h(A)$ for a homeomorphism h of R^n, if $A \subset B$ then $\dim A \leqslant \dim B$, and if A is definable then $\dim A$ coincides with the largest integer k, such that there exists a definable imbedding of R^k into A.

In the sense of the above notions of distance, direction set and dimension, we have the following:

Theorem 7.1. — *Let A, $B \subset R^n$ be definable set-germs at $0 \in R^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(A)$, $h(B)$ are also definable. Then we have*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

Let us show Theorem 7.1. Define Φ as in the Archimedean case. Let $A \subset R^n$ be definable such that $0 \in \overline{A}$. For $\theta \in \Phi$, we define the *sea-tangle neighbourhood* of A by

$$ST_\theta(A) := \{x \in R^n \mid dist(x, A) \leqslant \theta(\|x\|)\|x\|\},$$

which is definable. *ST-equivalence* is also defined as before.

Using the notions of Φ, sea-tangle neighbourhood and condition (SSP) as above, Lemma 4.8, Proposition 4.10, Lemma 4.11, Corollary 4.13, Proposition 4.15, Proposition 4.16, Theorem 4.17, Corollary 4.18, Theorem 4.19, Lemma 5.6, Lemma 5.7 and Proposition 5.8 are all proved for R and definable $A, B, h(A)$ and $h(B)$, in the same way as before by replacing sequences with filters. There are no essential modifications until §5, because those arguments work in the family of definable sets and their images under bi-Lipschitz homeomorphisms. However we need to modify §6, in particular, the volume arguments.

Theorem 7.1 follows from Proposition 6.1 for R as in the Archimedean case. We will refer to Proposition 6.1 for R as to the *Generalised Proposition* 6.1. If the germ of A at $0 \in R^n$ in the statement of the Generalised Proposition 6.1 is of dimension n, then $\dim h(LD(A)) = \dim LD(A) = n$ and the Generalised Proposition 6.1 holds true. Therefore we consider only subsets of $\{(x_1, \cdots, x_n) = (x', x_n) \in R^n \mid \|x'\| \leqslant cx_n, x_n \geqslant 0\}$, $c > 0 \in R$, of dimension less than n. For each $\epsilon \geqslant 0 \in R$, set

$$A_\epsilon := \{x' \in R^{n-1} \mid (x', \epsilon) \in A\}.$$

Then we can change the previous definition of $ST_\theta(A)$ to

$$\{(x', x_n) \in R^n \mid x_n \geqslant 0, dist(x', A_{x_n}) \leqslant \theta(x_n)x_n\},$$

which is easier to calculate.

The problem is how to define the volume of A. We do not have any good definition of volume. However the following definition is sufficient for our purpose. Assume that A is bounded and definable.

In the following we will define $VolA$. For the simplest case of an open box $O = (a_1, b_1) \times \cdots \times (a_n, b_n)$ in R^n with $a_i \leqslant b_i \in R$, set $VolO = \cup_{m\in\mathbb{N}}(-\infty,\, (b_1 - a_1)\cdots(b_n - a_n)(1 - 1/m)]$. The reason why we do not define $VolO$ to be $(b_1 - a_1)\cdots(b_n - a_n)$ or $(-\infty,\, (b_1 - a_1)\cdots(b_n - a_n)]$ becomes clear. Next consider the case where A is the following set:

$$\{(x_1, ..., x_n) \in R^n \mid \\ a_1 < x_1 < b_1, a_2 < x_2 < \phi_2(x_1), ..., a_n < x_n < \phi_n(x_1, ..., x_{n-1})\}$$

for $a_1, \cdots, a_n, b_1 \in R$ and bounded definable C^0 functions ϕ_2 on $(a_1,\, b_1), \cdots, \phi_n$ on

$$\{(x_1, \cdots, x_{n-1}) \in R^{n-1} \mid \\ a_1 < x_1 < b_1, a_2 < x_2 < \phi_2(x_1), \cdots, a_{n-1} < x_{n-1} < \phi_{n-1}(x_1, \cdots, x_{n-2})\}.$$

