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OBSTRUCTIONS FOR DEFORMATIONS OF COMPLEXES

by Frauke M. BLEHER & Ted CHINBURG (*)

Abstract. — We develop two approaches to obstruction theory for deformations of derived isomorphism classes of complexes of modules for a profinite group G over a complete local Noetherian ring A of positive residue characteristic.

Résumé. — Nous développons deux approches de la théorie de l'obstruction des déformations de classes d'isomorphisme dans la catégorie dérivée des complexes de $A[[G]]$-modules lorsque G est un groupe profini et A un anneau local, noethérien complet, de caractéristique positive résiduelle.

1. Introduction

Two basic tools of deformation theory are obstructions and parameterizations of infinitesimal deformations. Obstructions determine when an object has an infinitesimal deformation. When such an obstruction vanishes, one would like to parameterize all such infinitesimal deformations. In this paper we develop these tools in the context of deforming derived isomorphism classes of complexes $Z^\bullet$ of modules for a profinite group G over a complete local Noetherian ring A having a fixed residue field k of positive characteristic ℓ.

The infinitesimal deformation problem we consider has to do with lifting the isomorphism class of $Z^\bullet$ in the derived category $D^-(A[[G]])$ of bounded above complexes of pseudocompact $A[[G]]$-modules to a class in $D^-(A'[[G]])$ when $A' \to A$ is a surjection of complete local Noetherian

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rings whose kernel J has square 0. The precise definition of the deformation functor we consider is given in § 2.

We give two different approaches to obstruction theory. The first, more naive, method proceeds by first replacing $Z^\bullet$ by a bounded above complex of topologically free pseudocompact $A[[G]]$-modules. One can then separately lift each term of $Z^\bullet$ to an $A'[[G]]$-module. By considering the obstruction to lifting the boundaries of $Z^\bullet$ so as to obtain a complex of $A'[[G]]$-modules, one arrives at a lifting obstruction $\omega(Z^\bullet, A')$ in $\mathrm{Ext}^2_{D^-(A[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$. Here $\widehat{\otimes}^{\mathbf{L}}$ is the left derived tensor product discussed in Remark 2.5.

The second method uses a construction of Gabber and a suggestion of Illusie. This interprets the obstruction to lifting $Z^\bullet$ as the image of a certain canonical element under a boundary map in a spectral sequence which computes Ext groups over $D^-(A'[[G]])$ via Ext groups over $D^-(A[[G]])$ and $\mathrm{Tor}^{A'}$ complexes. We will describe this in more detail below.

When the lifting obstruction vanishes, each of the two above methods describes all local isomorphism classes of lifts of $Z^\bullet$ over A' as a principal homogeneous space for $\mathrm{Ext}^1_{D^-(A[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$. The spectral sequence method has the advantage of identifying this principal homogeneous space as a particular coset of $\mathrm{Ext}^1_{D^-(A[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$ inside $\mathrm{Ext}^1_{D^-(A'[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$. This identifies the local deformation functor as a subfunctor of a functor defined by Ext^1 groups over $D^-(A'[[G]])$. The spectral sequence also gives a natural filtration of $\mathrm{Ext}^1_{D^-(A'[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$. We obtain an interpretation of the last two terms in this filtration via exact sequences of complexes which satisfy additional conditions.

The spectral sequence we study is

$$(1.1)\quad E_2^{p,q} = \mathrm{Ext}^p_{D^-(A[[G]])}(\mathcal{H}^{-q}(A\widehat{\otimes}^{\mathbf{L}}_{A'} Z^\bullet), J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet) \Longrightarrow \mathrm{Ext}^{p+q}_{D^-(A'[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet).$$

Here $\mathcal{H}^{-q}(A\widehat{\otimes}^{\mathbf{L}}_{A'} Z^\bullet)$ is a Tor complex whose j^{th} term is $\mathrm{Tor}_q^{A'}(Z^j, A)$ (see Definition 3.5). We will show in Theorem 3.12 that the lifting obstruction $\omega(Z^\bullet, A')$ is the image under $d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}$ of a canonical element ι in $E_2^{0,1}$. In Theorem 3.9 (see also Lemmas 3.19 and 3.22), Gabber's construction will be shown to arise from the exact sequence of low degree terms

$$(1.2)\qquad 0 \to E_\infty^{1,0} \to F_{II}^0 \to E_2^{0,1}/W_2^{0,1} \xrightarrow{\overline{d_2^{0,1}}} E_2^{2,0}$$

where $F^0_{II} = F^0_{II}\ \mathrm{Ext}^1_{D^-(A'[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$ is the second to last term in the second filtration of the total cohomology of a bicomplex whose first total cohomology group is $\mathrm{Ext}^1_{D^-(A'[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$ and $E^{0,1}_\infty = \mathrm{Ker}(d^{0,1}_2)/W^{0,1}_2$ (see Definition 3.7). We will interpret F^0_{II} as the set of extension classes arising from short exact sequences of bounded above complexes of pseudocompact $A'[[G]]$-modules

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0 \tag{1.3}$$

in which $X^\bullet$ is annihilated by J and isomorphic to $J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet$ in $D^-(A[[G]])$. We will show in Lemma 3.10 that if $(Z^\bullet, \zeta)$ has a lift over A', then the local isomorphism class of every lift of $(Z^\bullet, \zeta)$ over A' contains a lift $(Y^\bullet, \upsilon)$ such that $Y^\bullet$ occurs as the middle term of a short exact sequence of the form (1.3). We will show in Theorem 3.12 that if a lift of $(Z^\bullet, \zeta)$ over A' exists, then the set of all local isomorphism classes of such lifts is in bijection with the full preimage of $\iota + W^{0,1}_2$ under the map $F^0_{II} \to E^{0,1}_2/W^{0,1}_2$ in (1.2). This proves that the set of all local isomorphism classes of such lifts is a principal homogeneous space for $E^{1,0}_\infty$ and it gives a description of the local isomorphism classes of lifts of $(Z^\bullet, \zeta)$ over A' in terms of classes in $F^0_{II} \subset \mathrm{Ext}^1_{D^-(A'[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$. Moreover, if a lift of $(Z^\bullet, \zeta)$ over A' exists, we will show that $E^{p,0}_2 = E^{p,0}_\infty$ for all p. This partial degeneration is stronger than what is implied by the naive method, which deals only with the case $p = 1$.

We now describe the sections of this paper.

In § 2 we recall the definitions and notations needed to state the main result of [1] concerning the existence of versal and universal deformations of derived isomorphism classes of bounded complexes $V^\bullet$ in $D^-(k[[G]])$. When $V^\bullet$ has only one non-zero term, this is the deformation theory of continuous G-modules developed by Mazur in [8] using work of Schlessinger in [10]. We also define local isomorphism classes of lifts over A' of complexes $Z^\bullet$ in $D^-(A[[G]])$ relative to a surjection of complete local Noetherian rings $A' \to A$ with residue field k having a square zero kernel.

The naive approach to obstruction theory is given in §3.1. An outline of the spectral sequence approach, beginning with the case of modules rather than complexes, is given in § 3.2. The details of this approach for complexes are developed in § 3.3 - § 3.8. The two methods are compared in § 3.9.

The results of this paper are used in [2] to study a new finiteness problem concerning deformations of arithmetically defined Galois modules. The particular result needed in [2] is Proposition 4.2, which shows that to determine versal deformations, one can take the quotient of G by any closed normal pro-prime-to-ℓ group which acts trivially on $V^\bullet$ where ℓ is the characteristic of k.

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2. Quasi-lifts and deformation functors

Let G be a profinite group, let k be a field of positive characteristic ℓ, and let W be a complete local commutative Noetherian ring with residue field k. Define $\widehat{\mathcal{C}}$ to be the category of complete local commutative Noetherian W-algebras with residue field k. The morphisms in $\widehat{\mathcal{C}}$ are continuous W-algebra homomorphisms that induce the identity on k. Let $\mathcal{C}$ be the subcategory of Artinian objects in $\widehat{\mathcal{C}}$. If $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$, let $R[[G]]$ be the completed group algebra of the usual abstract group algebra $R[G]$ of G over R, *i.e.* $R[[G]]$ is the projective limit of the ordinary group algebras $R[G/U]$ as U ranges over the open normal subgroups of G.

Definition 2.1. — *A topological ring Λ is called a* pseudocompact ring *if Λ is complete and Hausdorff and admits a basis of open neighborhoods of 0 consisting of two-sided ideals J for which Λ/J is an Artinian ring.*

Suppose Λ is a pseudocompact ring. A complete Hausdorff topological Λ-module M is said to be a pseudocompact Λ-module *if M has a basis of open neighborhoods of 0 consisting of submodules N for which M/N has finite length as Λ-module. We denote by $\mathrm{PCMod}(\Lambda)$ the category of pseudocompact Λ-modules. (If not stated otherwise, our modules are left modules.)*

A pseudocompact Λ-module M is said to be topologically free *on a set $X = \{x_i\}_{i\in I}$ if M is isomorphic to the product of a family $(\Lambda_i)_{i\in I}$ where $\Lambda_i = \Lambda$ for all i.*

Suppose R is a commutative pseudocompact ring. A complete Hausdorff topological ring Λ is called a pseudocompact R-algebra *if Λ is an R-algebra in the usual sense, and if Λ admits a basis of open neighborhoods of 0 consisting of two-sided ideals J for which Λ/J has finite length as R-module.*

Suppose Λ *is a pseudocompact* R*-algebra, and let* $\widehat{\otimes}_\Lambda$ *denote the completed tensor product in the category* $\mathrm{PCMod}(\Lambda)$ *(see* [3, §2]*). If* M *is a right (resp. left) pseudocompact* Λ*-module, then* $M\widehat{\otimes}_\Lambda -$ *(resp.* $-\widehat{\otimes}_\Lambda M$*) is a right exact functor. Moreover,* M *is said to be* topologically flat*, if the functor* $M\widehat{\otimes}_\Lambda -$ *(resp.* $-\widehat{\otimes}_\Lambda M$*) is exact.*

Remark 2.2. — Pseudocompact rings, algebras and modules have been studied, for example, in [4, 5, 3]. The following statements can be found in these references. Suppose Λ is a pseudocompact ring.

(i) The ring Λ is the projective limit of Artinian quotient rings having the discrete topology. A Λ-module is pseudocompact if and only if it is the projective limit of Λ-modules of finite length having the discrete topology. The category $\mathrm{PCMod}(\Lambda)$ is an abelian category with exact projective limits.

(ii) Every topologically free pseudocompact Λ-module is a projective object in $\mathrm{PCMod}(\Lambda)$, and every pseudocompact Λ-module is the quotient of a topologically free Λ-module. Hence $\mathrm{PCMod}(\Lambda)$ has enough projective objects.

(iii) Every pseudocompact R-algebra is a pseudocompact ring, and a module over a pseudocompact R-algebra has finite length if and only if it has finite length as R-module.

(iv) Suppose Λ is a pseudocompact R-algebra, and M and N are pseudocompact Λ-modules. Then we define the right derived functors $\mathrm{Ext}^n_\Lambda(M, N)$ by using a projective resolution of M.

(v) Suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$. Then R is a pseudocompact ring, and $R[[G]]$ is a pseudocompact R-algebra.

Remark 2.3. — Let R be an object in $\widehat{\mathcal{C}}$ and let m_R be its maximal ideal. Suppose $[(R/m_R^i)X_i]$ is an abstractly free (R/m_R^i)-module on the finite topological space X_i for all i, and that $\{X_i\}_i$ forms an inverse system. Define $X = \varprojlim_i X_i$ and $R[[X]] = \varprojlim_i [(R/m_R^i)X_i]$. Then $R[[X]]$ is a topologically free pseudocompact R-module on X. In particular, every topologically free pseudocompact $R[[G]]$-module is a topologically free pseudocompact R-module.

Remark 2.4. — Suppose R is an object in $\widehat{\mathcal{C}}$ and $\Lambda = R$ or $R[[G]]$. Let M be a pseudocompact right (resp. left) Λ-module.

(i) If M is finitely generated as a pseudocompact Λ-module, it follows from [3, Lemma 2.1(ii)] that the functors $M \otimes_\Lambda -$ and $M\widehat{\otimes}_\Lambda -$ (resp. $- \otimes_\Lambda M$ and $-\widehat{\otimes}_\Lambda M$) are naturally isomorphic.

(ii) By [3, Lemma 2.1(iii)] and [3, Prop. 3.1], M is topologically flat if and only if M is projective.
(iii) If $\Lambda = R$ and M is a pseudocompact R-module, it follows from [5, Proof of Prop. 0.3.7] and [5, Cor. 0.3.8] that M is topologically flat if and only if M is topologically free if and only if M is abstractly flat. In particular, if R is Artinian, a pseudocompact R-module is topologically flat if and only if it is abstractly free.

If Λ is a pseudocompact ring, let $C^-(\Lambda)$ be the abelian category of complexes of pseudocompact Λ-modules that are bounded above, let $K^-(\Lambda)$ be the homotopy category of $C^-(\Lambda)$, and let $D^-(\Lambda)$ be the derived category of $K^-(\Lambda)$. Let [1] denote the translation functor on $C^-(\Lambda)$ (resp. $K^-(\Lambda)$, resp. $D^-(\Lambda)$), *i.e.* [1] shifts complexes one place to the left and changes the sign of the differential. Note that a homomorphism in $C^-(\Lambda)$ is a quasi-isomorphism if and only if the induced homomorphisms on all the cohomology groups are bijective.

Remark 2.5. — Let $X^\bullet, Y^\bullet \in \mathrm{Ob}(K^-(R[[G]]))$ and consider the double complex $K^{\bullet,\bullet}$ of pseudocompact $R[[G]]$-modules with $K^{p,q} = (X^p \widehat{\otimes}_R Y^q)$ and diagonal G-action. We define the total tensor product $X^\bullet \widehat{\otimes}_R Y^\bullet$ to be the simple complex associated to $K^{\bullet,\bullet}$, *i.e.*

$$(X^\bullet \widehat{\otimes}_R Y^\bullet)^n = \bigoplus_{p+q=n} X^p \widehat{\otimes}_R Y^q$$

whose differential is $d(x \,\widehat{\otimes}\, y) = d_X(x) \,\widehat{\otimes}\, y + (-1)^x \, x \,\widehat{\otimes}\, d_Y(y)$ for $x \,\widehat{\otimes}\, y \in K^{p,q}$. Since homotopies carry over the completed tensor product, we have a functor

$$\widehat{\otimes}_R \colon K^-(R[[G]]) \times K^-(R[[G]]) \to K^-(R[[G]]).$$

Using [11, Thm. 2.2 of Chap. 2 §2], we see that there is a well-defined left derived completed tensor product $\widehat{\otimes}^{\mathbf{L}}_R$. Moreover, if $X^\bullet$ and $Y^\bullet$ are as above, then $X^\bullet \widehat{\otimes}^{\mathbf{L}}_R Y^\bullet$ may be computed in $D^-(R[[G]])$ in the following way. Take a bounded above complex $Y'^\bullet$ of topologically flat pseudocompact $R[[G]]$-modules with a quasi-isomorphism $Y'^\bullet \to Y^\bullet$ in $K^-(R[[G]])$. Then this quasi-isomorphism induces an isomorphism between $X^\bullet \widehat{\otimes}_R Y'^\bullet$ and $X^\bullet \widehat{\otimes}^{\mathbf{L}}_R Y^\bullet$ in $D^-(R[[G]])$.

Definition 2.6. — *We will say that a complex $M^\bullet$ in $K^-(R[[G]])$ has* finite pseudocompact R-tor dimension, *if there exists an integer N such that for all pseudocompact R-modules S, and for all integers $i < N$, $\mathrm{H}^i(S \widehat{\otimes}^{\mathbf{L}}_R M^\bullet) = 0$. If we want to emphasize the integer N in this definition, we say $M^\bullet$ has* finite pseudocompact R-tor dimension at N.

Remark 2.7. — Suppose $M^\bullet$ is a complex in $K^-([[RG]])$ of topologically flat, hence topologically free, pseudocompact R-modules that has finite pseudocompact R-tor dimension at N. Then the bounded complex $M'^\bullet$, which is obtained from $M^\bullet$ by replacing M^N by $M'^N = M^N/\delta^{N-1}(M^{N-1})$ and by setting $M'^i = 0$ if $i < N$, is quasi-isomorphic to $M^\bullet$ and has topologically free pseudocompact terms over R.

