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POSITIVITY ON SUBVARIETIES AND VANISHING OF HIGHER COHOMOLOGY

by Alex KÜRONYA (*)

ABSTRACT. — We study the relationship between positivity of restriction of line bundles to general complete intersections and vanishing of their higher cohomology. As a result, we extend classical vanishing theorems of Kawamata–Viehweg and Fujita to possibly non-nef divisors.

RÉSUMÉ. — Nous étudions la relation entre la positivité des restrictions aux intersections complètes générales de fibrés en droites et l'annulation de leur cohomologie supérieure. En application, nous étendons des théorèmes d'annulation classiques de Kawamata-Viehweg et Fujita à des diviseurs potentiellement non-nefs.

Introduction

Inspired by the recent paper [14] of Totaro, we investigate the relationship between ampleness of restrictions of line bundles to general complete intersections and the vanishing properties of higher cohomology groups. This train of thought eventually led us to a generalization of Fujita's vanishing theorem for big line bundles.

Vanishing theorems played a central role in algebraic geometry during the last fifty years. Results of this sort due to Serre, Kodaira, Kawamata–Viehweg among others are fundamental building blocks of complex geometry, and are indispensable to the successes of minimal model theory. Classically, vanishing theorems apply to ample or big and nef line bundles. However, there has been a recent shift of attention towards big line bundles, which, although possess less positivity, still turn out to share many of the good properties of ample ones (see [5] and the references therein).

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It has been known for some time that big line bundles have very small or vanishing cohomology groups roughly above the dimension of the stable base locus. An easy asymptotic version of this appeared in [10, Proposition 2.15], while Matsumura [12, Theorem 1.6] gave a partial generalization of the Kawamata–Viehweg vanishing theorem along these lines. In this work we will present similar results guaranteeing the vanishing of cohomology groups of high degree under various partial positivity conditions. We work over the complex numbers, divisors are meant to be Cartier unless otherwise mentioned.

Following the footsteps of Andreotti–Grauert [1] and Demailly–Peternell–Schneider [4], Totaro establishes a very satisfactory theory of line bundles with vanishing cohomology above a certain degree. One calls a line bundle L (naively) q-ample on X for a natural number q, if for every coherent sheaf $\mathcal{F}$ on X there exists an integer m_0 depending on $\mathcal{F}$ such that

$$H^i\left(X, \mathcal{F} \otimes \mathcal{O}_X(mL)\right) = 0 \quad \text{for all } i > q \text{ and } m \geqslant m_0.$$

It is immediate from the definition that 0-ampleness coincides with ampleness, while it is proved in [14, Theorem 10.1] that a divisor is $(n-1)$-ample exactly if it does not lie in the negative of the pseudo-effective cone in the Néron–Severi space. The notion of q-ampleness shares many important properties of traditional ampleness, for example it is open both in families and in the Néron–Severi space. In general, the behaviour of q-ample divisors remains mysterious.

Our motivation comes from the connection to geometric invariants describing partial positivity, in particular, to amplitude on restrictions to general complete intersection subvarieties. The main results of this work are vanishing theorems valid for not necessarily ample — oftentimes not even pseudo-effective — divisors. They all follow the same principle: positivity of restrictions of line bundles results in partial vanishing of higher cohomology groups. Our first statement of note is a generalization of Fujita's vanishing theorem.

Theorem A (Theorem 1.2). — *Let X be a complex projective scheme, L a Cartier divisor, $A_1, \ldots, A_q$ very ample Cartier divisors on X such that $L|_{E_1 \cap \cdots \cap E_q}$ is ample for general $E_j \in |A_j|$, $1 \leqslant j \leqslant q$. Then for any coherent sheaf $\mathcal{F}$ on X there exists an integer $m(L, A_1, \ldots, A_q, \mathcal{F})$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mL + N)) = 0$$

for all $i > q$, $m \geqslant m(L, A_1, \ldots, A_q, \mathcal{F})$ and all nef divisors N on X.

In particular, by setting $N = 0$ in the Theorem we obtain that L is q-ample. This way we recover a slightly weaker version of [4, Theorem 3.4]. In a few remarks we then relate Theorem A to invariants expressing partial positivity and the inner structure of various cones of divisors in the Néron–Severi space. Here again we go along lines similar to [4]; it turns out that sacrificing a certain amount of generality buys drastically simplified proofs.

An important feature of the above theorem is that it extends vanishing to not necessarily pseudo-effective divisors.

Once we extend the notion of stable and augmented base loci to schemes, the following more geometric version of our result becomes available.

Theorem B (Corollary 2.6). — *Let X be a complex projective scheme, L a Cartier divisor, $\mathcal{F}$ a coherent sheaf on X. Then there exists a positive integer $m_0(L,\mathcal{F})$ such that*

$$H^i\left(X, \mathcal{F} \otimes \mathcal{O}_X(mL + D)\right) = 0$$

for all $i > \dim \mathbf{B}_+(L)$, $m \geqslant m_0(L,\mathcal{F})$, and all nef divisors D on X.

Here $\mathbf{B}_+(L)$ denotes the augmented base locus of L introduced in [6]; this can be defined as the stable base locus of the $\mathbb{Q}$-Cartier divisor $L - A$ for any sufficiently small ample class A.

Last, we treat the case of vanishing for adjoint divisors; more precisely, a variant of the theorem of Kawamata and Viehweg.

Theorem C (Theorem 3.1). — *Let X be a smooth projective variety, L a divisor, A a very ample divisor on X. If $L|_{E_1 \cap \cdots \cap E_k}$ is big and nef for a general choice of $E_1, \ldots, E_k \in |A|$, then*

$$H^i\left(X, \mathcal{O}_X(K_X + L)\right) = 0 \quad \text{for } i > k.$$

The conditions of Matsumura hold true under our assumptions, hence we recover [12, Theorem 1.6].

A few words about the organization of the paper. Section 1 is devoted to Theorem A, and a discussion of invariants measuring partial positivity. Theorem B along with a short treatment of base loci on schemes takes up Section 2. The proof of Theorem C is treated in the last section.

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1. Ampleness on restrictions and cones in the Néron–Severi space

In this section we prove a Fujita–Serre type vanishing statement and consider an application to cone structures in $N^1(X)_{\mathbb{R}}$. This is where one can see most clearly the yoga of obtaining partial vanishing of higher cohomology groups by forcing ampleness on restrictions. A scheme is a separated scheme of finite type over the complex number field.

For future reference, we first state Fujita's vanishing theorem (see [8], or [11, 1.4.35] for proofs).