We write $A = A_{a_1,...,a_n,b_1,\phi_2,...,\phi_n}$. For each $k = 1, 2, \cdots$, let Vol_kA denote the superior limit of $\sum_i VolO_i$ where $\{O_i\}$ runs over all k disjoint open boxes contained in A and for subsets X and Y of R $X+Y$ denotes the set $\{x+y \mid x \in X, y \in Y\}$. Note that $\{Vol_k A\}_{k=1,2,\dots}$ is increasing. Set $VolA := \cup_k Vol_k A$. Then $VolA$ is connected and includes $(-\infty, 0]$ and $VolA = \{t - t/m |\ t \in VolA,\ m \in \mathbb{N}\}$. Let $\tilde{R}$ denote the family of subsets of R with these three properties.

Consider the set A of the form

$$\{(x_1, ..., x_n) \in R^n \mid$$
$$a_1<x_1<b_1, \psi_2(x_1)<x_2<\phi_2(x_1), ..., \psi_n(x_1, ..., x_{n-1})<x_n<\phi_n(x_1, ..., x_{n-1})\}$$

for $a_1, b_1 \in R$ and bounded definable C^0 functions ψ_i and ϕ_i on

$$\{(x_1, ..., x_{i-1}) \in R^{i-1} \mid a_1<x_1<b_1, ..., \psi_{i-1}(x_1, ..., x_{i-2})<x_{i-1}<\phi_{i-1}(x_1, ..., x_{i-2})\}$$

with $\psi_i < \phi_i,\ i = 2, \cdots, n$. We call A *of cell form* and write

$$A = A_{a_1, \psi_2, \dots, b_1, \phi_2, \dots}.$$

If we define $VolA$ for A of cell form in the same way as above, then our arguments do not work. For example, let $\epsilon > 0 \in R$ be smaller than any positive real number and set $A = A_{0,x,1,\epsilon+x}$ in R^2. Then

$$VolA = \{a \in R \mid \exists k \in \mathbb{N}\ a \leqslant k\epsilon^2\}.$$

However we expect $VolA = \cup_{m\in\mathbb{N}}(-\infty,\ \epsilon(1-1/m)]$, both from the context of the proofs in §6 and from the following arguments. Here we introduce an artificial different definition of volume. Choose $a_2, \cdots, a_n \in R$ so small that $a_2 < \psi_2, \cdots, a_n < \psi_n$. Then

$$A = A_{a_1,\dots,a_n,b_1,\phi_2,\dots,\phi_n} - \cup\overline{A_{a_1,\dots,a_n,b_1,\rho_2,\dots,\rho_n}},$$

where $\{\rho_2, ..., \rho_n\}$ satisfy $\rho_i = \psi_i$ for one i and $\rho_j = \phi_j$ for the other j. For such distinct families $\{\rho_i\}$ and $\{\rho'_i\}$ we have

$$A_{a_1,\dots,a_n,b_1,\rho_2,\dots,\rho_n} \cap A_{a_1,\dots,a_n,b_1,\rho'_2,\dots,\rho'_n} = A_{a_1,\dots,a_n,b_1,\min(\rho_2,\rho'_2),\dots,\min(\rho_n,\rho'_n)}.$$

If $R = \mathbb{R}$ then

$$VolA = \sum p_{\xi_2,\dots,\xi_n} VolA_{a_1,\dots,a_n,b_1,\xi_2,\dots,\xi_n}$$

in the usual sense of volume for some $p_{\xi_2,\dots,\xi_n} = \pm$ where $\{\xi_2, \cdots, \xi_n\}$ are so that for all i, $\xi_i = \psi_i$ or $\xi_i = \phi_i$. Here $p_{\xi_2,\dots,\xi_n}$ do not depend on the special choice of ψ_i and ϕ_i. For a general R, set

$$VolA := \sum p_{\xi_2,\dots,\xi_n} VolA_{a_1,\dots,a_n,b_1,\xi_2,\dots,\xi_n},$$

where for $X \in \tilde{X}$, $-X$ denotes $\{t \in R |\ \forall s \in VolA\ s < -t\}$.