Hypothesis 2.8. — *Throughout this paper, we assume that $V^\bullet$ is a complex in $D^-(k[[G]])$ that has only finitely many non-zero cohomology groups, all of which have finite k-dimension.*

Definition 2.9. — *A* quasi-lift *of $V^\bullet$ over an object R of $\widehat{\mathcal{C}}$ is a pair $(M^\bullet, \phi)$ consisting of a complex $M^\bullet$ in $D^-(R[[G]])$ that has finite pseudocompact R-tor dimension together with an isomorphism $\phi\colon k\widehat{\otimes}^{\mathbf{L}}_R M^\bullet \to V^\bullet$ in $D^-(k[[G]])$. Two quasi-lifts $(M^\bullet, \phi)$ and $(M'^\bullet, \phi')$ are* isomorphic *if there is an isomorphism $f\colon M^\bullet \to M'^\bullet$ in $D^-(R[[G]])$ with $\phi' \circ (k\widehat{\otimes}^{\mathbf{L}} f) = \phi$.*

Theorem 2.10. — *Suppose that $\mathrm{H}^i(V^\bullet) = 0$ unless $n_1 \leqslant i \leqslant n_2$. Every quasi-lift of $V^\bullet$ over an object R of $\widehat{\mathcal{C}}$ is isomorphic to a quasi-lift $(P^\bullet, \phi)$ for a complex $P^\bullet$ with the following properties:*

(i) *The terms of $P^\bullet$ are topologically free $R[[G]]$-modules.*
(ii) *The cohomology group $\mathrm{H}^i(P^\bullet)$ is finitely generated as an abstract R-module for all i, and $\mathrm{H}^i(P^\bullet) = 0$ unless $n_1 \leqslant i \leqslant n_2$.*
(iii) *One has $\mathrm{H}^i(S\widehat{\otimes}^{\mathbf{L}}_R P^\bullet) = 0$ for all pseudocompact R-modules S unless $n_1 \leqslant i \leqslant n_2$.*

Proof. — Part (i) follows from [1, Lemma 2.9]. Assume now that the terms of $P^\bullet$ are topologically free $R[[G]]$-modules, which means in particular that the functors $-\widehat{\otimes}^{\mathbf{L}}_R P^\bullet$ and $-\widehat{\otimes}_R P^\bullet$ are naturally isomorphic. Let m_R denote the maximal ideal of R, and let n be an arbitrary positive integer. By [1, Lemmas 3.1 and 3.8], $\mathrm{H}^i((R/m_R^n)\widehat{\otimes}_R P^\bullet) = 0$ for $i > n_2$ and $i < n_1$. Moreover, for $n_1 \leqslant i \leqslant n_2$, $\mathrm{H}^i((R/m_R^n)\widehat{\otimes}_R P^\bullet)$ is a subquotient of an abstractly free (R/m_R^n)-module of rank $d_i = \dim_k \mathrm{H}^i(V^\bullet)$, and $(R/m_R^n)\widehat{\otimes}_R P^\bullet$ has finite pseudocompact (R/m_R^n)-tor dimension at $N = n_1$. Since $P^\bullet \cong \varprojlim_n (R/m_R^n)\widehat{\otimes}_R P^\bullet$ and since by Remark 2.2(i), the category $\mathrm{PCMod}(R)$ has exact projective limits, it follows that for all pseudocompact R-modules S

$$\mathrm{H}^i(S\widehat{\otimes}_R P^\bullet) = \varprojlim_n \mathrm{H}^i\left((S/m_R^n S)\widehat{\otimes}_{R/m_R^n}\left((R/m_R^n)\widehat{\otimes}_R P^\bullet\right)\right)$$

for all i. Hence Theorem 2.10 follows. □

Definition 2.11. — *Let* $\widehat{F} = \widehat{F}_{V^\bullet}\colon \widehat{\mathcal{C}} \to \mathrm{Sets}$ *be the functor which sends an object* R *of* $\widehat{\mathcal{C}}$ *to the set* $\widehat{F}(R)$ *of all isomorphism classes of quasi-lifts of* $V^\bullet$ *over* R*, and which sends a morphism* $\alpha\colon R \to R'$ *in* $\widehat{\mathcal{C}}$ *to the set map* $\widehat{F}(R) \to \widehat{F}(R')$ *induced by* $M^\bullet \mapsto R'\widehat{\otimes}^{\mathbf{L}}_{R,\alpha} M^\bullet$*. Let* $F = F_{V^\bullet}$ *be the restriction of* $\widehat{F}$ *to the subcategory* $\mathcal{C}$ *of Artinian objects in* $\widehat{\mathcal{C}}$*.*

Let $k[\varepsilon]$*, where* $\varepsilon^2 = 0$*, denote the ring of dual numbers over* k*. The set* $F(k[\varepsilon])$ *is called the* tangent space *to* F*, denoted by* t_F*.*

Definition 2.12. — *A profinite group* G *has* finite pseudocompact cohomology*, if for each discrete* $k[[G]]$*-module* M *of finite* k*-dimension, and all integers* j*, the cohomology group* $\mathrm{H}^j(G, M) = \mathrm{Ext}^j_{k[[G]]}(k, M)$ *has finite* k*-dimension.*

Theorem 2.13 ([1],Thm. 2.14). — *Suppose that* G *has finite pseudocompact cohomology.*

(i) *The functor* F *has a pro-representable hull* $R(G, V^\bullet) \in \mathrm{Ob}(\widehat{\mathcal{C}})$ *(c.f.* [10, Def. 2.7] *and* [9, §1.2]*), and the functor* $\widehat{F}$ *is continuous (cf.* [9]*).*

(ii) *There is a* k*-vector space isomorphism* $h\colon t_F \to \mathrm{Ext}^1_{D^-(k[[G]])}(V^\bullet, V^\bullet)$*.*

(iii) *If* $\mathrm{Hom}_{D^-(k[[G]])}(V^\bullet, V^\bullet) = k$*, then* $\widehat{F}$ *is represented by* $R(G, V^\bullet)$*.*

Remark 2.14. — By Theorem 2.13(i), there exists a quasi-lift $(U(G, V^\bullet), \phi_U)$ of $V^\bullet$ over $R(G, V^\bullet)$ with the following property. For each $R \in \mathrm{Ob}(\widehat{\mathcal{C}})$, the map $\mathrm{Hom}_{\widehat{\mathcal{C}}}(R(G, V^\bullet), R) \to \widehat{F}(R)$ induced by $\alpha \mapsto R\widehat{\otimes}^{\mathbf{L}}_{R(G,V^\bullet),\alpha} U(G, V^\bullet)$ is surjective, and this map is bijective if R is the ring of dual numbers $k[\varepsilon]$ over k where $\varepsilon^2 = 0$.

In general, the isomorphism type of the pro-representable hull $R(G, V^\bullet)$ is unique up to non-canonical isomorphism. If $R(G, V^\bullet)$ represents $\widehat{F}$, then $R(G, V^\bullet)$ is uniquely determined up to canonical isomorphism.

Definition 2.15. — *Using the notation of Theorem 2.13 and Remark 2.14, we call* $R(G, V^\bullet)$ *the* versal deformation ring *of* $V^\bullet$ *and* $(U(G, V^\bullet), \phi_U)$ *a* versal deformation *of* $V^\bullet$*.*

If $R(G, V^\bullet)$ *represents* $\widehat{F}$*, then* $R(G, V^\bullet)$ *will be called the* universal deformation ring *of* $V^\bullet$ *and* $(U(G, V^\bullet), \phi_U)$ *will be called a* universal deformation *of* $V^\bullet$*.*

Remark 2.16. — If $V^\bullet$ consists of a single module V_0 in dimension 0, the versal deformation ring $R(G, V^\bullet)$ coincides with the versal deformation ring studied by Mazur in [8, 9]. In this case, Mazur assumed only that G satisfies a certain finiteness condition (Φ_p), which is equivalent to the requirement that $\mathrm{H}^1(G, M)$ have finite k-dimension for all discrete $k[[G]]$-modules M of finite k-dimension. Since the higher G-cohomology enters into determining lifts of complexes $V^\bullet$ having more than one non-zero cohomology group, the condition that G have finite pseudocompact cohomology is the natural generalization of Mazur's finiteness condition in this context.

We also need to set up some notation concerning local deformation functors.

Definition 2.17. — *Let $V^\bullet$ be as in Hypothesis 2.8, let A be in $\widehat{\mathcal{C}}$, and let $(Z^\bullet,\zeta)$ be a quasi-lift of $V^\bullet$ over A. Let $A'\to A$ in $\widehat{\mathcal{C}}$ be a surjective morphism in $\widehat{\mathcal{C}}$ whose kernel is an ideal J with $J^2=0$.*

*A (*local*) *quasi-lift* of $(Z^\bullet,\zeta)$ over A' is a pair $(Y^\bullet,\upsilon)$ consisting of a complex $Y^\bullet$ in $D^-(A'[[G]])$ that has finite pseudocompact A'-tor dimension together with an isomorphism $\upsilon\colon A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet\to Z^\bullet$ in $D^-(A[[G]])$. Note that if $(Y^\bullet,\upsilon)$ is a quasi-lift of $(Z^\bullet,\zeta)$ over A', then $(Y^\bullet,\zeta\circ(k\widehat{\otimes}^{\mathbf{L}}\upsilon))$ is a quasi-lift of $V^\bullet$ over A'.*

Two quasi-lifts $(Y^\bullet,\upsilon)$ and $(Y'^\bullet,\upsilon')$ of $(Z^\bullet,\zeta)$ over A' are said to be locally isomorphic *if there exists an isomorphism $f\colon Y^\bullet\to Y'^\bullet$ in $D^-(A'[[G]])$ with $\upsilon'\circ(A\widehat{\otimes}^{\mathbf{L}}f)=\upsilon$.*

3. Obstructions

Let $V^\bullet$ be as in Hypothesis 2.8, let A be in $\widehat{\mathcal{C}}$, and let $(Z^\bullet,\zeta)$ be a quasi-lift of $V^\bullet$ over A. Let $A'\to A$ in $\widehat{\mathcal{C}}$ be a surjective morphism in $\widehat{\mathcal{C}}$ whose kernel is an ideal J with $J^2=0$. In this section, we develop the two different approaches described in the introduction to finding a lifting obstruction

$$\omega(Z^\bullet,A')\in \mathrm{Ext}^2_{D^-(A[[G]])}(Z^\bullet, J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet)$$

which vanishes if and only if $(Z^\bullet,\zeta)$ can be lifted to A'. The naive approach is given in § 3.1 while the spectral sequence approach is developed in § 3.2 - § 3.8. The two methods are compared in § 3.9. More precisely, we show that the lifting obstruction from either method can be obtained from the other by composing with suitable automorphisms of $Z^\bullet$ and $J\widehat{\otimes}^{\mathbf{L}}_A Z^\bullet[2]$, respectively, in $D^-(A[[G]])$.

Using the results from §2, we can make the following assumption concerning $V^\bullet$ and $Z^\bullet$.

Hypothesis 3.1. — *Assume $V^\bullet$ is as in Hypothesis 2.8 with $\mathrm{H}^i(V^\bullet)=0$ unless $-p_0\leqslant i\leqslant -1$. Suppose $0\to J\to A'\to A\to 0$ is an extension of objects A',A in $\widehat{\mathcal{C}}$ with $J^2=0$. Let $B'=A'[[G]]$ and $B=A[[G]]$.*

Let $(Z^\bullet,\zeta)$ be a quasi-lift of $V^\bullet$ over A. By Theorem 2.10 and Remark 2.7, we can make the following assumptions: The complex $Z^\bullet$ is a bounded complex of pseudocompact B-modules whose terms Z^i are zero unless $-p_0 \leqslant i \leqslant -1$. The terms Z^i are topologically flat, hence projective, pseudocompact B-modules for $i \neq -p_0$, and Z^{-p_0} is topologically flat, hence topologically free, over A.

Remark 3.2. — The functors $A\widehat{\otimes}_{A'}-$ and $B\widehat{\otimes}_{B'}-$ are naturally isomorphic functors $\mathrm{PCMod}(B') \to \mathrm{PCMod}(B)$. Similarly to Remark 2.5, one obtains a well-defined left derived completed tensor product $B\widehat{\otimes}^{\mathbf{L}}_{B'}-$. The functors $A\widehat{\otimes}^{\mathbf{L}}_{A'}-$ and $B\widehat{\otimes}^{\mathbf{L}}_{B'}-$ are naturally isomorphic functors $D^-(B') \to D^-(B)$.

3.1. A naive approach

In this subsection we describe a naive approach to obstruction theory. We assume Hypothesis 3.1. Let $(\widetilde{Z}^\bullet,\tilde{\zeta})$ be a quasi-lift of $V^\bullet$ over A that is isomorphic to the quasi-lift $(Z^\bullet,\zeta)$ such that $\widetilde{Z}^\bullet$ is concentrated in degrees $\leqslant -1$ and all terms of $\widetilde{Z}^\bullet$ are topologically free pseudocompact B-modules. For each $j \in \mathbb{Z}$, let Y^j be a topologically free pseudocompact B'-module which is a lift of $\widetilde{Z}^j$ over A' and let $a_Y^j : Y^j \to \widetilde{Z}^j$ be the composition of the natural surjection $Y^j \to A\widehat{\otimes}_{A'}Y^j$ followed by $A\widehat{\otimes}_{A'}Y^j \xrightarrow{\cong} \widetilde{Z}^j$. Moreover, let $c_Y^j : Y^j \to Y^{j+1}$ be a homomorphism of pseudocompact B'-modules such that $a_Y^{j+1}\circ c_Y^j = d^j_{\widetilde{Z}}\circ a_Y^j$ for all j. In particular, $Y^j=0$ for $j \geqslant 0$, and $c_Y^j = 0$ for $j \geqslant -1$. Note that $c_Y^{j+1}\circ c_Y^j$ may be non-zero so that $(Y^j, c_Y^j)_j$ is not necessarily a complex. However, $(JY^j, c_Y^j\big|_{JY^j})_j$ defines a complex $JY^\bullet$ in $C^-(B)$ which is isomorphic to $J\widehat{\otimes}_A\widetilde{Z}^\bullet$ in $C^-(B)$. For all $j\in\mathbb{Z}$, define $\tilde{\omega}^j : \widetilde{Z}^j \to JY^{j+2}$ by

$$\tilde{\omega}^j(a_Y^j(y)) = c_Y^{j+1}(c_Y^j(y)) \tag{3.1}$$

for all $y \in Y^j$. Then $\tilde{\omega} \in \mathrm{Hom}_{C^-(B)}(\widetilde{Z}^\bullet, JY^\bullet[2])$. Let $\omega_0(Z^\bullet, A')$ be the corresponding morphism in $\mathrm{Ext}^2_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet) \cong \mathrm{Hom}_{K^-(B)}(\widetilde{Z}^\bullet, JY^\bullet[2])$.

We will show in §3.9 that $\omega_0(Z^\bullet, A')$ is independent of choices by showing that $\omega_0(Z^\bullet, A')$ can be obtained from the lifting obstruction defined by a spectral sequence by composing with suitable automorphisms of $Z^\bullet$ and $J\widehat{\otimes}_A Z^\bullet[2]$, respectively, in $D^-(B)$ (see Proposition 3.14).

In particular, by using a fixed versal deformation of $V^\bullet$ over $R = R(G,V^\bullet)$ whose terms are topologically free pseudocompact $R[[G]]$-modules, we can assume that if there exists a quasi-lift of $(Z^\bullet,\zeta)$ over A', then it is locally isomorphic to a quasi-lift $(\widetilde{Y}^\bullet,\tilde{\upsilon})$ of $(Z^\bullet,\zeta)$ over A' satisfying $\widetilde{Y}^j = Y^j$ for all j.

Since $\omega_0(Z^\bullet,A')=0$ in $D^-(B)$ if and only if $\tilde{\omega}$ is homotopic to zero in $C^-(B)$, we see the following. If there exists a quasi-lift $(\widetilde{Y}^\bullet,\tilde{\upsilon})$ of $(Z^\bullet,\zeta)$ over A' such that $\widetilde{Y}^j=Y^j$ for all j, then the homotopy $h^j\colon \widetilde{Z}^j\to JY^{j+1}=J\widetilde{Y}^{j+1}$ defined by $h^j\circ a_Y^j = c_Y^j - d_{\widetilde{Y}}^j$ for all j can be used to show that $\tilde{\omega}=0$ in $K^-(B)$. On the other hand, if $\tilde{\omega}$ is homotopic to zero in $C^-(B)$, then the corresponding homotopy can be used to correct the maps c_Y^j to obtain a complex $(Y^\bullet,d_Y)$ in $C^-(B')$ which defines a quasi-lift of $(Z^\bullet,\zeta)$ over A'.