Theorem 1.1 (Fujita). — *Let X be a complex projective scheme, L an ample divisor, $\mathcal{F}$ a coherent sheaf on X. Then there exists a positive number $m_0 = m_0(L, \mathcal{F})$ such that*

$$H^i\left(X, \mathcal{O}_X(mL + N) \otimes \mathcal{F}\right) = 0$$

for all $i > 0$, all $m \geqslant m_0$, and all nef divisors N.

Here is our generalization.

Theorem 1.2. — *Let X be a complex projective scheme, L a Cartier divisor, $A_1, \ldots, A_q$ very ample Cartier divisors on X such that $L|_{E_1 \cap \cdots \cap E_q}$ is ample for general $E_j \in |A_j|$. Then for any coherent sheaf $\mathcal{F}$ on X there exists an integer $m(L, A_1, \ldots, A_q, \mathcal{F})$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mL + N + \sum_{j=1}^{q} k_j A_j)) = 0$$

for all $i > q$, $m \geqslant m(L, A_1, \ldots, A_q, \mathcal{F})$, $k_j \geqslant 0$, and all nef divisors N.

Proof. — For every $1 \leqslant j \leqslant q$, pick a general element $E_j \in |A_j|$ ($1 \leqslant j \leqslant q$), and let N be an arbitrary nef divisor on X. Consider the set of standard exact sequences

$$
\begin{aligned}
0 &\to \mathcal{F}\otimes \mathcal{O}_{Y_j}(mL+N+\sum_{l=1}^{q} k_lA_l) \\
&\to \mathcal{F}\otimes \mathcal{O}_{Y_j}(mL+N+\sum_{l=1}^{q} k_lA_l + A_{j+1}) \\
&\to \mathcal{F}\otimes \mathcal{O}_{Y_{j+1}}(mL+N+\sum_{l=1}^{q} k_lA_l + A_{j+1}) \to 0
\end{aligned}
\tag{1.1}
$$

for all $0 \leqslant j \leqslant q-1$, all m, and all $k_1,\dots,k_q \geqslant 0$. Here $Y_j \stackrel{\text{def}}{=} E_1\cap\dots\cap E_j$ for all $1\leqslant j\leqslant q$, for the sake of completeness set $Y_0 \stackrel{\text{def}}{=} X$. Take a look at the sequences with $j=q-1$.

Fujita's vanishing theorem on $Y_q = E_1\cap\dots\cap E_q$ applied to the ample divisor $L|_{Y_q}$ gives that

$$
H^i(Y_q,\mathcal{F}\otimes\mathcal{O}_{Y_q}(mL+N+\sum_{l=1}^{q}k_lA_l)) \;=\; 0
$$

for all $i\geqslant 1$, $m\geqslant m(\mathcal{F},L,A_1,\dots,A_q,Y_q)$, all $k_1,\dots,k_q\geqslant 0$, and all nef divisors N on X.

This implies that first and the last group in the exact sequence

$$
\begin{aligned}
&H^{i-1}(Y_q,\mathcal{F}\otimes\mathcal{O}_{Y_q}(mL+N+\sum_{l=1}^{q}k_lA_l+A_q)) \\
&\qquad\to H^i(Y_{q-1},\mathcal{F}\otimes\mathcal{O}_{Y_{q-1}}(mL+N+\sum_{l=1}^{q}k_lA_l)) \\
&\qquad\to H^i(Y_{q-1},\mathcal{F}\otimes\mathcal{O}_{Y_{q-1}}(mL+N+\sum_{l=1}^{q}k_lA_l+A_q)) \\
&\qquad\to H^i(Y_q,\mathcal{F}\otimes\mathcal{O}_{Y_q}(mL+N+\sum_{l=1}^{q}k_lA_l+A_q))
\end{aligned}
$$

vanish for $i\geqslant 2$, $m\geqslant m(\mathcal{F},L,A_1,\dots,A_q,Y_q)$, $k_1,\dots,k_q\geqslant 0$, and all nef divisors N. Consequently,

$$
\begin{aligned}
&H^i(Y_{q-1},\mathcal{F}\otimes\mathcal{O}_{Y_{q-1}}(mL+N+\sum_{l=1}^{q}k_lA_l)) \\
&\qquad\simeq H^i(Y_{q-1},\mathcal{F}\otimes\mathcal{O}_{Y_{q-1}}(mL+N+\sum_{l=1}^{q}k_lA_l+kA_q))
\end{aligned}
$$

for all $i \geqslant 2$, $m \geqslant m(\mathcal{F}, L, A_1, \ldots, A_q, Y_q)$, N nef, $k \geqslant 0$, and $k_1, \ldots, k_q \geqslant 0$. Then

$$H^i(Y_{q-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{q-1}}(mL + N + \sum_{l=1}^{q} k_l A_l + kA_q)) \;=\; 0$$

follows for all $k \geqslant 0$ from Serre vanishing applied to the ample divisor $A_q|_{Y_{q-1}}$. By the semicontinuity theorem and the general choice of the E_j's we can drop the dependence on Y_q.

Next, we will work backwards along the cohomology sequences associated to the cohomology long exact sequences (1.1). We obtain by descending induction on j from the fragment

$$\begin{aligned} H^{i-1}(Y_j, \mathcal{F} \otimes \mathcal{O}_{Y_j}(mL + N + \sum_{l=1}^{q} k_l A_l + A_q)) \\ \to H^i(Y_{j-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{j-1}}(mL + N + \sum_{l=1}^{q} k_l A_l)) \\ \to H^i(Y_{j-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{j-1}}(mL + N + \sum_{l=1}^{q} k_l A_l + A_q)) \\ \to H^i(Y_j, \mathcal{F} \otimes \mathcal{O}_{Y_j}(mL + N + \sum_{l=1}^{q} k_l A_l + A_q)) \end{aligned}$$

that

$$\begin{aligned} H^i(Y_{j-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{j-1}}(mL + N + \sum_{l=1}^{q} k_l A_l)) \\ \simeq H^i(Y_{j-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{j-1}}(mL + N + \sum_{l=1}^{q} k_l A_l + kA_q)) \end{aligned}$$

for all $i > q-j$, $m \geqslant m(\mathcal{F}, L, A_1, \ldots, A_q, Y_j)$, N nef, $k \geqslant 0$, and $k_1, \ldots, k_q \geqslant 0$.

Arguing as above with the help of Serre's vanishing theorem, we arrive at

$$H^i(Y_j, \mathcal{F} \otimes \mathcal{O}_{Y_j}(mL + N + \sum_{l=1}^{q} k_l A_l)) \;=\; 0$$

for $i > q - j$, $m \gg 0$, and all $k_1, \ldots, k_q \geqslant 0$. This gives the required result for $j = 0$. □

Remark 1.3. — We point out that the proof works under the less restrictive assumption that A_i is ample, globally generated, and not composed of a pencil for all $1 \leqslant i \leqslant q$. This is a consequence of the base-point free Bertini theorem.

