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REGENERATING HYPERBOLIC CONE 3-MANIFOLDS FROM DIMENSION 2

by Joan PORTI (*)

Abstract. — We prove that a closed 3-orbifold that fibers over a hyperbolic polygonal 2-orbifold admits a family of hyperbolic cone structures that are viewed as regenerations of the polygon, provided that the perimeter is minimal.

Résumé. — On prouve qu'une 3-orbifold close qui fibre sur une 2-orbifold hyperbolique et polygonale admet une famille de structures coniques hyperboliques qu'on voit comme une régénérescence du polygone, pourvu que son périmètre soit minimal.

1. Introduction

The space of hyperbolic cone 3-manifolds with fixed topological type and with cone angles less than π is well understood by [26], but the boundary of this space is not. The aim of this paper is to establish a regeneration result, that goes from a hyperbolic 2-orbifold, viewed as a collapsed 3-orbifold, to a family of hyperbolic cone 3-manifolds with decreasing cone angles, starting at π. Here, for instance, we determine which hyperbolic 2-orbifold in the Teichmüller space is obtained as limit of degenerating cone 3-manifolds, according to the speed of the cone angles.

Let $\mathcal{O}^3$ be a closed and orientable 3-orbifold, which is Seifert fibered over a Coxeter two orbifold P^2:

$$S^1 \to \mathcal{O}^3 \to P^2.$$

Keywords: orbifold, hyperbolic cone 3-manifold, degeneration, hyperbolic polygon, perimeter.
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The branching locus of $\mathcal{O}^3$ is a link or a trivalent graph $\Sigma_{\mathcal{O}^3}$. Its edges and circles are grouped in two, horizontal (if they are transverse to the fibers) or vertical (if they are fibers):

$$\Sigma_{\mathcal{O}^3} = \Sigma_{\mathcal{O}^3}^{hor} \cup \Sigma_{\mathcal{O}^3}^{vert}.$$

Points in $\Sigma_{\mathcal{O}^3}^{hor}$ project to the mirror and dihedral points of P^2. Assume that the orbifold fundamental group of P^2 is a *hyperbolic* Coxeter group, generated by reflections on a hyperbolic polygon whose angles are π over an integer. Thus P^2 is a polygon with mirror points at the edges, and dihedral points at the vertices. We may assume also that P^2 has possibly a single cone point in its interior. For instance, S^3 with branching locus a Montesinos link, other than a two-bridge link, is an example of such fibration.

We view the Seifert fibration as a transversely hyperbolic foliation, hence with a developing map

$$D_0 \colon \tilde{\mathcal{O}}^3 \to \mathbf{H}^2$$

that factors through the universal covering of P^2.

According to [23] there is a unique point in the Teichmüller space that minimizes the perimeter of P^2 (this also follows from Kerckhoff's proof of Nielsen conjecture [19]). Let

$$P^2_{min}$$

denote the orbifold equipped with this hyperbolic structure.

The main result of this paper is the following:

Theorem 1.1. — *Assume that P^2 has at most one cone point in its interior. There exists a family of hyperbolic cone manifold structures $C(\alpha)$ on $|\mathcal{O}^3|$, with singular locus $\Sigma_{\mathcal{O}^3}$ and cone angle $\alpha \in (\pi-\varepsilon, \pi)$ on $\Sigma_{\mathcal{O}^3}^{hor}$ and constant angles (the orbifold ones) on $\Sigma_{\mathcal{O}^3}^{vert}$, so that*

$$\lim_{\alpha \to \pi^-} C(\alpha) = P^2_{min}$$

for the Gromov-Hausdorff convergence. Moreover the developing maps converge to the developing map of the transversely hyperbolic foliation.

This result is generalized in two ways: by allowing vertical angles that are not integer divisors of 2π and by changing the speed of the horizontal angles. Regarding the generalization on the vertical angles, choose n I-fibers of $\mathcal{O}^3$,

$$\Sigma^{vert} = \{f_1, \ldots, f_n\}$$

that include all singular I-fibers. Let $q_1, \ldots, q_n \in \mathbf{N}$ denote their respective indices in the fibration. In particular $q_i = 1$ if and only if f_i is a regular

I-fiber. Fix angles $\vartheta_1, \dots, \vartheta_n \in (0, 2\pi]$ so that

$$\vartheta_i / q_i \leqslant \pi,$$

for $i = 1, \dots, n$. We require that

$$\sum_{i=1}^{n} (\pi - \vartheta_i/(2q_i)) > 2\pi.$$

This implies that the polygon Q with angles $\vartheta_i/(2q_i)$ is hyperbolic.

Regarding the speed of the horizontal angles, we introduce weights. Let k be the number of circles and edges of the horizontal branching locus Σ^{hor}. Choose weights $w_1, \dots, w_k \in \mathbf{R}_+ = \{x \in \mathbf{R} \mid x > 0\}$ on these horizontal components. Those induce weights

$$\mathcal{W} = \{\overline{w}_1, \dots, \overline{w}_n\}$$

on the edges of Q, just by adding the two weights of the mirror points of the fiber of any interior point of the edge, ie. $\overline{w}_i = w_{j_i} + w_{k_i}$. Of course if $k = 1$ (ie. when Σ^{hor} is a circle), then $\mathcal{W}$ is the constant weight. If $e_1, \dots, e_n$ are the edges of Q, with lengths $|e_1|, \dots, |e_n|$, the $\mathcal{W}$-perimeter is defined as

$$\overline{w}_1 |e_1| + \cdots + \overline{w}_n |e_n|.$$

It follows from the argument in [23] (cf. Proposition 1.5) that in the set of hyperbolic structures on a polygon with ordered angles $\vartheta_1/2q_1, \dots, \vartheta_n/2q_n$ there is a unique minimizer of the $\mathcal{W}$-perimeter, that we denote $Q_{\mathcal{W}-min}$.

Let $|\mathcal{O}^3|$ denote the underlying manifold of $\mathcal{O}^3$, $\Sigma^{hor} = \Sigma^{hor}_{\mathcal{O}}$ and $\Sigma^{vert} = \{f_1, \dots, f_n\}$.

Theorem 1.2. — *There exists a family of hyperbolic cone manifold structures $C(\alpha_1, \dots, \alpha_k)$ on $|\mathcal{O}^3|$, singular locus $\Sigma^{hor} \cup \Sigma^{vert}$, with cone angles $\alpha_i = \pi - w_i t$, for $i = 1, \dots, k$ and $t \in (0, \varepsilon)$ on Σ^{hor}, and constant angles ϑ_i on $f_i \subset \Sigma^{vert}$, such that*

$$\lim_{t \to 0^+} C(\alpha_1, \dots, \alpha_k) = Q_{\mathcal{W}-min}$$

for the Gromov-Hausdorff converge.

Theorem 1.1 was stated in Hodgson's thesis [17] in a more general context. In particular, Hodgson showed that the minimizer of the perimeter corresponds to a singularity in the variety of representations of $\mathcal{O}^3 \setminus \Sigma_{\mathcal{O}^3}$. However, further work is required to construct a path of representations, and this is done here in the orbifold context.

The 3-orbifold is Seifert fibered, and those cone manifolds appear in the proof of the orbifold theorem; however, none of the approaches shows the

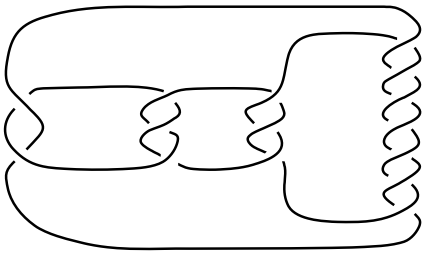

Figure 1.1. Example of fibered orbifold: S^3 with branching locus this link. The base is a quadrilateral.

explicit collapse, as the Seifert fibration is constructed by other methods [6, 4, 3, 2].

Theorem 1.1 needs to assume that there is at most one cone point in the interior of the polygon, otherwise $\mathcal{O}^3 \setminus \Sigma_{\mathcal{O}^3}$ would contain an essential torus, contradicting the existence of hyperbolic cone structures.

We make the following assumption along the paper:

Remark 1.3. — We may assume that P^2 has no interior cone point: If the interior of P^2 has one cone point, then we consider the orbifold covering that unfolds this point and work equivariantly.

Concerning the equivariance, the structure of P^2 that minimizes the perimeter is unique, hence equivariant. The structure of the hyperbolic cone manifolds is also equivariant because it is unique, by Weiss' global rigidity [27]. Notice that two or more cone points would not produce a polygon when unfolding, but an orbifold with more complicated underlying space.

Let us show a consequence of Theorem 1.2 for hyperbolic polyhedra. Fix n positive real numbers

$$0 < \beta_1, \ldots, \beta_n \leqslant \pi/2,$$

satisfying $\sum(\pi - \beta_i) > \pi$. By Andreev theorem, for any choice of $\alpha_1, \ldots, \alpha_n$, $\alpha'_1, \ldots, \alpha'_n$ satisfying

$$0 < \alpha_i, \alpha'_i < \pi/2, \qquad \alpha_i + \alpha_{i+1} > \pi - \beta_i, \quad \text{and} \quad \alpha'_i + \alpha'_{i+1} > \pi - \beta_i,$$

there exists a unique hyperbolic polyhedron with the combinatorial type of a prism with an n-edged polygonal base, with dihedral angles at the "vertical" edges $\beta_1, \cdots, \beta_n$, angles $\alpha_1, \ldots, \alpha_n$ at the respective n "horizontal" edges of the top face, and $\alpha'_1, \ldots, \alpha'_n$ at the respective n horizontal edges of the bottom face. They are arranged so that the edges with angles α_i, β_i, α'_i and β_{i+1} bound a quadrilateral face. See Figure 1.2.

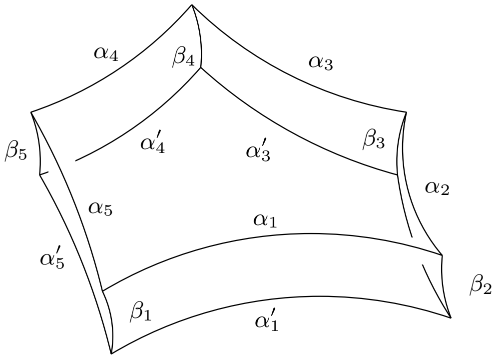

Figure 1.2. A hyperbolic prism as in Corollary 1.4.

Now if we choose weights $w_1, \ldots, w_n > 0$ for the bottom horizontal edges and $w_1', \ldots, w_n' > 0$ for the top ones, assume that

$$\begin{aligned} \alpha_i &= \pi/2 - w_i t, \\ \alpha_i' &= \pi/2 - w_i' t. \end{aligned}$$

Keep $\beta_1, \ldots, \beta_n$ fixed and let $t \searrow 0$ (hence $\alpha_i, \alpha_i' \nearrow \pi/2$). Set

$$\mathcal{W} = \{w_1 + w_1', \ldots, w_n + w_n'\}$$

and recall that the $\mathcal{W}$-perimeter of a polygon is the addition of its edge lengths multiplied by the weights.

Corollary 1.4. — *When $t \searrow 0$, the prism converges to the n-edged polygon with angles $\beta_1, \ldots, \beta_n$ of minimal $\mathcal{W}$-perimeter, where $\mathcal{W} = \{w_1 + w_1', \ldots, w_n + w_n'\}$.*

Theorem 1.2 and Corollary 1.4, require the following proposition:

Proposition 1.5. — *For $\mathcal{W} = \{w_1, \ldots, w_n\}$, with $w_i > 0$, the $\mathcal{W}$-perimeter has a unique minimum among all polygons with given angles $0 < \vartheta_1, \ldots, \vartheta_n \leqslant \pi/2$ with $\sum(\pi - \vartheta_i) > 2\pi$.*

In addition, this is the only polygon with those angles having a point p in its interior so that $\frac{1}{w_i} \sinh(d(e_i, p))$ is independent of the edges e_i of the polygon. In particular the (unweighted) perimeter is minimized by the polygon with an inscribed circle.

Proposition 1.5 is proved in [23] when all weights are constant. The same proof applies here, noticing that since angles are $\leqslant \pi/2$, the space of hyperbolic polygons with those given angles has no boundary.

The proof of Theorem 1.1 has two parts: first to construct a curve of representations of the smooth part of $\mathcal{O}^3$ and second to prove that these representations are holonomy structures of the cone manifolds by constructing developing maps.

For the construction of the curve, we have to choose the structure that minimizes the perimeter. Using Goldman's symplectic structure of the variety of representations of $\partial\mathcal{N}(\Sigma_{\mathcal{O}})$ [12], the Hamiltonian vector field of the perimeter is essentially the direction to regenerate in the variety of representations of $\partial\mathcal{N}(\Sigma_{\mathcal{O}})$. Being a critical point for the perimeter implies that this direction is induced from deformations of $\mathcal{O}^3 \setminus \mathcal{N}(\Sigma_{\mathcal{O}})$.

The construction of the curve of representations is quite involved, because, in general, the Teichmüller space of the base P^2 and the cone manifold structures may lie in different irreducible components of the variety of characters, and their intersection gives a singularity.

Once we have the existence of the curve of representations, we construct developing maps. For this we use the fibration: vertices of the base correspond to rational tangles, edges to I-fibered strips, and the interior points to regular fibers. We construct the developing map first for the tangles, then for the strips that connect them, and finally for the regular points. In particular the union of tangles and neighborhoods of the strips is a solid torus, and the underlying space of the orbifold is a generalized lens space. Previous to this construction, we must analyze the infinitesimal deformation of the fiber, and the corresponding Killing vector field, which happens to be perpendicular to the developing map of the two dimensional polygon.

Notice that when the orbifold is small the analysis of Paiva-Barreto [1] applies here, to prove that the limit of cone manifolds is the 2-dimensional orbifold. We must also remark the recent work of Danciger about the transition geometry called half-pipe [9]. With his work, one can understand the transition to anti de Sitter singular structures.

The proof of Theorem 1.2 follows exactly the same scheme as Theorem 1.1, just by adding the weights, and by adapting some arguments from orbifolds to cone manifolds. To simplify, we discuss first Theorem 1.1, while the existence of representations for Theorem 1.2 is proved in Section 3.3.

The paper is organized as follows. In Section 2 we state the existence of the required one parameter deformation of representations, and we give some preliminary material to prove it in Section 3. The developing maps corresponding to these representations are constructed in Section 4. Finally, Appendix A is devoted to some results about infinitesimal isometries, used mainly in Subsection 4.2.

2. Varieties of representations

We start with the holonomy representation of the hyperbolic orbifold

$$\mathrm{hol}\colon \pi_1(P^2) \to PGL_2(\mathbf{R}) = \mathrm{Isom}(\mathbf{H}^2),$$

where the elements preserve or reverse the orientation of $\mathbf{H}^2$ according to the sign of the determinant. Notice that

$$PGL_2(\mathbf{R}) = PSL_2(\mathbf{R}) \sqcup PSL_2(\mathbf{iR}) < PSL_2(\mathbf{C}) = \mathrm{Isom}^+(\mathbf{H}^3).$$

Let

$$M = |\mathcal{O}^3| \setminus \mathcal{N}(\Sigma_{\mathcal{O}^3})$$

denote the smooth part of the orbifold. By [7, 15] the induced representation on M can be lift to

$$\rho_0 : \pi_1(M) \to SL_2(\mathbf{C}).$$

The goal of Sections 2 and 3 is to prove the following result.

Proposition 2.1. — *If ρ_0 is the holonomy of the perimeter minimizing polygon, then there exists $\{\rho_t\}_{t\in(-\varepsilon,\varepsilon)}$ an analytic path of representations of M in $SL_2(\mathbf{C})$ such that ρ_0 is as above and for each $t \in (0,\varepsilon)$, ρ_t of a vertical meridian is constant, and ρ_t of a horizontal meridian is a rotation of angle (independent of the meridian)*

$$\pi - t^r,$$

for some $r \in \mathbf{Z}$, $r > 0$.

It is convenient to fix the orientation of the singular edges and their meridians to distinguish a rotation of angle $\pi - t^r$ from $\pi + t^r$, $t > 0$.