Suppose now that $\omega_0(Z^\bullet,A')=0$, and let $(Y_0^\bullet,\upsilon_0)$ and $(Y'^\bullet,\upsilon')$ be two quasi-lifts of $(Z^\bullet,\zeta)$ over A'. As seen above, we can assume without loss of generality that $Y_0^j=Y^j=Y'^j$ for all j. For all $j\in\mathbb{Z}$, define $\tilde{\beta}^j_{Y'}\colon \widetilde{Z}^j\to JY^{j+1}$ by

$$\tilde{\beta}^j_{Y'}(a_Y^j(y)) = d^j_{Y'}(y) - d^j_{Y_0}(y) \tag{3.2}$$

for $y\in Y_0^j=Y^j=Y'^j$. Then $\tilde{\beta}_{Y'}\in \mathrm{Hom}_{C^-(B)}(\widetilde{Z}^\bullet, JY^\bullet[1])$. Let $\beta_{Y'}$ be the corresponding morphism in

$$\mathrm{Ext}^1_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)\cong \mathrm{Hom}_{K^-(B)}(\widetilde{Z}^\bullet, JY^\bullet[1]).$$

We will show later that this can be used to prove that the set of all local isomorphism classes of quasi-lifts of $(Z^\bullet,\zeta)$ over A' is a principal homogeneous space for $\mathrm{Ext}^1_{D^-(B)}(Z^\bullet,J\widehat{\otimes}_A Z^\bullet)$, by relating this to the corresponding result obtained from the spectral sequence method. More precisely, we will show that if the local isomorphism classes of the quasi-lifts $(Y_0^\bullet,\upsilon_0)$ and $(Y'^\bullet,\upsilon')$ of $(Z^\bullet,\zeta)$ over A' correspond to the classes η_0 and η', respectively, in $\mathrm{Ext}^1_{D^-(B')}(Z^\bullet,J\widehat{\otimes}_A Z^\bullet)$ by the spectral sequence method, then the difference $\eta'-\eta_0$ in $\mathrm{Ext}^1_{D^-(B')}(Z^\bullet,J\widehat{\otimes}_A Z^\bullet)$ is uniquely determined by $\beta_{Y'}$ (see Proposition 3.14).

3.2. Outline of the spectral sequence approach

In this subsection we introduce the spectral sequence approach to obstruction theory by discussing the case of modules and by then indicating what adjustments must be made for complexes. This method goes back to Illusie in [7, §3.1]. It requires more effort than the naive approach, but as indicated in the introduction, it places the local lifting problem in the context of studying Ext^1 groups.

Let Z be a pseudocompact B-module which is (abstractly) free and finitely generated over A. We have a convergent spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_B^p(\mathrm{Tor}_q^{A'}(Z,A), J\widehat{\otimes}_A Z) \Longrightarrow \mathrm{Ext}_{B'}^{p+q}(Z, J\widehat{\otimes}_A Z). \tag{3.3}$$

This arises in the following way. To find the groups $\mathrm{Tor}_q^{A'}(Z,A)$, one chooses a resolution $P^\bullet$ of Z by projective pseudocompact B'-modules. Then $\mathrm{Tor}_q^{A'}(Z,A) = \mathrm{H}^{-q}(A\widehat{\otimes}_{A'}P^\bullet)$, and the group $\mathrm{Ext}_{B'}^{p+q}(Z, J\widehat{\otimes}_A Z)$ is the group $\mathrm{H}^{p+q}(\mathrm{Hom}_{B'}(P^\bullet, J\widehat{\otimes}_A Z))$. The key observation is that since $J\widehat{\otimes}_A Z$ is a B-module, the complex $\mathrm{Hom}_{B'}(P^\bullet, J\widehat{\otimes}_A Z)$ is canonically isomorphic to the complex $\mathrm{Hom}_B(A\widehat{\otimes}_{A'}P^\bullet, J\widehat{\otimes}_A Z)$. A Cartan-Eilenberg resolution $M^{\bullet,\bullet}$ of $A\widehat{\otimes}_{A'}P^\bullet$ is a double complex of projective pseudocompact B-modules which gives a resolution of each term of $A\widehat{\otimes}_{A'}P^\bullet$ which is compatible with boundary maps and has some additional splitting properties (see [6, §(11.7) of Chap. 0]). One arrives at a double complex $L^{\bullet,\bullet}$ of B-modules given by $L^{q,p} = \mathrm{Hom}_B(M^{-q,-p}, J\widehat{\otimes}_A Z)$ such that

$$\mathrm{H}^{p+q}(\mathrm{Tot}(L^{\bullet,\bullet})) = \mathrm{H}^{p+q}(\mathrm{Hom}_B(A\widehat{\otimes}_{A'}P^\bullet, J\widehat{\otimes}_A Z)) = \mathrm{Ext}_{B'}^{p+q}(Z, J\widehat{\otimes}_A Z).$$

The spectral sequence (3.3) is then the spectral sequence of $L^{\bullet,\bullet}$ relative to the second filtration of the total complex $\mathrm{Tot}(L^{\bullet,\bullet})$. We obtain the following exact sequence of low degree terms associated to the spectral sequence (3.3):

$$0 \to E_2^{1,0} \to \mathrm{Ext}_{B'}^1(Z, J\widehat{\otimes}_A Z) \to E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \tag{3.4}$$

We now sketch Gabber's approach to realizing the obstruction to lifting Z from A to A' via the spectral sequence (3.3). We can find an exact sequence

$$0 \to T \xrightarrow{\delta} P^0 \xrightarrow{\epsilon} Z \to 0 \tag{3.5}$$

in which P^0 is a finitely generated projective pseudocompact B'-module. Applying the functor $A\widehat{\otimes}_{A'}-$ to (3.5), we obtain a Tor sequence

$$0 \to \mathrm{Tor}_1^{A'}(A,Z) \xrightarrow{\sigma} A\widehat{\otimes}_{A'}T \xrightarrow{A\widehat{\otimes}_{A'}\delta} A\widehat{\otimes}_{A'}P^0 \xrightarrow{A\widehat{\otimes}_{A'}\epsilon} Z \to 0. \tag{3.6}$$

Applying the functor $-\widehat{\otimes}_{A'}Z$ to the exact sequence

$$0 \to J \to A' \to A \to 0,$$

we obtain a canonical isomorphism

$$\iota\colon \mathrm{Tor}_1^{A'}(A,Z) \to J\widehat{\otimes}_{A'}Z = J\widehat{\otimes}_A Z \tag{3.7}$$

since $A\widehat{\otimes}_{A'}Z = Z$. Combining (3.6) and (3.7) gives an exact sequence

$$0 \to J\widehat{\otimes}_A Z \xrightarrow{\sigma\circ\iota^{-1}} A\widehat{\otimes}_{A'}T \xrightarrow{A\widehat{\otimes}_{A'}\delta} A\widehat{\otimes}_{A'}P^0 \xrightarrow{A\widehat{\otimes}_{A'}\epsilon} Z \to 0. \tag{3.8}$$

Let $\omega(Z,A')$ be the class of (3.8) in $\mathrm{Ext}^2_B(Z, J\widehat{\otimes}_A Z)$. Using the fact that $E_2^{p,q} = H^p_{II}(H^q_I(L^{\bullet,\bullet}))$ one can show that $\omega(Z,A')$ is the image of

$$\iota \in \mathrm{Hom}_B(\mathrm{Tor}_1^{A'}(A,Z), J\widehat{\otimes}_A Z) = E_2^{0,1}$$

under the boundary map

$$d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}$$

associated to the spectral sequence (3.3).

We now sketch why $\omega(Z,A')$ is the obstruction to lifting Z to a pseudocompact B'-module Y which is (abstractly) free and finitely generated over A' such that $A\widehat{\otimes}_{A'}Y \cong Z$. If such a lift Y exists, one has an exact sequence of B'-modules

$$0 \to X \to Y \to Z \to 0 \tag{3.9}$$

in which X is isomorphic to $JY = J\widehat{\otimes}_A Z$. The associated Tor sequence

$$0 \to \mathrm{Tor}_1^{A'}(A,Z) \xrightarrow{f} A\widehat{\otimes}_{A'}X \to A\widehat{\otimes}_{A'}Y \xrightarrow{\upsilon} Z \to 0 \tag{3.10}$$

has the property that υ is an isomorphism, so f is an isomorphism. Thus (3.10) has trivial extension class. By constructing a map from (3.6) to (3.10) which is an identity on the leftmost and rightmost terms we see $\omega(Z,A') = 0$. Conversely, suppose that $\omega(Z,A') = 0$. Define D to be the kernel of the homomorphism $A\widehat{\otimes}_{A'}\epsilon$ in (3.6). By dimension shifting, $\omega(Z,A') = 0$ implies that the exact sequence

$$0 \to J\widehat{\otimes}_A Z \xrightarrow{\sigma\circ\iota^{-1}} A\widehat{\otimes}_{A'}T \xrightarrow{A\widehat{\otimes}_{A'}\delta} D = \mathrm{Image}(A\widehat{\otimes}_{A'}\delta) \to 0 \tag{3.11}$$

is split by a homomorphism $\kappa\colon A\widehat{\otimes}_{A'}T \to J\widehat{\otimes}_A Z$ of pseudocompact B-modules. We now define Y to be the pushout of $T \xrightarrow{\delta} P^0$ in (3.5) and the composition $T \to A\widehat{\otimes}_{A'}T \xrightarrow{\kappa} J\widehat{\otimes}_A Z$. One then has an exact sequence of the form (3.9) with $X = J\widehat{\otimes}_A Z$. On identifying f in the resulting sequence (3.10) with $\kappa\circ\sigma = \iota$, one sees that f is an isomorphism. Therefore υ in (3.10) is an isomorphism, which shows Y is a lift of Z.

It follows from the sequence (3.4) of low degree terms that if there exists a lift of Z over A', *i.e.* if $\omega(Z,A') = 0$, then the set of all local isomorphism classes of lifts of Z over A' is in bijection with the full preimage of ι in $\mathrm{Ext}^1_{B'}(Z, J\widehat{\otimes}_A Z)$ and is therefore a principal homogeneous space for $E_2^{1,0} = \mathrm{Ext}^1_B(Z, J\widehat{\otimes}_A Z)$.

We now describe the counterpart of the spectral sequence (3.3) for a complex $Z^\bullet$ in place of Z. Assume Hypothesis 3.1. The main point of assuming that $\mathrm{H}^i(V^\bullet) = 0$ unless $-p_0 \leqslant i \leqslant -1$ is that this allows us to work in the abelian categories $C_0(B)$ and $C_0(B')$ of bounded above complexes that are concentrated in degrees $\leqslant 0$. Moreover, by insisting that $\mathrm{H}^0(V^\bullet)$ is zero, we can make sure there exists an acyclic complex of projective pseudocompact B'-modules $P^{0,\bullet}$ in $C_0(B')$ together with a morphism $\epsilon\colon P^{0,\bullet}\to Z^\bullet$ in $C_0(B')$ that is surjective on terms. One can now generalize the spectral sequence (3.3) by choosing a projective resolution $P^{\bullet,\bullet}$ of $Z^\bullet$ of projective objects in $C_0(B')$ such that $P^{0,\bullet}$ has the nice properties above. We then work with a triple complex $M^{\bullet,\bullet,\bullet}$ which is a Cartan-Eilenberg resolution of $A\widehat{\otimes}_{A'}P^{\bullet,\bullet}$. The double complex $L^{\bullet,\bullet}$ of B-modules which leads to the spectral sequence we require is a partial total complex of the quadruple complex $\mathrm{Hom}_B(M^{\bullet,\bullet,\bullet}, J\widehat{\otimes}_A Z^\bullet)$. The spectral sequence which results has the form

(3.12)
$$E_2^{p,q}=\mathrm{Ext}^p_{D^-(B)}(\mathrm{H}_I^{-q}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet) \implies \mathrm{Ext}^{p+q}_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$$

(see also (3.16)). As in the module case, we obtain an exact sequence of low degree terms, which looks slightly more complicated than the sequence (3.4):

$$(3.13)\qquad 0\to E_2^{1,0}/W_2^{1,0}\to F^0_{II}\ \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))\to E_2^{0,1}/W_2^{0,1}\xrightarrow{\overline{d_2^{0,1}}} E_2^{2,0}$$

(see also (3.24)). Here $E_\infty^{1,0}=E_2^{1,0}/W_2^{1,0}$, $E_\infty^{0,1}=\mathrm{Ker}(d_2^{0,1})/W_2^{0,1}$ and $F^0_{II}\ \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ is the second to last term in the second filtration of $\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))=\mathrm{Ext}^1_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$. The details of the set-up of the spectral sequence (3.12) and the sequence of low degree terms (3.13) for complexes $Z^\bullet$ are explained in § 3.3.

To define lifting obstructions, we follow the outlined construction in the module case given by equations (3.5) – (3.8). We assume as before that $P^{0,\bullet}$ is an acyclic complex of projective pseudocompact B'-modules in $C_0(B')$. In particular, ι is an isomorphism in $C^-(B)$ and our candidate for the lifting obstruction $\omega(Z^\bullet, A')$ is an element of $\mathrm{Ext}^2_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$. Using the definition of $L^{\bullet,\bullet}$ and the projective Cartan-Eilenberg resolution $M^{\bullet,\bullet,\bullet}$ of $A\widehat{\otimes}_{A'}P^{\bullet,\bullet}$, we see, similarly to the module case, that $\omega(Z^\bullet, A')$ is the image of ι under the boundary map $d_2^{0,1}$ associated to the spectral sequence (3.12) (see Lemma 3.17).

A complication in the case of complexes compared to the module case is that in the sequence of low degree terms (3.13) the term $F^0_{II} = F^0_{II}\ \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ is usually a proper subspace of $\mathrm{Ext}^1_{B'}(Z^\bullet, J\widehat{\otimes}_A Z)$. Therefore, we analyze in § 3.5 this subspace F^0_{II}. We use Gabber's ideas to see that F^0_{II} consists precisely of those elements in $\mathrm{Ext}^1_{B'}(Z^\bullet, J\widehat{\otimes}_A Z)$ which can be realized by short exact sequences in $C^-(B')$ of the form

$$\xi\colon\quad 0\to X^\bullet\to Y^\bullet\to Z^\bullet\to 0$$

where the terms of $X^\bullet$ are annihilated by J and there exists an isomorphism $h_\xi\colon X^\bullet \to J\widehat{\otimes}_A Z^\bullet$ in $D^-(B)$. A crucial step in showing this is to rewrite the elements of F^0_{II} in terms of morphisms $\kappa \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ (see Definition 3.18 and Lemma 3.19). We then use the definition of $L^{\bullet,\bullet}$ and in particular the triple complex $M^{\bullet,\bullet,\bullet}$ to represent the class in $\mathrm{Ext}^1_{B'}(Z^\bullet, J\widehat{\otimes}_A Z)$ given by (ξ, h_ξ) explicitly as an element in $L^{1,0}$, and hence as an element in F^0_{II}. Finally we analyze the image of $E^{1,0}_\infty = E^{1,0}_2/W^{1,0}_2$ in F^0_{II} in (3.13) and describe the map $F^0_{II} \to E^{0,1}_2/W^{0,1}_2$ in (3.13) to show that every element in F^0_{II} can be represented by a short exact sequence ξ and an isomorphism h_ξ as above. These steps are carried out in the proof of Lemma 3.21.

The proof that $\omega(Z^\bullet, A') = 0$ if and only if $Z^\bullet$ has a quasi-lift over A' is then done in a very similar way to the module case (see Lemmas 3.22 and 3.25).

Another complication in the complex case is that the left most term in the sequence (3.13) is $E^{1,0}_\infty = E^{1,0}_2/W^{1,0}_2$ rather than $E^{1,0}_2 = \mathrm{Ext}^1_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$. As in the module case, we can directly use (3.13) together with our analysis of F^0_{II} to show that if $\omega(Z^\bullet, A') = 0$ then the set of all local isomorphism classes of quasi-lifts of $Z^\bullet$ over A' is a principal homogeneous space for $E^{1,0}_\infty$. We then show that the existence of a quasi-lift of $Z^\bullet$ over A' implies that the spectral sequence (3.12) partially degenerates. More precisely, we show that the inflation map

$$\mathrm{Inf}_B^{B'}\colon\ \mathrm{Ext}^p_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet) \to \mathrm{Ext}^p_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$$

is injective for all p if $\omega(Z^\bullet, A') = 0$. This is carried out in the proof of Lemma 3.25.