The next step is to connect up with q-ampleness. Partial positivity was studied in the form of uniform q-ampleness in [4], where among many other achievements it was established that uniform q-ampleness respects numerical equivalence of Cartier divisors. For the sake of completeness we briefly recall Totaro's main result on q-ample line bundles; this serves as a definition as well.

Theorem 1.4. — [14, Theorem 8.1] *Let X be a projective scheme over a field of characteristic zero, A a very ample divisor on X, $0 \leqslant q \leqslant n = \dim X$ an integer. Then there exists a natural number m_0 such that for all Cartier divisors L on X the following properties are equivalent.*

(1) *There exists a natural number n_0 such that $H^i\left(X, \mathcal{O}_X(n_0L - jA)\right) = 0$ for all $i > q$ and $1 \leqslant j \leqslant m_0$.*
(2) *(L is naively q-ample) For every coherent sheaf $\mathcal{F}$ on X there exists an integer $m(L, \mathcal{F})$ such that $H^i\left(X, \mathcal{F} \otimes \mathcal{O}_X(mL)\right) = 0$ for all $i > q$ and $m \geqslant m(L, \mathcal{F})$.*
(3) *(L is uniformly q-ample) There exists a constant $\lambda > 0$ such that for all $i > q$,$j > 0$ and $\frac{m}{j} \geqslant \lambda$ the cohomology groups $H^i\left(X, \mathcal{O}_X(mL - jA)\right)$ vanish.*

The first consequence of Theorem 1.2 is the following claim, which was also proved by Demailly–Peternell–Schneider under the more general assumption that L is $(n-q)$-flag ample (see [4, Definition 3.1]). Their proof however requires considerably more effort.

Corollary 1.5. — *With notation as above, if $L|_{E_1 \cap \cdots \cap E_q}$ is ample for general $E_j \in |A_j|$, then L is q-ample.*

The above vanishing result provides a birational variant for the higher asymptotic cohomology $\widehat{h}^i\left(X, L\right)$ of L. We remind that

$$\widehat{h}^i\left(X, L\right) \stackrel{\text{def}}{=} \limsup_m \frac{h^i\left(X, \mathcal{O}_X(mL)\right)}{m^n/n!} \ ;$$

note that $\widehat{h}^0(X, L)$ gives the volume of L. For properties of higher asymptotic cohomology the reader is referred to [7, 10], or Demailly's paper [3] in the analytic setting.

Corollary 1.6. — *Let X be an irreducible projective variety, L a Cartier divisor on X. Assume that there exists a proper birational morphism $\pi : Y \to X$, a natural number q, and very ample divisors $A_1, \ldots, A_q$*

on Y such that $\pi^*L|_{E_1\cap\cdots\cap E_q}$ is ample for general elements $E_i\in|A_i|$. Then

$$\widehat{h}^i\,(X,L)\;=\;0\quad\textit{for } i>q.$$

Proof. — As a consequence of Theorem 1.2, one has $H^i\,(Y,\pi^*\mathcal{O}_X(mL))=0$ for all $i>q$ and $m\gg 0$. This gives $\widehat{h}^i\,(Y,\pi^*L)=0$ for all $i>q$. By the birational invariance of asymptotic cohomology [10, Corollary 2.10]

$$\widehat{h}^i\,(X,L)\;=\;\widehat{h}^i\,(Y,\pi^*L)\;=\;0\;\;\text{for all } i>q.$$

□

We move on to building a connection to the interior structure of the Néron–Severi space. For an arbitrary integral Cartier divisor L

$$q(L)\stackrel{\text{def}}{=}\min\{q\in\mathbb{N}\,|\,L\text{ is }q\text{-ample}\}$$

is an interesting numerical invariant, which was probably first defined in [4, Definition 1.1]. To put it into perspective, let us briefly recall some other ways of expressing partial ampleness associated to big divisors (these were discussed in an earlier version of [7]).

$$\begin{aligned}
a(L;A_1,\ldots,A_n) &\stackrel{\text{def}}{=} \min\{k\mid L|_{E_1\cap\cdots\cap E_k}\text{ is ample}\\
&\qquad\qquad\qquad\text{for very general } E_i\in|A_i|\big\},\\
b(L) &\stackrel{\text{def}}{=} \dim\mathbf{B}_+(L),\\
c(L) &\stackrel{\text{def}}{=} \max\big\{i\mid\widehat{h}^i\text{ is not identically zero}\\
&\qquad\qquad\qquad\text{in any neighborhood of } [L]\in N^1(X)_{\mathbb{R}}\big\},
\end{aligned}$$

where $A_1,\ldots,A_n$ are very ample divisors on X. The minimum of all $a(L;A_1,\ldots,A_n)$ (over all sequences of very ample divisors of length $n=\dim X$) is closely related to $\sigma_+(L)$ defined in [4]. For the definition and properties of the augmented base locus $\mathbf{B}_+(L)$ see either Section 2 or [6].

The quantities $q(L)$, $a(L,A)$, $b(L)$, and $c(L)$ depend only on the numerical equivalence class of L, and make good sense for $\mathbb{Q}$-divisors as well. They all express how far a given divisor is from being ample, with smaller numbers corresponding to more positivity.

Corollary 1.7. — *With notation as above,*

$$c(L)\;\leqslant\;q(L)\;\leqslant\;a(L;A_1,\ldots,A_n)\;\leqslant\;b(L)$$

for all sequences of very ample divisors $A_1,\ldots,A_n$.

Proof. — The first inequality comes from observing the definition of $\widehat{h}^i$ and the openness of q-ampleness in the Néron–Severi space (see [14, Theorem 8.2]). The second one is [4, 3.4], at the same time, it is immediate from Theorem 1.2.

The inequality $a(L; A_1, \ldots, A_n) \leqslant b(L)$ comes from the observation that the restriction of a Cartier divisor to a general very ample divisor strictly reduces $\dim \mathbf{B}_+(L)$, and the fact that a divisor with empty augmented base locus is ample. □

As it was noticed on [4, p. 167.], one does not have equality in $q(L) \leqslant a(L; A_1, \ldots, A_n)$ in general. Here we present another simple example (borrowed again from an early version of [7]) exhibiting this property. More precisely, we give an example of a divisor L that is 1-ample, and a very ample divisor A such that $L|_E$ is not ample for general $E \in |A|$.