Note that for $X, Y \in \tilde{R}$, $X - Y$ $(\overset{\text{def}}{=} X + (-Y)) \in \tilde{R}$ if $X - Y \supset (-\infty, 0]$ by easy calculations and then $Vol_k A, VolA \in \tilde{R}$; $Vol_k A$ and $VolA$ do not depend on choice of $a_2, \dots, a_n$; $VolA_{0,x,1,\epsilon+x} = \cup_{m\in\mathbb{N}}(-\infty,\ \epsilon - \epsilon/m]$; if A is of cell form and is a disjoint union of a definable set of dimension less than n and finitely many definable sets A_i of cell form, then $VolA = \sum_i VolA_i$ for the following reason.

We need to show that $\sum_i VolA_i$ does not change after we sub-decompose $\{A_i\}$. For that it suffices to see that for $X, Y \in \tilde{R}$ with $X \subsetneq Y$, $(X - X) + Y = Y$, i.e., for each $t \in Y$ there exists $t_1 \in Y$, $s_1 \in X$ and $s_2 \in -X$ such that $t = t_1 + (s_1 + s_2)$. We can assume $t \notin X$ because $X \subsetneq Y$ and $(X - X) + Y \in \tilde{R}$. There are two cases to consider: $nX \supset (-\infty, t)$ for some $n \in \mathbb{N}$ or not.

The former case. By definition of $\tilde{R}$ there exist $t_2 \in Y$ and $n \in \mathbb{N}$ such that $t = t_2 - t_2/n$ and $nX \supset (-\infty, t)$. Set $\mathbb{Q}_1 = \{q \in \mathbb{Q} |\ qt_2 \in X\}$. Then we have $u \in \mathbb{R}$ such that $\mathbb{Q}_1 = \{q \in \mathbb{Q} |\ q < u\}$ or $\mathbb{Q}_1 = \{q \in \mathbb{Q} |\ q \leqslant u\}$. Choose $q_1 \in \mathbb{Q}_1$ and $q_2 \in \mathbb{Q} - \mathbb{Q}_1$ so that $q_2 - q_1 < 1/n$. Then

$$q_1t_2 \in X,\ q_2t_2 \notin X,\ \text{i.e.,}\ -q_2t_2 \in -X,$$
$$t = t - (q_1t_2 - q_2t_2) + (q_1t_2 - q_2t_2),$$
$$t - (q_1t_2 - q_2t_2) < t + t_2/n = t_2 \in Y.$$

Thus $t_1 = t - (q_1t_2 - q_2t_2)$, $s_1 = q_1t_2$ and $s_2 = -q_2t_2$ satisfy the requirements. The latter case is clear because for $t_2 \in Y$ and $n \in \mathbb{N}$ with $t = t_2 - t_2/n$, $(-\infty, t_2/n) \supsetneq X$, $-t_2/n \in -X$ and hence $t = t_2 + (0 - t_2/n)$ is the required description of t.

Let A be a general bounded definable set in R^n. Then, by the Cell Decomposition Theorem, A is a disjoint union of a definable set of dimension less than n and finitely many definable sets A_i of cell form. Set $VolA = \sum_i VolA_i$. Then $VolA$ is well-defined as an element of $\tilde{R}$ by the above note; $VolA \neq (\infty, 0]$ if and only if A has inner points in R^n; if B is bounded and definable and contains A, then $VolA \leqslant VolB$; for disjoint bounded definable sets A_1 and A_2 we have $Vol(A_1 \cup A_2) = VolA_1 + VolA_2$, where $VolA \leqslant VolB$ and $VolA < VolB$ mean $VolA \subset VolB$ and $VolA \subsetneq VolB$, respectively. However there exist bounded definable sets A_1 and A_2 such that $VolA_1 + VolA_2 = VolA_1$ and $VolA_2 \neq (-\infty, 0]$. Hence we cannot call $VolA$ a measure of A.

This is the definition of a sort of volume of bounded definable subsets of R^n.