3.3. A spectral sequence

In this subsection we describe the spectral sequence we will use for the obstructions. The definition of this spectral sequence follows (the dual of) Grothendieck's construction in [6, §(11.7) of Chap. 0]. The following remark describes certain subcategories of $C^-(B')$ and $C^-(B)$ which play an important role in this construction.

Remark 3.3. — Suppose $\Lambda = B'$ or B. Let $C^-_0(\Lambda)$ be the full subcategory of $C^-(\Lambda)$ whose objects are bounded above complexes $M^\bullet$ with $M^i = 0$ for $i > 0$. Then $C^-_0(\Lambda)$ is an abelian category with enough projective objects. More precisely, we have the following result which provides a slight correction of [6, Lemma 11.5.2.1], but which is proved in a similar fashion.

Let $\mathcal{P}$ be the set of all complexes $P^\bullet = (P^{-n})_{n\geqslant 0}$ in $C_0^-(\Lambda)$ having the following properties: Every P^{-n} is projective, $\mathrm{B}^{-n}(P^\bullet)$ is a direct summand of P^{-n} for $n \geqslant 0$, and $\mathrm{B}^{-n}(P^\bullet) = \mathrm{Z}^{-n}(P^\bullet)$ for $n \geqslant 1$. Then

(i) $\mathcal{P}$ is the set of projective objects in $C_0^-(\Lambda)$, and
(ii) every $M^\bullet$ in $C_0^-(\Lambda)$ is a homomorphic image of a complex in $P^\bullet \in \mathcal{P}$.

Note that $P^\bullet \in \mathcal{P}$ is not acyclic in general, but that $\mathrm{H}^{-n}(P^\bullet) = 0$ for $n \geqslant 1$ and $\mathrm{H}^0(P^\bullet)$ is a projective pseudocompact Λ-module.

We will use a projective resolution $P^{\bullet,\bullet}$ of $Z^\bullet$ in the category $C_0(B')$ of the following kind.

Definition 3.4. — *Choose a resolution of $Z^\bullet$ by projective objects in $C_0^-(B')$*

$$\cdots \to P^{-2,\bullet} \to P^{-1,\bullet} \to P^{0,\bullet} \xrightarrow{\epsilon} Z^\bullet \to 0 \tag{3.14}$$

such that $P^{-x,-y} = 0$ unless $x \geqslant 0$ and $0 \leqslant y \leqslant p_0$.

Note that $P^{\bullet,\bullet}$ has commuting differentials d'_P and d''_P. We use the same convention as in [6, §(11.3) of Chap. 0] *with respect to the differential of the total complex* $\mathrm{Tot}(P^{\bullet,\bullet})$. *Namely,* $\mathrm{Tot}(P^{\bullet,\bullet})^{-n} = \bigoplus_{-x-y=-n} P^{-x,-y}$ *and the differential is given by $d\,a = d'_P\,a + (-1)^x\,d''_P\,a$ for $a \in P^{-x,-y}$.*

Define the map $\pi_P\colon \mathrm{Tot}(P^{\bullet,\bullet}) \to Z^\bullet$ by letting $\pi_P^{-n}\colon \mathrm{Tot}(P^{\bullet,\bullet})^{-n} \to Z^{-n}$ be the composition of the natural projection $\mathrm{Tot}(P^{\bullet,\bullet})^{-n} \to P^{0,-n}$ with $\epsilon^{-n}\colon P^{0,-n} \to Z^{-n}$. Then π_P defines a quasi-isomorphism in $C_0^-(B')$ that is surjective on terms.

Using the projective resolution $P^{\bullet,\bullet}$ of $Z^\bullet$ in $C_0(B')$, we can describe the spectral sequence as follows.

Definition 3.5. — *Assume the notation of Definition 3.4. Taking the contravariant functor*

$$\mathrm{Hom}_B(-, J\widehat{\otimes}_A Z^\bullet)\colon\ \mathrm{PCMod}(B) \to C^-(B),$$

one shows similarly to [6, §(11.7) of Chap. 0] *that there is a convergent spectral sequence*

$$\mathrm{H}^p(\mathbf{R}\,\mathrm{Hom}^\bullet_B(\mathrm{H}_I^{-q}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet)) \Longrightarrow \mathrm{H}^{p+q}(\mathbf{R}\,\mathrm{Hom}^\bullet_B(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}, J\widehat{\otimes}_A Z^\bullet)). \tag{3.15}$$

Here $\mathrm{H}_I^{-q}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})$ *is the complex resulting from taking the* $-q^{\mathrm{th}}$ *cohomology in the first direction of* $A\widehat{\otimes}_{A'}P^{\bullet,\bullet}$. *Using that* $\mathbf{R}\,\mathrm{Hom}^\bullet_B(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}, J\widehat{\otimes}_A Z^\bullet) \cong \mathbf{R}\,\mathrm{Hom}^\bullet_{B'}(P^{\bullet,\bullet}, J\widehat{\otimes}_A Z^\bullet)$, *the spectral sequence* (3.15) *becomes*

$$E_2^{p,q} = \mathrm{Ext}^p_{D^-(B)}(\mathrm{H}_I^{-q}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet) \Longrightarrow \mathrm{Ext}^{p+q}_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet). \tag{3.16}$$

Note that $\mathrm{H}_I^{-q}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})$ *is the Tor complex* $\mathcal{H}^{-q}(A\widehat{\otimes}^{\mathbf{L}}_{A'}Z^\bullet)$ *from* (1.1).

The proof of the convergence of the spectral sequence (3.15) relies on the existence of a projective Cartan-Eilenberg resolution $M^{\bullet,\bullet,\bullet}$ of $A\widehat{\otimes}_{A'}P^{\bullet,\bullet}$. Moreover, the triple complex $M^{\bullet,\bullet,\bullet}$ allows us to realize the spectral sequence (3.16) as a spectral sequence of a double complex $L^{\bullet,\bullet}$ relative to the second filtration of $\mathrm{Tot}(L^{\bullet,\bullet})$. We now give the definition of $M^{\bullet,\bullet,\bullet}$ and $L^{\bullet,\bullet}$.

Definition 3.6. — *Let* $P^{\bullet,\bullet}$ *be as in Definition 3.4. As described in* [6, §(11.7) of Chap. 0], $A\widehat{\otimes}_{A'}P^{\bullet,\bullet}$ *admits a projective Cartan-Eilenberg resolution* $M^{\bullet,\bullet,\bullet} = (M^{-x,-y,-z})$ *where* $x, z \geqslant 0$ *and* $0 \leqslant y \leqslant p_0$. *This means that the terms* $M^{-x,-y,-z}$ *are projective pseudocompact* B*-modules, and for all* x, $M^{-x,\bullet,\bullet}$ *(resp.* $\mathrm{B}_I^{-x}(M^{\bullet,\bullet,\bullet})$*, resp.* $\mathrm{Z}_I^{-x}(M^{\bullet,\bullet,\bullet})$*, resp.* $\mathrm{H}_I^{-x}(M^{\bullet,\bullet,\bullet})$*) forms a projective resolution of* $A\widehat{\otimes}_{A'}(P^{-x,\bullet})$ *(resp.* $\mathrm{B}_I^{-x}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})$*, resp.* $\mathrm{Z}_I^{-x}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})$*, resp.* $\mathrm{H}_I^{-x}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})$*) in the abelian category* $C_0^-(B)$*. In particular,* $M^{-x,-y,\bullet} \to A\widehat{\otimes}_{A'}P^{-x,-y} \to 0$ *is a projective resolution in the category* $\mathrm{PCMod}(B)$ *for all* x, y*. The Cartan-Eilenberg property implies that we have for all* x, z *split exact sequences of complexes in* $C_0^-(B)$

$$0 \to \mathrm{B}_I^{-x}(M^{\bullet,\bullet,-z}) \to \mathrm{Z}_I^{-x}(M^{\bullet,\bullet,-z}) \to \mathrm{H}_I^{-x}(M^{\bullet,\bullet,-z}) \to 0, \tag{3.17}$$

$$0 \to \mathrm{Z}_I^{-x}(M^{\bullet,\bullet,-z}) \to M^{-x,\bullet,-z} \xrightarrow{d_{M,x}} \mathrm{B}_I^{-x+1}(M^{\bullet,\bullet,-z}) \to 0. \tag{3.18}$$

Since $M^{\bullet,\bullet,\bullet}$ *has commuting differentials* $d_{M,x}$*,* $d_{M,y}$ *and* $d_{M,z}$*, we use again the convention in* [6, §(11.3) of Chap. 0] *with respect to the differential of the total complex* $\mathrm{Tot}(M^{\bullet,\bullet,\bullet})$*. Define the map* $\pi_M\colon \mathrm{Tot}(M^{\bullet,\bullet,\bullet}) \to \mathrm{Tot}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})$ *by letting* π_M^{-n} *be the composition of the natural projection* $\mathrm{Tot}(M^{\bullet,\bullet,\bullet})^{-n} \to \bigoplus_{-x-y=-n} M^{-x,-y,0}$ *with the direct sum of the surjections* $M^{-x,-y,0} \to A\widehat{\otimes}_{A'}P^{-x,-y}$*. Then* π_M *defines a quasi-isomorphism in* $C_0^-(B')$ *that is surjective on terms.*

Define a double complex $L^{\bullet,\bullet}$ *of* B*-modules by*

$$L^{q,p} = \bigoplus_{-i+y+z=p} \mathrm{Hom}_B(M^{-q,-y,-z}, J\widehat{\otimes}_A Z^{-i}). \tag{3.19}$$

Since $1 \leqslant i \leqslant p_0$, $0 \leqslant y \leqslant p_0$ *and* $z \geqslant 0$, *it follows that for each integer* p, *there are only finitely many triples* (y, z, i) *with* $-i + y + z = p$. *So we could also have used* $\prod$ *instead of* $\bigoplus$ *in defining* $L^{q,p}$. *Note that* $L^{q,p} = 0$ *unless* $q \geqslant 0$ *and* $p \geqslant -p_0$. *In particular, for each integer* n *there are only finitely many pairs* (q, p) *with* $q + p = n$ *and* $L^{q,p} \neq 0$. *The differentials*

$$d_I^{q,p} \colon L^{q,p} \to L^{q+1,p} \qquad \text{and} \qquad d_{II}^{q,p} \colon L^{q,p} \to L^{q,p+1}$$

are described as follows:

$$d_I^{q,p}(g) = g \circ d_{M,x}^{-q-1,-y,-z}, \tag{3.20}$$

$$d_{II}^{q,p}(g) = g \circ d_{M,y}^{-q,-y-1,-z} + (-1)^y\, g \circ d_{M,z}^{-q,-y,-z-1} + (-1)^{p+1}\, d^{-i}_{J\widehat{\otimes}_A Z} \circ g \tag{3.21}$$

for $g \in \mathrm{Hom}_B(M^{-q,-y,-z}, J\widehat{\otimes}_A Z^{-i})$. *Since* d_I *and* d_{II} *commute, the total complex of* $L^{\bullet,\bullet}$ *whose* n^{th} *term is*

$$\mathrm{Tot}(L^{\bullet,\bullet})^n = \bigoplus_{q+(-i+y+z)=n} \mathrm{Hom}_B(M^{-q,-y,-z}, J\widehat{\otimes}_A Z^{-i})$$

has differential d *with* $d\,g = d_I^{q,p}(g) + (-1)^q\, d_{II}^{q,p}(g)$ *for* $g \in \mathrm{Hom}_B(M^{-q,-y,-z}, J\widehat{\otimes}_A Z^{-i})$. *Note that* $\mathrm{Tot}(L^{\bullet,\bullet})$ *is the total Hom complex corresponding to the quadruple complex* $\big(\mathrm{Hom}_B(M^{-q,-y,-z}, J\widehat{\otimes}_A Z^{-i})\big)_{q,y,z,i}$.

The following definition pertains to realizing the spectral sequence (3.16) as the spectral sequence of $L^{\bullet,\bullet}$ relative to the second filtration of $\mathrm{Tot}(L^{\bullet,\bullet})$. This then leads to the sequences of low degree terms corresponding to (3.16).

Definition 3.7. — *Assume the notation of Definitions 3.4 – 3.6. Let* $(F_{II}^r(\mathrm{Tot}(L^{\bullet,\bullet})))_{r\in\mathbb{Z}}$ *be the filtration of the total complex* $\mathrm{Tot}(L^{\bullet,\bullet})$ *defined by*

$$F_{II}^r(\mathrm{Tot}(L^{\bullet,\bullet}))^n = \bigoplus_{q+p=n,\, p\geqslant r} L^{q,p}. \tag{3.22}$$

Define $F_{II}^r\ \mathrm{H}^n(\mathrm{Tot}(L^{\bullet,\bullet}))$ *to be the image in* $\mathrm{H}^n(\mathrm{Tot}(L^{\bullet,\bullet}))$ *of the* n*-cocycles in* $F_{II}^r(\mathrm{Tot}(L^{\bullet,\bullet}))$, *i.e. of the elements in* $F_{II}^r(\mathrm{Tot}(L^{\bullet,\bullet}))^n$ *that are in the kernel of the* n^{th} *differential of* $\mathrm{Tot}(L^{\bullet,\bullet})$.

The spectral sequence (3.16) coincides with the spectral sequence of the double complex $L^{\bullet,\bullet}$ *relative to the filtration* $(F_{II}^r(\mathrm{Tot}(L^{\bullet,\bullet})))_{r\in\mathbb{Z}}$ *of* $\mathrm{Tot}(L^{\bullet,\bullet})$ *in (3.22). In particular,*

$$E_2^{p,q} = \mathrm{H}_{II}^p(\mathrm{H}_I^q(L^{\bullet,\bullet}))$$

and

$$\mathrm{H}^{p+q}(\mathrm{Tot}(L^{\bullet,\bullet})) = \mathrm{Ext}^{p+q}_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet).$$

We have a short exact sequence of low degree terms

$$0 \to E_\infty^{1,0} \xrightarrow{\psi^0_{II}} F^0_{II}\ \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) \xrightarrow{\varphi^0_{II}} E_\infty^{0,1} \to 0. \tag{3.23}$$

Here $E_\infty^{1,0}$ *is the quotient of* $E_2^{1,0}$ *by the subgroup* $W_2^{1,0}$ *which is defined as the sum of the preimages in* $E_2^{1,0}$ *of the successive images of* $d_2^{-1,1}, d_3^{-2,2}, \ldots$. *Similarly* $E_\infty^{0,1}$ *is the quotient of* $\mathrm{Ker}(d_2^{0,1})$ *by the subgroup* $W_2^{0,1}$ *which is defined as the sum of the preimages in* $\mathrm{Ker}(d_2^{0,1})$ *of the successive images of* $d_2^{-2,2}, d_3^{-3,3}, \ldots$. *Since* $d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}$ *sends* $W_2^{0,1}$ *identically to zero, the short exact sequence* (3.23) *results in an exact sequence of low degree terms*

$$0 \to E_\infty^{1,0} \xrightarrow{\psi^0_{II}} F^0_{II}\ \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) \xrightarrow{\tilde{\varphi}^0_{II}} E_2^{0,1}/W_2^{0,1} \xrightarrow{\overline{d_2^{0,1}}} E_2^{2,0}. \tag{3.24}$$

3.4. Obstruction results

In this subsection we list the main results concerning the obstruction to lifting $(Z^\bullet, \zeta)$ to A'. A key ingredient is a careful analysis of the exact sequence of low degree terms in (3.23). The following definition is used to relate the term $F^0_{II}\ \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ in (3.23) to extension classes arising from short exact sequences of bounded above complexes of pseudocompact B'-modules.

Definition 3.8. — *In* $\mathrm{Ext}^1_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$ *let* $\widetilde{F}^0_{II}$ *be the subset of classes represented by short exact sequences in* $C^-(B')$

$$\xi\colon 0 \to X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{v_\xi} Z^\bullet \to 0 \tag{3.25}$$

such that the terms of $X^\bullet$ *are annihilated by* J, *and there is an isomorphism* $h_\xi\colon X^\bullet \to J\widehat{\otimes}_A Z^\bullet$ *in* $D^-(B)$. *Note that* h_ξ *defines an isomorphism in* $D^-(B')$. *The triangle associated to the sequence* ξ *in* (3.25) *has the form*

$$X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{v_\xi} Z^\bullet \xrightarrow{w_\xi} X^\bullet[1] \tag{3.26}$$

where $\eta_\xi = h_\xi[1] \circ w_\xi \in \mathrm{Hom}_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet[1]) = \mathrm{Ext}^1_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$ *is the class represented by* (ξ, h_ξ). *Applying the functor* $A\widehat{\otimes}_{A'}-$ *to* (3.25) *gives the long exact Tor sequence in* $C^-(B)$

(3.27)

$$\cdots \to \mathrm{Tor}_1^{A'}(Y^\bullet, A) \to \mathrm{Tor}_1^{A'}(Z^\bullet, A) \xrightarrow{f_\xi} X^\bullet \to A\widehat{\otimes}_{A'} Y^\bullet \to Z^\bullet \to 0$$

where $\mathrm{Tor}_1^{A'}(Z^\bullet, A) = \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet})$ *since* $P^{\bullet,\bullet}$ *in Definition 3.4 is a projective resolution of* $Z^\bullet$.