Example 1.8. — Let $X = \mathbb{F}_1 \times \mathbb{P}^1$, where $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at a point, and denote by $p : X \to \mathbb{F}_1$ and $q : X \to \mathbb{P}^1$ the two projections. Let $E \subset \mathbb{F}_1$ be the exceptional curve of the blow-up, $F \subset \mathbb{F}_1$ be a fiber of the ruling, and let $H \subset \mathbb{P}^1$ be a point. We consider the divisors

$$L = p^*(\lambda E + F) + q^* H \quad \text{and} \quad A = p^*(E + \mu F) + q^* H \quad \text{for some} \quad \lambda, \mu \in \mathbb{Z}_{\geqslant 2}.$$

Note that A is very ample and L is big. The stable base locus of L coincides with its augmented base locus and is equal to $B \stackrel{\text{def}}{=} p^{-1}(E)$. In particular $b(L) = 2$.

On the other hand, the Künneth's formula for asymptotic cohomology (see [10, Remark 2.14]), and the fact that L is not ample imply that $c(L) = 1$. Fix a general element $Y \in |A|$ cutting out a smooth divisor D on B. Note that $\mathcal{O}_B(D) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu - 1, 1)$ via the isomorphism $B = E \times \mathbb{P}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Therefore, since D is smooth, it must be irreducible; moreover, p induces an isomorphism $D \cong \mathbb{P}^1$. We observe that the base locus of $L|_Y$ is contained in the restriction of the base locus of L, hence in D, and that $\mathcal{O}_D(L|_Y) \cong \mathcal{O}_{\mathbb{P}^1}(\mu - \lambda)$. We conclude that

$$a(L, A) \;=\; \begin{cases} 2 = b(L) & \text{if } \lambda \geqslant \mu, \\ 1 = c(L) & \text{if } \lambda < \mu. \end{cases}$$

For $\lambda < \mu$, we have $1 = c(L) \leqslant q(L) \leqslant a(L; A) = 1$, therefore $q(L) = 1$. On the other hand there exist very ample divisors on X such that $L|_A$ is not ample.

Totaro asks in [14, Question 12.1] whether $c(L) = q(L)$ always holds. As a result of the discussion so far, we can see that the respective behaviours of $q(L)$ and $c(L)$ with respect to $a(L; A_1, \ldots, A_n)$ have similarities, which furnishes some evidence that the answer to Totaro's question is affirmative.

We give a reformulation of these concepts in terms of cones of divisors. This leads to a connection with strongly movable curves as defined in [2, Definition 1.3].

Definition 1.9. — *For a natural number q and very ample divisors $A_1, \ldots, A_q$ on a projective variety X set*

$$\mathcal{C}_{A_1,\ldots,A_q}(X) \stackrel{\text{def}}{=} \left\{ \alpha \in N^1(X)_{\mathbb{R}} \mid \ \alpha|_{E_1 \cap \cdots \cap E_q} \text{ ample for } E_i \in |A_i| \text{ general} \right\} .$$

In addition, let $\mathrm{Amp}_q(X)$ *denote the open (but not necessarily convex) cone of q-ample divisor classes.*

It is immediate that $\mathcal{C}_{A_1,\ldots,A_n}(X) \subseteq N^1(X)_{\mathbb{R}}$ is a convex cone, it is also open by [14, Section 9]; for simplicity we set $\mathcal{C}_{\emptyset}(X) = \mathrm{Amp}(X)$.

Remark 1.10. — In the important special case $q = n-1$ a general codimension $n-1$ complete intersection is an irreducible curve, and one has

$$\mathcal{C}_{A_1,\ldots,A_{n-1}}(X) = \left\{ \alpha \in N^1(X)_{\mathbb{R}} \,|\, (\alpha \cdot A_1 \cdot \cdots \cdot A_{n-1}) > 0 \right\} .$$

Corollary 1.7 can then be rephrased in the following way.

Corollary 1.11. —

$$\bigcup_{A_1,\ldots,A_q} \mathcal{C}_{A_1,\ldots,A_q}(X) \subseteq \mathrm{Amp}_q(X)$$

Remark 1.12. — The question arises naturally whether the two sides of Corollary 1.11 are equal in general. This is quickly seen to be true on surfaces. By [14, Theorem 9.1] L is 1-ample if and only if $-L$ is not pseudoeffective, which is equivalent to the existence of a very ample divisor A for which $(-L \cdot A) < 0$. This latter holds precisely when $L|_E$ is ample for $E \in |A|$ general.

In general one obstruction that is easy to foresee is the existence of strongly movable curves on X that are not in the boundary of the cone spanned by complete intersection curves.

Proposition 1.13. — *Let X be an irreducible projective variety of dimension n. Then*

$$\bigcup_{A_1,\ldots,A_{n-1}} \mathcal{C}_{A_1,\ldots,A_{n-1}}(X) = \mathrm{Amp}_{n-1}(X)$$

exactly if every strongly movable curve is the limit of elements of the convex cone spanned by complete intersection curves coming from very ample divisors.

Among other results, the proof utilizes a characterization of the pseudo-effective due from [2], which we now state for reference.

Theorem 1.14 ([2], Theorem 2.2). — *Let X be a projective variety. A class $\alpha \in N^1(X)_{\mathbb{R}}$ is pseudo-effective if and only if it is in the dual cone of the cone of strongly movable curves.*

Proof. — For a Cartier divisor L on X, $L|_{E_1\cap\cdots\cap E_{n-1}}$ is ample if and only if $(L\cdot A_1\cdot\cdots\cdot A_{n-1}) > 0$. Consequently, $L \in \cup_{A_1,\ldots,A_{n-1}}\mathcal{C}_{A_1,\ldots,A_{n-1}}(X)$ holds exactly if there exists a complete intersection curve coming from very ample divisors intersected by L positively.

By [14, Theorem 9.1], a line bundle is $(n-1)$-ample precisely if $-L$ is not pseudo-effective. The equality of the Proposition is equivalent via Theorem 1.14 to the property that a divisor intersecting every complete intersection curve of very ample divisors positively necessarily intersects every movable curve positively. This happens precisely if the cone spanned by complete intersection curves on X is dense in the cone of moving curves. □

Example 1.15. — In [13, Example 3.2.4] Neumann constructs a smooth projective threefold X on which the cone spanned by complete intersection curves is not dense in the movable cone. The space he constructs is a double blow-up of $\mathbb{P}^3$: first one blows up a line in $\mathbb{P}^3$, then a point on the exceptional divisor. The work [13] gives all the details. By Proposition 1.13

$$\bigcup_{A_1,\ldots,A_{n-1}} \mathcal{C}_{A_1,\ldots,A_{n-1}}(X) \subsetneq \mathrm{Amp}_{n-1}(X)\ .$$

Duality links the following question to the relationship between complete intersection curves and strongly movable curves.

Question 1.16. — *Let X be an irreducible projective variety. Under what condition does*

$$\bigcup_{A_1,\ldots,A_q} \mathcal{C}_{A_1,\ldots,A_q}(X) = \mathrm{Amp}_q(X)$$

hold for all $0 \leqslant q \leqslant n-1$?