We prove this proposition in Section 3. It happens that $r = 1$: in some cases we shall prove it directly, in general this will follow from the regeneration result in Section 4. When we know that $r = 1$, the regeneration results of Danciger [9] apply, not only to the regeneration of hyperbolic cone manifolds but to the regeneration of AdS structures with tachyon singularities.

In the remainder of the section, we state some properties of the variety of representations and characters.

2.1. Preliminaries on representations and characters

For an orbifold or a manifold Z, the *variety of representations* of $\pi_1(Z)$ in $SL_2(\mathbf{C})$ is
$$R(Z) = \hom(\pi_1(Z), SL_2(\mathbf{C})).$$
This is a complex affine set of $SL_2(\mathbf{C})^n \subset \mathbf{C}^{4n}$ defined over $\mathbf{Q}$.

For a representation $\rho \in R(Z)$, its character is the map
$$\begin{array}{rcl} \chi_\rho : \pi_1(Z) & \to & \mathbf{C} \\ \gamma & \mapsto & \mathrm{Trace}(\rho(\gamma)). \end{array}$$
The *variety of characters* $X(Z)$ is the set of all characters of $R(Z)$, and it is also a complex affine set over $\mathbf{Q}$. The embedding in $\mathbf{C}^N$ is given by trace functions of N elements of $\pi_1(Z)$ [8, 15].

A representation $\rho \in R(Z)$ is called *irreducible* if no proper subspace of $\mathbf{C}^2$ is $\rho(\pi_1(Z))$-invariant. The representations we are considering are always irreducible. The set of irreducible representations is Zariski open, and so is the set of irreducible characters [8]. We denote them by $R^{irr}(Z)$ and $X^{irr}(Z)$ respectively.

Lemma 2.2 ([8]). — *The projection*
$$\begin{array}{rcl} R(Z) & \to & X(Z) \\ \rho & \mapsto & \chi_\rho \end{array}$$
is the quotient in geometric invariant theory of the action by conjugation. In particular it is surjective. Moreover $R^{irr}(Z) \to X^{irr}(Z)$ is a local fibration with fiber the orbit by conjugation.

Since orbifolds have torsion, sometimes we need to work with representations in $PSL_2(\mathbf{C})$, because they may not lift to $SL_2(\mathbf{C})$. This does not make any difference for the local structure of the variety of representations and characters at the representations we are interested in, cf. [16]. The varieties of $PSL_2(\mathbf{C})$-representations and characters are denoted by
$$R_{PSL_2(\mathbf{C})}(Z) \quad \text{and} \quad X_{PSL_2(\mathbf{C})}(Z).$$
By Weil's construction [25], the Zariski tangent space to $R(Z)$ at ρ is naturally identified with the space of cocycles (or crossed morphisms):
$$Z^1(\pi_1(Z), Ad\rho) = \{ d : \pi_1(Z) \to \mathfrak{sl}_2(\mathbf{C}) \mid d(\gamma_1\gamma_2) = d(\gamma_1) + Ad_{\rho(\gamma_1)} d(\gamma_2) \\ \forall \gamma_1, \gamma_2 \in \pi_1(Z) \},$$
so that $d \in Z^1(\pi_1(Z), Ad\rho)$ corresponds to the infinitesimal deformation
$$\gamma \mapsto (1 + \varepsilon d(\gamma))\rho(\gamma) + o(\varepsilon^2), \qquad \forall \gamma \in \mathbf{Z}.$$

Under this identification, the tangent space to the orbit by conjugation corresponds to the coboundaries (or inner crossed morphisms):

$$B^1(\pi_1(Z), Ad\rho) = \{d : \pi_1(Z) \to \mathfrak{sl}_2(\mathbf{C}) \mid \exists m \in \mathfrak{sl}_2(\mathbf{C}) \text{ s.t. } d(\gamma) = Ad_{\rho(\gamma)}(m) - m,\ \forall \gamma \in \pi_1(Z)\}.$$

Let $H^1(\pi_1(Z), Ad\rho) = Z^1/B^1$ denote the first cohomology group of $\pi_1(Z)$ with coefficients in the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ twisted by the adjoint representation $Ad\rho$.

Lemma 2.3 ([25, 20, 16]). — *If $\rho \in R(Z)$ is irreducible, then*

$$T^{Zar}_{\chi_\rho} X(Z) \cong H^1(\pi_1(Z), Ad\rho),$$

where T^{Zar} means the Zariski tangent space as a scheme (not necessarily reduced).

If $\rho \in R_{PSL_2(\mathbf{C})}(Z)$ does not preserve a subset of $\partial_\infty \mathbf{H}^3$ of cardinality $\leqslant 2$, then

$$T^{Zar}_{\chi_\rho} X_{PSL_2(\mathbf{C})}(Z) \cong H^1(\pi_1(Z), Ad\rho).$$

The hypothesis that ρ does not preserve a subset of $\partial_\infty \mathbf{H}^3$ of cardinality $\leqslant 2$ is equivalent to say that ρ is irreducible and that it does not preserve any unoriented geodesic. See [8, 15, 12, 16, 20] for more results about the varieties of representations and characters.

2.2. Relative character variety

Let Z be a compact aspherical 3-manifold with boundary, for instance the exterior of the singular locus $M = \mathcal{O} \setminus \mathcal{N}(\Sigma_{\mathcal{O}})$. One way to work with manifolds instead of orbifolds is to use relative character varieties of manifolds. This is convenient for working also with cone manifolds.

Definition 2.4. — *Let $\Gamma = \{\gamma_1, \dots, \gamma_k\} \subset \pi_1(Z)$ be a finite subset. The* relative character variety *with respect to the values $a_1, \dots, a_k \in \mathbf{C} \setminus \{\pm 2\}$ is*

$$X(Z, \Gamma) = \{\chi \in X(Z) \mid \chi(\gamma_i) = a_i \text{ for } \gamma_i \in \Gamma\}.$$

The role of the parameters $a_1, \dots, a_k \in \mathbf{C} \setminus \{\pm 2\}$ is not important, and they are not included in the notation. Usually, these values are clear from the context.

Lemma 2.5. — *Let $\chi = \chi_\rho \in X(Z)$ be an irreducible character such that $\chi(\gamma) \neq \pm 2$ for $\gamma \in \Gamma$.*

(1) *The Zariski tangent space to* $X(Z,\Gamma)$ *is:*

$$T^{Zar}_{\chi} X(Z,\Gamma) \cong \ker(H^1(Z;Ad\rho) \to \oplus_{\gamma\in\Gamma} H^1(\gamma, Ad\rho)) \\ \cong \mathrm{Im}(H^1(Z,\Gamma;Ad\rho) \to H^1(Z;Ad\rho)).$$

(2) *The Zariski cotangent space to* $X(Z,\Gamma)$ *is:*

$$(T^{Zar}_{\chi})^* X(Z,\Gamma) \cong H_1(Z;Ad\rho)/\,\mathrm{Im}(\oplus_{\gamma\in\Gamma} H_1(\gamma, Ad\rho)) \to H_1(Z;Ad\rho)) \\ \cong \mathrm{Im}(H_1(Z;Ad\rho) \to H_1(Z,\Gamma;Ad\rho)).$$

The lemma follows from Lemma 2.3 by analyzing tangent and cotangent induced maps of the morphism induced by inclusion,

$$X(Z) \to X(\gamma_1) \times \cdots \times X(\gamma_k),$$

and from the long exact sequence in cohomology of the pair (Z,Γ). See [18, §8.8] for more results on relative character varieties.

2.3. The symplectic structure of the variety of characters

As in the previous section, Z is a compact aspherical 3-manifold with boundary. Let $X(\partial Z)$ denote the product of character varieties of components of ∂Z:

$$X(\partial Z) = X(\partial_1 Z) \times \cdots \times X(\partial_r Z),$$

where $\partial Z = \partial_1 Z \cup \cdots \cup \partial_r Z$ is the splitting in connected components.

Choose a *pants decomposition* for the components of ∂Z. This is, a collection of disjoint simple closed curves in ∂Z,

$$\gamma_1, \ldots, \gamma_k,$$

that cut ∂Z into pairs of pants or cylinders, and the family has minimal cardinality. Here $k = -\frac{3}{2}\chi(\partial N) + k_0$, where k_0 is the number of components of ∂Z that are tori.

For $j = 1, \ldots, k$, let μ_j denote twice the logarithm of the eigenvalue of the j-th meridian γ_j, so that μ_j has real part the translation length and imaginary part the rotation angle of $\rho(\gamma_j)$. Let λ_j denote the twist parameter. Algebraically, when we cut along the (non-separating) meridian and write the fundamental group of the surface as an HNN-extension, λ_j is twice the logarithm of the eigenvalue of the element of the extension. (In the separating case, it is twice the logarithm of the eigenvalue of the conjugating factor). Notice that λ_j is only defined after normalization (Cf. [14] fore more details).

When γ_j is the meridian of a cone manifold, by [26], λ_j can be chosen so that its real part is the length of the corresponding singular edge. For representations of $M = \mathcal{O} \setminus \mathcal{N}(\Sigma_{\mathcal{O}})$ that factor through P^2, the real part of $\sum \lambda_j$ is twice the perimeter of P^2.

Proposition 2.6 (Fenchel-Nielsen local coordinates). — *Let $\chi \in X(\partial Z)$ be such that $\chi(\gamma_j) \neq \pm 2$, and χ restricted to each pant of $\partial Z \setminus \cup_i \gamma_i$ is irreducible. Then the parameters*

$$(\mu_1, \dots, \mu_k, \lambda_1, \dots, \lambda_k)$$

define local coordinates for $X(\partial Z)$ around χ.

Though it is well known, we give a proof in this algebraic setting for completeness.

Proof of Proposition 2.6. — Since the representation restricted to each pant is irreducible, it is locally parametrized by the trace of its boundary curves. Namely, an irreducible character in $SL_2(\mathbf{C})$ of a free group with two generators a and b is parametrized by the traces of a, b and ab (see for instance [15]), that are precisely the boundary curves of a pair of pants. We also use that, since $\chi(\gamma_j) \neq \pm 2$, the value of a character at γ_j is locally parametrized by μ_j, because $\chi(\gamma_j) = 2\cosh(\mu_j/2) \neq \pm 2$. When the curve γ_i is in a torus, the cutoff of this component is a cylinder, and its conjugacy class is parametrized by μ_j. Hence the $\mu_1, \dots, \mu_k$ are local coordinates for the restrictions to pants and cylinders. The coordinates are completed by adding the parameters of amalgamation along the curves γ_i, namely the $\lambda_1, \dots, \lambda_k$. □

Definition 2.7. — *The tangent vectors $\{\partial_{\mu_1}, \dots, \partial_{\mu_k}, \partial_{\lambda_1}, \dots, \partial_{\lambda_k}\}$ are the coordinate vectors of a parametrization as in Proposition 2.6.*

Notice that $\{\partial_{\mu_1}, \dots, \partial_{\mu_k}, \partial_{\lambda_1}, \dots, \partial_{\lambda_k}\}$ is a $\mathbf{C}$-basis for $H^1(\partial Z, Ad\rho)$.

Consider the pairing that consists in combining the usual cup product with the Killing form (see Appendix A):

$$B : \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}) \to \mathbf{C},$$

to get a 2-cocycle with values in $\mathbf{C}$, see [12, 13, 14]. We still denote by $\cup$ this paring:

$$\cup : H^1(\partial Z; Ad\rho) \times H^1(\partial Z; Ad\rho) \to H^2(\partial Z; \mathbf{C}) \to \mathbf{C}. \tag{2.1}$$

Here the last arrow is just the composition of the isomorphism $H^2(\partial_i Z, \mathbf{C}) \cong \mathbf{C}$ for each boundary component $\partial_i Z$ with the addition of the coordinates $\mathbf{C} \times \cdots \times \mathbf{C} \to \mathbf{C}$.

Theorem 2.8 (Goldman [12, 13]). — *The product (2.1) defines a* $\mathbf{C}$*-symplectic structure on* $X(\partial Z)$*. Moreover* ∂_{λ_j} *is the Hamiltonian vector field of* μ_j*:*

$$d\mu_j = \partial_{\lambda_j} \cup -.$$

Corollary 2.9. — *Let* F *be a holomorphic function of an open subset of* $X(\partial M)$*, and* $H_F \in T_\rho X(\partial Z)$ *its Hamiltonian, the vector that satisfies* $dF = H_F \cup -$*. Then*

$$d\mu_j(H_F) = -\frac{\partial F}{\partial \lambda_j}.$$

Proof. — By Theorem 2.8,

$$d\mu_j(H_F) = \partial_{\lambda_j} \cup H_F = -H_F \cup \partial_{\lambda_j} = -\frac{\partial F}{\partial \lambda_j}.$$

□

Theorem 2.10 (Duality Formula). — *Let* $\chi \in X(Z)$ *and* $\Gamma = \{\gamma_1, \dots, \gamma_k\} \subset \pi_1(\partial Z)$ *satisfy the hypothesis of Proposition 2.6 and let* $a_1, \dots, a_k \in \mathbf{C}$*. There exists a tangent vector* $v \in H^1(Z, Ad\rho)$ *such that*

$$d\mu_i(v) = a_i, \qquad \text{for } i = 1, \dots, k,$$

if and only if $a_1 d\lambda_1 + \dots + a_k d\lambda_k$ *vanishes in the cotangent space to* $X(Z, \Gamma)$*.*

Proof. — Assume first that there exists a tangent vector $v \in H^1(Z, Ad\rho) = T_\chi X(Z)$ such that $d\mu_i(v) = a_i$, for $i = 1, \dots, k$. Let $i : \partial Z \to Z$ denote the inclusion, and $i^* : T_\chi X(Z) \to T_\chi X(\partial Z)$ the induced map in cohomology. Let F be a function linear in μ_i and λ_i such that $i^*(v) = H_F$ at χ. Then by Corollary 2.9

$$a_j = d\mu_j(H_F) = -\frac{\partial F}{\partial \lambda_j}.$$

Hence

$$F = -a_1\lambda_1 - \dots - a_k\lambda_k + \sum b_i \mu_i,$$

for some $b_i \in \mathbf{C}$.

For every $w \in T_\chi X(Z)$,

$$dF \circ i^*(w) = H_F \cup i^*(w) = i^*(v) \cup i^*(w) = 0,$$

because the image of i^* is isotropic. Thus $i_*(dF) = 0$ in $T^*_{\chi_\rho} X(Z)$ and, by Lemma 2.5 (2), $a_1 d\lambda_1 + \dots + a_k d\lambda_k$ vanishes in the cotangent space

$$(T^{Zar}_{\chi_\rho})^* X(Z, \Gamma).$$

To prove the converse, start assuming that $a_1 d\lambda_1 + \dots + a_k d\lambda_k$ vanishes in the cotangent space to $X(Z, \Gamma)$. Thus $i_*(a_1 d\lambda_1 + \dots + a_k d\lambda_k) \in$

$\bigoplus_{i=1}^k H_1(\gamma_i; Ad\rho)$, by Lemma 2.5 (2). Hence there exist $b_1, \dots, b_k \in \mathbf{C}$ such that

$$i_*(a_1 d\lambda_1 + \cdots + a_k d\lambda_k) = i_*(-b_1\, d\mu_1 - \cdots - b_k\, d\mu_k) \in H_1(\partial Z; Ad\rho).$$

Setting $F = -(a_1\lambda_1 + \cdots + a_k\lambda_k + b_1\mu_1 + \cdots + b_k\mu_k)$ we have $i_*(dF) = 0$. Working in cohomology, consider the image of

$$i^* : H^1(Z; Ad\rho) \to H^1(\partial Z; Ad\rho)$$

which is a Lagrangian subspace of $H^1(\partial Z; Ad\rho)$, by a standard argument using Poincaré duality (cf. [17]). Moreover, since $i_*(dF) = 0$, dF vanishes on all infinitesimal deformations of ∂Z induced from deformations of Z: $dF(\mathrm{Im}(i^*)) = 0$. Let $H_F \in H^1(\partial Z; Ad\rho)$ be the "Hamiltonian vector of F":

$$H_F \cup - = dF.$$

In particular $(H_F)^\perp = \ker dF$ contains $\mathrm{Im}(i^*)$. Since $\mathrm{Im}(i^*)$ is Lagrangian for the symplectic pairing,

$$H_F \in \mathrm{Im}(i^*),$$

otherwise $\mathrm{Im}(i^*) \oplus \langle H_F \rangle$ would contradict the maximality of $\mathrm{Im}(i^*)$ among isotropic subspaces.