Theorem 3.9. — *Assume Hypothesis 3.1 and the notation introduced in Definitions 3.4 – 3.8. The short exact sequence* (3.23) *has the following properties.*

(i) *The group* $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ *equals the subset* $\widetilde{F}^0_{II}$ *from Definition 3.8.*

(ii) *The image of* $E^{1,0}_\infty$ *under* ψ^0_{II} *in* $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \widetilde{F}^0_{II}$ *is equal to the subset of* $\widetilde{F}^0_{II}$ *consisting of classes represented by short exact sequences as in* (3.25) *where* $Y^\bullet$ *is in* $C^-(B)$.

(iii) *The map* $\varphi^0_{II}\colon F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) \to E^{0,1}_\infty$ *is defined in the following way. Represent a class in* $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \widetilde{F}^0_{II}$ *by* (ξ, h_ξ) *as in Definition 3.8. Let* $f_\xi : \mathrm{Tor}_1^{A'}(Z^\bullet, A) = \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to X^\bullet$ *be as in* (3.27). *Then* (ξ, h_ξ) *is sent to the class of* $h_\xi \circ f_\xi$ *in* $E^{0,1}_\infty$.

We obtain the following connection between the local isomorphism classes of quasi-lifts of $(Z^\bullet,\zeta)$ over A' and the classes in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \widetilde{F}^0_{II}$ defined by short exact sequences ξ as in (3.25).

Lemma 3.10. — *Assume the hypotheses of Theorem 3.9. If* $(Z^\bullet,\zeta)$ *has a quasi-lift over* A', *then the local isomorphism class of every quasi-lift of* $(Z^\bullet,\zeta)$ *over* A' *contains a quasi-lift* $(Y^\bullet,\upsilon)$ *such that* $Y^\bullet$ *occurs as the middle term of a short exact sequence* ξ *as in* (3.25).

The obstruction $\omega(Z^\bullet, A')$ to lifting $(Z^\bullet,\zeta)$ to A' is defined in terms of the following natural homomorphism in $C^-(B)$.

Definition 3.11. — *Let* $\iota\colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) = \mathrm{Tor}_1^{A'}(Z^\bullet, A) \to J\widehat{\otimes}_A Z^\bullet$ *be the natural homomorphism in* $C^-(B)$ *resulting from tensoring the short exact sequence* $0\to J\to A'\to A\to 0$ *with* $Z^\bullet$ *over* A'. *Because the terms of* $Z^\bullet$ *are topologically flat* A*-modules by Hypothesis 3.1 , we get an exact sequence in* $C^-(B)$

$$0 \to \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \xrightarrow{\iota} J\widehat{\otimes}_{A'}Z^\bullet \to A'\widehat{\otimes}_{A'}Z^\bullet \xrightarrow{\cong} A\widehat{\otimes}_{A'}Z^\bullet \to 0. \tag{3.28}$$

Hence ι *is an isomorphism in* $C^-(B)$.

Theorem 3.12. — *Assuming the hypotheses of Theorem 3.9, let* $\iota\colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to J\widehat{\otimes}_A Z^\bullet$ *be the isomorphism in* $C^-(B)$ *from Definition 3.11. If* $[\iota]$ *is the class of* ι *in* $E_2^{0,1}/W_2^{0,1}$, *let* $\omega = \omega(Z^\bullet, A')$ *be the class* $\omega = \overline{d_2^{0,1}}([\iota]) = d_2^{0,1}(\iota) \in E_2^{2,0} = \mathrm{Ext}^2_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$.

(i) *The class* ω *is zero if and only if there is a quasi-lift* $(Y^\bullet,\upsilon)$ *of* $(Z^\bullet,\zeta)$ *over* A'.

(ii) *If* $\omega = 0$, *then* $[\iota]\in E^{0,1}_\infty$ *and the set of all local isomorphism classes of quasi-lifts of* $(Z^\bullet,\zeta)$ *over* A' *is in bijection with the full preimage* *of* $[\iota]$ *in* $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \widetilde{F}^0_{II}$ *under* φ^0_{II}. *In other words, the set of all local isomorphism classes of quasi-lifts of* $(Z^\bullet, \zeta)$ *over* A' *is a principal homogeneous space for* $E^{1,0}_\infty$.

(iii) *If* $\omega = 0$, *then* $E^{p,0}_2 = E^{p,0}_\infty$ *for all* p, *i.e. the spectral sequence* (3.16) *partially degenerates.*

We will see in Remark 3.26 that if the lifting obstruction $\omega(Z^\bullet, A') \neq 0$, then $E^{1,0}_\infty$ is a proper quotient of $E^{1,0}_2$ in general.

With respect to automorphisms of quasi-lifts, we get the following result.

Lemma 3.13. — *Assume the notation of Theorem 3.12, and suppose that* $\omega(Z^\bullet, A') = 0$. *Let* $(Y^\bullet, \upsilon)$ *be a quasi-lift of* $(Z^\bullet, \zeta)$ *over* A'. *Define* $\mathrm{Aut}^0_{D^-(B')}(Y^\bullet)$ *to be the group of automorphisms* θ *of* $Y^\bullet$ *in* $D^-(B')$ *for which* $\upsilon \circ (A\widehat{\otimes}^{\mathbf{L}}_{A'}\theta) = \upsilon$ *in* $D^-(B)$, *i.e.* $A\widehat{\otimes}^{\mathbf{L}}_{A'}\theta$ *is equal to the identity on* $A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet$ *in* $D^-(B)$. *Then*

$$\mathrm{Aut}^0_{D^-(B')}(Y^\bullet) \cong \mathrm{Hom}_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)/\,\mathrm{Image}(\mathrm{Ext}^{-1}_{D^-(B)}(Z^\bullet, Z^\bullet)).$$

Here $\mathrm{Image}(\mathrm{Ext}^{-1}_{D^-(B)}(Z^\bullet, Z^\bullet))$ *is the image of* $\mathrm{Ext}^{-1}_{D^-(B)}(Z^\bullet, Z^\bullet)$ *in* $\mathrm{Hom}_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$ *under the map which is induced by the homomorphism* $A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet[-1] \to J\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet$ *in the triangle* $A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet[-1] \to J\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet \to A'\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet \to A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet$ *in* $D^-(B')$.

We obtain the following connection between the lifting obstruction $\omega(Z^\bullet, A')$ of Theorem 3.12 and the lifting obstruction $\omega_0(Z^\bullet, A')$ resulting from the naive approach described in § 3.1.

Proposition 3.14. — *Assume the notation of § 3.1 and Theorem 3.12. There exists an automorphism* u *(resp.* v*) of* $Z^\bullet$ *(resp.* $J\widehat{\otimes}_A Z^\bullet$*) in* $D^-(B)$ *such that* $\omega_0(Z^\bullet, A') = v[2] \circ \omega(Z^\bullet, A') \circ u$ *in* $D^-(B)$.

Suppose $\omega(Z^\bullet, A') = 0$. *There exists an automorphism* u' *(resp.* v'*) of* $Z^\bullet$ *(resp.* $J\widehat{\otimes}_A Z^\bullet$*) in* $D^-(B')$ *with the following property: Let* $(Y^\bullet_0, \upsilon_0)$ *and* $(Y'^\bullet, \upsilon')$ *be two quasi-lifts of* $(Z^\bullet, \zeta)$ *over* A' *whose local isomorphism classes correspond to* η_{ξ_0} *and* $\eta_{\xi'}$, *respectively, in* $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \widetilde{F}^0_{II}$ *according to Lemma 3.10 and Theorem 3.12(ii). Then* $\eta_{\xi'} - \eta_{\xi_0} = v'[1] \circ \varphi^0_{II}(\beta_{Y'}) \circ u'$ *in* $D^-(B')$.

The proofs of Theorems 3.9, 3.12, Lemma 3.13 and Proposition 3.14 are carried out in several sections.

3.5. Gabber's construction

In this subsection we prove a result due to Gabber which is the key to relating the term $F_{II}^0\ \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ from the sequence (3.24) to the set $\widetilde{F}_{II}^0$ from Definition 3.8.

Definition 3.15. — *Assume Hypothesis 3.1 and the notation introduced in Definitions 3.4 – 3.8. We have a short exact sequence in $C^-(B')$*

$$0 \to T^\bullet \xrightarrow{\delta} P^{0,\bullet} \xrightarrow{\epsilon} Z^\bullet \to 0 \tag{3.29}$$

where $T^\bullet = \mathrm{Ker}(\epsilon)$ and δ is inclusion. Recall that $P^{0,\bullet}$ is a projective object in $C_0^-(B')$. Since $Z^0 = 0$, we can, and will, assume that $P^{0,\bullet}$ is an acyclic complex of projective pseudocompact B'-modules. Tensoring (3.29) with A over A' gives an exact sequence of complexes in $C^-(B)$

$$0 \to \mathrm{Tor}_1^{A'}(Z^\bullet, A) \xrightarrow{\sigma} A\widehat{\otimes}_{A'}T^\bullet \xrightarrow{A\widehat{\otimes}_{A'}\delta} A\widehat{\otimes}_{A'}P^{0,\bullet} \xrightarrow{A\widehat{\otimes}_{A'}\epsilon} Z^\bullet \to 0 \tag{3.30}$$

where $\mathrm{Tor}_1^{A'}(Z^\bullet, A) = \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})$. Write (3.30) as the Yoneda composition of two short exact sequences in $C^-(B)$

$$0 \to D^\bullet \xrightarrow{\delta_D} A\widehat{\otimes}_{A'}P^{0,\bullet} \xrightarrow{A\widehat{\otimes}_{A'}\epsilon} Z^\bullet \to 0, \tag{3.31}$$

$$0 \to \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \xrightarrow{\sigma} A\widehat{\otimes}_{A'}T^\bullet \xrightarrow{\tau} D^\bullet \to 0. \tag{3.32}$$

Then the triangles in $D^-(B)$ associated to (3.31) and to (3.32) have the form

$$D^\bullet \xrightarrow{\delta_D} A\widehat{\otimes}_{A'}P^{0,\bullet} \xrightarrow{A\widehat{\otimes}_{A'}\epsilon} Z^\bullet \xrightarrow{\alpha_1} D^\bullet[1], \tag{3.33}$$

$$\mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \xrightarrow{\sigma} A\widehat{\otimes}_{A'}T^\bullet \xrightarrow{\tau} D^\bullet \xrightarrow{\alpha_2} \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})[1]. \tag{3.34}$$

We first express the differential $d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}$ in terms of the morphisms α_1 and α_2 in the triangles (3.33) and (3.34) in $D^-(B)$.

Remark 3.16. — By (3.16) and Definition 3.7,

$$E_2^{p,q} = \mathrm{Ext}^p_{D^-(B)}(\mathrm{H}_I^{-q}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet) = \mathrm{H}^p_{II}(H^q_I(L^{\bullet,\bullet})). \tag{3.35}$$

Thus the elements in $E_2^{p,q}$ are represented by elements $\beta \in L^{q,p}$ satisfying $d_I^{q,p}(\beta) = 0$ and $d_{II}^{q,p}(\beta) \in \mathrm{Image}(d_I^{q-1,p+1})$. It follows from (3.19) that

$$L^{q,p} = \bigoplus_j \mathrm{Hom}_B(\mathrm{Tot}(M^{-q,\bullet,\bullet})^{-j}, J\widehat{\otimes}_A Z^{-j+p}), \tag{3.36}$$

which is equal to the 0^{th} term in the total Hom complex $\mathrm{Hom}_B^\bullet(\mathrm{Tot}(M^{-q,\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet[p])$.

Lemma 3.17. — *Assume the notation of Definition 3.15 and Remark 3.16, and in particular the notation of* (3.33), (3.34) *and* (3.35). *If*

$$f \in E_2^{0,1} = \mathrm{Hom}_{D^-(B)}(\mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet),$$

then $d_2^{0,1}(f) = f[2]\circ\alpha_2[1]\circ\alpha_1 \in \mathrm{Hom}_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet[2]) = \mathrm{Ext}^2_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet) = E_2^{2,0}$.

Proof. — It follows from Remark 3.16 that if $\beta_f \in L^{1,0}$ represents $f \in E_2^{0,1}$, then there exists $\gamma_f \in L^{0,1}$ with $d_{II}^{1,0}(\beta_f) = d_I^{0,1}(\gamma_f)$. Hence $d_2^{0,1}(f) \in E_2^{2,0}$ is represented by $d_{II}^{0,1}(\gamma_f) \in L^{0,2}$. A calculation using (3.20) and (3.21) shows that $d_{II}^{0,1}(\gamma_f)$ also represents $f[2]\circ\alpha_2[1]\circ\alpha_1 \in E_2^{2,0}$. In carrying out this calculation, it is useful to represent α_1 explicitly in (3.33) using a quasi-isomorphism between the mapping cone of δ_D and $Z^\bullet$, and similarly for α_2 in (3.34). □

The next definition gives a connection between morphisms κ in $\mathrm{Hom}_{D^-(B)}$ $(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ and elements in $\widetilde{F}^0_{II}$. This is the key to relating $\widetilde{F}^0_{II}$ to F^0_{II} $\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$.

Definition 3.18. — *Assume the notation of Definition 3.15, so that in particular,* $P^{0,\bullet}$ *is an acyclic complex of projective pseudocompact* B'*-modules. Suppose* $\kappa : A\widehat{\otimes}_{A'}T^\bullet \to J\widehat{\otimes}_A Z^\bullet$ *is a homomorphism in* $D^-(B)$. *Then* κ *can be represented as*

$$\kappa = s^{-1}\circ\tilde{\kappa} \tag{3.37}$$

for suitable homomorphisms $s\colon J\widehat{\otimes}_A Z^\bullet \to X^\bullet$ *and* $\tilde{\kappa}\colon A\widehat{\otimes}_{A'}T^\bullet \to X^\bullet$ *in* $C^-(B)$ *such that* s *is a quasi-isomorphism. We obtain a pushout diagram in* $C^-(B')$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^\bullet & \xrightarrow{\delta} & P^{0,\bullet} & \xrightarrow{\epsilon} & Z^\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_T} & & \Big\downarrow & & \Big\| & & \\
 & & A\widehat{\otimes}_{A'}T^\bullet & & {\scriptstyle\lambda} & & & & \\
 & & \downarrow{\scriptstyle\tilde{\kappa}} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & X^\bullet & \xrightarrow{u_\xi} & Y^\bullet & \xrightarrow{v_\xi} & Z^\bullet & \longrightarrow & 0
\end{array} \tag{3.38}$$

where $a_T\colon T^\bullet \to A\widehat{\otimes}_{A'}T^\bullet$ *is the natural homomorphism in* $C^-(B')$. *Let* ξ *be the bottom row of* (3.38) *and let* $h_\xi = s^{-1}$ *in* $D^-(B)$. *Then* (ξ, h_ξ) *represents a class* $\eta_\xi \in \widetilde{F}^0_{II}$ *as in Definition 3.8. Considering the triangles associated to the top and bottom rows of* (3.38), *we obtain a commutative* *diagram in* $D^-(B')$

$$\begin{array}{ccccccc} T^\bullet & \xrightarrow{\delta} & P^{0,\bullet} & \xrightarrow{\epsilon} & Z^\bullet & \xrightarrow{\eta_T} & T^\bullet[1] \\ \downarrow{\scriptstyle\tilde\lambda} & & \downarrow{\scriptstyle\lambda} & & \| & & \downarrow{\scriptstyle\tilde\lambda[1]} \\ X^\bullet & \xrightarrow{u_\xi} & Y^\bullet & \xrightarrow{v_\xi} & Z^\bullet & \xrightarrow{w_\xi} & X^\bullet[1] \end{array} \tag{3.39}$$

where $\tilde\lambda = \tilde\kappa \circ a_T$. *Hence* $\eta_\xi = h_\xi[1] \circ w_\xi = s^{-1}[1] \circ \tilde\kappa[1] \circ a_T[1] \circ \eta_T = \kappa[1] \circ a_T[1] \circ \eta_T$. *Thus the class* $\eta_\xi \in \widetilde{F}^0_{II}$ *is independent of the choice of the triple* $(X^\bullet, s, \tilde\kappa)$ *used to represent* κ, *and we denote this class by* η_κ. *In particular,*

$$\eta_\kappa = \kappa[1] \circ a_T[1] \circ \eta_T. \tag{3.40}$$

Since $P^{0,\bullet}$ *is acyclic, it follows that* $\eta_T \colon Z^\bullet \to T^\bullet[1]$ *is an isomorphism in* $D^-(B')$. *Therefore it follows from* (3.40) *that if* $\kappa, \kappa' \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$, *then* $\eta_\kappa = \eta_{\kappa'}$ *if and only if* $\kappa \circ a_T = \kappa' \circ a_T$ *in* $D^-(B')$.