Let A be a bounded definable subset of R^n. Consider a definable decomposition $A = \cup_i A_i$ such that $h(A) = \cup_i h(A_i)$ is a cell decomposition for the map $h : R^n \ni (x_1, \dots, x_n) \to (x_n, x_1, \dots, x_{n-1}) \in R^n$. Then for each $\epsilon \in R$ with $\epsilon \geqslant 0$, $A_\epsilon = \cup_i A_{i\epsilon}$ is a cell decomposition and $VolA_\epsilon = \cup_i VolA_{i\epsilon}$ is well-defined, here $VolA_{i\epsilon}$ is calculated by regarding $A_{i\epsilon}$ as a subset of R^{n-1}. Moreover, for another decomposition $A = \cup_{i'} A'_{i'}$ with the same properties, $\cup_i Vol A_{i\epsilon} = \cup_{i'} Vol A_{i'\epsilon}$. Hence we have a well-defined correspondence $[0, \infty) \ni \epsilon \to Vol A_\epsilon \in \tilde{R}$. Let Vol_A denote the correspondence. Thus we obtain a map from $[0, \infty)$ to $\tilde{R}$ and we regard Vol_A as a map $[0, \infty)$ to the power set $\mathfrak{B}(R)$. Let f and g be maps from $[0, \infty)$ to $\mathfrak{B}(R)$. If there are $K > 0$ and $\delta > 0$ in R such that

$$f(\epsilon) \subset Kg(\epsilon) \subset K^2 f(\epsilon) \quad \text{for} \quad \epsilon \in [0, \delta],$$

we write $f \approx g$. By $\lim_{\epsilon\to 0} \frac{g(\epsilon)}{f(\epsilon)} = 0$, we mean that for any $\epsilon \in R$ with $\epsilon > 0$ there exists $\delta \in R$ such that $\delta > 0$ and $g(\epsilon_1) \subset \epsilon f(\epsilon_1) \neq \{0\}$ for any $\epsilon_1 \in (0, \delta]$.

For subsets X and Y of R^n, $Vol_X \approx Vol_Y$ means that there are definable sets A_1, A_2, B_1 and B_2 in R^n such that

$$A_1 \subset X \subset A_2,\ B_1 \subset Y \subset B_2 \text{ and } Vol_{A_1} \approx Vol_{A_2} \approx Vol_{B_1} \approx Vol_{B_2}.$$

In the same way we define $\lim_{\epsilon\to 0} \frac{Vol_X(\epsilon)}{Vol_Y(\epsilon)} = 0$.

Because of Lemma 6.2, we may expect $Vol_{ST_{c\theta}(X)} \approx Vol_{ST_\theta(X)}$, for $X \subset R^n$, $\theta \in \Phi$ and $c > 0 \in R$. However we do not know whether this is the case. We first prove:

LEMMA 7.2. — *Let A be a definable cone at $0 \in R^n$. Then*

$$Vol_{ST_{c\theta}(A)} \approx Vol_{ST_\theta(A)}$$

for $\theta \in \Phi$ and $c > 0 \in R$.

Proof. — Assume $c > 1$, and let A be the cone with base $C \times \{1\} \subset R^{n-1} \times R$, where we assume that C is closed. Since C admits a finite stratification into definable C^1 manifolds, we only need to prove the lemma for the cones with vertex 0 and base each of the strata. In addition, as in §II.1 in [17], we can choose the stratification so that for each stratum C_1 of dimension k, there exist $n_1 < \cdots < n_k$ in $\{1, \cdots, n\}$ (assume, for the simplicity of notation, that $n_i = i$, $i = 1, \cdots, k$), such that the restriction to $\overline{C_1}$ (not only to C_1) of the projection $p : R^{n-1} \ni (x_1, \cdots, x_{n-1}) \to (x_1, \cdots, x_k) \in R^k$ is injective, i.e., $\overline{C_1}$ is the graph of some C^0 map $\alpha = (\alpha_1, \cdots, \alpha_{n-1-k}) : p(\overline{C_1}) \to R^{n-1-k}$, $\alpha|_{p(C_1)}$ is of class C^1 and $\|\mathrm{grad}\, \alpha_i|_{p(C_1)}\| \leqslant 1$ for $i = 1, \cdots, n-1-k$.