2. Base loci on schemes

Here we treat base loci of line bundles on arbitrary schemes, and discuss a geometric generalization of Fujita's vanishing theorem for big divisors. Although a large part of what we do is straightforward, the topic has not been investigated much so far, and there is no suitable reference available. In the course of this section X is an arbitrary scheme unless otherwise mentioned.

It is customary (see [11, Section 1.1.B] for example) to define the base ideal sheaf of a Cartier divisor L on a complete algebraic scheme (where algebraic means separated of finite type over an algebraically closed field) over $\mathbb{C}$ to be

$$\mathfrak{b}(L) \stackrel{\text{def}}{=} \operatorname{im}\left(H^0\left(X, \mathcal{O}_X(L)\right) \otimes_{\mathbb{C}} \mathcal{O}_X(-L) \xrightarrow{\text{eval}_L} \mathcal{O}_X\right) .$$

Since for a short while we will be dealing with schemes instead of varieties, we set $\operatorname{Bs}(L)$ to be the closed subscheme of X given by $\mathfrak{b}(L)$ (and not the associated closed subset). Then it is usual to denote

$$\mathbf{B}(L) \stackrel{\text{def}}{=} \bigcap_{m=1}^{\infty} \operatorname{Bs}(mL)_{\text{red}} \subseteq X$$

as a closed subset. We can nevertheless define the base locus of an invertible sheaf in full generality.

Definition 2.1. — *Let X be a scheme, $\mathcal{L}$ an invertible sheaf on X. Let $\mathcal{F}_{\mathcal{L}}$ denote the quasi-coherent subsheaf of $\mathcal{L}$ generated by $H^0\left(X, \mathcal{L}\right)$. With this notation set*

$$\mathfrak{b}(\mathcal{L}) \stackrel{\text{def}}{=} \operatorname{ann}_{\mathcal{O}_X}(\mathcal{L}/\mathcal{F}_{\mathcal{L}}) ,$$

define $\operatorname{Bs}(\mathcal{L})$ to be the closed subscheme corresponding to $\mathfrak{b}(\mathcal{L})$, and let

$$\mathbf{B}(\mathcal{L}) \stackrel{\text{def}}{=} \bigcap_{m=1}^{\infty} \operatorname{Bs}(\mathcal{L}^{\otimes m})_{red} \subseteq X$$

as a closed subset of the topological space associated to X.

It is immediate that we recover the usual definition in the case X is complete and algebraic.

Lemma 2.2. — *Let X, Y be schemes, $f : Y \to X$ a map of schemes, $\mathcal{L}$ an invertible sheaf on X. Then*

$$\mathfrak{b}(\mathcal{L}) \cdot \mathcal{O}_Y \subseteq \mathfrak{b}(f^*\mathcal{L}) .$$

In particular, if $Y \subseteq X$ is a closed subscheme, then $\operatorname{Bs}(\mathcal{L}|_Y) \subseteq \operatorname{Bs}(\mathcal{L}) \cap Y$.

Proof. — Observe that to any quasi-coherent subsheaf $\mathcal{F}$ of an invertible sheaf $\mathcal{L}$ one can associate a (quasi-coherent) sheaf of ideals

$$\mathcal{I}(\mathcal{F}) \stackrel{\text{def}}{=} \operatorname{ann}_{\mathcal{O}_X} \mathcal{L}/\mathcal{F} .$$

The definition implies that $\mathcal{I}(\mathcal{F}) \subseteq \mathcal{I}(\mathcal{F}')$ whenever $\mathcal{F} \subseteq \mathcal{F}'$.

Considering the map $H^0(X, \mathcal{L}) \to H^0(Y, f^*\mathcal{L})$ obtained by pulling back sections, one observes that the map

$$f^*(\mathcal{F}_{\mathcal{L}}) \longrightarrow f^*\mathcal{L}$$

factors through the sheaf of modules $\mathcal{F}_{f^*\mathcal{L}}$. Consequently,

$$\mathfrak{b}(\mathcal{L}) \cdot \mathcal{O}_Y = \mathcal{I}(\operatorname{im} f^*(\mathcal{F}_{\mathcal{L}})) \subseteq \mathcal{I}(\mathcal{F}_{f^*\mathcal{L}}) = \mathfrak{b}(f^*\mathcal{L})$$

as we wanted. □

Corollary 2.3. — *If X is a scheme, $\mathcal{L}$ an invertible sheaf on X, and $Y \subseteq X$ a closed subscheme, then*

$$\dim \mathbf{B}(\mathcal{L}|_Y) \leqslant \dim \mathbf{B}(\mathcal{L}) ,$$

which immediately extends to the case of $\mathbb{Q}$-Cartier divisors.

Remark 2.4. — We point out that the proofs of both [11, Example 1.1.9] and [11, Proposition 2.1.21] go through unchanged when X is a noetherian scheme. This holds since both proofs depend only on the property

$$\mathfrak{b}(\mathcal{L}^{\otimes m}) \cdot \mathfrak{b}(\mathcal{L}^{\otimes k}) \subseteq \mathfrak{b}(\mathcal{L}^{\otimes (m+k)}) \text{ for all } m, k \geqslant 1 ,$$

which follows from the fact that one can multiply global sections. Therefore we obtain that $\mathbf{B}(\mathcal{L})$ is the unique minimal member of the collection of closed subsets $\{\operatorname{Bs}(\mathcal{L}^{\otimes m})_{\text{red}} \mid m \geqslant 1\}$. Moreover, just as in the reduced and irreducible case there exists $m_0 \in \mathbb{N}$ with the property that $\mathbf{B}(\mathcal{L}) = \operatorname{Bs}(\mathcal{L}^{\otimes p m_0})_{\text{red}}$ for all natural numbers p.

As a consequence, $\mathbf{B}(\mathcal{L}) = \mathbf{B}(\mathcal{L}^{\otimes m})$ for all positive integers m, and we are allowed to define the stable base locus for $\mathbb{Q}$-Cartier divisors by taking the stable base locus of a Cartier multiple.

From now on we will assume that X is projective. Following [6, Remark 1.3], we define the augmented base locus of a $\mathbb{Q}$-Cartier divisor L via

$$\mathbf{B}_+(L) \stackrel{\text{def}}{=} \bigcap_A \mathbf{B}(L - A)$$

where A runs through all ample $\mathbb{Q}$-Cartier divisors. For more information on augmented base loci the reader is referred to [6].