As $H_F \in \mathrm{Im}(i^*)$, there exists $v \in H^1(Z, Ad\rho)$ whose restriction to ∂Z is H_F, and therefore, by Corollary 2.9,

$$d\mu_i(v) = d\mu_i(H_F) = -\frac{\partial F}{\partial \lambda_i} = a_i,$$

for $i = 1, \dots, k$. □

3. Constructing the path of representations

Here we prove Proposition 2.1. We divide the proof in two cases. First we assume that all singular fibers are in the branching locus; in this case the variety of characters is smooth. The general case is deduced from this one by a deformation argument. In general the variety of characters may be singular, because the holonomies of cone manifold structures and the Teichmüller space of the polygon may lie in different irreducible components.

3.1. All singular fibers are in the branching locus

In this subsection we make the following assumption, that we will remove in Subsection 3.2:

Assumption 3.1. — *All singular I-fibers are in the branching locus of $\mathcal{O}^3$.*

Under this assumption, we have:

Remark 3.2. — The smooth part

$$H = \mathcal{O}^3 \setminus \mathcal{N}(\Sigma_{\mathcal{O}^3}),$$

is a handlebody of genus $n+1$.

Consider

$$\Gamma = \{\gamma_1, \dots, \gamma_{3n}\} \subset \pi_1(\partial H)$$

the (oriented) meridian curves for $\mathcal{O}^3$, one for each singular arc of $\Sigma_{\mathcal{O}^3}$. In particular they give a pants decomposition of ∂H. Order them so that:

- $\Gamma^{hor} = \{\gamma_1, \dots, \gamma_{2n}\}$ is the set of *horizontal* meridians (around $\Sigma^{hor}_{\mathcal{O}^3}$), and
- $\Gamma^{vert} = \{\gamma_{2n+1}, \dots \gamma_{3n}\}$ is the set of *vertical* meridians (around $\Sigma^{vert}_{\mathcal{O}^3}$).

Lemma 3.3. — *We have the following local isomorphisms:*

$$X^{irr}_{PSL_2(\mathbf{C})}(P^2) \cong_{LOC} X^{irr}_{PSL_2(\mathbf{C})}(\mathcal{O}^3) \cong_{LOC} X^{irr}_{PSL_2(\mathbf{C})}(H,\Gamma) \cong_{LOC} X^{irr}(H,\Gamma). \tag{3.1}$$

The first isomorphism follows from the fact that (a power of) the fiber of $\mathcal{O}^3$ is mapped to a central element, and an irreducible representation has no center. The second isomorphism can be proved easily from the definition of the relative character variety, by imposing that the meridians are rotations of the same order as the branching index. Finally, the third isomorphism follows from the fact that all representations of a free group to $PSL_2(\mathbf{C})$ lift to $SL_2(\mathbf{C})$.

Recall that

$$\chi_0 \in X_{PSL_2(\mathbf{C})}(\mathcal{O}^3)$$

is the $PSL_2(\mathbf{C})$-character induced by the perimeter minimizing hyperbolic metric of P^2.

We have an inclusion $X(H,\Gamma) \subset X(H,\Gamma^{vert})$. There exists a neighborhood of $U \subset X(H,\Gamma^{vert})$ of χ_0, so that if we define:

$$\mu = (\mu_1, \dots, \mu_{2n}) : U \subset X(H,\Gamma^{vert}) \to \mathbf{C}^{2n},$$

where the μ_i are in Subsection 2.3, then

$$X(H,\Gamma)\cap U=\mu^{-1}(\pi\mathbf{i},\dots,\pi\mathbf{i}).$$

Lemma 3.4. — (1) *The pair $(U,X(H,\Gamma)\cap U)$ is biholomorphic to a neighborhood of the origin in $(\mathbf{C}^{2n},\mathbf{C}^{n-3})$.*

(2) *The tangent map*

$$\mu_*:T_\chi U\to T_{(\pi\mathbf{i},\dots,\pi\mathbf{i})}\mathbf{C}^{2n}$$

is injective on the normal bundle to $X(H,\Gamma)\cap U$, $\forall\chi\in X(H,\Gamma)\cap U$.

Proof. — We first prove that U is biholomorphic to an open set of $\mathbf{C}^{2n}$. The 2-orbifold $P'=P^2\setminus vertices(P^2)$ obtained by removing the vertices of P^2 can be deformed by changing the angles of the vertices. This gives $\nu_1,\dots,\nu_n$ tangent vectors to the variety of characters of P', one for each cone angle (keeping the other angles fixed). Let $\bar\nu_i$ denote the induced vectors in the variety of characters of H. The trace functions of γ_{2n+i} satisfy $d\ \mathrm{Trace}_{\gamma_{2n+i}}(\bar\nu_j)=\delta_{ij}$. Thus, viewing $X(H,\Gamma^{vert})$ as the fiber of the (local) submersion of the traces of the vertical meridians

$$(\mathrm{Trace}_{\gamma_{2n+1}},\dots,\mathrm{Trace}_{\gamma_{3n}}):X(H)\to\mathbf{C}^n,$$

$X(H,\Gamma^{vert})$ is smooth at χ_0 and has dimension $2n$.

By Lemma 3.3, $X(H,\Gamma)\cap U$ is biholomorphic to an open subset of $X_{PSL_2(\mathbf{C})}(P^2)$, which is the complexification of the the Teichmüller space of P^2, hence it is biholomorphic to a open subset of $\mathbf{C}^{n-3}$, cf. [23]. To conclude the proof of Assertion 1, we may use the fact that the Teichmüller space of P^2 is locally parametrized by $n-3$ length functions: by considering the corresponding $\lambda_{i_1},\cdots,\lambda_{i_{n-3}}$ defined in Subsection 2.3, we deduce that the inclusion $X(H,\Gamma)\cap U\subset U$ is a holomorphic immersion.

The Teichmüller space of P^2 embeds in $\mathbf{R}^n$, with coordinates edge lengths, and it is a smooth submanifold of codimension 3, cf. [23]. Let $W_\chi\subset\mathbf{R}^n$ denote the normal space to the Teichmüller space at $\chi\in X(H,\Gamma)\cap U$ and $W_\chi\otimes\mathbf{C}\subset\mathbf{C}^n$ be its complexification. Assuming that γ_i and γ_{n+i} project to the same edge of P^2, $i=1,\dots,n$, let

$$W'_\chi=\{(a_1,\dots,a_{2n})\in\mathbf{C}^{2n}\mid(a_1+a_{n+1},\dots,a_n+a_{2n})\in W_\chi\otimes\mathbf{C}\}\cong\mathbf{C}^{n+3}.$$

Thus, if $(a_1,\dots,a_{2n})\in W'_\chi$, then $a_1d\lambda_1+\dots+a_{2n}d\lambda_{2n}$ vanishes on the cotangent space

$$(T^{Zar}_\chi)^*X_{PSL_2(\mathbf{C})}(P^2)\cong(T^{Zar}_\chi)^*X(H,\Gamma).$$

By the duality formula (Thm. 2.10), W'_χ is contained in the image of $\mu_*:T_\chi U\to T_{(\pi\mathbf{i},\dots,\pi\mathbf{i})}\mathbf{C}^{2n}$. Therefore $\mathrm{rank}(\mu_*)\geqslant n+3$. Since the dimension of $X^{irr}_{PSL_2(\mathbf{C})}(P^2) \cong_{LOC} X^{irr}(H,\Gamma)$ is $n-3$, the rank of $\mu_* : T_\chi U \to T_{(\pi\mathbf{i},\ldots,\pi\mathbf{i})}\mathbf{C}^{2n}$ is constant equal to $n+3$, showing Assertion 2. $\square$

Let $V \subset \mathbf{C}^{2n}$ be a neighborhood of $\mu(\chi_0) = (\pi\mathbf{i},\ldots,\pi\mathbf{i})$. Let $\check{U}$ be the blow-up of U at the submanifold $X(H,\Gamma)\cap U$, and $\check{V}$, the blow-up of V at the point $(\pi\mathbf{i},\ldots,\pi\mathbf{i}) \in V$. The respective exceptional divisors are denoted by $E_U \subset \check{U}$ and $E_V \subset \check{V}$.

Lemma 3.5. — *The map μ lifts to the blow-up, so that the following diagram commutes:*

$$\begin{tikzcd} (\check{U}, E_U) \arrow[r, "\check{\mu}"] \arrow[d, "pr_U"'] & (\check{V}, E_V) \arrow[d, "pr_V"] \\ (U, X(H,\Gamma)\cap U) \arrow[r, "\mu"] & (V, (\pi\mathbf{i},\ldots,\pi\mathbf{i})) \end{tikzcd}$$

Proof. — This is a consequence of Lemma 3.4 (2), that applies to every $\chi \in X(H,\Gamma)\cap U$, because the condition to lift is that μ_* is injective on the normal bundle. $\square$

Lemma 3.6. — *The character χ_0 is an isolated critical point in $X_{PSL_2(\mathbf{C})}(P^2)$ of the complex function $\lambda_1 + \cdots + \lambda_{2n}$. In addition, for any choice of local coordinates, the determinant of the Hessian at χ_0 does not vanish.*

Proof. — The results in [23] can be extended by using [24, Theorem B_{dS}] to prove that the perimeter has a positive definite Hessian at χ_0 in the deformation space of a polygon with given angles. Notice that since χ_0 is a critical point, the sign of the Hessian is independent of the coordinates. By complexifying, it follows that it is an isolated critical point in $X_{PSL_2(\mathbf{C})}(P^2)$. $\square$

By using the duality theorem (Thm. 2.10), and since χ_0 minimizes the perimeter of P^2, we can make the following definition:

Definition 3.7. — *We denote by $v_0 \in T_{\chi_0}X(H,\Gamma^{vert})$ a vector that satisfies*

$$d\mu_i(v_0) = \begin{cases} 1, & \text{for } i = 1,\ldots,2n \quad (\text{ie. } \gamma_i \in \Gamma^{hor}), \\ 0, & \text{for } i = 2n+1,\ldots,3n \quad (\text{ie. } \gamma_i \in \Gamma^{vert}). \end{cases}$$

Elements of the exceptional divisor $E_U \subset \check{U}$ are directions of vectors v normal to $X(H,\Gamma)$, denoted by $\langle v\rangle$.

Proposition 3.8. — *The map $\check{\mu}$ restricts to a biholomorphism between a neighborhood of $\langle v_0\rangle$ in $\check{U}$ and a neighborhood of $\langle(1,\ldots,1)\rangle$ in $\check{V}$.*

Proof. — We prove first that $\check{\mu}$ is a bijection between a neighborhood of $\langle v_0\rangle$ in the exceptional divisor E_U and a neighborhood of $\langle(1,\ldots,1)\rangle$ in E_V. Consider $\xi=(\xi_1,\ldots,\xi_{n-3})\in\mathbf{C}^{n-3}$ local coordinates for $X(H,\Gamma)\cap U$, write $a=(a_1,\ldots,a_{2n})$ and define, for $i=1,\ldots,n-3$, $F_i : X(H,\Gamma)\cap U\to\mathbf{C}$ as:

$$F_i(\xi,a)=\sum_{j=1}^{2n}a_j\frac{\partial\lambda_j}{\partial\xi_i}.$$

By Lemma 3.6, the determinant of the Hessian of $\lambda_1+\cdots+\lambda_{2n} : X(H,\Gamma)\cap U\to\mathbf{C}$ at χ_0 does not vanish. Hence we may apply the implicit function theorem to $F_1=\cdots=F_{n-3}=0$, to write ξ as a function of a. Thus for any $(a_1,\ldots,a_{2n})\in\mathbf{C}^{2n}$ in a neighborhood of $(1,\ldots,1)$, there exists a unique $\chi\in X(H,\Gamma)\cap U$ such that χ is a critical point of $a_1d\lambda_1+\cdots+a_{2n}d\lambda_{2n}$. In particular, by Theorem 2.10 there exists a unique $\chi\in X(H,\Gamma)\cap U$ such that its tangent space contains a vector $v\in T_\chi U$ with $\mu_*(v)=(a_1,\ldots,a_{2n})$. Moreover, this v is unique in the normal bundle, by Lemma 3.4 (2). This proves that $\check{\mu}$ is a bijection between neighborhoods in the exceptional divisors E_U and E_V. By holomorphicity, this implies that $\check{\mu}$ is a biholomorphism between the neighborhoods in E_U and E_V. By construction $\check{\mu}_*$ is injective in the normal direction to E_U ($(pr_U)_*$ is injective in the normal direction), hence the inverse function theorem applies. □

Corollary 3.9. — *There exists an algebraic* $\mathbf{C}$*-curve* $\mathcal{C}\subset X(H,\Gamma^{vert})$ *containing* χ_0*, such that* $\mu|_{\mathcal{C}}$ *is a biholomorphism between a neighborhood of* χ_0 *and a neighborhood of the diagonal* $\mu_1=\cdots=\mu_{2n}$.

Remark 3.10. — In this way we obtain a path of representations satisfying Proposition 2.1 with $r=1$, just by considering the path $\mu_1=\cdots=\mu_{2n}=\mathbf{i}(\pi-t)$ and lifting it to a deformation of representations ρ_t.

For every horizontal meridian γ_i, the deformation of Remark 3.10 satisfies, up to conjugation

$$\rho_t(\gamma_i)=\pm\begin{pmatrix}e^{\mu_i(t)/2} & 0\\ 0 & e^{-\mu_i(t)/2}\end{pmatrix}. \tag{3.2}$$

Remark 3.11. — Replacing t by $-t$ in the previous choice changes the sign of the trace. This corresponds to changing the orientation because when we take the complex conjugate, the sign of the trace of $\rho_t(\gamma_i)$ in Equation (3.2) is changed, but also the sign of

$$\rho_0(\gamma_i)=\pm\begin{pmatrix}e^{\mathbf{i}\pi/2} & 0\\ 0 & e^{-\mathbf{i}\pi/2}\end{pmatrix},$$

hence the sign in the relation $tr(\rho_t(\gamma_i))=\pm2\cos(\alpha_i(t)/2)$.

3.2. Constructing a curve of representations of M

The goal of this subsection is to prove Proposition 2.1 without Assumption 3.1. The argument of Subsection 3.1 does not apply here, as the involved character varieties may be singular. Instead, we use a deformation argument and apply the previous subsection.

Proof of Proposition 2.1. — The case where all singular fibers are contained in $\Sigma_{\mathcal{O}}$ is discussed in the previous section. To simplify, we assume that $\Sigma_{\mathcal{O}}$ is a link, so that no I-fiber is in the branching locus (ie. $\Sigma_{\mathcal{O}}^{vert} = \emptyset$). We shall reduce to the previous case by deforming the angles on the singular I-fibers. In the general case we should only deform some of the I-fibers.

Choose a increasing sequence $t_k \in (0,1)$, $t_k \nearrow 1$, and consider the degenerate cone manifold $C(t_k)$ whose singular locus is the union of $\Sigma_{\mathcal{O}}$ and the singular I-fibers. The cone angle in the horizontal singular locus is π and the cone angle in the vertical singular locus is $2\pi t_k$. We view it as a collapsed hyperbolic structure, namely as a hyperbolic polygon with angles t_k times the original angles of the polygon. The character of the hyperbolic structure with those angles that minimizes the perimeter is denoted by χ_k.