What we want to prove is that there exists $K > 0 \in R$ such that

$$Vol\{x' \in R^{n-1} \mid dist(x', C_1) \leqslant cc'\} \leqslant K Vol\{x' \in R^{n-1} \mid dist(x', C_1) \leqslant c'\}$$

for small $c' \geqslant 0 \in R$. Proceeding by induction on k, we can reduce the problem to

$$\begin{aligned} &Vol\{x' \in p^{-1}(p(C_1)) \mid dist(x', C_1) \leqslant cc'\} \\ &\leqslant K Vol\{x' \in p^{-1}(p(C_1)) \mid dist(x', C_1) \leqslant c'\}. \end{aligned} \tag{7.1}$$

By the definition of volume

$$\begin{aligned}&Vol\{x' \in p^{-1}(p(C_1)) \mid dist(x', C_1) \leqslant cc'\}\\ &\leqslant Vol\{x' \in p^{-1}(p(C_1)) \mid dist(x', p(C_1) \times \{0\}^{n-1-k}) \leqslant cc'\}.\end{aligned}$$

On the other hand, since $\|\text{grad } \alpha_i|_{p(C_1)}\| \leqslant 1$,

$$\begin{aligned}&Vol\{x' \in p^{-1}(p(C_1)) \mid dist(x', p(C_1) \times \{0\}^{n-1-k}) \leqslant c'\}\\ &\leqslant 2^{n-1-k} Vol\{x' \in p^{-1}(p(C_1)) \mid dist(x', C_1) \leqslant c'\}\end{aligned}$$

up to multiplication by natural number. Thus we can replace C_1 in (7.1) with $p(C_1) \times \{0\}^{n-1-k}$. Clearly (7.1) for $p(C_1) \times \{0\}^{n-1-k}$ holds true for $K = c^{n-1-k}$. □

We generalise the above lemma as follows.

Lemma 7.2′. — *Let A be a definable set-germ at $0 \in R^n$. Then there exists $\theta_1 \in \Phi$ such that*

$$Vol_{ST_{c\theta}(A)} \approx Vol_{ST_\theta(A)}$$

for $\theta \in \Phi$ and $c > 0 \in R$ with $\theta \geqslant \theta_1$.

Proof. — By Theorem 4.17 A is ST-equivalent to $LD(A)$. Therefore there exist $\theta_2, \theta_3 \in \Phi$ such that

$$LD(A) \subset ST_{\theta_2}(A) \text{ and } A \subset ST_{\theta_3}(LD(A)).$$

Set $\theta_1 = 2\max(\theta_2, \theta_3)$. Then, as in the proof of Proposition 6.6, we have

$$ST_{\frac{\theta}{2}}(LD(A)) \subset ST_\theta(A) \subset ST_{2\theta}(LD(A))$$

for $\theta \in \Phi$ with $\theta \geqslant \theta_1$. Since $LD(A)$ is a definable cone with vertex $0 \in R^n$, by Lemma 7.2

$$Vol_{ST_{\frac{\theta}{2}}(LD(A))} \approx Vol_{ST_{\frac{c\theta}{2}}(LD(A))} \approx Vol_{ST_{2c\theta}(LD(A))} \approx Vol_{ST_{2\theta}(LD(A))}.$$

Hence $$Vol_{ST_{c\theta}(A)} \approx Vol_{ST_\theta(A)}.$$ □

By the above arguments the following lemma is clear; it corresponds to Lemma 6.3.

Lemma 7.3. — *Let α, β be linear subspaces of R^{n-1} with $\dim \alpha < \dim \beta$. Let α_1 and β_1 denote the cones in R^n with vertex $0 \in R^n$ and bases $\{x' \in \alpha \mid \|x'\| \leqslant 1\} \times \{1\}$ and $\{x' \in \beta \mid \|x'\| \leqslant 1\} \times \{1\}$, respectively. Then, for $\theta \in \Phi$,*

$$\lim_{\epsilon \to 0} \frac{Vol_{ST_\theta(\alpha_1)}(\epsilon)}{Vol_{ST_\theta(\beta_1)}(\epsilon)} = 0.$$

Using Lemmas 7.2 and 7.3, we can show the following proposition in the same way as in the proof of Proposition 6.4.