Remark 2.5. — Let us assume that X be defined over $\mathbb{C}$. Arguing as in the proof of [6, Proposition 1.5] we can see that for a given $\mathbb{Q}$-divisor L one can always find $\epsilon > 0$ such that

$$\mathbf{B}_+(L) \;=\; \mathbf{B}(L-A)$$

for any ample $\mathbb{Q}$-divisor with $\|A\| < \epsilon$ (with respect to an arbitrary norm on the Néron–Severi space). Exploiting Corollary 2.3 this implies that

$$\dim \mathbf{B}_+(L|_Y) \;\leqslant\; \dim \mathbf{B}_+(L)$$

for any closed subscheme Y in X.

If A is very ample Cartier divisor, and $E \in |A|$ is a general element, then it follows from Lemma 2.2 that

$$\dim \mathbf{B}_+(L|_E) \;<\; \dim \mathbf{B}_+(L)\ .$$

These simple observations result in the following statement.

Corollary 2.6. — *Let X be a complex projective scheme, L a Cartier divisor, $\mathcal{F}$ a coherent sheaf on X. Then there exists a positive integer $m_0(L,\mathcal{F})$ such that*

$$H^i\left(X, \mathcal{F}\otimes \mathcal{O}_X(mL+D)\right) \;=\; 0$$

for all $i > \dim \mathbf{B}_+(L)$, $m \geqslant m_0(L,\mathcal{F})$, and all nef divisors D on X.

Proof. — If $\dim \mathbf{B}_+(L) = \dim X$, then the claim holds for dimension reasons. Assume that $\dim \mathbf{B}_+(L) < \dim X$, and fix a very ample divisor A on X. Set $q = \dim \mathbf{B}_+(L)$.

Having chosen general elements $E_1,\ldots,E_q \in |A|$, we observe that $\mathbf{B}_+(L|_{E_1,\cap\cdots\cap E_q}) = \emptyset$ by Remark 2.5, which means that the restriction $L|_{E_1,\cap\cdots\cap E_q}$ is ample.

Thus, we have shown that L satisfies the conditions of Theorem 1.2 with $q = \dim \mathbf{B}_+(L)$ with respect to A. Then Theorem 1.2 gives the required vanishing. □

It turns out that above result can also be obtained in a technically more involved way following Fujita's original proof. The main tools are Corollary 3.2, reduction to the case of varieties, and Proposition 2.7, which is of independent interest. To relate Corollary 2.6 to the integral case, one imitates the proof of the claim that a Cartier divisor is ample precisely if it is ample when restricted to the irreducible components of the corresponding reduced subscheme.

Recall the fact that any coherent sheaf becomes globally generated after twisting with a high enough multiple of an ample line bundle. We point out to what extent this remains true for big line bundles.

Proposition 2.7. — *Let X be a projective scheme, L a Cartier divisor on X. Then $\mathbf{B}_+(L)$ is the smallest subset V of X with the property that for all coherent sheaves $\mathcal{F}$ on X there exists a possibly infinite sequence of sheaves of the form*

$$\cdots \to \bigoplus_{i=1}^{r_i} \mathcal{O}_X(-m_iL) \to \cdots \to \bigoplus_{i=1}^{r_1} \mathcal{O}_X(-m_1L) \to \mathcal{F} \ ,$$

which is exact off V.

Proof. — If L is a non-big divisor, then $\mathbf{B}_+(L) = X$, and the statement is obviously true. Hence we can assume without loss of generality that L is big.

First we prove the following claim: let $\mathcal{F}$ be an arbitrary coherent sheaf on X; then there exist positive integers r,m, and a map of sheaves

$$\bigoplus_{i=1}^{r} \mathcal{O}_X(-mL) \xrightarrow{\phi} \mathcal{F} \ , \tag{2.1}$$

which is surjective away from $\mathbf{B}_+(L)$.

Fix an arbitrary ample divisor A on X. The sheaves $\mathcal{F} \otimes \mathcal{O}_X(m'A)$ are globally generated for m' sufficiently large. According to [6, Proposition 1.5] $\mathbf{B}_+(L) = \mathbf{B}(L - \epsilon A)$ for any rational $\epsilon > 0$ small enough. Pick such an ϵ, set $L' \stackrel{\text{def}}{=} L - \epsilon A$ and let $m \gg 0$ be a positive integer such that $m' \stackrel{\text{def}}{=} m\epsilon$ is an integer, and

$$\mathrm{Bs}(mL') \ = \ \mathbf{B}(mL') \ = \ \mathbf{B}_+(L) \ .$$

By picking m large enough, we can in addition assume that $\mathcal{F} \otimes \mathcal{O}_X(m'A)$ is globally generated. As a consequence,

$$\mathcal{F} \otimes \mathcal{O}_X(m'A) \otimes \mathcal{O}_X(mL') \simeq \mathcal{F} \otimes \mathcal{O}_X(m'A + mL')$$

is globally generated away from $\mathrm{Bs}(mL') = \mathbf{B}_+(L)$. On the other hand

$$mL' + m'A \ = \ m(L - \epsilon A) + (m\epsilon)A \ = \ mL \ ,$$

hence we have found $m \gg 0$ such that $\mathcal{F} \otimes \mathcal{O}_X(mL)$ is globally generated away from $\mathbf{B}_+(L)$. Thanks to the map

$$H^0\left(X, \mathcal{F} \otimes \mathcal{O}_X(m'A)\right) \otimes H^0\left(X, \mathcal{O}_X(mL')\right) \longrightarrow H^0\left(X, \mathcal{F} \otimes \mathcal{O}_X(mL)\right)$$

one can find a finite set of sections giving rise to a map of sheaves

$$\bigoplus_{i=1}^{r} \mathcal{O}_X \longrightarrow \mathcal{F} \otimes \mathcal{O}_X(mL)$$

surjective away from $\mathbf{B}_+(L)$. Tensoring by $\mathcal{O}_X(-mL)$ gives the map in (2.1).

Next we will prove that $\mathbf{B}_+(L)$ satisfies that property described in the Proposition. Let $\mathcal{G}$ be the kernel of the map $\bigoplus_{i=1}^{r_1} \mathcal{O}_X(-m_1 L) \xrightarrow{\phi_1} \mathcal{F}$ coming from (2.1). Applying (2.1) to $\mathcal{G}$ we obtain a map

$$\bigoplus_{i=1}^{r_2} \mathcal{O}_X(-mL) \xrightarrow{\phi_2} \mathcal{G}$$

surjective off $\mathbf{B}_+(L)$, hence a two-term sequence

$$\bigoplus_{i=1}^{r_2} \mathcal{O}_X(-m_2 L) \to \bigoplus_{i=1}^{r_1} \mathcal{O}_X(-m_1 L) \to \mathcal{F}$$

exact away from the closed subset $\mathbf{B}_+(L)$. Continuing in this fashion we arrive at a possibly infinite sequence of the required type.