Set $H = C(t_k) \setminus \Sigma_{C(t_k)}$. Recall that $\Gamma^{vert} = \{\gamma_{2n+1}, \dots, \gamma_{3n}\} \subset \pi_1(H)$ denotes the set of vertical meridians, ie. meridians of the singular components of $C(t_k)$ corresponding to singular I-fibers, and $\Gamma^{hor} = \{\gamma_1, \dots \gamma_{2n}\} \subset \pi_1(H)$ denote the set of horizontal ones. In particular

$$\pi_1(M) \cong \pi_1(H) \Big/ \langle \Gamma^{vert} \rangle,$$

where $\langle \Gamma^{vert} \rangle$ denote the subgroup normally span by Γ^{vert}.

By Proposition 1.5, $\chi_k \to \chi_0 = \chi_{\rho_0}$. The character χ_k satisfies $\chi_k(\gamma_i) = 0$ for $i = 1, \dots, 2n$ (ie. $\gamma_i \in \Gamma^{hor}$). The characters of the curve of Corollary 3.9 with all cone angles equal satisfy $\chi(\gamma_i) = \pm\chi(\gamma_j)$, and the sign depends on the lift of the holonomy of $C(t_k)$ to $SL_2(\mathbf{C})$. Namely, a rotation of angle π that fixes the oriented axis in the upper half space model for $\mathbf{H}^3$ that goes from 0 to ∞ is

$$\pm \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} = \pm \begin{pmatrix} e^{\mathbf{i}\pi/2} & 0 \\ 0 & e^{-\mathbf{i}\pi/2} \end{pmatrix}.$$

Thus decreasing the angle π affects differently the sign of the trace: since we work with half angle, it depends on whether we start with $\pi/2$ or $3\pi/2$. We will assume that, after having extracted a subsequence, for every k, the deformed representations satisfy $\chi(\gamma_i) = \chi(\gamma_j)$, ie. we are able to make the same choice of lift for for all k. Otherwise, some equalities $\chi(\gamma_i) = \chi(\gamma_j)$ have to be replaced by $\chi(\gamma_i) = -\chi(\gamma_j)$.

Consider the algebraic subset of $X(H)$ defined by the equations:

$$\begin{cases} \chi(\gamma_i) = \chi(\gamma_j) & \text{for } i, j = 1, \dots 2n, \text{ (ie. } \gamma_i, \gamma_j \in \Gamma^{hor}); \\ \chi(\gamma_{2n+i}) = \chi(\gamma_{2n+j}) & \text{for } i, j = 1, \dots n, \text{ (ie. } \gamma_{2n+i}, \gamma_{2n+j} \in \Gamma^{vert}). \end{cases}$$

By construction, this algebraic subset contains χ_k and the deformations of characters of Corollary 3.9. Again up to subsequence, we may assume that it has an irreducible component $\mathcal{S}$ that contains all of them.

Lemma 3.12. — *The variety $\mathcal{S}$ satisfies:*

i) $\dim_{\mathbf{C}} \mathcal{S} \geqslant 2$.
ii) $\chi_0 = \chi_{\rho_0} \in \mathcal{S}$.
iii) $\{\chi \in \mathcal{S} \mid \chi(\gamma_{2n+1}) = \pm 2\}$ *is a proper subset of* $\mathcal{S}$.
iv) $\{\chi \in \mathcal{S} \mid \chi(\gamma_1) = 0\}$ *is a proper subset of* $\mathcal{S}$.

Proof. — Assertion i) follows from the fact that $\mathcal{S}$ is a component of an algebraic subset of $X(H)$ defined by $3n-2$ equations. Since $\chi_k \to \chi_0 = \chi_{\rho_0}$, and $\chi_k \in \mathcal{S}$, ii) is clear. To prove Assertion iii), use that $\chi_k \in \mathcal{S}$ and $\chi_k(\gamma_{2n+1}) = \pm 2\cos(2\pi t_k)$. Finally iv) is a consequence of the fact that $\mathcal{S}$ contains the deformations provided by Corollary 3.9. ☐

Remark 3.13. — Once we know that the deformations correspond to hyperbolic cone structures, it will follow from local rigidity results for cone manifolds, that $\dim_{\mathbf{C}} \mathcal{S} = 2$, by [28].

Consider $\mathcal{E}$ a $\mathbf{C}$-irreducible component of

$$\{\chi \in \mathcal{S} \mid \chi(\gamma_{2n+1}) = 2\}$$

that contains $\chi_0 = \chi_{\rho_0}$.

By Lemma 3.12, $\mathcal{E}$ is not the whole $\mathcal{S}$ and $\dim \mathcal{E} \geqslant 1$. Notice that $\chi(\gamma_{2n+1}) = 2$ does not imply that χ is the character of a representation ρ trivial on γ_{2n+1}, because $\rho(\gamma_{2n+1})$ could be a parabolic element. Thus we need the following lemma.

Lemma 3.14. — *If $\rho \in R(H)$ is a representation close to ρ_0 and $\chi_\rho \in \mathcal{E}$, then $\rho(\gamma_{2n+i})$ is the identity matrix for all $i = 1, \dots, n$. In particular it factors to a representation of M.*

Proof. — Let γ_{2n+1} be a vertical meridian. Each endpoint of this edge meets the endpoints of two more horizontal singular edges, with respective meridians ς and $\tilde{\varsigma}$ in $\pi_1(H)$. They satisfy

$$\varsigma \gamma_{2n+1} = \tilde{\varsigma}$$

and ς and $\tilde{\varsigma}$ project both to the same element σ_1 in $\pi_1(M)$ (cf. Figure 3.1). Since ρ is close to ρ_0 and $\rho_0(\sigma_1)$ is a rotation of angle π, we may assume

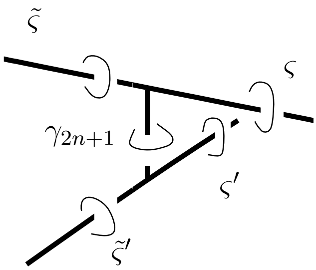

Figure 3.1. The loops of $\pi_1(H)$ around a vertical edge

that $\rho(\varsigma)$ and $\rho(\tilde\varsigma)$ are both conjugate to diagonal matrices with (equal) eigenvalues $\lambda^{\pm 1} \neq \pm 1$. We write

$$\rho(\varsigma) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } \rho(\gamma_{2n+1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $ad-bc=1$, $a+d=2$. Since $\chi_\rho \in \mathcal{S}$, $a\lambda + d\lambda^{-1} = \lambda+\lambda^{-1} \neq \pm 2$. Thus $a=d=1$ and either b or c vanishes. This means that if $\rho(\gamma_{2n+1})$ is not the identity but parabolic, then the fixed point of $\rho(\gamma_{2n+1})$ has to be one of the endpoints of the axis of $\rho(\varsigma)$. Let σ_1' be the meridian of the opposite edge in the tangle, so that the tangle group is the free group on σ_1 and σ_1'. The axis of $\rho_0(\sigma_1)$ and $\rho_0(\sigma_1')$ form an angle, hence the endpoints of their axis are far, and the previous argument for σ_1' instead of σ_1 gives a contradiction with the hypothesis that $\rho(\gamma_{2n+1})$ is not the identity. $\square$

Lemma 3.15. — *The trace of the meridian γ_1 is not constant along $\mathcal{E}$.*

Proof. — By contradiction, assume that it is constant, then this trace is zero. Take a character $\bar\chi \in \mathcal{E}$ close to χ_0. Lemma 3.14 implies that $\bar\chi$ induces a character of M.

Claim 3.16. — *There exists a tangent vector $v \in T_{\bar\chi}X(M)$ that satisfies*

$$d\mu_j(v) = 1, \text{ for } j = 1, \dots, 2n.$$

We continue the proof of Lemma 3.15 assuming Claim 3.16, that we prove later. By Claim 3.16 and Theorem 2.10, $\bar\chi$ is a critical point of $\lambda_1+\cdots+\lambda_{2n}$ in

$$X^{irr}(M,\Gamma') \cong_{LOC} X^{irr}_{PSL_2(\mathbf{C})}(\mathcal{O}^3) \cong X^{irr}_{PSL_2(\mathbf{C})}(P^2),$$

where $\Gamma' \subset \pi_1(M)$ is a collection of meridians for the singular components of $\mathcal{O}^3$. This contradicts the analogue of Lemma 3.6, that the perimeter minimizer is an isolated critical point of $\lambda_1+\cdots+\lambda_{2n}$ in $X_{PSL_2(\mathbf{C})}(P^2)$. $\square$

End of the proof of Proposition 2.1. By Lemma 3.14, $\mathcal{E}$ gives a curve of representations of M. In addition, by Lemma 3.15, the trace of the meridian on this curve is nonconstant. A nonconstant complex map is open, thus by looking at the inverse image of points with real trace, we find the path of representations we are looking for. $\square$

Remark 3.17. — Once we have Proposition 4.25 below about regeneration of hyperbolic cone structures, the trace of the meridian on $\mathcal{E}$ is a local diffeomorphism around χ_0.

In fact, the trace of the meridian on $\mathcal{E}$ cannot be a ramified covering, because this would contradict global rigidity of hyperbolic cone manifolds. Namely, we only can have one inverse image of the real line, that gives two branches, corresponding to the two complex conjugate representations, one with trace $2\cos(\alpha)$Âů and the other one with trace $-2\cos(\alpha)$.

Proof of Claim 3.16. — Let $\overline{\mathcal{E}}$ denote the projection of $\mathcal{E}$ to $X_{PSL_2(\mathbf{C})}(M)$. Since we assume $\mathcal{E} \subset \{\chi \in \mathcal{S} \mid \chi(\gamma_1) = 0\}$, we have that $\overline{\mathcal{E}} \subset X_{PSL_2(\mathbf{C})}(P^2)$. By Lemma 3.12(iv), $\mathcal{S}$ is not contained in $\{\chi \in \mathcal{S} \mid \chi(\gamma_1) = 0\}$ and therefore the projection of $\mathcal{S}$ to $X_{PSL_2(\mathbf{C})}(H)$, $\overline{\mathcal{S}}$, is not contained in $X_{PSL_2(\mathbf{C})}(P')$, where $P' = P^2 \setminus \mathit{vertices}(P^2)$. In particular there exist a curve that deforms $\bar{\chi}$ in $\overline{\mathcal{S}}$ away from $X_{PSL_2(\mathbf{C})}(P')$. We lift this curve from the variety of characters to the variety of representations. We obtain in this way an analytic path of representations $\bar{\rho}_s$ in $\overline{\mathcal{S}}$, with $\bar{\rho}_0 = \bar{\rho}$ a representation whose character is $\chi_{\bar{\rho}} = \bar{\chi}$. Let $l \geqslant 0$ be maximal such that the power expansion

$$\bar{\rho}_s(\gamma) = \pm(1+sa_1(\gamma)+\cdots+s^l a_l(\gamma)+s^{l+1}a_{l+1}(\gamma)+\cdots)\bar{\rho}(\gamma), \qquad \forall\gamma \in \pi_1(H),$$

is a representation in $PSL_2(\mathbf{C}[s]/(s^{l+1}))$ that factors through $\pi_1(P')$, but as a representation in $PSL_2(\mathbf{C}[s]/(s^{l+2}))$ does not factor through $\pi_1(P')$.

Namely, up to conjugation we may assume that $a_i : \pi_1(H) \to M_2(\mathbf{C})$ factors through $\pi_1(P')$, for $i = 1, \ldots, l$, but a_{l+1} does not factor. Since the variety of representations of P' is smooth, there exists $b : \pi_1(H) \to M_2(\mathbf{C})$ such that

$$\gamma \mapsto (1 + sa_1(\gamma) + \cdots + s^l a_l(\gamma) + s^{l+1}b(\gamma))\bar{\rho}(\gamma)$$

is a representation of $\pi_1(P')$ in $PSL_2(\mathbf{C}[s]/(s^{l+2}))$. The compatibility relations to be a representation imply that

$$d := a_{l+1} - b$$

is a group cocycle of $\pi_1(H)$ taking values in the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$. In addition by maximality of l, $d = a_{l+1} - b$ is nontrivial on horizontal meridians, and it takes the same value on all of them, because by construction it is a cocycle tangent to the set of characters χ that are equal on horizontal meridians. Hence we obtain a cohomology element $v := [d] = [a_{l+1} - b] \in H^1(H; Ad\bar\rho)$ that satisfies

$$d\mu_i(v) = 1, \qquad \text{for } i = 1, \dots, 2n.$$

Next we want to show that d can be assumed to vanish on all vertical meridians γ_{2n+i}, $i = 1, \dots, n$, so that it induces an element of $H^1(M, Ad\bar\rho)$. We claim first that d can be assumed to satisfy

$$B(d(\gamma_{2n+i}), d(\gamma_{2n+i})) = 0, \quad \text{for } i = 1, \dots, n, \tag{3.3}$$

where $B : \mathfrak{sl}_2(\mathbf{C}) \times \mathfrak{sl}_2(\mathbf{C}) \to \mathbf{C}$ denotes the $\mathbf{C}$-Killing form (cf. Appendix A). We can add to d any infinitesimal deformation d_1 induced by $\pi_1(P')$, because d_1 vanishes on horizontal meridians. By deforming the curves of P' that bound an orbifold disk around the missing dihedral vertices, we may obtain d_1 satisfying

$$d_1(\gamma_{2n+i}) = x_i \mathfrak{a}_i, \tag{3.4}$$

for any $x_i \in \mathbf{C}$, where $\mathfrak{a}_i \in \mathfrak{sl}_2(\mathbf{C})$ is an infinitesimal rotation around the axis of $\mathbf{H}^3$ perpendicular to the edges of P^2 that meet at the i-th vertex. Notice that since $\rho_0(\gamma_{2n+i})$ is trivial, each coboundary evaluated at γ_{2n+i} vanishes, hence $d_1(\gamma_{2n+i})$ is necessarily as Equality (3.4). Since the Killing form B is $\mathbf{C}$-bilinear and $B(\mathfrak{a}_i, \mathfrak{a}_i) \neq 0$, x_i can be chosen so that, $B((d + d_1)(\gamma_{2n+i}), (d + d_1)(\gamma_{2n+i})) = 0$. Thus we replace d by $d + d_1$, so that (3.3) is satisfied.

Next we claim that $d(\gamma_{2n+i}) = 0$, for $i = 1, \dots, n$, by using an infinitesimal version of Lemma 3.14. Namely, let γ_{2n+i} be a vertical meridian and ς and $\tilde\varsigma$ in $\pi_1(H)$ horizontal meridians as in Lemma 3.14, in particular they satisfy $\varsigma\gamma_{2n+i} = \tilde\varsigma$. Since we assume $\bar\chi(\gamma_1) = 0$ and $\bar\chi(\gamma_{2n+i}) = 2$, by Lemma 3.14, up to conjugation

$$\bar\rho(\varsigma) = \bar\rho(\tilde\varsigma) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \bar\rho(\gamma_{2n+i}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Write

$$d(\varsigma) = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \quad \text{and} \quad d(\gamma_{2n+i}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

for some $x, y, z, a, b, c \in \mathbf{C}$. Using the cocycle relation

$$d(\varsigma) + Ad_{\bar\rho(\varsigma)} d(\gamma_{2n+i}) = d(\tilde\varsigma),$$

it follows that if D_d denotes the directional derivative in the direction of the infinitesimal deformation d, then

$$D_d \operatorname{Trace}_\varsigma = 2ix \quad \text{and} \quad D_d \operatorname{Trace}_{\tilde\varsigma} = 2i(x+a).$$

Since in $\mathcal{S}$ we impose $\operatorname{Trace}_\varsigma = \operatorname{Trace}_{\tilde\varsigma}$, it follows that $a = 0$. Moreover, the vanishing of the Killing form in Equality (3.3) implies $bc = 0$. Thus either $d(\gamma_{2n+i})$ vanishes, either it is an infinitesimal parabolic transformation whose fixed point at infinity is fixed by $\bar\rho(\varsigma)$. By the same reason as in Lemma 3.14, it follows that $d(\gamma_{2n+i}) = 0$. □

3.3. Weights

To prove Theorem 1.2, we need the following version of Proposition 2.1.