Lemma 3.19 (O. Gabber). — *Assume the notation of Definition 3.15, so that in particular,* $P^{0,\bullet}$ *is an acyclic complex of projective pseudocompact* B'*-modules.*

(i) *Let* (ξ, h_ξ) *represent a class* η_ξ *in* $\widetilde{F}^0_{II}$ *as in Definition* 3.8, *and let* $f_\xi \colon \mathrm{Tor}_1^{A'}(Z^\bullet, A) = \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to X^\bullet$ *be as in* (3.27). *Then* $d_2^{0,1}(h_\xi \circ f_\xi) = 0$. *Moreover, there exists* $\kappa_\xi \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ *such that* $h_\xi \circ f_\xi = \kappa_\xi \circ \sigma$ *and* $\eta_\xi = \eta_{\kappa_\xi}$, *where* σ *is as in* (3.32) *and* η_{κ_ξ} *is the class in* $\widetilde{F}^0_{II}$ *defined by* κ_ξ *as in Definition 3.18.*

(ii) *Conversely, suppose* $f \in E_2^{0,1} = \mathrm{Hom}_{D^-(B)}(\mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet)$ *satisfies* $d_2^{0,1}(f) = 0$. *Then there exists* $\kappa \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ *such that* $\kappa \circ \sigma = f$. *Moreover, if* $\kappa' \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ *also satisfies* $\kappa' \circ \sigma = f$, *then there exists* $\alpha \in \mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_A Z^\bullet) \cong \mathrm{Ext}^1_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet) = E_2^{1,0}$ *with* $\kappa - \kappa' = \alpha \circ \tau$. *Let* $\eta_\kappa \in \widetilde{F}^0_{II}$ *be the class defined by* κ *as in Definition 3.18 and let* (ξ, h_ξ) *be a representative. Then the corresponding morphism* $f_\xi \colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to X^\bullet$ *from* (3.27) *satisfies* $h_\xi \circ f_\xi = f$.

(iii) *Let* $\widetilde{F}^1_{II}$ *be the subset of* $\widetilde{F}^0_{II}$ *consisting of classes represented by short exact sequences as in* (3.25) *where* $Y^\bullet$ *is in* $C^-(B)$. *Then* $\widetilde{F}^1_{II}$ *is equal to the set of all classes* η_{κ_α} *in* $\widetilde{F}^0_{II}$ *defined by* $\kappa_\alpha = \alpha \circ \tau$ *as in Definition 3.18 as* α *varies over all elements in* $\mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_A Z^\bullet) \cong E_2^{1,0}$.

Proof. — Since $P^{0,\bullet}$ is acyclic, the morphism $\alpha_1\colon Z^\bullet \to D^\bullet[1]$ from (3.33) is an isomorphism in $D^-(B)$. Thus $\mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_A Z^\bullet) \cong \mathrm{Ext}^1_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet) = E_2^{1,0}$. Moreover, using Lemma 3.17, we have that $d_2^{0,1}(f) = 0$ if and only if $f[1]\circ\alpha_2 = 0$ in $D^-(B)$.

For part (i), let (ξ, h_ξ) be as in Definition 3.8, where $\xi\colon 0 \to X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{v_\xi} Z^\bullet \to 0$. Since $P^{0,\bullet}$ is a projective object in $C^-(B')$, there exists a commutative diagram in $C^-(B')$ of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^\bullet & \xrightarrow{\delta} & P^{0,\bullet} & \xrightarrow{\epsilon} & Z^\bullet & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\tilde\lambda} & & \Big\downarrow{\scriptstyle\lambda} & & \Big\| & & \\ 0 & \longrightarrow & X^\bullet & \xrightarrow{u_\xi} & Y^\bullet & \xrightarrow{v_\xi} & Z^\bullet & \longrightarrow & 0. \end{array} \tag{3.41}$$

Because J annihilates the terms of $X^\bullet$, $\tilde\lambda$ factors as $\tilde\lambda = (A\widehat{\otimes}_{A'}\tilde\lambda)\circ a_T$. Hence ξ is the bottom row of a pushout diagram as in (3.38) with $\tilde\kappa = A\widehat{\otimes}_{A'}\tilde\lambda$. Letting $\kappa_\xi = h_\xi \circ (A\widehat{\otimes}_{A'}\tilde\lambda)$ gives $\eta_\xi = \eta_{\kappa_\xi}$ by Definition 3.18. Tensoring (3.41) with A over A' and using 3.30 shows that $(A\widehat{\otimes}_{A'}\tilde\lambda)\circ\sigma = f_\xi$. Since α_2 and $\sigma[1]$ are consecutive maps in the triangle obtained by shifting (3.34), this implies that $f_\xi[1]\circ\alpha_2 = 0$ in $D^-(B)$, and hence $d_2^{0,1}(h_\xi\circ f_\xi) = 0$. Moreover, $h_\xi\circ f_\xi = \kappa_\xi\circ\sigma$.

For part (ii), assume $d_2^{0,1}(f) = 0$. Applying the functor $\mathrm{Hom}_{D^-(B)}(-, J\widehat{\otimes}_A Z^\bullet)$ to the triangle (3.34), we obtain a long exact Hom sequence. By the first paragraph of the proof, $f\circ\alpha_2[-1] = 0$, which shows that there exists $\kappa \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ with $\kappa\circ\sigma = f$. Let $\eta_\kappa \in \widetilde{F}^0_{II}$ be the class defined by κ as in Definition 3.18 and let (ξ, h_ξ) be a representative. In particular, $\kappa = h_\xi\circ\tilde\kappa$ where $\tilde\kappa$ is as in (3.38) and ξ is the bottom row of (3.38). Tensoring (3.38) with A over A' and using (3.30) shows that $\tilde\kappa\circ\sigma = f_\xi$. This implies that $h_\xi\circ f_\xi = \kappa\circ\sigma = f$.

For part (iii), let first $\alpha \in \mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_A Z^\bullet)$ and let $\kappa = \alpha\circ\tau$. Following the construction of the class $\eta_\kappa \in \widetilde{F}^0_{II}$ in Definition 3.18 which has representative (ξ, h_ξ), we see that we can choose $\tilde\kappa$ in (3.37) and in (3.38) to be of the form $\tilde\kappa = \tilde\mu\circ\tau$ for a suitable $\tilde\mu\colon D^\bullet \to X^\bullet$ in $C^-(B)$. Using the definitions of δ_D and τ in (3.31) and (3.32), it follows that ξ is the bottom row of a pushout diagram in $C^-(B)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^\bullet & \xrightarrow{\delta_D} & A\widehat{\otimes}_{A'}P^{0,\bullet} & \xrightarrow{A\widehat{\otimes}_{A'}\epsilon} & Z^\bullet & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\tilde\mu} & & \Big\downarrow{\scriptstyle\mu} & & \Big\| & & \\ 0 & \longrightarrow & X^\bullet & \xrightarrow{u_\xi} & Y^\bullet & \xrightarrow{v_\xi} & Z^\bullet & \longrightarrow & 0 \end{array} \tag{3.42}$$

where the first row is given by (3.31). This implies that $\eta_\kappa = \eta_\xi$ lies in $\widetilde{F}^1_{II}$. To prove the converse direction, one takes the representative (ξ, h_ξ) of a class in $\widetilde{F}^1_{II}$ and uses that $A\widehat{\otimes}_{A'}P^{0,\bullet}$ is a projective object in $C^-(B)$ to realize ξ as the bottom row of a diagram as in (3.42). Letting $\tilde{\kappa} = \tilde{\mu} \circ \tau$, it follows that ξ is also the bottom row of a pushout diagram as in (3.38). Define $\kappa = h_\xi \circ \tilde{\kappa}$ and $\alpha = h_\xi \circ \tilde{\mu} \in \mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_A Z^\bullet)$. Then $\kappa = \alpha \circ \tau$ and $\eta_\xi = \eta_\kappa$. □

3.6. Proof of Theorem 3.9

In this subsection we prove Theorem 3.9 by proving Lemma 3.21 given below. We use the following notation.

Definition 3.20. — *Suppose $\Lambda = B'$ or B, and $M_1^\bullet$ and $M_2^\bullet$ are complexes in $C^-(\Lambda)$. We say a homomorphism $f \in \mathrm{Hom}_{D^-(\Lambda)}(M_1^\bullet, M_2^\bullet)$ is represented by a homomorphism $f' \colon M_1'^\bullet \to M_2'^\bullet$ in $C^-(\Lambda)$ (resp. in $D^-(\Lambda)$) if there exist isomorphisms $s_i \colon M_i^\bullet \to M_i'^\bullet$ in $D^-(\Lambda)$ for $i = 1, 2$ with $f = s_2^{-1} \circ f' \circ s_1$ in $D^-(\Lambda)$.*

Lemma 3.21. — *Assume the notation of Definition 3.15, so that in particular, $P^{0,\bullet}$ is an acyclic complex of projective pseudocompact B'-modules. Let (ξ, h_ξ) represent a class η_ξ in $\widetilde{F}^0_{II}$ as in Definition 3.8, where $\xi \colon 0 \to X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{v_\xi} Z^\bullet \to 0$. Let $w_\xi \in \mathrm{Hom}_{D^-(B')}(Z^\bullet, X^\bullet[1])$ be as in (3.26), and let $f_\xi \colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to X^\bullet$ be the connecting homomorphism as in (3.27).*

(i) *The class $\eta_\xi = h_\xi[1] \circ w_\xi$ lies in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$. More precisely, η_ξ defines an element $\beta_\xi \in L^{1,0}$ which lies in the kernel of the first differential of $\mathrm{Tot}(L^{\bullet,\bullet})$. This identifies $\widetilde{F}^0_{II}$ with a subset of $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$.*
(ii) *The map $\varphi^0_{II} : F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) \to E^{0,1}_\infty$ in (3.23) sends $\eta_\xi = h_\xi[1] \circ w_\xi$ to the class of $h_\xi \circ f_\xi$ in $E^{0,1}_\infty$. This gives a surjection $\widetilde{F}^0_{II} \to E^{0,1}_\infty$.*
(iii) *The image of $E^{1,0}_\infty$ in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ under ψ^0_{II} is equal to the subset $\widetilde{F}^1_{II}$ of $\widetilde{F}^0_{II}$ consisting of classes represented by short exact sequences as in (3.25) where $Y^\bullet$ is in $C^-(B)$.*
(iv) *Fix an element $f \in \mathrm{Ker}(d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0})$ as in Lemma 3.19(ii). Let κ vary over all choices of elements of $\mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ for which $\kappa \circ \sigma = f$. Then the classes η_κ in $\widetilde{F}^0_{II}$, as defined in Definition 3.18, form a coset of $\psi^0_{II}(E^{1,0}_\infty)$ in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$.*

In particular, $\widetilde{F}^0_{II} = F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$.

Proof. — Let $\kappa_\xi \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ be as in Lemma 3.19(i). It follows from (3.40) that we can write η_ξ as

$$\eta_\xi = \kappa_\xi[1] \circ a_T[1] \circ \eta_T \tag{3.43}$$

where $\eta_T \in \mathrm{Hom}_{D^-(B')}(Z^\bullet, T^\bullet[1])$ is as in (3.39) and $a_T : T^\bullet \to A\widehat{\otimes}_{A'}T^\bullet$ is the natural homomorphism in $C^-(B')$.

To prove part (i), one uses the projective Cartan-Eilenberg resolution $M^{\bullet,\bullet,\bullet}$ of $A\widehat{\otimes}_{A'}P^{\bullet,\bullet}$ from Definition 3.6 to represent κ_ξ by the homotopy class in $K^-(B)$ of a homomorphism

$$\kappa_{\xi,\sharp} : \frac{\mathrm{Tot}(M^{-1,\bullet,\bullet})}{\mathrm{Tot}(\mathrm{B}^{-1}_I(M^{\bullet,\bullet,\bullet}))} \to J\widehat{\otimes}_A Z^\bullet \tag{3.44}$$

in $C^-(B)$. To find $\kappa_{\xi,\sharp}$, one first identifies $T^\bullet$ with $\mathrm{B}^0_I(P^{\bullet,\bullet})$ by (3.29) and then shows that there are quasi-isomorphisms

$$\begin{aligned}\frac{\mathrm{Tot}(M^{-1,\bullet,\bullet})}{\mathrm{Tot}(\mathrm{B}^{-1}_I(M^{\bullet,\bullet,\bullet}))} \xrightarrow{\overline{\pi}^{-1,\bullet}_M} \frac{A\widehat{\otimes}_{A'}P^{-1,\bullet}}{\mathrm{B}^{-1}_I(A\widehat{\otimes}_{A'}P^{\bullet,\bullet})} &\xrightarrow{\overline{A\widehat{\otimes}_{A'}d'_P}} \\ A\widehat{\otimes}_{A'}\,\mathrm{B}^0_I(P^{\bullet,\bullet}) &= A\widehat{\otimes}_{A'}T^\bullet\end{aligned} \tag{3.45}$$

in $C^-(B)$, where $\overline{\pi}^{-1,\bullet}_M$ is induced by the quasi-isomorphism π_M from Definition 3.6. Using (3.17) and (3.18), it follows that $\mathrm{Tot}(M^{-1,\bullet,\bullet})\,/\,\mathrm{Tot}(\mathrm{B}^{-1}_I(M^{\bullet,\bullet,\bullet}))$ is a bounded above complex of projective pseudocompact B-modules. Hence the composition of κ_ξ with the quasi-isomorphisms in (3.45) represents κ_ξ in $D^-(B)$ and is given by the homotopy class in $K^-(B)$ of a homomorphism $\kappa_{\xi,\sharp}$ as in (3.44).

Let $\pi_{\mathrm{B}^{-1}_I} : \mathrm{Tot}(M^{-1,\bullet,\bullet}) \to \mathrm{Tot}(M^{-1,\bullet,\bullet})\,/\,\mathrm{Tot}(\mathrm{B}^{-1}_I(M^{\bullet,\bullet,\bullet}))$ be the natural projection in $C^-(B)$ and define $\beta_{\xi,j} \in \mathrm{Hom}_B(\mathrm{Tot}(M^{-1,\bullet,\bullet})^{-j}, J\widehat{\otimes}_A Z^{-j})$ by

$$\beta_{\xi,j} = \kappa^{-j}_{\xi,\sharp} \circ \pi^{-j}_{\mathrm{B}^{-1}_I} \tag{3.46}$$

for all j. By (3.36), $\beta_\xi = (\beta_{\xi,j})$ defines an element in $L^{1,0}$. It follows from the construction that $d^{1,0}_I(\beta_\xi) = 0 = d^{1,0}_{II}(\beta_\xi)$. By considering the effect of making a different choice of κ_ξ in (3.43), one sees that the class $[\beta_\xi]$ in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ only depends on $\eta_\xi \in \widetilde{F}^0_{II}$. Hence the map $\eta_\xi \mapsto [\beta_\xi]$ shows that $\widetilde{F}^0_{II} \subseteq F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$.

Part (ii) is proved by considering the restriction of the homomorphism $\kappa_{\xi,\sharp}$ in $C^-(B)$ from (3.44) to $\mathrm{Tot}(\mathrm{H}^{-1}_I(M^{\bullet,\bullet,\bullet}))$. By Lemma 3.19(i), we have $h_\xi \circ f_\xi = \kappa_\xi \circ \sigma$, where $\sigma : \mathrm{H}^{-1}_I(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to A\widehat{\otimes}_{A'}\,\mathrm{B}^0_I(P^{\bullet,\bullet}) = A\widehat{\otimes}_{A'}T^\bullet$ is the homomorphism from (3.32). Using the projective Cartan-Eilenberg resolution $M^{\bullet,\bullet,\bullet}$, one sees that σ is represented in $D^-(B)$ by the restriction of the composition of the quasi-isomorphisms in (3.45) to $\mathrm{Tot}(\mathrm{H}_I^{-1}(M^{\bullet,\bullet,\bullet}))$. Since $\kappa_{\xi,\sharp}$ represents the composition of κ_ξ with the quasi-isomorphisms in (3.45), it follows that the restriction of $\kappa_{\xi,\sharp}$ to $\mathrm{Tot}(\mathrm{H}_I^{-1}(M^{\bullet,\bullet,\bullet}))$ represents $\kappa_\xi \circ \sigma = h_\xi \circ f_\xi$. This implies that φ_{II}^0 sends η_ξ to the class of $h_\xi \circ f_\xi$ in $E_\infty^{0,1}$. It follows from Lemma 3.19(ii) that the restriction of φ_{II}^0 to $\widetilde{F}_{II}^0$ gives a surjection $\widetilde{F}_{II}^0 \to E_\infty^{0,1}$.