Last, if $x \in \mathbf{B}_+(L)$ then for all $\epsilon \in \mathbb{Q}^{\geqslant 0}$ and all $m \geqslant 1$ such that $m\epsilon \in \mathbb{Z}$, all global sections of $\mathcal{O}_X(m(L-\epsilon A))$ vanish at x. By taking $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{O}_X(-A)$,

$$\mathcal{F} \otimes \mathcal{O}_X(mL) \;=\; \mathcal{O}_X(mL-A)$$

will then have all global sections vanishing at $x \in X$. Therefore $\mathbf{B}_+(L)$ is indeed the smallest subset of X with the required property. $\square$

3. A Kawamata–Viehweg type vanishing for non-pseudo-effective divisors

Independently of the discussion so far, we show that the ideas leading to Theorem 1.2 also provide a partial vanishing theorem for adjoint divisors $K_X + L$ on smooth projective varieties, where L is not necessarily pseudo-effective.

It has been common knowledge that cohomology groups of big line bundles tend to vanish in degrees roughly above the dimension of the stable base locus (see [10, Proposition 2.15] for an early example). Matsumura in [12, Theorem 1.6] proved that

$$H^i\left(X, \mathcal{O}_X(K_X+L)\right) \;=\; 0 \;\; \text{for } i > \dim \mathbf{B}_-(L)$$

for a big line bundle L. Here $\mathbf{B}_-(L)$ denotes the restricted base locus of L, which is defined as

$$\mathbf{B}_-(L) \;=\; \bigcup_A \mathbf{B}(L+A)\;,$$

with A running through all ample $\mathbb{Q}$-divisors (see [6, Definition 1.12]).

In Theorem 3.1 we present a variant which works without the bigness assumption, and in addition provides vanishing in a wider range of degrees thanks to the fact that $a(L, A)$ can be strictly smaller than the dimension of the stable base locus (see Example 1.8).

The proof of Theorem 3.1 relies on the existence of a certain resolution defined in [10, Section 4]. Let D be an arbitrary Cartier divisor, A a very ample Cartier divisor on an irreducible projective variety X of dimension n. Upon choosing general elements $E_1, \ldots, E_r \in |A|$, one obtains an exact sequence

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D+rA) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{E_i}(D+rA) \longrightarrow \tag{3.1}$$

$$\longrightarrow \bigoplus_{1\leqslant i_1<i_2\leqslant r} \mathcal{O}_{E_{i_1}\cap E_{i_2}}(D+rA) \longrightarrow \ldots \longrightarrow \tag{3.2}$$

$$\longrightarrow \bigoplus_{1\leqslant i_1<i_2<\cdots<i_n\leqslant r} \mathcal{O}_{E_{i_1}\cap\cdots\cap E_{i_n}}(D+rA) \longrightarrow 0\ .$$

Given a coherent sheaf $\mathcal{F}$ one can also assume that the sequence (3.1) remains exact after tensoring by $\mathcal{F}$ by the general position of the effective divisors E_i.

Although strictly speaking it would not be necessary, it helps in the book-keeping process to chop up the above resolution into short exact sequences

$$\begin{aligned}
&0 \to \mathcal{F}\otimes\mathcal{O}_X(D) \to \mathcal{F}\otimes\mathcal{O}_X(D+rA) \to \mathcal{C}_1 \to 0\\
&0 \to \mathcal{C}_1 \to \bigoplus_{i=1}^{r} \mathcal{F}\otimes\mathcal{O}_{E_i}(D+rA) \to \mathcal{C}_2 \to 0\\
&\vdots\\
&0 \to \mathcal{C}_{n-1} \to \bigoplus_{1\leqslant i_1<i_2<\cdots<i_{n-1}\leqslant r} \mathcal{F}\otimes\mathcal{O}_{E_{i_1}\cap\cdots\cap E_{i_{n-1}}}(D+rA)\\
&\to \bigoplus_{1\leqslant i_1<i_2<\cdots<i_n\leqslant r} \mathcal{F}\otimes\mathcal{O}_{E_{i_1}\cap\cdots\cap E_{i_n}}(D+rA) \to 0\ .
\end{aligned}$$

Theorem 3.1. — *Let X be a smooth projective variety, L a divisor, A a very ample divisor on X. If $L|_{E_1\cap\cdots\cap E_q}$ is big and nef for a general choice of $E_1, \ldots, E_q \in |A|$, then*

$$H^i\left(X, \mathcal{O}_X(K_X+L)\right) = 0 \quad \textit{for } i > q.$$

Proof. — We will prove the statement by induction on the codimension of the complete intersections we restrict to; the case $q = 0$ is the Kawamata–Viehweg vanishing theorem. Let $E_1, \ldots, E_n \in |A|$ be elements such that the intersection of any combination of them is smooth of the expected dimension, and irreducible when it has positive dimension. As the E_i's are assumed to be general, this can clearly be done via the base-point free Bertini theorem, which works for $\dim X \geqslant 2$. In the remaining cases (when $\dim X \leqslant 1$) the statement of the proposition is immediate.

Consider the exact sequence (3.1) with $D = K_X + L + mA$ and $r = q$. First we show that it suffices to verify

$$H^i\left(X, \mathcal{C}_1^{(m,\underline{E})}\right) = 0 \quad \text{for all } m \geqslant 0 \text{ and } i > q-1,$$

where the upper index of $\mathcal{C}$ is used to emphasize the explicit dependence on m and the sequence $\underline{E} = (E_1, \ldots, E_n)$. Grant this for the moment, and see how this helps up to prove the statement of the proposition.

Take the following part of the long exact sequence associated to the first piece above

$$H^{i-1}\left(X, \mathcal{C}_1^{(m,\underline{E})}\right) \to H^i\left(X, \mathcal{O}_X(K_X + L + mA)\right) \\ \to H^i\left(X, \mathcal{O}_X(K_X + L + (m+q)A)\right) \to H^i\left(X, \mathcal{C}_1^{(m,\underline{E})}\right).$$

By assumption the cohomology groups on the two sides vanish for all m whenever $i > q$, hence

$$H^i\left(X, \mathcal{O}_X(K_X + L + mA)\right) \\ \simeq H^i\left(X, \mathcal{O}_X(K_X + L + (m+q)A)\right) \quad \text{for all } m \geqslant 0 \text{ and } i > q.$$

These groups are zero however for m sufficiently large by Serre vanishing, hence

$$H^i\left(X, \mathcal{O}_X(K_X + L)\right) = 0 \quad \text{for all } i > q,$$

as we wanted.