PROPOSITION 3.18. — *Under the hypothesis of Theorem 1.2, if ρ_0 is the holonomy of the $\mathcal{W}$-perimeter minimizing polygon, then there exists $\{\rho_t\}_{t\in(-\varepsilon,\varepsilon)}$ an analytic path of representations of M in $SL_2(\mathbf{C})$ such that ρ_0 is as above and for each $t \in (0,\varepsilon)$, ρ_t of a vertical meridian is constant, and ρ_t of a horizontal meridian is a rotation of angle*

$$\pi - w_i t^r + O(t^{r+1})$$

for some $r \in \mathbf{Z}$, $r > 0$, independent of the meridian.

The arguments of Subsection 3.1 work exactly the same, just by replacing the vector $(1,\dots,1)$ by $(w_1,\dots,w_n,w'_1,\dots,w'_n)$. In Subsection 3.2 one has to work with analytic sets instead of algebraic ones, but all results apply.

4. Developing maps

In this section we construct developing maps whose holonomy structures are the representations given by Proposition 2.1. Firstly, in Subsection 4.1, we consider the Seifert fibration structure of the orbifold, secondly in Subsection 4.2 we prove the existence of a Killing vector field corresponding to the infinitesimal deformation of the fiber. The developing maps are then constructed in Subsection 4.3.

4.1. The fibration of the orbifold

The orbifold $\mathcal{O}^3$ is Seifert fibered over P^2:

$$S^1 \to \mathcal{O}^3 \xrightarrow{p} P^2.$$

We distinguish three kinds of points of P^2: interior points of the underlying space $|P^2|$, interior points of the mirror edges, and vertices. Each interior point of P^2 has a neighborhood U such that $p^{-1}(U)$ is a fibered solid torus. By hypothesis, there is at most one cone point in the interior. Such a point has a neighborhood $U \subset P^2$, such that $p^{-1}(U)$ can have a singular core, a singular Seifert fibration, or both. By Remark 1.3, we may assume that there is no such interior cone point. Points in the boundary of $|P^2|$ have a neighborhood with inverse image an orbifold with topological underlying space a ball, and with branching locus two unknotted arcs of order 2, possibly linked by a segment, giving a graph with H-shape. For points in the interior of the edges, the fibration is nonsingular, but for vertices, the fiber is either singular, or in the branching locus, or both. The singularity and the branching determine the angle, see [5]. More precisely, there is a rational number $p/q \in \mathbf{Q}$, $p, q \in \mathbf{Z}$ coprime, describing the singular fibration, and the angle at the vertex of P^2 is $\pi/(mq)$, cf. Figure 4.1, where $m \geqslant 1$ is the branching index (not branched for $m = 1$).

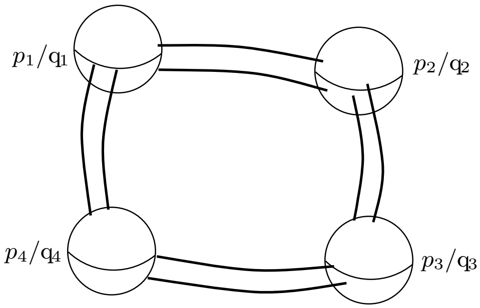

Figure 4.1. Fibration with 4 vertices The p_i/q_i-tangles around the I-singular fibers are inside the balls of the picture

We *orient* the components of $\Sigma^{hor}_{\mathcal{O}^3}$. The fiber of the interior of each edge of P^2 contains two subsegments of $\Sigma^{hor}_{\mathcal{O}^3}$, that project homeomorphically to the edge. The segments of $\Sigma^{hor}_{\mathcal{O}^3}$ may induce the same or opposite orientations.

Remark 4.1. — The orientations induced by $\Sigma^{hor}_{\mathcal{O}^3}$ can be chosen to be either compatible for every edge of P^2, or opposite for every edge.

It suffices to prove this remark when $\Sigma_{\mathcal{O}^3}$ is a link. Notice that when at least one of the indices q_i of the I-singular fibers of the vertices is even, then the orientations of all pairs of edges are opposite. When all singular indices are odd and $\Sigma_{\mathcal{O}^3}$ is connected, then the orientations are compatible. Finally, when all singular indices are odd and $\Sigma_{\mathcal{O}^3}$ is not connected, then $\Sigma_{\mathcal{O}^3}$ has two components and the orientations can be chosen compatible or opposite.

Set

$$M = \mathcal{O}^3 \setminus \mathcal{N}(\Sigma_{\mathcal{O}^3}).$$

We choose elements and subgroups of the fundamental group of M according to the fibration. We fix a base point $x_0 \in M$ that projects to an interior point of the polygon P^2.

The vertices of P^2 are denoted by $v_1, \ldots, v_n$, and the edges, $e_1, \ldots, e_n$, so that the endpoints of e_i are v_i and v_{i+1}, with coefficients modulo n. We distinguish the following elements of $\pi_1(M, x_0)$:

- Let $f \in \pi_1(M, x_0)$ be an element represented by the fiber through x_0. In particular f projects to the center of an index two subgroup of $\pi_1(\mathcal{O}^3)$.
- For each edge e_i of P^2, let e_i and e_i' denote still the components of the singular locus of $\mathcal{O}^3$ that project to it. We choose meridians m_i and m_i' by joining x_0 to e_i and e_i' along a path that projects to an interior path of P^2, and then turn around the respective axis, so that m_i and m_i' differ only in a neighborhood of the I-fiber. We orient m_i and m_i' accordingly to the orientation of the edges. Thus, when the orientations of the edges are compatible, we require that

$$m_i m_i' = f,$$

(Figure 4.2). When they are opposite,

$$m_i (m_i')^{-1} = f.$$

- For each vertex v_i of P^2 we choose V_i a neighborhood of the corresponding singular I-fiber and we call $\pi_1(V_i \setminus (\Sigma_{\mathcal{O}} \cap V_i))$ the *i-th tangle group*. We distinguish two cases.

  If the singular I-fiber is not in the branching locus of the orbifold, then the tangle group is the free group on two meridians $\sigma_i, \sigma_i' \in \pi_1(M)$. We choose a point in the middle of the singular fiber, and from there we consider both loops (Figure 4.3 left).

  When the singular I-fiber is in the branching locus of the orbifold, the tangle group is isomorphic to the fundamental group of a sphere with 4 punctures. We choose generators $\varsigma_i, \bar\varsigma_i, \varsigma_i', \bar\varsigma_i' \in \pi_1(V_i \setminus (\Sigma_{\mathcal{O}} \cap V_i))$ such that $\varsigma_i \bar\varsigma_i^{-1} = (\varsigma_i')^{-1}\bar\varsigma_i'$ is a meridian for the singular I-fiber. We choose the loops similarly (Figure 4.3 right).

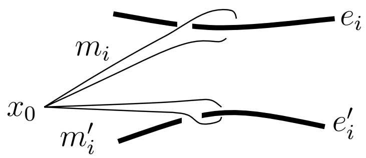

Figure 4.2. The loops for the meridians around regular I-fibers (i-th edge))

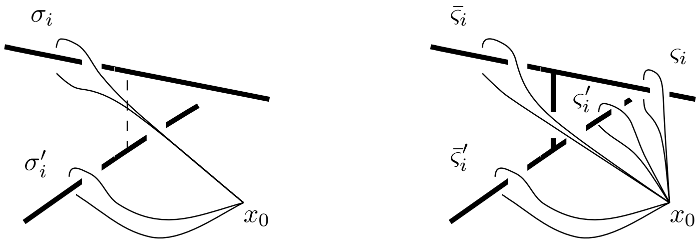

Figure 4.3. The loops for the meridians around the i-th tangle. When the singular I-fiber is smooth in $\mathcal{O}$ (on the left) and when it is in the branching locus (on the right)

For a singular I-fiber, the product $\sigma_i\sigma_i'$ ($\varsigma_i\varsigma_i'$ when the fiber is in the branching locus) projects in $\pi_1(\mathcal{O}^3)$ to a root of $f^{\pm 1}$, but not in $\pi_1(M)$. On the other hand, if e_i and e_{i+1} are the edges adjacent to the i-th vertex, then

$$m_i, m_i', m_{i+1}, m_{i+1}' \in \pi_1(V_i \setminus (\Sigma_{\mathcal{O}} \cap V_i)).$$

For an elliptic element $a \in \mathrm{Isom}^+(\mathbf{H}^3)$, let $\mathrm{A}(a) \subset \mathbf{H}^3$ denote its fixed point set (or its axis).

When the I-fiber is not in the branching locus, the angle between the axis of $\rho_0(\sigma_i)$ and $\rho_0(\sigma_i')$ is

$$\angle(\mathrm{A}(\rho_0(\sigma_i)), \mathrm{A}(\rho_0(\sigma_i'))) = \frac{p_i}{q_i}\pi \tag{4.1}$$

with $p_i, q_i \in \mathbf{Z}$ coprime, $0 < p_i < q_i$. This rational number p_i/q_i describes the singularity of the fiber, that has order q_i. The angle of P^2 at the corresponding vertex is π/q_i.

When the I-fiber is in the branching locus, the angle between the axis of $\rho_0(\varsigma_i)$ and $\rho_0(\varsigma_i')$ is

$$\angle(\mathrm{A}(\rho_0(\varsigma_i)), \mathrm{A}(\rho_0(\varsigma_i'))) = \frac{p_i}{2q_i}\vartheta_i \tag{4.2}$$

with $p_i, q_i \in \mathbf{Z}$ coprime, $0 < p_i < q_i$ as above and $0 < \vartheta_i < 2\pi$ the cone angle. (For an orbifold, $\vartheta_i = 2\pi/m_i$, where $m_i \geqslant 2$ is the branching order). The angle of P^2 at the corresponding vertex is $\frac{\vartheta_i}{2q_i}$.

Definition 4.2. — a) *The* Euclidean model *is the metric orbifold*

$$E(p_i/q_i) = \mathbf{R}^3/D_\infty,$$

where D_∞ is the infinite dihedral group generated by two rotations of order 2, whose axis are at distance one and have an angle (after parallel transport) equal to $\frac{p_i}{q_i}\pi$.

b) *The* singular Euclidean model *is the cone manifold*

$$E(p_i/q_i, \vartheta_i) = \mathbf{R}^3(\vartheta_i)/D_\infty,$$

where $\mathbf{R}^3(\vartheta_i) = \mathbf{R}^2(\vartheta_i) \times \mathbf{R}$ and $\mathbf{R}^2(\vartheta_i)$ is the Euclidean plane with a singular point of angle $0 < \vartheta_i < 2\pi$. Here D_∞ is generated by two rotations of order 2, with axis at distance one perpendicular to the singular axis of $\mathbf{R}^3(\vartheta_i)$, and forming an angle (after parallel transport) equal to $\frac{p_i}{2q_i}\vartheta_i$.

Remark 4.3. — The orbifold $E(p_i/q_i)$ and the cone manifold $E(p_i/q_i, \vartheta_i)$ have a natural fibration, that gives precisely the fibration of a neighborhood of the i-th singular vertex. This is the fibration by parallel lines of $\mathbf{R}^3$, in the direction of the translation vector of the index two subgroup $\mathbf{Z} < D_\infty$.

An alternative way of describing $E(p_i/q_i)$ is by considering fundamental domains, cf. Figure 4.4. Consider a region of $\mathbf{R}^3$ bounded by two parallel planes at distance one. On each plane there is a rotation axis, one for each generator, and $E(p_i/q_i)$ is obtained from identifying half of each face with the other half after folding. For $E(p_i/q_i, \vartheta_i)$, a similar fundamental domain is constructed in $\mathbf{R}^3(\vartheta_i) = \mathbf{R}^2(\vartheta_i) \times \mathbf{R}$.

The fibers come from the vertical segments (say the planes are horizontal), the singular I-fiber is the minimizing segment between the rotation axis. It is its soul, in the Cheeger-Gromoll sense.

Definition 4.4. — *A sequence of pointed metric spaces (X_n, x_n)* converges *to (X_∞, x_∞) for the* pointed bi-Lipschitz *topology if, $\forall R > 0$ and $\varepsilon > 0$, there exists n_0 such that, for $n \geqslant n_0$, $B(x_\infty, R)$ is $(1+\varepsilon)$-bi-Lipschitz to a neighborhood $U \subset X_n$ of x_n that satisfies $B(x_n, R-\varepsilon) \subseteq U \subseteq B(x_n, R+\varepsilon)$.*

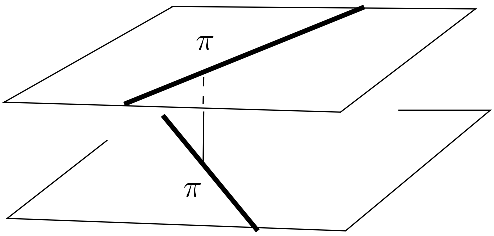

Figure 4.4. The model $E(p_i/q_i)$ via its fundamental domain.

When constructing the developing maps in Subsection 4.3, we will use the following lemma for the transition between singular and regular I-fibers:

Lemma 4.5. — *Let x_n be a sequence of points in the singular locus of $E(p_i/q_i)$. If $x_n \to \infty$, then $(E(p_i/q_i), x_n)$ converges to another Euclidean model with parallel singular axis, for the pointed bi-Lipschitz topology. In addition, the distance between the axis is q_i, the order of the singular fiber.*

The same statement holds true for the cone manifold $E(p_i/q_i, \vartheta_i)$ and points in the horizontal singular locus.

Proof. — We prove it for $E(p_i/q_i)$, the proof for $E(p_i/q_i, \vartheta_i)$ being similar. The lift of the branching locus of $E(p_i/q_i)$ to the universal covering (isometric to $\mathbf{R}^3$) is a countable family of lines, all of them perpendicular to a given axis that minimize the distance between any pair of the lines. From each line, we obtain the next one by a screw motion. This screw motion has axis the line perpendicular to all the lifts, translation length one and rotation angle $\frac{p_i}{q_i}\pi$. In this way, if x_n goes to infinity along one of the lines, the closest singular component will be parallel and at distance q_i. Then the convergence follows easily. $\square$

4.2. The Killing vector field

In this section we prove a result about Killing vector fields that will be used in the construction of developing maps.

Consider $P^2 \setminus \Sigma_{P^2}$ the smooth part of P^2 (ie. we remove the boundary of the underlying polygon). Via the developing map of the transversely hyperbolic foliation, the closure

$$\mathcal{P} = \overline{D_0(P^2 \setminus \Sigma_{P^2})}$$

is a polygon in $\mathbf{H}^2 \subset \mathbf{H}^3$. Let $m_i, m_i' \in \pi_1(M, x_0)$ be as in Section 4.1, for $i = 1, \dots, n$. Let $\tilde{m}_i$ and $\tilde{m}_i'$ be the corresponding paths lifted to the universal covering. We may assume that the path $D_0(\tilde{m}_i)$ starts at the base point $D_0(\tilde{x}_0)$ in the interior, crosses the boundary of $\mathcal{P}$, and follows along $\rho_0(m_i)(\mathcal{P}) = \overline{D_0(m_i P)}$ until $\rho_0(m_i)(D_0(\tilde{x}_0))$, and similarly for m_i', σ_j and σ_j'. Recall that

$$f = m_i (m_i')^{\pm 1},$$

where the sign of the power $(m_i')^{\pm 1}$ depends on the compatibility of orientations at a given axis.

By analyticity, if $\rho_t(f)$ is nontrivial, then there is a natural way to associate a Killing vector field F to the deformation of $\rho_t(f)$. Namely, as $\rho_0(f) = \pm\,\mathrm{Id}$,

$$\rho_t(f) = \pm \exp(t^s \mathfrak{f} + O(t^{s+1}))$$

for some $\mathfrak{f} \in \mathfrak{sl}_2(\mathbf{C})$, $\mathfrak{f} \neq 0$. The Killing vector field F associated to the infinitesimal isometry $\mathfrak{f}$ is then

$$F_x = \lim_{t \to 0} \frac{\rho_t(f)(x) - x}{t^s} \qquad \forall x \in \mathbf{H}^3.$$

By Proposition 1.5, $\mathcal{P}$ has an *incenter*: a point whose distance to *every* edge of the polygon is the same.