To prove part (iii), one relates the elements of $\widetilde{F}_{II}^1$ and of $F_{II}^1\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ to elements in $L^{0,0}$, using the differentials in (3.20) and (3.21). Let first (ξ, h_ξ) represent a class in $\widetilde{F}_{II}^1$. By Lemma 3.19(iii), there exists a morphism $\alpha_\xi \in \mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_{A'} Z^\bullet)$ such that $\eta_\xi = \eta_{\kappa_\xi}$ for $\kappa_\xi = \alpha_\xi \circ \tau$. By analyzing the construction of $\beta_\xi = (\beta_{\xi,j}) \in L^{1,0}$ in (3.46) for $\kappa_\xi = \alpha_\xi \circ \tau$, one shows that there exists $\gamma = (\gamma_j) \in L^{0,0}$ such that

$$[\beta_\xi] = [-d_{II}^{0,0}(\gamma)] \tag{3.47}$$

in $F_{II}^0\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$. To construct $\gamma = (\gamma_j)$, one represents α_ξ by a homomorphism of complexes

$$\alpha_{\xi,\sharp} \colon \mathrm{Tot}(\mathrm{B}_I^0(M^{\bullet,\bullet,\bullet})) \to J\widehat{\otimes}_A Z^\bullet \tag{3.48}$$

in $C^-(B)$ and defines $\gamma_j \in \mathrm{Hom}_B(\mathrm{Tot}(M^{0,\bullet,\bullet})^{-j}, J\widehat{\otimes}_A Z^{-j})$ by

$$\gamma_j = \alpha_{\xi,\sharp}^{-j} \circ \mathrm{proj}_{0,j}, \tag{3.49}$$

where $\mathrm{proj}_{0,j} : \mathrm{Tot}(M^{0,\bullet,\bullet})^{-j} \to \mathrm{Tot}(\mathrm{B}_I^0(M^{\bullet,\bullet,\bullet}))^{-j}$ is induced by the projections $M^{0,-y,-z} \to \mathrm{B}_I^0(M^{\bullet,-y,-z})$ for all y, z with $y + z = j$ coming from the split exact sequences (3.17) and (3.18). Using (3.20), one checks that $[\beta_\xi] = [d_I^{0,0}(\gamma)]$ in $F_{II}^0\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$, which implies (3.47) because $[d^0_{\mathrm{Tot}(L)}(\gamma)] = 0$. Hence $[\beta_\xi]$ is equal to an element in $F_{II}^1\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \psi_{II}^0(E_\infty^{1,0})$.

Conversely, suppose $\beta = (\beta_j) \in L^{0,1} = \bigoplus_j \mathrm{Hom}_B(\mathrm{Tot}(M^{0,\bullet,\bullet})^{-j}, J\widehat{\otimes}_A Z^{-j+1})$ represents a class in $F_{II}^1\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \psi_{II}^0(E_\infty^{1,0})$. One uses β to construct a representative (ξ, h_ξ) in $\widetilde{F}_{II}^1$ such that the corresponding element $\beta_\xi = (\beta_{\xi,j}) \in L^{1,0}$ defined by (3.46) satisfies

$$[\beta] = [\beta_\xi] \tag{3.50}$$

in $F_{II}^0\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$. To find (ξ, h_ξ), one first shows that there exists an element $\gamma_\beta = (\gamma_{\beta,j}) \in L^{0,0}$ with

$$[\beta] = [-d_I^{0,0}(\gamma_\beta)] \tag{3.51}$$

in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$. To define γ_β, let $f_\beta\colon\ \mathrm{Tot}(M^{0,\bullet,\bullet}) \to J\widehat{\otimes}_A Z^\bullet[1]$ be the map given by $f_\beta^{-j} = \beta_j$ for all j. Because $(\beta_j) \in L^{0,1}$, it follows that f_β is a homomorphism in $C^-(B)$ that factors through $\mathrm{Tot}(\mathrm{H}^0_I(M^{\bullet,\bullet,\bullet})) = \mathrm{Tot}(M^{0,\bullet,\bullet}) \,/\, \mathrm{Tot}(\mathrm{B}^0_I(M^{\bullet,\bullet,\bullet}))$. Let

$$\overline{f_\beta} : \mathrm{Tot}(\mathrm{H}^0_I(M^{\bullet,\bullet,\bullet})) \to J\widehat{\otimes}_{A'} Z^\bullet[1] \tag{3.52}$$

be the induced homomorphism in $C^-(B)$. Since $P^{0,\bullet}$ is acyclic, the morphism $\alpha_1 : Z^\bullet \to D^\bullet[1]$ from (3.33) is an isomorphism in $D^-(B)$, and we can use the projective Cartan-Eilenberg resolution $M^{\bullet,\bullet,\bullet}$ to represent the inverse of α_1 by a quasi-isomorphism

$$\psi_1\colon\ \mathrm{Tot}(\mathrm{B}^0_I(M^{\bullet,\bullet,\bullet}))[1] \to \mathrm{Tot}(\mathrm{H}^0_I(M^{\bullet,\bullet,\bullet})) \tag{3.53}$$

in $C^-(B)$. Define $\gamma_\beta = (\gamma_{\beta,j}) \in L^{0,0}$ by

$$\gamma_{\beta,j} = \overline{f_\beta}^{\,-j-1} \circ \psi_1^{-j-1} \circ \mathrm{proj}_{0,j} \tag{3.54}$$

where $\mathrm{proj}_{0,j}$ is as in (3.49). Using (3.21), one checks that $[\beta] = [d^{0,0}_{II}(\gamma_\beta)]$ in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$, which implies (3.51). Define $\widehat{\alpha}_\beta\colon\ \mathrm{Tot}(\mathrm{B}^0_I(M^{\bullet,\bullet,\bullet})) \to J\widehat{\otimes}_A Z^\bullet$ in $C^-(B)$ by

$$\widehat{\alpha}_\beta = -\overline{f_\beta}[-1] \circ \psi_1[-1]. \tag{3.55}$$

It follows that $\widehat{\alpha}_\beta$ represents a morphism $\alpha \in \mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_A Z^\bullet)$. By Lemma 3.19(iii), $\alpha \circ \tau$ defines a class in $\widetilde{F}^1_{II}$. Let (ξ, h_ξ) be a representative of this class. Since $\eta_\xi = \eta_{\kappa_\xi}$ for $\kappa_\xi = \alpha \circ \tau$, one can take the morphism α_ξ from the beginning of the proof of part (iii) to be $\alpha_\xi = \alpha$. This implies that in (3.48) one can take $\alpha_{\xi,\sharp} = \widehat{\alpha}_\beta$. Using (3.47) and comparing γ_j in (3.49) to $\gamma_{\beta,j}$ in (3.54), one sees (3.50).

Part (iv) follows from part (iii) above and from parts (ii) and (iii) of Lemma 3.19. □

3.7. Proof of Lemma 3.10 and Theorem 3.12

In this subsection we prove Lemma 3.10 and Theorem 3.12 by proving Lemmas 3.24 and 3.25 below. The proof relies on Lemmas 3.19 and 3.21 and the following result.

Lemma 3.22 (O. Gabber). — *Assume Hypothesis 3.1, and suppose we have a short exact sequence in $C^-(B')$*

$$\xi\colon 0 \to X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{v_\xi} Z^\bullet \to 0 \tag{3.56}$$

where the terms of $X^\bullet$ *are annihilated by* J*. Let* $f_\xi\colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) = \mathrm{Tor}_1^{A'}(Z^\bullet, A) \to X^\bullet$ *be the homomorphism in* $C^-(B)$ *resulting from tensoring* ξ *with* A *over* A'*. Then* f_ξ *is an isomorphism in* $D^-(B)$ *if and only if the homomorphism* $\upsilon\colon A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet \to Z^\bullet$ *induced by* $A\widehat{\otimes}^{\mathbf{L}}_{A'}-$ *is an isomorphism in* $D^-(B)$*.*

Remark 3.23. — The homomorphism $\upsilon\colon A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet \to Z^\bullet$ in $D^-(B)$ in Lemma 3.22 is given as follows. Let $Q^\bullet$ be a bounded above complex of projective pseudocompact B'-modules such that there is a quasi-isomorphism $\rho\colon Q^\bullet \to Y^\bullet$ in $C^-(B')$ that is surjective on terms. Then υ is represented in $D^-(B)$ by a homomorphism $\upsilon_Q\colon A\widehat{\otimes}_{A'}Q^\bullet \to Z^\bullet$ in $C^-(B)$ which is the composition

$$A\widehat{\otimes}_{A'}Q^\bullet \xrightarrow{A\widehat{\otimes}_{A'}\rho} A\widehat{\otimes}_{A'}Y^\bullet \xrightarrow{A\widehat{\otimes}_{A'}v_\xi} A\widehat{\otimes}_{A'}Z^\bullet = Z^\bullet. \tag{3.57}$$

Proof. — Let $Q^\bullet$, ρ and υ_Q be as in Remark 3.23 so that υ_Q represents υ. We obtain a commutative diagram in $C^-(B')$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J\widehat{\otimes}_{A'}Q^\bullet & \longrightarrow & Q^\bullet & \longrightarrow & A\widehat{\otimes}_{A'}Q^\bullet & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mu_Y\circ(J\widehat{\otimes}_{A'}\rho)} & & \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\upsilon_Q} & & \\ 0 & \longrightarrow & X^\bullet & \xrightarrow{u_\xi} & Y^\bullet & \xrightarrow{v_\xi} & Z^\bullet & \longrightarrow & 0 \end{array} \tag{3.58}$$

where $\mu_Y\colon J\widehat{\otimes}_{A'}Y^\bullet \to X^\bullet$ is the composition of the natural homomorphisms $J\widehat{\otimes}_{A'}Y^\bullet \to JY^\bullet \to X^\bullet$. By tensoring the diagram (3.58) with A over A', and by also tensoring $\upsilon_Q\colon A\widehat{\otimes}_{A'}Q^\bullet \to Z^\bullet$ with $0 \to J \to A' \to A \to 0$ over A', one sees that in $C^-(B)$

$$\mu_Y \circ (J\widehat{\otimes}_{A'}\rho) = f_\xi \circ \iota^{-1} \circ (J\widehat{\otimes}_{A}\upsilon_Q), \tag{3.59}$$

where $\iota\colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to J\widehat{\otimes}_A Z^\bullet$ is the isomorphism in $C^-(B)$ from Definition 3.11.

To prove the lemma, suppose first that υ, and hence υ_Q, is an isomorphism in $D^-(B)$. Since ρ is a quasi-isomorphism in $C^-(B')$, one sees, using (3.58), that $\mu_Y\circ(J\widehat{\otimes}_{A'}\rho)$ is a quasi-isomorphism in $C^-(B)$. By (3.59), this implies that f_ξ is an isomorphism in $D^-(B)$.

Conversely, suppose that f_ξ is an isomorphism in $D^-(B)$. Rewriting (3.58) with the aid of (3.59), one obtains a commutative diagram with exact rows in $C^-(B')$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J\widehat{\otimes}_{A'}Q^{\bullet} & \longrightarrow & Q^{\bullet} & \longrightarrow & A\widehat{\otimes}_{A'}Q^{\bullet} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle J\widehat{\otimes}_A \upsilon_Q} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \upsilon_Q} & & \\ & & J\widehat{\otimes}_{A}Z^{\bullet} & \xrightarrow{u'_\xi} & Y^{\bullet} & \xrightarrow{\upsilon_\xi} & Z^{\bullet} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & C(J\widehat{\otimes}_A \upsilon_Q)^{\bullet} & \longrightarrow & C(\rho)^{\bullet} & \longrightarrow & C(\upsilon_Q)^{\bullet} & & \end{array} \tag{3.60}$$

where $u'_\xi = u_\xi \circ f_\xi \circ \iota^{-1}$. Because $f_\xi \circ \iota^{-1} \colon J\widehat{\otimes}_A Z^{\bullet} \to X^{\bullet}$ is an isomorphism in $D^-(B')$, the rows in (3.60) represent triangles in $D^-(B')$. Using the triangle corresponding to the last row in (3.60), one argues inductively that $C(\upsilon_Q)^{\bullet}$ is acyclic. To make this argument, one uses that $C(\rho)^{\bullet}$ is acyclic, that the terms of $C(\upsilon_Q)^{\bullet}$ are topologically free over A and that all complexes involved are bounded above. The acyclicity of $C(\upsilon_Q)^{\bullet}$ implies that υ_Q, and hence υ, is an isomorphism in $D^-(B)$. □

We also need the following result which relates quasi-lifts of $(Z^{\bullet}, \zeta)$ over A' to short exact sequences ξ in $C^-(B')$ as in Definition 3.8.

Lemma 3.24. — *Assume Hypothesis 3.1 and the notation introduced in Definition 3.8. Suppose $(Y^{\bullet}, \upsilon)$ is a quasi-lift of $(Z^{\bullet}, \zeta)$ over A'. Then there exists a quasi-lift $(Y'^{\bullet}, \upsilon')$ of $(Z^{\bullet}, \zeta)$ over A' which is locally isomorphic to $(Y^{\bullet}, \upsilon)$ with the following properties:*

(a) *There is a short exact sequence $\xi' \colon 0 \to X'^{\bullet} \to Y'^{\bullet} \to Z^{\bullet} \to 0$ in $C^-(B')$ as in Definition 3.8, i.e. the terms of $X'^{\bullet}$ are annihilated by J and there is an isomorphism $X'^{\bullet} \to J\widehat{\otimes}_A Z^{\bullet}$ in $D^-(B)$.*

(b) *The isomorphism $\upsilon' \colon A\widehat{\otimes}^{\mathbf{L}}_{A'} Y'^{\bullet} \to Z^{\bullet}$ is the homomorphism in $D^-(B)$ from Lemma 3.22 which is induced by $A\widehat{\otimes}^{\mathbf{L}}_{A'} -$ relative to ξ'.*

Proof. — Using Theorem 2.10 and Remark 2.7, we may assume that the terms Y^i of $Y^{\bullet}$ are zero for $i < -p_0$ and $i > -1$, they are projective pseudocompact B'-modules for $-p_0 < i \leqslant -1$, and Y^{-p_0} is topologically free over A'. Since the terms Z^i of $Z^{\bullet}$ are projective pseudocompact B-modules for $i > -p_0$, it follows that the inverse of the isomorphism $\upsilon : A\widehat{\otimes}_{A'} Y^{\bullet} \to Z^{\bullet}$ in $D^-(B)$ can be represented by a quasi-isomorphism $\chi \colon Z^{\bullet} \to A\widehat{\otimes}_{A'} Y^{\bullet}$ in $C^-(B)$. We obtain a pullback diagram in $C^-(B')$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J\widehat{\otimes}_{A'}Y^\bullet & \longrightarrow & Y'^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \chi_Y} & & \downarrow{\scriptstyle \chi} & & \\ 0 & \longrightarrow & J\widehat{\otimes}_{A'}Y^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & A\widehat{\otimes}_{A'}Y^\bullet & \longrightarrow & 0. \end{array} \tag{3.61}$$

It follows that χ_Y is a quasi-isomorphism in $C^-(B')$. Letting $X'^\bullet = J\widehat{\otimes}_{A'}Y^\bullet$, the top row of (3.61) defines a short exact sequence ξ' as in part (a). To prove part (b), let $\upsilon'\colon A\widehat{\otimes}^{\mathbf{L}}_{A'}Y'^\bullet \to Z^\bullet$ be the homomorphism in $D^-(B)$ from Lemma 3.22, which is induced by $A\widehat{\otimes}^{\mathbf{L}}_{A'}-$ relative to the top row ξ' of (3.61). By representing υ' by a homomorphism in $C^-(B)$ as in Remark 3.23, one sees that in $D^-(B)$

$$A\widehat{\otimes}^{\mathbf{L}}_{A'}\chi_Y = \chi\circ\upsilon' = \upsilon^{-1}\circ\upsilon'.$$