As for the vanishing of the cohomology groups $H^i\left(X, \mathcal{C}_1^{(m,\underline{E})}\right)$ for $m \geqslant 0$ and $i > q-1$, it is quickly checked inductively. Observe that for all $1 \leqslant j \leqslant q$ we have

$$K_X + L + (m+q)A|_{E_1\cap\cdots\cap E_j} = K_{E_1\cap\cdots\cap E_j} + (L + (m + (q-j))A)|_{E_1\cap\cdots\cap E_j}$$

by adjunction, and $(L + (m + (q-j))A)|_{E_1\cap\cdots\cap E_j}$ becomes ample when restricted to the intersection with $E_{j+1} \cap \cdots \cap E_q$. Induction on j gives

$$H^i\left(E_1 \cap \cdots \cap E_j, K_X + L + (m+q)A|_{E_1\cap\cdots\cap E_j}\right) = 0 \\ \text{for all } m \geqslant 0 \text{ and } i > q-j\,. \quad (3.3)$$

The above vanishing applied to the cohomology long exact sequence associated to

$$0 \to \mathcal{C}_{n-1}^{(m,\underline{E})} \to \bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_{n-1} \leqslant r} \mathcal{O}_{E_{i_1} \cap \cdots \cap E_{i_{n-1}}}(K_X + L + (m+q)A)$$
$$\to \bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant r} \mathcal{O}_{E_{i_1} \cap \cdots \cap E_{i_n}}(K_X + L + (m+q)A) \to 0 \ .$$

results in

$$H^i\left(X, \mathcal{C}_{n-1}^{(m,\underline{E})}\right) = 0 \quad \text{for all } m \geqslant 0 \text{ and } i > q-(n-1).$$

By chasing through the long exact sequence associated to

$$0 \to \mathcal{C}_{j-1}^{(m,\underline{E})} \to \bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_{j-1} \leqslant r} \mathcal{O}_{E_{i_1} \cap \cdots \cap E_{i_{j-1}}}(K_X + L + (m+q)A)$$
$$\to \mathcal{C}_j^{(m,\underline{E})} \to 0 \ ,$$

more precisely the segment

$$\bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_{j-1} \leqslant r} H^{i-1}\left(E_1 \cap \cdots \cap E_{j-1}, K_X + L + (m+q)A|_{E_1 \cap \cdots \cap E_{j-1}}\right)$$
$$\to H^{i-1}\left(X, \mathcal{C}_j^{(m,\underline{E})}\right) \to H^i\left(X, \mathcal{C}_{j-1}^{(m,\underline{E})}\right)$$
$$\to \bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_{j-1} \leqslant r} H^i\left(E_1 \cap \cdots \cap E_{j-1}, K_X + L + (m+q)A|_{E_1 \cap \cdots \cap E_{j-1}}\right)$$

we obtain by (3.3) that

$$H^i\left(X, \mathcal{C}_{j-1}^{(m,\underline{E})}\right)$$
$$\simeq H^{i-1}\left(X, \mathcal{C}_j^{(m,\underline{E})}\right) = 0 \text{ for all } m \geqslant 0 \text{ and } i > q-(j-1).$$

In summary, we arrive at

$$H^i\left(X, \mathcal{C}_j^{(m)}\right) = 0 \quad \text{for all } m \geqslant 0 \text{ and } i > q-j.$$

For $j = 1$ this is the required vanishing. □

Corollary 3.2. — *Let X be an irreducible projective variety, L a line bundle, A a very ample line bundle on X. Let q be a natural number having the property that $L|_{E_1 \cap \cdots \cap E_q}$ is big and nef for general elements $E_i \in |A|$ general ($1 \leqslant i \leqslant q$). If $\pi : Y \to X$ is a proper birational morphism from a smooth variety, B a nef divisor on Y, then*

$$H^i\left(Y, \mathcal{O}_Y(K_Y + \pi^*L + B)\right) = 0$$

for all $i > q$.

Proof. — This is in fact a corollary of the proof of Theorem 3.1. We point out the necessary modifications. Assuming $\dim X \geqslant 2$, Lemma 3.3 makes sure that we can consider restrictions to intersections of general elements of $|\pi^*A|$ just as we did in the proof of Theorem 3.1; we also obtain that the generic restriction $\pi^*L|_{E'_1\cap\cdots\cap E'_q}$ is still big and nef for $E'_i \in |\pi^*A|$ general.

Next, run the proof on Y, with $D = K_Y + \pi^*L + B + m\pi^*A$, and $r = q$. The task that remains is to show that the cohomology groups

$$H^i\left(Y, \mathcal{O}_Y(K_Y + \pi^*L + B + m\pi^*A)\right) \simeq H^i\left(Y, \mathcal{O}_Y(K_Y + \pi^*L + B + (m+q)\pi^*A)\right)$$

vanish for all $m \geqslant 0$ and $i > q$. By the given isomorphisms, it suffices to prove this for $m \gg 0$. Serre vanishing no longer applies, since π^*A is only big and nef; luckily we can use the classical Kawamata–Viehweg theorem to our advantage. Namely, observe that

$$\pi^*L + B + m\pi^*A \;=\; \pi^*(L + m_0A) + B + (m - m_0)\pi^*A$$

for all integers m, m_0 with $m \geqslant m_0$. If m_0 is suitably large then $L + m_0A$ itself is ample, therefore $\pi^*(L + m_0A) + B + (m - m_0)\pi^*A$ is big and nef, and the required vanishing follows. □

LEMMA 3.3. — *Let $\pi : Y \to X$ a proper birational morphism of irreducible projective varieties of dimension $n \geqslant 2$, L a Cartier divisor, A a very ample Cartier divisor on X. If $L|_{E_1\cap\cdots\cap E_k}$ is big and nef for some $k \geqslant 1$ and general elements $E_1, \ldots, E_k$ from $|A|$, then $\pi^*L|_{E'_1\cap\cdots\cap E'_k}$ is big and nef, where $E'_1, \ldots, E'_k$ are general elements from $|\pi^*A|$ with E'_i mapping to E_i for all $1 \leqslant i \leqslant k$.*

Proof. — As π^*A is big and globally generated and $\dim Y \geqslant 2$, a general element $E' \in |\pi^*A|$ maps to a general element of $|A|$ by the base-point free Bertini theorem. Moreover, by the same token, the intersection $E'_1 \cap \cdots \cap E'_k$ of general elements $|\pi^*A|$ is irreducible, and $\pi|_{E'_1\cap\cdots\cap E'_k}$ is a proper birational morphism onto its image, which is the intersection of k general elements of $|A|$, say $E_1 \cap \cdots \cap E_k$.

Consequently, $\pi^*(L|_{E_1\cap\cdots\cap E_k})$ is a big and nef divisor on $E'_1 \cap \cdots \cap E'_k$. However,

$$\pi^*(L|_{E_1\cap\cdots\cap E_k}) \;=\; (\pi^*L)|_{E'_1\cap\cdots\cap E'_k}\ ,$$

hence the latter is big and nef as we wanted. □

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