The goal of this section is to prove the following:

Proposition 4.6. — *The Killing vector field F is a field of infinitesimal purely loxodromic translations along an axis that meets perpendicularly $\mathcal{P}$ at its incenter. So F is perpendicular to $\mathcal{P}$. In addition, it has the same orientation as the fiber of the Seifert fibration of $\mathcal{O}^3$ restricted to the interior of $\mathcal{P}$.*

Notice that the interior of $\mathcal{P}$ is orientable because the mirror points are in $\partial\mathcal{P}$, thus it makes sense to talk about the induced orientation of the fiber in $\mathcal{O}^3$ and of the Killing field on $\mathbf{H}^3$.

Before proving the proposition, we need to show that $\rho_t(f)$ is nontrivial.

For a representation ρ_t, a pseudodeveloping map is a ρ_t-equivariant map $D_t \colon \tilde{M} \to \mathbf{H}^3$, such that around the singular locus it is like the developing map around a cone singularity (ie. conical in a tubular neighborhood). This D_t can be used to define a volume of ρ_t [11].

Lemma 4.7. — *For ρ_t satisfying the conclusion of Proposition 2.1, there exists a uniform constant $C > 0$ such that, for $t > 0$ close to 0:*

$$\mathrm{Vol}(\rho_t) \geqslant C t^r.$$

Proof. — Schläfli's formula applied to cone manifolds [22] gives:

$$\mathrm{Vol}(\rho_t) = -\frac{1}{2} \int_0^t \sum_e \mathrm{length}(e)\, d\alpha_e,$$

where the sum runs over all singular edges or components. In our case, as the length is bounded below, and the cone angles are $\pi - t^r + O(t^{r+1})$, the lemma is straightforward. □

Lemma 4.8. — *For small $t > 0$, $\rho_t(f)$ is nontrivial.*

Proof. — Seeking a contradiction, assume that $\rho_t(f)$ is trivial. Then, $\rho_t(m_i') = \rho_t(m_i)^{\pm 1}$.

We first claim that for small values of $t > 0$, ρ_t of the i-th tangle group is elliptic, ie. for a tubular neighborhood V_i of the i-th I-fiber, $\rho_t(\pi_1(V_i \setminus (V_i \cap \Sigma_{\mathcal{O}})))$ is elliptic. We deal first with the case where the singular I-fiber is not in the branching locus of the orbifold. Again by contradiction, assume that the axis of $\rho_t(\sigma_i)$ and $\rho_t(\sigma_i')$ are disjoint. Then there is a minimizing segment between the axis of $\rho_t(\sigma_i)$ and $\rho_t(\sigma_i')$, because the axis of $\rho_0(\sigma_i)$ and $\rho_0(\sigma_i')$ meet at one point with angle $\frac{\pi}{q_i}$. By rescaling the hyperbolic space in such a way that the length of this segment is one, and by taking the pointed limit with base point the midpoint of this segment, we look at the limits of the axis $\mathrm{A}(\rho_t(\sigma_i))$ and $\mathrm{A}(\rho_t(\sigma_i'))$ after rescaling: we obtain two Euclidean lines at distance one and forming an angle, as in the Euclidean model of Definition 4.2. In this model, the axis of the m_i and m_i' are parallel but different, by Lemma 4.5. This contradicts that $\rho_t(m_i') = \rho_t(m_i)^{\pm 1}$, and hence $\mathrm{A}(\rho_t(\sigma_i))$ and $\mathrm{A}(\rho_t(\sigma_i'))$ meet at one point. When the singular I-fiber is in the branching locus of the orbifold, then a similar argument tells that the segment between $\mathrm{A}(\rho_t(\varsigma_i)) \cap \mathrm{A}(\rho_t(\bar\varsigma_i))$ and $\mathrm{A}(\rho_t(\varsigma_i')) \cap \mathrm{A}(\rho_t(\bar\varsigma_i'))$ has length zero.

Construct a pseudodeveloping map $D_t : \tilde{M} \to \mathbf{H}^3$ as follows. Start by mapping a tubular neighborhood of the singularity to a tubular neighborhood of the axis of the corresponding elements via ρ_t. Now, since ρ_t of any tangle group is elliptic, the singular I-fiber can be mapped to a neighborhood of this point. Similarly, as $\mathrm{A}(\rho_t(m_i)) = \mathrm{A}(\rho_t(m_i'))$, the regular I-fibers can be mapped to a δ-neighborhood of the axis, for $\delta > 0$ arbitrarily small. The boundary of the neighborhood of the I-fibers is a torus, and since $\rho_t(f)$ is trivial, this torus can be deformed ρ_t-equivariantly to a circle, in a neighborhood of radius 2δ. Extend D_t by collapsing the rest of the manifold to a disk. Thus $\rho(t)$ has arbitrary small volume, by choosing $\delta > 0$ small enough, contradicting Lemma 4.7. □

Recall that the Killing vector field F is the corresponding field of the infinitesimal isometry $\mathfrak{f} \in \mathfrak{sl}_2(\mathbf{C})$, $\mathfrak{f} \neq 0$,where $\rho_t(f) = \pm\exp(t^s \mathfrak{f} + O(t^{s+1}))$. By Lemma 4.8, $\mathfrak{f} \neq 0$ is well defined.

LEMMA 4.9. — *If $\rho_t(f) = \exp(t^s \mathfrak{f} + O(t^{s+1}))$, and $r \in \mathbf{N}$ is as in Proposition 2.1, then*

$$s \leqslant r.$$

Proof. — By Lemma 4.7, $\mathrm{Vol}(\rho_t) \geqslant Ct^r$ for some uniform constant $C > 0$.

On the other hand, the displacement function of $\rho_t(f)$ in a compact neighborhood U of $\mathcal{P}$ is $\leqslant C_0 t^s$. In particular, the Hausdorff distance between $\mathrm{A}(\rho_t(m_i)) \cap U$ and $\mathrm{A}(\rho_t(m_i')) \cap U$ is $\leqslant C_1 t^s$.

We want to construct a pseudodeveloping map with volume $\leqslant C' t^s$. We start by constructing a developing map around the singular locus, by taking a small radius of the tube, with arbitrarily small volume, say $\leqslant t^s$. Moreover as the Hausdorff distance between $\mathrm{A}(\rho_t(m_i)) \cap U$ and $\mathrm{A}(\rho_t(m_i')) \cap U$ is $\leqslant C_1 t^s$, we can develop a solid torus that is a neighborhood of the I-fibers with volume $\leqslant C_2 t^s$, and so that the length of the fiber is $\leqslant 3C_1 t^s$. The exterior of this torus in $\mathcal{O}^3$ is a solid torus without singularity V, and since the displacement function of $\rho_t(f)$ in U is $\leqslant C_0 t^s$, the pseudodeveloping map can be extended to V with a volume contribution $\leqslant C_3 t^s$. Thus, the volume of the pseudodeveloping map, and of ρ_t, is $\leqslant C' t^s$. Comparing both inequalities for the volume:

$$Ct^r \leqslant \mathrm{Vol}(\rho_t) \leqslant C' t^s,$$

for small values of $t > 0$. Thus $s \leqslant r$. □

Before proving Proposition 4.6, we still need a further computation. Let

$$B : \mathfrak{sl}_2(\mathbf{C}) \times \mathfrak{sl}_2(\mathbf{C}) \to \mathbf{C}$$

denote the complex Killing form, see Appendix A. For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbf{C})$,

$$B(\mathfrak{a}, \mathfrak{b}) = \mathrm{Trace}(Ad_{\mathfrak{a}} \circ Ad_{\mathfrak{b}}) = 4\,\mathrm{Trace}(\mathfrak{a}\mathfrak{b}).$$

DEFINITION 4.10. — *We say that an infinitesimal isometry $\mathfrak{a} \in \mathfrak{sl}_2(\mathbf{C})$ has* complex length $l \in \mathbf{C}$ *if $\exp(t\mathfrak{a})$ has complex length $t\,l$.*

LEMMA 4.11. — *Let $\mathfrak{d}_i \in \mathfrak{sl}_2(\mathbf{C})$ denote infinitesimal rotation of complex length $\pi\mathbf{i}$ around the i-the oriented axis of $\mathcal{P}$. Then*

$$B(\mathfrak{d}_i, \mathfrak{f}) = \begin{cases} 0, & \text{if } s < r. \\ 4, & \text{if } s = r. \end{cases}$$

In particular $B(\mathfrak{d}_i, \mathfrak{f})$ is independent of i.

We will show later in Lemma 4.13 that only the case $B(\mathfrak{d}_i, \mathfrak{f}) = 4$ occurs, in particular $s = r$.

Proof. — We can find $C_t, C'_t \in SL_2(\mathbf{C})$ that depend analytically on $t^{1/2}$, $t \in (0,\varepsilon)$ so that $C_0 = C'_0 = \mathrm{Id}$, and that maps the axis at time zero to the axis at time t:

$$\mathrm{A}(\rho_t(m_i)) = C_t(\mathrm{A}(\rho_0(m_i))) \qquad \text{and} \qquad \mathrm{A}(\rho_t(m'_i)) = C'_t(\mathrm{A}(\rho_0(m'_i))).$$

Those matrices C_t and C'_t are obtained by solving the characteristic polynomials for $\rho_t(m_i)$ and $\rho_t(m'_i)$, hence they are analytic on $t^{1/2}$.

Assume that the axis of $\rho_0(m_i)$ is $\overline{0\infty}$ in the upper half space model of $\mathbf{H}^3$. If the orientations of m_i and m'_i are compatible, then.

$$\begin{aligned}\rho_t(m_i) &= \pm C_t \begin{pmatrix} e^{i\alpha_i/2} & 0 \\ 0 & e^{-i\alpha_i/2}\end{pmatrix} C_t^{-1}, \\ \rho_t(m'_i) &= \pm C'_t \begin{pmatrix} e^{i\alpha'_i/2} & 0 \\ 0 & e^{-i\alpha'_i/2}\end{pmatrix} (C'_t)^{-1}.\end{aligned}$$

Notice that since $\rho_0(f) = \pm\,\mathrm{Id}$, if $\rho_t(f) = \pm\exp(t^s\mathfrak{f} + O(t^{s+1}))$, then

$$C_t^{-1}\rho_t(f)C_t = \pm\exp(t^s\mathfrak{f} + O(t^{s+1})),$$

hence we may assume $C_t = \mathrm{Id}$ (after replacing C'_t by $C_t^{-1}C'_t$). Let

$$C'_t = \begin{pmatrix} 1+at^\nu & bt^\nu \\ ct^\nu & 1-at^\nu\end{pmatrix} + O(t^{\nu+1/2}),$$

with $a,b,c \in \mathbf{C}$, $\nu \in \frac{1}{2}\mathbf{N}$. Since $\alpha_i(t) = \alpha'_i(t) = \pi - t^r + O(t^{r+1})$,

$$\rho_t(f) = \rho_t(m_i)\rho_t(m'_i) = \pm\begin{pmatrix} -1+\mathbf{i}t^r & 2t^\nu b \\ 2t^\nu c & -1-\mathbf{i}t^r\end{pmatrix} + O(t^{\min(r,\nu)+1/2}). \quad (4.3)$$

When m_i and m'_i have opposite orientation, then

$$\rho_t(m'_i) = \pm C'_t \begin{pmatrix} e^{-i\alpha'_i/2} & 0 \\ 0 & e^{i\alpha'_i/2}\end{pmatrix} (C'_t)^{-1},$$

and since $f = m_i(m'_i)^{-1}$, (4.3) also holds true.

Let $\mathfrak{d}_i$ be the infinitesimal rotation around the oriented axis of $\rho_0(m_i)$ of complex length $\pi\mathbf{i}$. In this model:

$$\mathfrak{d}_i = \begin{pmatrix} \mathbf{i}/2 & 0 \\ 0 & -\mathbf{i}/2\end{pmatrix}.$$

From (4.3) we distinguish two cases:

1) If $\nu < r$, then $s = \nu < r$ and

$$\mathfrak{f} = \begin{pmatrix} 0 & -2b \\ -2c & 0\end{pmatrix}. \quad (4.4)$$

2) If $\nu \geqslant r$, then $s = r$ and

$$\mathfrak{f} = \begin{pmatrix} -\mathbf{i} & -2b \\ -2c & \mathbf{i} \end{pmatrix}. \tag{4.5}$$

This includes the case $\nu > r$, with $b = c = 0$.

Then the formula follows from $B(\mathfrak{d}_i, \mathfrak{f}) = 4\,\mathrm{Trace}(\mathfrak{d}_i \mathfrak{f})$. □

Lemma 4.12. — *The Killing vector field F is perpendicular to the axis* $\mathrm{A}(\rho_0(m_i))$.

Proof. — This follows from Equalities (4.4) and (4.5), because in both cases the real part of the diagonal of $\mathfrak{f}$ vanishes, and the axis is $\mathrm{A}(\rho_0(m_i)) = \overline{0\infty}$. □

Lemma 4.13. — *$r = s$.*

Proof. — Assume that $s < r$, hence $B(\mathfrak{f}, \mathfrak{d}_i) = 0$ for each $i = 1, \ldots, n$. Using the formulas of Appendix A, we shall find a contradiction. When $\mathfrak{f}$ is non parabolic, let $\mathrm{A}(\mathfrak{f}) \subset \mathbf{H}^3$ denote the axis of $\mathfrak{f}$, which is the minimizing set for the norm of the Killing vector field $|F|$. If f is non parabolic, then by Proposition A.1 the complex distance between $\mathrm{A}(\mathfrak{f})$ and $\mathrm{A}(\mathfrak{d}_i)$ is $\pm\frac{\pi}{2}\mathbf{i}$, hence $\mathrm{A}(\mathfrak{f})$ must meet perpendicularly all edges of $\mathcal{P}$, which is impossible. So we assume that $\mathfrak{f}$ is parabolic. In this case, Proposition A.3 tells that the point at ∞ fixed by $\mathfrak{f}$ is an endpoint of all (infinite) geodesics containing an edge of $\mathcal{P}$, which is again impossible. □

Proof of Proposition 4.6. — By Lemmas 4.13 and 4.11, $r = s$ and $B(\mathfrak{f}, \mathfrak{d}_i) = 4$ for each $i = 1, \ldots, n$. We discuss again the possibilities for $\mathfrak{f}$. If $\mathfrak{f}$ was parabolic, then Corollary A.5 would tell that all (infinite) edges of $\mathcal{P}$ are tangent to a given horosphere, and that their tangent vectors are parallel in this horosphere, which is again impossible. Hence we are left with the case that $\mathfrak{f}$ is nonparabolic and has an axis whose complex distance to all oriented edges of $\mathcal{P}$ is the same (by Proposition A.1).

Notice that by Lemma 4.12, the Killing vector field F is perpendicular to every edge of $\mathcal{P}$. Hence, at the vertices of $\mathcal{P}$, F is perpendicular to the plane containing $\mathcal{P}$, and since it is a Killing vector field, F is perpendicular to $\mathcal{P}$. Thus $\mathfrak{f}$ is either an infinitesimal rotation with axis coplanar to $\mathcal{P}$ or an infinitesimal translation with axis perpendicular to $\mathcal{P}$. If $\mathfrak{f}$ is an infinitesimal rotation then by Remark A.2 (Equation A.1) the complex distance between $\mathrm{A}(\mathfrak{f})$ and every oriented axis of $\mathcal{P}$ is the same, but this is impossible in a coplanar configuration. Thus $\mathfrak{f}$ is an infinitesimal translation, and its axis meets $\mathcal{P}$ perpendicularly and is equidistant to all edges of $\mathcal{P}$.