Proposition 7.4. — *Let $\alpha, \beta \subset R^n$ be definable cones at $0 \in R^n$. Suppose that $\dim \alpha < \dim \beta$. Then, for $\theta \in \Phi$,*

$$\lim_{\epsilon \to 0} \frac{Vol_{ST_\theta(\alpha)}(\epsilon)}{Vol_{ST_\theta(\beta)}(\epsilon)} = 0.$$

The following lemma is clear.

Lemma 7.5. — *Let $A \subset R^n$ be a definable set-germ at $0 \in R^n$. Then we have $\dim LD(A) \leqslant \dim A$.*

We do not know whether Proposition 6.6 holds for general R. However, under some assumption, using Lemma 7.2$'$ as in the proof of Proposition 6.6, we can prove the following.

Proposition 7.6. — *Let A, B be set-germs at $0 \in R^n$. Suppose that A and B are ST-equivalent and A is definable. Then there exists $\theta_1 \in \Phi$ such that*

$$Vol_{ST_\theta(A)} \approx Vol_{ST_\theta(B)}$$

for any $\theta \in \Phi$ with $\theta \geqslant \theta_1$.

The following corollary is also clear.

Corollary 7.7. — *Let $\alpha \subset R^n$ be a definable set-germ at $0 \in R^n$, and let $\beta \subset R^n$ be a definable cone at $0 \in R^n$. Suppose that $\dim \alpha < \dim \beta$. Then there is $\theta_1 \in \Phi$ such that*

$$\lim_{\epsilon \to 0} \frac{Vol_{ST_\theta(\alpha)}(\epsilon)}{Vol_{ST_\theta(\beta)}(\epsilon)} = 0$$

for any $\theta \in \Phi$ with $\theta \geqslant \theta_1$.

We need to modify Lemma 6.8 as follows.

Lemma 7.8. — *Let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism, and let A be a definable set-germ at $0 \in R^n$. Suppose that $h(A)$ is definable. Set $E = LD(A)$ and $F = h(E)$. Then $\dim LD(F) \leqslant \dim E$.*

Proof. — In the proof of Lemma 6.8, (6.4) was a consequence of Lemmas 4.8 and 6.2. For a general R, (6.4) corresponds to

$$Vol_{ST_\theta(F)} \approx Vol_{ST_\theta(E)} \quad \text{for} \quad \theta \in \Phi. \tag{7.2}$$

If F is definable, (7.2) follows from Lemmas 4.8 and 7.2$'$. However F is not necessarily definable or we do not know whether (7.2) holds for any θ.

We will find $\theta_1 \in \Phi$ such that (7.2) holds for $\theta \in \Phi$ with $\theta \geqslant \theta_1$. Such a restriction does not yield any trouble in our proof.

Since A and E are ST-equivalent, by Proposition 7.6 there exists $\theta_1 \in \Phi$ such that

$$Vol_{ST_\theta(A)} \approx Vol_{ST_\theta(E)}$$

for any $\theta \in \Phi$ with $\theta \geqslant \theta_1$. By the same reason as above and Lemma 5.6 we can assume that

$$Vol_{ST_\theta(h(A))} \approx Vol_{ST_\theta(F)}.$$

On the other hand, as in the proof of Lemma 6.8, by Lemmas 4.8 and 7.2′ we have

$$Vol_{ST_\theta(A)} \approx Vol_{ST_\theta(h(A))}.$$

Hence (7.2) holds for $\theta \geqslant \theta_1 \in \Phi$.

The other arguments in the proof of Lemma 6.8 continue to work. □

The Generalised Proposition 6.1 and then Theorem 7.1 are proved in the same way as in §6.

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Satoshi KOIKE
Department of Mathematics,
Hyogo University of Teacher Education,
Kato, Hyogo 673-1494,
Japan
koike@hyogo-u.ac.jp

Ta Lê LOI
Department of Mathematics,
University of Dalat,
Dalat,
Vietnam
loitl@dlu.edu.vn

Laurentiu PAUNESCU
School of Mathematics,
University of Sydney,
Sydney, NSW, 2006,
Australia
laurent@maths.usyd.edu.au

Masahiro SHIOTA
Graduate School of Mathematics,
Nagoya University,
Furo-cho, Chigusa-ku, Nagoya 464-8602,
Japan
shiota@math.nagoya-u.ac.jp