Finally, the assertion about orientations follows from the next lemma. □

Lemma 4.14. — *If the cone angles decrease with t, then the orientation of the Killing vector field F is the same as the orientation of the fiber in $\mathcal{O}^3$.*

Proof. — By Lemma 4.7, the volume of the representation is positive, $\mathrm{Vol}(\rho_t) > 0$ for $t > 0$. On the other hand, if the orientation of the Killing vector field was the wrong one, we would be able to construct a pseudodeveloping map with negative volume, following the strategy of Lemma 4.8. □

Corollary 4.15. — *For $t \in (0, \varepsilon)$, $\rho_t(f)$ is loxodromic (ie. not elliptic nor parabolic).*

Proof. — Assume first that $\rho_0(f) = \mathrm{Id}$. Since $\mathfrak{f}$ is purely loxodromic, then the first nonzero derivative of the trace of $\rho_t(f)$ is real positive, in particular for small values of $t > 0$ it is not contained in $[-2, 2]$. A similar argument applies when $\rho_0(f) = -\mathrm{Id}$. □

4.3. Constructing developing maps

Along this subsection, assume that $\{\rho_t\}_{t\in[0,\varepsilon)}$ is a path of representations that satisfies the conclusion of Proposition 2.1. The goal is to construct developing maps with holonomy ρ_t.

We construct the developing maps in three steps. Firstly, in a neighborhood of the vertices of P^2, that correspond to tangles of the orbifold, or singular interval fibers. Secondly, on the edges, and finally on the interior.

We start with the vertices of P^2, ie. the tangles of $\mathcal{O}^3$.

We assume for the moment that the I-fiber of the i-th vertex is not in the branching locus of the orbifold. (See Remark 4.20 when it is in the branching locus of the orbifold). Let σ_i and σ_i' in $\pi_1(M)$ denote the meridians corresponding to the i-th tangle, as in Subsection 4.1 (Figure 4.2).

Lemma 4.16. — *For $t > 0$,*

$$\mathrm{dist}(\mathrm{A}(\rho_t(\sigma_i)), \mathrm{A}(\rho_t(\sigma_i')) > 0.$$

Moreover, there is a shortest segment $\nu_i(t)$ between both axis that converges to the i-th vertex of the polygon as $t \to 0^+$.

Proof. — By contradiction, assume that $\mathrm{dist}(\mathrm{A}(\rho_t(\sigma_i)), \mathrm{A}(\rho_t(\sigma_i'))) = 0$ for small values of $t > 0$. Since

$$\angle(\mathrm{A}(\rho_0(\sigma_i)), \mathrm{A}(\rho_0(\sigma_i')) = \pi p_i/q_i,$$

$\langle \rho_t(\sigma_i), \rho_t(\sigma_i') \rangle$ is an elliptic group that fixes a point close to the initial vertex in $\mathbf{H}^3$. In particular, since $m_i, m_i' \in \langle \sigma_i, \sigma_i' \rangle$, $\rho_t(f^{\pm 1}) = \rho_t(m_i)\, \rho_t(m_i')^{\pm 1}$ is either trivial or elliptic, which contradicts Corollary 4.15.

The existence of the shortest segment $\nu_i(t)$ comes from the fact that $\mathrm{A}(\rho_0(\sigma_i))$ and $\mathrm{A}(\rho_0(\sigma_i'))$ meet at one point with angle $\pi \frac{p_i}{q_i}$, so the distance function between both axis is a proper convex function on $\mathrm{A}(\rho_0(\sigma_i)) \times \mathrm{A}(\rho_0(\sigma_i'))$ and has a minimum. Therefore, for small $t > 0$ it is also a proper convex function on $\mathrm{A}(\rho_t(\sigma_i)) \times \mathrm{A}(\rho_t(\sigma_i'))$ and has a minimum. □

The idea now is to construct a double roof $\mathcal{R}_i(t) \subset \mathbf{H}^3$ around $\nu_i(t)$ as follows. Consider an embedding of both axis $\mathrm{A}(\rho_t(\sigma_i))$ and $\mathrm{A}(\rho_t(\sigma_i'))$ and the common perpendicular $\nu_i(t)$ in $\mathbf{H}^3$. Now consider two sectors, one with axis $\mathrm{A}(\rho_t(\sigma_i))$ and angle $\alpha_i(t)$, another one with axis $\mathrm{A}(\rho_t(\sigma_i'))$ and angle $\alpha_i'(t)$. (Here $\alpha_i(t)$ and $\alpha_i'(t)$ are the respective rotation angles of $\rho_t(\sigma_i)$ and $\rho_t(\sigma_i')$). Choose the sectors so that $\nu_i(t)$ is bisector to both of them, and consider the intersection (Figure 4.5).

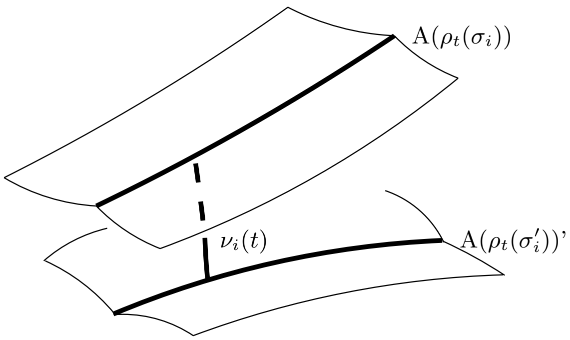

Figure 4.5. The double roof. The tubular neighborhood here is $\mathcal{R}_i(t)$.

The boundary of these sectors may intersect. Let $r_i(t) > 0$ be the maximal radius such that the tubular neighborhood $\mathcal{N}_{r_i(t)}(\nu_i(t))$ does not meet the intersection of the sides of the sectors. We denote $\mathcal{R}_i(t) = \mathcal{N}_{r_i(t)/2}(\nu_i(t))$ the tubular neighborhood of $\nu_i(t)$ in this double roof. Notice that possibly $r_i(t) \to 0$ as $t \to 0^+$.

Lemma 4.17. —
$$\lim_{t \to 0^+} \frac{r_i(t)}{|\nu_i(t)|} = +\infty.$$

Proof. — We cut the double roof along the hyperplane perpendicular to $\nu_i(t)$ that contains its midpoint, and consider each roof separately. We bound below the distance from $\nu_i(t)$ to the intersection of each piece of the roof with this hyperplane, and it suffices to discuss the argument for one of the edges, say σ_i. Let $\alpha_i(t)$ denote the cone angle, which is the angle of the roof. By comparison with the Euclidean right triangle (Figure 4.6):

$$\frac{r_i(t)}{|\nu_i(t)|/2} \geqslant \tan\frac{\alpha_i(t)}{2} \to +\infty \quad \text{as } t \to 0^+,$$

because $\alpha_i(0) = \pi$. $\square$

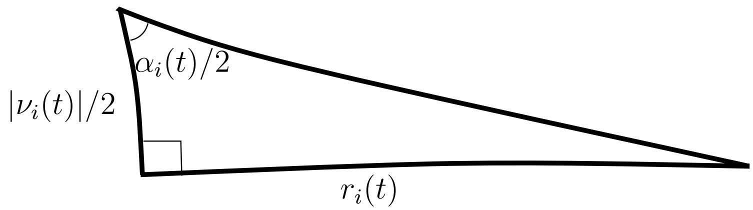

Figure 4.6. The hyperbolic triangle approximated by a Euclidean one.

Let $x_i(t)$ denote the midpoint of $\nu_i(t)$. Let $\overline{\mathcal{R}}_i(t)$ be the result of identifying the sides of each roof of $\mathcal{R}_i(t)$ by a rotation around its edge, so that the edges become interior points.

From Lemma 4.17, we get:

Corollary 4.18. — *For the pointed bi-Lipschitz topology:*

$$\lim_{t\to 0^+} \frac{1}{|\nu_i(t)|}(\overline{\mathcal{R}}_i(t), x_i(t)) = (E(p_i/q_i), x_\infty).$$

Next corollary deals with points of $\overline{\mathcal{R}}_i(t)$ away from the center.

Corollary 4.19. — *There exist $R_0 > 0$ and $t_0 > 0$ such that for $0 < t \leqslant t_0$ and $x \in \overline{\mathcal{R}}_i(t)$ that it is singular and $R_0|\nu_i(t)| \leqslant d(x, x_i(t)) < \frac{1}{2} r_i(t)$, the following hold. Let $\delta(x)$ be the distance between x and the other singular component. Then the rescaled ball*

$$\frac{1}{\delta(x)} B(x, 10\delta(x))$$

is $3/2$-bi-Lipschitz to the corresponding ball in $E(0)$.

Proof. — By Corollary 4.18, it is sufficient to prove it for the Euclidean models $E(p_i/q_i)$. Then the corollary follows from Lemma 4.5. $\square$

Remark 4.20. — When the I-fiber of the i-th vertex is in the branching locus of the orbifold, then one needs to consider the double roofs $\mathcal{R}_i(t)$ and the corresponding neighborhoods $\overline{\mathcal{R}_i(t)}$ with a singular core $\nu_i(t)$ of cone angle ϑ_i. Lemma 4.17 and Corollaries 4.18 and 4.19 apply in this case.

Next we deal with the edges of P^2. We shall construct locally the hyperbolic structures in pieces $\overline{\mathcal{S}}(q)$ and study its behavior and compatibility in Corollary 4.23 and Lemma 4.24. Before that, we need few technical results about the edges $\mathrm{A}(\rho_t(m_i))$ and $\mathrm{A}(\rho_t(m_i'))$.

To simplify notation, set $i = 1$. The endpoints of the segment e_1 of $\mathcal{P}^2$ at time $t = 0$ are v_1 and v_2. But for $t > 0$, we consider two segments $e_1(t)$ and $e_1'(t)$ that are contained in $\mathrm{A}(\rho_t(m_1))$ and $\mathrm{A}(\rho_t(m_1'))$, respectively, and whose endpoints are given by the σ's or ς's: ie. the endpoints of the corresponding conjugates of $\nu_1(t)$ and $\nu_2(t)$.

Let $p_1(t)$ and $p_2(t)$ denote the endpoints of $e_1(t)$. For $q \in e_1(t)$, let $q' \in \mathrm{A}(\rho_t(m_1'))$ be the point that realizes the distance between q and $\mathrm{A}(\rho_t(m_1'))$ (cf. Fig. 4.7). Define, for $q \in e_1$:

$$\delta_t(q) = d(q, q') = d(q, \mathrm{A}(\rho_t(m_1'))).$$

Lemma 4.21. — *Let $q \in e_1(t)$ and $q' \in \mathrm{A}(\rho_t(m_1'))$ be as above.*

(1) *The distance $\delta_t(q) = d(q, q')$ converges to zero uniformly on $q \in e_1(t)$:*

$$\lim_{t\to 0^+} \sup_{q\in e_1(t)} \delta_t(q) = 0.$$

(2) *Let $v_{q,t} \in T_q\mathbf{H}^3$ be the parallel transport of the tangent vector to $e_1'(t)$ along the segment $\overline{q'q}$. Then*

$$\lim_{t\to 0^+} \sup_{q\in e_1(t)} \angle_q e_1(t) v_{q,t} = 0.$$

(3) *Let $R_0 > 0$ be as in Corollary 4.19. There exists $t_0 > 0$ such that, for $0 < t < t_0$, $q \in e_1(t)$ satisfies $d(q, p_1(t)) > R_0\,|\nu_1(t)|$ and $d(q, p_2(t)) > R_0\,|\nu_2(t)|$, then:*

$$q' \in e_1'(t).$$

Proof. — By convexity of the distance function in hyperbolic space, we have, for $q \in e_1(t)$:

$$\delta_t(q) \leqslant \max\{\delta_t(p_1(t)), \delta_t(p_2(t))\}, \tag{4.6}$$

because $p_1(t)$ and $p_2(t)$ are the endpoints of $e_1(t)$. This proves Assertion 1 of the lemma.

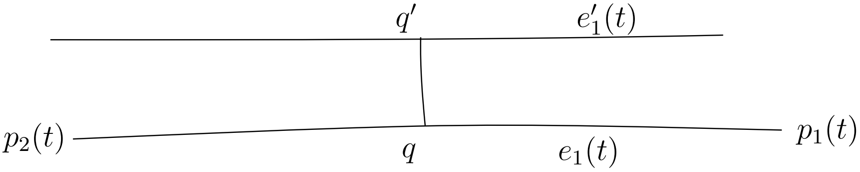

Figure 4.7

In order to prove Assertion 3, if $d(q, p_1(t)) = R_0\,|\nu_1(t)|$ or if $d(q, p_2(t)) = R_0\,|\nu_2(t)|$, then the assertion holds true for these q, because of Corollary 4.19. As those q are extremal, for other q the assertion follows from Inequality (4.6) and elementary arguments.

Next we prove Assertion 2. Up to permuting p_1 with p_2, we may assume that $d(q, p_1(t)) > \frac{1}{3}|e_1(0)|$, where $|e_1(0)|$ denotes the length of $e_1(0)$. Let $\beta_q(t)$ be the angle between $v_{q,t}$ and $e_1(t)$. By the triangle inequality in spherical space, the angle $\beta_q(t)$ satisfies: $0 \leqslant \beta_q(t) \leqslant \beta_1 + \beta_2$, where β_1 is the angle between $v_{q,t}$ and qp_1', β_2 is the angle between qp_1' and $qp_1 \subset e_1(t) \subset \mathrm{A}(\rho_t(m_1))$, and $p_1' \in \mathrm{A}(\rho_t(m_1'))$ realizes $d(p_1, \mathrm{A}(\rho_t(m_1'))) = d(p_1, p_1')$, cf. Figure 4.8.

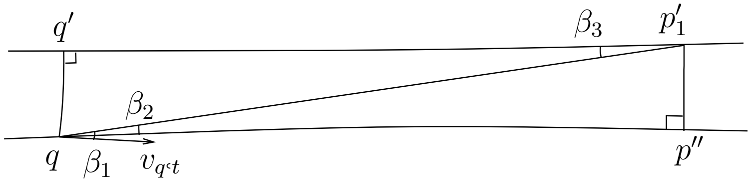

Figure 4.8. Triangles in the proof of Assertion 2 of Lemma 4.21.

Let $p'' \in \mathrm{A}(\rho_t(m_1))$ realize the distance from p_1' to $\mathrm{A}(\rho_t(m_1))$, so that q, p_1' and p'' form a triangle with angles β_2 at q, and $\pi/2$ at p''. Then

$$\tan \beta_2 = \frac{\tanh d(p_1', p'')}{\sinh d(q, p'')} \leqslant \frac{\tanh d(p_1', p'')}{\sinh(\frac{1}{3}|e_1(0)| - d(p_1, p''))}$$

which converges to zero uniformly on q. Consider now the triangle q, q' and p_1'. By the same argument as before the angle β_3 of this triangle at p_1' converges to zero. The angles of the triangle satisfy:

$$(\frac{\pi}{2} - \beta_1) + \frac{\pi}{2} + \beta_3 = \pi - \mathrm{Area}(qq'p_1').$$

In addition, the area of this triangle converges to zero uniformly on q, by Assertion 1 of the lemma. Thus

$$\beta_1 = \beta_3 + \mathrm{Area}(qq'p_1') \to 0, \qquad \text{uniformly on } q.$$

□

We define, for $0 < t < t_0$ as in Assertion 3 of Lemma 4.21:

$$\hat{e}_1(t) = \{q \in e_1(t) \mid d(q, p_1(t)) \geqslant R_0 \, |\nu_1(t)| \text{ and } d(q, p_2(t)) \geqslant R_0 \, |\nu_2(t)|\}.$$

Using also Lemma 4.21, construct a double roof from the segment between q and q', with edges determined by $\mathrm{A}(\rho_t(m_1))$ and $\mathrm{A}(\rho_t(m_1'))$, and with dihedral angles the respective rotation angles of $\rho_t(m_1)$ and $\rho_t(m_1')$, $\alpha_1(t)$ and $\alpha_1'(t)$, as before. Let $\mathcal{S}(q,t) = B(q, s(q,t)/2)$ be the ball in this double roof, with $s(q,t)$ maximal such that the sides of the roof do not meet. As in Lemma 4.17:

Lemma 4.22. —

$$\lim_{t \to 0^+} \frac{s(q,t)}{d(q,q')} = +\infty$$

uniformly on $q \in \hat{e}_1(t)$.

The proof of this limit is the same as Lemma 4.17, using the uniform limits of Lemma 4.21.

Identifying the sides of $\mathcal{S}(q,t)$ by the rotations corresponding to its edges, we obtain $\overline{\mathcal{S}}(q,t)$. From Lemmas 4.22 and 4.21, we get:

Corollary 4.23. — *For any choice of $q(t) \in \hat{e}_1(t)$ and for the pointed bi-Lipschitz topology:*

$$\lim_{t \to 0^+} \frac{1}{\delta_t(q)} (\overline{\mathcal{S}}(q,t), q(t)) = (E(0), q_\infty),$$

uniformly on q.

Recall that $\delta_t(q) = d(q,q') = d(q, \mathrm{A}(\rho_t(m_i')))$.

Lemma 4.24. — *Let $r \in \overline{\mathcal{S}}(q,t)$ belong to the same connected component of the singular locus as q. Let r' and q' be the corresponding closest points in the other singular components. If $d(q,r) \leqslant 10\delta_t(q)$, then the angle between qq' and rr' after parallel transport (along any of both singular components) is $\leqslant \gamma(t)$, for some uniform $\gamma(t) \to 0$.*

This lemma follows easily from the estimates of Lemma 4.21 and elementary trigonometric arguments.

Proposition 4.25. — *Let ρ_t be as in Proposition 2.1. There exists $\varepsilon > 0$ such that for $t \in (0, \varepsilon)$ there exists $D_t : \tilde{M} \to \mathbf{H}^3$ the developing map of a cone structure on $(|\mathcal{O}^3|, \Sigma_{\mathcal{O}^3})$ with holonomy ρ_t. In addition, when $t \to 0$, D_t converges to D_0, the developing map of the transverse hyperbolic foliation.*

Proof. — Fix $0 < t < t_0$, where $t_0 > 0$ is as in Assertion 3 of Lemma 4.21. The edge $\hat{e}_1(t)$ is covered by balls $B(q, 2\delta_t(q))$. Choose a finite covering of such balls, with centers q in $\hat{e}_1(t)$. We claim that the model $\overline{\mathcal{S}}(q,t)$ of each ball matches with the next one: this is a consequence of Lemma 4.24, because the segments between q and the opposite singular edge vary continuously with q, and they are almost parallel (the difference with the parallel transport is uniformly small in $B(q, 10\delta_t(q)))$. Notice also that the position of the singular edges is determined by the isometries $\rho_t(m_1)$ and $\rho_t(m_1')$. This gives a metric structure for a neighborhood of the edges.

By Lemma 4.5, when $q \in \partial \hat{e}_1(t)$, then the $\overline{\mathcal{S}}(q,t)$ match with the corresponding $\overline{\mathcal{R}}_i(t)$. In this way we put a geometric structure on a solid torus that contains the singular locus, made of the union of 0-cells (the $\overline{\mathcal{R}}_i(t)$ for the singular vertices of the polygon) and 1-cells (the union of $\overline{\mathcal{S}}(q,t)$ for the edges of the polygon). Let D_t be the corresponding developing map of this solid torus that contains the singular locus.

Notice that the orientation is globally preserved, by Proposition 4.6, and because it depends on the displacement of $\rho_t(f)$.

Recall that we assume that there is no singular fiber in the interior of the orbifold. Consider the 2-torus that bounds the previous tubular neighborhood of the singularity. Now the developing map of the universal covering of the 2-torus factors to a map from the 2-torus to the hyperbolic solid torus $\mathbf{H}^3/\rho_t(f)$, ($\rho_t(f)$ is loxodromic by Corollary 4.15). By Proposition 4.6, this map is injective on the intersection of the 2-torus and each model $\overline{\mathcal{S}}(q,t)$ and $\overline{\mathcal{R}}_i(t)$. In addition, the models are either far apart or their intersection is well understood, by the previous discussion, hence it is an embedding of the torus.

Since it is not contained in a ball, this 2-torus must bound a solid torus in $\mathbf{H}^3/\rho_t(f)$, with meridian the curve that has trivial holonomy. This 2 torus is fibered over a curve that converges to the singular locus. Thus we extend D_t to the universal covering of the corresponding solid torus V in the smooth part of $\mathcal{O}^3$. The map D_t restricted to each compact subset of $\partial\tilde{V}$ converges to $\partial\mathcal{P}$, coherently with the fibration. Then we choose D_t so that restricted to compact subsets of $\tilde{V}$ converges to the D_0. □

Appendix A. Appendix: Infinitesimal isometries

The Lie algebra of infinitesimal isometries of $\mathbf{H}^3$ is $\mathfrak{sl}_2(\mathbf{C})$. It is naturally equipped with the complex Killing form

$$B:\mathfrak{sl}_2(\mathbf{C}) \times \mathfrak{sl}_2(\mathbf{C}) \to \mathbf{C}$$

defined by $B(\mathfrak{a},\mathfrak{b}) = \mathrm{Trace}(Ad_{\mathfrak{a}} \circ Ad_{\mathfrak{b}}) = 4\,\mathrm{Trace}(\mathfrak{a}\mathfrak{b})$. Thus,

$$B\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix}\right) = 8ax + 4bz + 4cy.$$

Let $\mathfrak{a}$ be an infinitesimal isometry of complex length l (ie. $\exp(t\mathfrak{a})$ has complex length $t\,l$). Then

$$l = \pm\frac{1}{\sqrt{2}}\sqrt{B(\mathfrak{a},\mathfrak{a})}.$$

In particular $B(\mathfrak{a},\mathfrak{a}) = 0$ iff $\mathfrak{a}$ is trivial or parabolic.

The Killing vector field corresponding to $\mathfrak{a} \in \mathfrak{sl}_2(\mathbf{C})$ is the field tangent to the orbits of $\exp(t\mathfrak{a})$, the one-parameter group of diffeomorphism of $\mathbf{H}^3$. We notice that for $x \in \mathbf{H}^3$, the Killing vector field evaluated at x is the translational part of $\mathfrak{a}$ at x. When $\mathfrak{a}$ is not parabolic, then $\exp(t\mathfrak{a})$ has an invariant axis, that it is also the minimizing locus for the norm of the Killing vector field. This axis is denoted by $\mathrm{A}(\mathfrak{a})$. The Killing vector field has nonempty vanishing locus iff $\mathfrak{a}$ is an infinitesimal rotation, then it vanishes precisely at $\mathrm{A}(\mathfrak{a})$.

When $\mathfrak{a}$ is parabolic, then $\exp(t\mathfrak{a})$ fixes a point at ∞, that we denote by $\mathrm{A}_\infty(\mathfrak{a}) \in \partial_\infty \mathbf{H}^3$.

Following Fenchel [10], we denote by $d_{\mathbf{C}}$ the complex distance between two geodesics, ie. the real part is the metric distance and the imaginary part the rotation angle.

Proposition A.1. — *Let $\mathfrak{a},\mathfrak{b} \in \mathfrak{sl}_2(\mathbf{C})$ be two nonzero and nonparabolic infinitesimal isometries. Then*

$$\frac{B(\mathfrak{a},\mathfrak{b})^2}{B(\mathfrak{a},\mathfrak{a})B(\mathfrak{b},\mathfrak{b})} = \cosh^2 d_{\mathbf{C}}(\mathrm{A}(\mathfrak{a}),\mathrm{A}(\mathfrak{b})).$$

Proof. — Notice that $B(\mathfrak{a},\mathfrak{a}) = -8\det(\mathfrak{a})$. Thus since traceless matrices in $SL_2(\mathbf{C})$ are π-rotations in $\mathbf{H}^3$,

$$\sqrt{\frac{-8}{B(\mathfrak{a},\mathfrak{a})}}\,\mathfrak{a} \in SL_2(\mathbf{C})$$

is a rotation of angle π around $\mathrm{A}(\mathfrak{a})$. Hence, the product

$$\sqrt{\frac{-8}{B(\mathfrak{a},\mathfrak{a})}}\,\mathfrak{a}\sqrt{\frac{-8}{B(\mathfrak{b},\mathfrak{b})}}\,\mathfrak{b}=\frac{\pm 8}{\sqrt{B(\mathfrak{a},\mathfrak{a})B(\mathfrak{b},\mathfrak{b})}}\,\mathfrak{a}\mathfrak{b}$$

is an isometry of complex length $2d_{\mathbf{C}}(\mathrm{A}(\mathfrak{a}),\mathrm{A}(\mathfrak{b}))$. Thus

$$\mathrm{Trace}(\frac{\pm 8}{\sqrt{B(\mathfrak{a},\mathfrak{a})B(\mathfrak{b},\mathfrak{b})}}\,\mathfrak{a}\mathfrak{b})=\pm 2\cosh d_{\mathbf{C}}(\mathrm{A}(\mathfrak{a}),\mathrm{A}(\mathfrak{b}))$$

The proposition follows from this formula and $B(\mathfrak{a},\mathfrak{b})=4\,\mathrm{Trace}(\mathfrak{a}\mathfrak{b})$. □

The idea of considering elements of the Lie algebra that are also in $SL_2(\mathbf{C})$ as rotations of angle π is taken from the so called line geometry in Marden's book [21, Ch. 7].

Remark A.2. — In the previous proposition, if $\mathfrak{a},\mathfrak{b}\in\mathfrak{sl}_2(\mathbf{C})$ are infinitesimal rotations of respective angles α and β, and if we have $\alpha,\beta>0$, then it makes sense to talk about orientation of their axis. In this case we have:

$$B(\mathfrak{a},\mathfrak{b})=-2\alpha\beta\cosh d_{\mathbf{C}}(\mathrm{A}(\mathfrak{a}),\mathrm{A}(\mathfrak{b})). \tag{A.1}$$

This remark follows immediately from Proposition A.1 and a continuity argument, by deforming first β to α and then by moving one of the oriented edges to the other, because $B(\mathfrak{a},\mathfrak{a})=-2\alpha^2$.

Given four points $z_1,z_2,z_3,z_4\in\partial\mathbf{H}^3=\mathbf{C}\cup\infty$, the *cross ratio* is

$$[z_1:z_2:z_3:z_4]=\frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}\in\mathbf{C}\cup\{\infty\}.$$

Proposition A.3. — *Let $\mathfrak{a},\mathfrak{b}\in\mathfrak{sl}_2(\mathbf{C})$ be two nonzero infinitesimal isometries. Assume that $\mathfrak{a}$ is parabolic and $\mathfrak{b}$ is not. Then*

(1) *$B(\mathfrak{a},\mathfrak{b})=0$ iff $\mathrm{A}_\infty(\mathfrak{a})$ is an endpoint of $\mathrm{A}(\mathfrak{b})$.*
(2) *Let $p_+,p_-\in\mathbf{C}\cup\{\infty\}$ be the endpoints of $\mathrm{A}(\mathfrak{b})$, and assume that they are both different from $\mathrm{A}_\infty(\mathfrak{a})$. Then*

$$\frac{B(\mathfrak{a},\mathfrak{b})^2}{B(\mathfrak{b},\mathfrak{b})}=\frac{8}{t^2}[p_+:e^{t\mathfrak{a}}(p_-):e^{t\mathfrak{a}}(p_+):p_-].$$

Proof. — Up to conjugacy, we may assume that $\mathrm{A}_\infty(\mathfrak{a})=\infty$ in the upper half space model. Then

$$\mathfrak{a}=\begin{pmatrix}0&a\\0&0\end{pmatrix}\text{ and }\mathfrak{b}=\begin{pmatrix}u&v\\w&-u\end{pmatrix}.$$

With this expressions, $B(\mathfrak{a},\mathfrak{b})=4aw$, and the first assertion of the proposition follows from the fact that $w=0$ iff ∞ is an endpoint of the axis of

$\mathfrak{b}$. To prove Assertion 2, we may assume up to further conjugation that the axis of $\mathfrak{b}$ has endpoints $p_\pm = \pm x \in \mathbf{R} \setminus \{0\}$. Hence

$$\mathfrak{a} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ and } \mathfrak{b} = \begin{pmatrix} 0 & yx \\ y/x & 0 \end{pmatrix}$$

with $B(\mathfrak{b},\mathfrak{b}) = 8y^2$. Thus:

$$\frac{B(\mathfrak{a},\mathfrak{b})^2}{B(\mathfrak{b},\mathfrak{b})} = \frac{(4ay/x)^2}{8y^2} = 2\frac{a^2}{x^2}.$$

On the other hand,

$$[p_+ : e^{t\mathfrak{a}}(p_-) : e^{t\mathfrak{a}}(p_+) : p_-] = [x : -x + t\,a : x + t\,a : -x] = \frac{t^2 a^2}{4x^2},$$

and the formula follows. □

Using Proposition A.3 and the computations in its proof we have the following remark:

Remark A.4. — Assume $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbf{C})$ satisfy $B(\mathfrak{a},\mathfrak{b}) \neq 0$, $\mathfrak{a}$ is parabolic, and $\mathfrak{b}$ is not. The horosphere centered at $\mathrm{A}_\infty(\mathfrak{a}) \in \partial_\infty \mathbf{H}^3$ and tangent to the axis $\mathrm{A}(\mathfrak{b})$ has a natural complex structure, up to homothety. Fix a complex structure in which $1 \in \mathbf{C}$ is the unit tangent vector to $\mathrm{A}(\mathfrak{b})$, and suppose that $tz \in \mathbf{C}$ is the translation vector of $e^{t\mathfrak{a}}$ in this horosphere. Then:

$$\frac{B(\mathfrak{a},\mathfrak{b})^2}{B(\mathfrak{b},\mathfrak{b})} = 2z^2.$$

By looking at the homothety factor of the complex structure on different horospheres with the same center, we get:

Corollary A.5. — *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{sl}_2(\mathbf{C})$ be three nonzero infinitesimal isometries. Assume that $\mathfrak{a}$ is parabolic, but $\mathfrak{b}$ and $\mathfrak{c}$ are not. If*

$$\frac{B(\mathfrak{a},\mathfrak{b})^2}{B(\mathfrak{b},\mathfrak{b})} = \frac{B(\mathfrak{a},\mathfrak{c})^2}{B(\mathfrak{c},\mathfrak{c})} \neq 0,$$

then the axis $\mathrm{A}(\mathfrak{b})$ and $\mathrm{A}(\mathfrak{c})$ are tangent to the same horosphere centered at $\mathrm{A}_\infty(\mathfrak{a}) \in \partial_\infty \mathbf{H}^3$. Moreover, their tangent directions are parallel in the Euclidean structure of the horosphere.

Finally we deal with the case where $\mathfrak{a}$ and $\mathfrak{b}$ are both parabolic.

Proposition A.6. — *Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbf{C})$ be two nonzero infinitesimal parabolic isometries, with respective fixed points at infinity $\mathrm{A}_\infty(\mathfrak{a})$ and $\mathrm{A}_\infty(\mathfrak{b})$. Then:*

(1) $B(\mathfrak{a},\mathfrak{b}) = 0$ *iff* $\mathrm{A}_\infty(\mathfrak{a}) = \mathrm{A}_\infty(\mathfrak{b})$.

(2) *When* $\mathrm{A}_\infty(\mathfrak{a}) \neq \mathrm{A}_\infty(\mathfrak{b})$,

$$B(\mathfrak{a},\mathfrak{b}) = \frac{4}{t^2}[\mathrm{A}_\infty(\mathfrak{a}) : \mathrm{A}_\infty(\mathfrak{b}) : e^{t\mathfrak{b}}(\mathrm{A}_\infty(\mathfrak{a})) : e^{t\mathfrak{a}}(\mathrm{A}_\infty(\mathfrak{b}))].$$

Proof. — The first assertion is an elementary computation, with a proof analogous to the first statement of Proposition A.3. For the second one, assume $\mathrm{A}_\infty(\mathfrak{a}) = \infty$ and $\mathrm{A}_\infty(\mathfrak{b}) = 0$. Hence

$$\mathfrak{a} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \text{ and } \mathfrak{b} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}.$$

Then $B(\mathfrak{a},\mathfrak{b}) = 4xy$. On the other hand, $e^{t\mathfrak{a}}(0) = tx$ and $e^{t\mathfrak{b}}(\infty) = 1/(ty)$, and the formula is straightforward. $\square$

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