Hence υ' is an isomorphism in $D^-(B)$, and χ_Y defines a local isomorphism between the quasi-lifts $(Y^\bullet,\upsilon)$ and $(Y'^\bullet,\upsilon')$ of $(Z^\bullet,\zeta)$ over A'. □

Lemma 3.25. — *Assume the notation of Definition 3.15, so that in particular, $P^{0,\bullet}$ is an acyclic complex of projective pseudocompact B'-modules. Let $\iota : \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to J\widehat{\otimes}_A Z^\bullet$ be the isomorphism in $C^-(B)$ from Definition 3.11, so $\iota \in E_2^{0,1}$. Let $\omega = \omega(Z^\bullet, A')$ be the class $\omega = d_2^{0,1}(\iota) \in E_2^{2,0} = \mathrm{Ext}^2_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$.*

(i) *Suppose $(Z^\bullet,\zeta)$ has a quasi-lift $(Y^\bullet,\upsilon)$ over A'. Then $\omega = 0$.*

(ii) *Conversely, suppose that $\omega = 0$.*

(a) *There exists $\kappa \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ with $\kappa\circ\sigma = \iota$. Let (ξ, h_ξ) represent the class η_κ in $\widetilde{F}^0_{II}$, as defined in Definition 3.18, where $\xi\colon 0 \to X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{v_\xi} Z^\bullet \to 0$. Let $\upsilon\colon A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet \to Z^\bullet$ be the homomorphism in $D^-(B)$ from Lemma 3.22 relative to ξ. Then $(Y^\bullet,\upsilon)$ is a quasi-lift of $(Z^\bullet,\zeta)$ over A', which we denote by $(Y^\bullet_\kappa,\upsilon_\kappa)$.*

(b) *Let Ξ be the set of the classes η_κ in $\widetilde{F}^0_{II}$ as κ varies over all choices of elements of $\mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ with $\kappa\circ\sigma = \iota$. Then the map $\eta_\kappa \mapsto [(Y^\bullet_\kappa,\upsilon_\kappa)]$ defines a bijection between Ξ and the set Υ of all local isomorphism classes of quasi-lifts of $(Z^\bullet,\zeta)$ over A'.*

(c) *Let $[\iota]$ be the class of ι in $E_\infty^{0,1}$. The set of all local isomorphism classes of quasi-lifts of $(Z^\bullet,\zeta)$ over A' is in bijection with the full preimage of $[\iota]$ in $F^0_{II}\,\mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet})) = \widetilde{F}^0_{II}$ under φ^0_{II}. In other words, the set of all local isomorphism classes* *of quasi-lifts of* $(Z^\bullet, \zeta)$ *over* A' *is a principal homogeneous space for* $E_\infty^{1,0}$*. The set of all* $\kappa \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ *with* $\kappa \circ \sigma = \iota$ *is a principal homogeneous space for* $E_2^{1,0} = \mathrm{Ext}^1_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$*.*

(d) *We have* $E_2^{p,0} = E_\infty^{p,0}$ *for all* p*, i.e. the spectral sequence* (3.16) *partially degenerates.*

Proof. — For part (i), suppose $(Y^\bullet, \upsilon)$ is a quasi-lift of $(Z^\bullet, \zeta)$ over A'. Using Theorem 2.10, we may assume that the terms of $Y^\bullet$ are projective pseudocompact B'-modules. Moreover, by adding an acyclic complex of topologically free pseudocompact B'-modules to $Y^\bullet$ if necessary, we can assume that $\upsilon\colon A\widehat{\otimes}_{A'}Y^\bullet \to Z^\bullet$ is given by a quasi-isomorphism of complexes in $C^-(B)$ that is surjective on terms. Hence we a have a short exact sequence in $C^-(B')$ of the form

$$0 \to K^\bullet \to Y^\bullet \xrightarrow{\upsilon_Y} Z^\bullet \to 0, \tag{3.62}$$

where υ_Y is the composition $Y^\bullet \to A\widehat{\otimes}_{A'}Y^\bullet \xrightarrow{\upsilon} Z^\bullet$ and $K^\bullet = \mathrm{Ker}(\upsilon_Y)$. Note that $K^\bullet$ may or may not be annihilated by J. Since $P^{0,\bullet}$ is a projective object in $C^-(B')$, we obtain a commutative diagram in $C^-(B')$ whose top (resp. bottom) row is given by (3.29) (resp. (3.62)). Tensoring this diagram with A over A', we get a commutative diagram in $C^-(B)$ with exact rows

(3.63)

$$\begin{array}{ccccccccc} 0 \to & \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) & \xrightarrow{\sigma} & A\widehat{\otimes}_{A'}T^\bullet & \xrightarrow{A\widehat{\otimes}_{A'}\delta} & A\widehat{\otimes}_{A'}P^{0,\bullet} & \xrightarrow{A\widehat{\otimes}_{A'}\epsilon} & Z^\bullet & \to 0 \\ & \| & & \downarrow & & \downarrow & & \| & \\ 0 \to & \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) & \xrightarrow{f_Y} & A\widehat{\otimes}_{A'}K^\bullet & \longrightarrow & A\widehat{\otimes}_{A'}Y^\bullet & \xrightarrow{\upsilon} & Z^\bullet & \to 0 \end{array}$$

whose top row is given by (3.30). Using the definition of α_1 and α_2 in (3.33) and (3.34), one sees that the top row of (3.63) defines the class $\alpha_2[1] \circ \alpha_1$ in $\mathrm{Ext}^2_{D^-(B)}(Z^\bullet, \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}))$. Because υ is an isomorphism in $D^-(B)$, the bottom row of (3.63) shows that f_Y is also an isomorphism in $D^-(B)$. Therefore, $\alpha_2[1] \circ \alpha_1 = 0$ in $D^-(B)$. Since $\omega = d_2^{0,1}(\iota) = \iota[2] \circ \alpha_2[1] \circ \alpha_1$ by Lemma 3.17, this implies $\omega = 0$ in $D^-(B)$.

For part (ii), assume that $\omega = d_2^{0,1}(\iota) = 0$. By Lemma 3.19(ii), there exists $\kappa : A\widehat{\otimes}_{A'}T^\bullet \to J\widehat{\otimes}_A Z^\bullet$ in $D^-(B)$ with $\kappa \circ \sigma = \iota$. Let (ξ, h_ξ) and υ be as in the statement of part (ii)(a), where $\xi\colon 0 \to X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{\upsilon_\xi} Z^\bullet \to 0$. Let $f_\xi\colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to X^\bullet$ be the homomorphism in $C^-(B)$ resulting from tensoring ξ with A over A'. By Lemma 3.19(ii), $h_\xi \circ f_\xi = \iota$, which implies that f_ξ is an isomorphism in $D^-(B)$. Hence by Lemma 3.22, $\upsilon\colon A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet \to Z^\bullet$ is an isomorphism in $D^-(B)$. Using the isomorphism υ

together with the fact that $Z^\bullet$ has finite pseudocompact A-tor dimension, it follows that there exists an integer N such that $\mathrm{H}^i(S\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet) = 0$ for all $i < N$ and for all pseudocompact A-modules S. Since for all pseudocompact A'-modules S' we have that JS' and S'/JS' are annihilated by J and thus pseudocompact A-modules, one sees that $\mathrm{H}^i(S'\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet) = 0$ for all $i < N$. Hence $(Y^\bullet, \upsilon)$ is a quasi-lift of $(Z^\bullet, \zeta)$ over A', which we denote by $(Y^\bullet_\kappa, \upsilon_\kappa)$.

Let Ξ and Υ be as in the statement of part (ii)(b). We need to show that the map

$$(3.64)\qquad \begin{array}{ccc} \Xi & \longrightarrow & \Upsilon \\ \eta_\kappa & \longmapsto & [(Y^\bullet_\kappa, \upsilon_\kappa)] \end{array}$$

is a bijection. This map is well-defined, since, as seen at the end of Definition 3.18, $\eta_\kappa = \eta_{\kappa'}$ if and only if $\kappa \circ a_T = \kappa' \circ a_T$ in $D^-(B')$ and the construction in Definition 3.18 shows that $\kappa \circ a_T$ determines the local isomorphism class $[(Y^\bullet_\kappa, \upsilon_\kappa)]$.

We first prove that (3.64) is surjective. Given a quasi-lift $(Y^\bullet, \upsilon)$ of $(Z^\bullet, \zeta)$ over A', we may assume by Lemma 3.24 that there is a short exact sequence $\xi\colon 0 \to X^\bullet \xrightarrow{u_\xi} Y^\bullet \xrightarrow{v_\xi} Z^\bullet \to 0$ in $C^-(B')$ as in Definition 3.8 and that the isomorphism $\upsilon\colon A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet \to Z^\bullet$ is the homomorphism in $D^-(B)$ from Lemma 3.22 relative to ξ. Since υ is an isomorphism in $D^-(B)$, it follows from Lemma 3.22 that the homomorphism f_ξ : $\mathrm{H}^{-1}_I(A\widehat{\otimes}_{A'}P^{\bullet,\bullet}) \to X^\bullet$ is an isomorphism in $D^-(B)$. Letting $h_\xi = \iota \circ f_\xi^{-1}$, it follows that (ξ, h_ξ) represents a class η_ξ in $\widetilde{F}^0_{II}$. By Lemma 3.19(i), there exists $\kappa \in \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'}T^\bullet, J\widehat{\otimes}_A Z^\bullet)$ such that $\kappa \circ \sigma = h_\xi \circ f_\xi = \iota$ and $\eta_\xi = \eta_\kappa$ in $\widetilde{F}^0_{II}$. Following the definition of $(Y^\bullet_\kappa, \upsilon_\kappa)$, one sees that $(Y^\bullet_\kappa, \upsilon_\kappa)$ and $(Y^\bullet, \upsilon)$ are locally isomorphic quasi-lifts of $(Z^\bullet, \zeta)$ over A'.

To prove that (3.64) is injective, let $\eta_\kappa, \eta_{\kappa'} \in \Xi$ be such that $(Y^\bullet_\kappa, \upsilon_\kappa)$ and $(Y^\bullet_{\kappa'}, \upsilon_{\kappa'})$ are locally isomorphic quasi-lifts of $(Z^\bullet, \zeta)$ over A'. This means that there exists an isomorphism $\theta : Y^\bullet_\kappa \to Y^\bullet_{\kappa'}$ in $D^-(B')$ with $\upsilon_{\kappa'} \circ (A\widehat{\otimes}^{\mathbf{L}}_{A'}\theta) = \upsilon_\kappa$. Consider the triangle in $D^-(B')$

$$(3.65)\qquad J\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet_\kappa \longrightarrow A'\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet_\kappa \longrightarrow A\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet_\kappa \xrightarrow{\eta^{\mathbf{L}}_\kappa} J\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet_\kappa[1],$$

which is associated to the short exact sequence obtained by applying the functor $-\widehat{\otimes}^{\mathbf{L}}_{A'}Y^\bullet_\kappa$ to the sequence $0 \to J \to A' \to A \to 0$. Using the definition of υ_κ, one sees that $\eta_\kappa \circ \upsilon_\kappa = (J\widehat{\otimes}^{\mathbf{L}}_A \upsilon_\kappa[1]) \circ \eta^{\mathbf{L}}_\kappa$. On replacing κ by κ', one obtains a similar equation relating $\eta_{\kappa'}$ and $\eta^{\mathbf{L}}_{\kappa'}$. Since $(J\widehat{\otimes}^{\mathbf{L}}_{A'}\theta[1]) \circ \eta^{\mathbf{L}}_\kappa = \eta^{\mathbf{L}}_{\kappa'} \circ (A\widehat{\otimes}^{\mathbf{L}}_{A'}\theta)$, this implies that $\eta_\kappa = \eta_{\kappa'}$.

The first statement of part (ii)(c) follows from part (ii)(b) above and from Lemma 3.21(iv). For the second statement of part (ii)(c), one notes that since $\omega = \iota[2] \circ \alpha_2[1] \circ \alpha_1 = 0$ and ι and α_1 are isomorphisms in $D^-(B)$, one has $\alpha_2 = 0$. Replacing $\alpha_2 = 0$ in the triangle (3.34) and applying the functor $\mathrm{Hom}_{D^-(B)}(-, J\widehat{\otimes}_A Z)$, one obtains a short exact sequence of abelian groups

$$(3.66)\quad 0 \longrightarrow \mathrm{Hom}(D^\bullet, J\widehat{\otimes}_A Z^\bullet) \xrightarrow{\tau^*} \mathrm{Hom}(A\widehat{\otimes}_{A'} T^\bullet, J\widehat{\otimes}_A Z^\bullet) \xrightarrow{\sigma^*} \mathrm{Hom}(\mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet}), J\widehat{\otimes}_A Z^\bullet) \longrightarrow 0,$$

where Hom stands for $\mathrm{Hom}_{D^-(B)}$. Since

$$\mathrm{Hom}_{D^-(B)}(D^\bullet, J\widehat{\otimes}_A Z^\bullet) \cong \mathrm{Ext}^1_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet),$$

part (ii)(c) follows.

To prove part (ii)(d), we show that for all p the inflation map

$$\mathrm{Inf}_B^{B'} \colon \mathrm{Ext}^p_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet) \longrightarrow \mathrm{Ext}^p_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$$

is injective, which implies that $E_\infty^{p,0} = E_2^{p,0} = \mathrm{Ext}^p_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$. Let $(Y^\bullet, \upsilon)$ be a quasi-lift of $(Z^\bullet, \zeta)$ such that $Y^\bullet$ is a bounded above complex of topologically free pseudocompact B'-modules. Let $a_Y : Y^\bullet \to A\widehat{\otimes}_{A'} Y^\bullet$ be the natural homomorphism in $C^-(B')$, and let $\pi_P \colon \mathrm{Tot}(P^{\bullet,\bullet}) \to Z^\bullet$ be the quasi-isomorphism in $C^-(B')$ from Definition 3.4. Then $g = \pi_P^{-1} \circ \upsilon \circ a_Y \in \mathrm{Hom}_{D^-(B')}(Y^\bullet, \mathrm{Tot}(P^{\bullet,\bullet})) = \mathrm{Hom}_{K^-(B')}(Y^\bullet, \mathrm{Tot}(P^{\bullet,\bullet}))$. Suppose $f \in \mathrm{Ext}^p_{D^-(B)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$ and $\mathrm{Inf}_B^{B'}(f) = 0$ in $\mathrm{Ext}^p_{D^-(B')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$. Since $A\widehat{\otimes}_{A'} Y^\bullet$ is a bounded above complex of topologically free pseudocompact B-modules, it follows that $f \circ \upsilon \in \mathrm{Hom}_{K^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, J\widehat{\otimes}_A Z^\bullet[p])$. Since $\mathrm{Inf}_B^{B'}(f) = 0$ and π_P is a quasi-isomorphism in $C^-(B')$, it follows that $F = f \circ \pi_P \colon \mathrm{Tot}(P^{\bullet,\bullet}) \to J\widehat{\otimes}_A Z^\bullet[p]$ is homotopic to zero in $C^-(B')$. Then $(f\circ\upsilon)\circ a_Y = (f\circ\pi_P)\circ(\pi_P^{-1}\circ\upsilon\circ a_Y) = (f\circ\pi_P)\circ g = F\circ g$, which implies that $(f \circ \upsilon) \circ a_Y$ is homotopic to zero in $C^-(B')$. Applying $A\widehat{\otimes}_{A'}-$ shows that $f \circ \upsilon$ is homotopic to zero in $C^-(B)$. Since υ is an isomorphism in $D^-(B)$ and $\mathrm{Hom}_{K^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, J\widehat{\otimes}_A Z^\bullet[p]) = \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, J\widehat{\otimes}_A Z^\bullet[p])$, it follows that $f = 0$ in $D^-(B)$ which proves part (ii)(d). □

Remark 3.26. — If $\omega = \omega(Z^\bullet, A') \neq 0$, *i.e.* if there is no quasi-lift of $(Z^\bullet, \zeta)$ over A', then $E_\infty^{1,0}$ is a proper quotient of $E_2^{1,0}$ in general. For example, let $k = \mathbb{Z}/2$, $A = k[t]/(t^4)$, $A' = k[t]/(t^6)$ and let $\pi \colon A' \to A$ be the natural surjection. Let G be the trivial group, so that $B = A$ and $B' = A'$. Suppose $V^\bullet = k \xrightarrow{0} k \xrightarrow{0} k$ and $Z^\bullet = A \xrightarrow{t^3} A \xrightarrow{t} A$ are both concentrated in degrees $-3, -2, -1$. Then $J\widehat{\otimes}_A Z^\bullet = A/t^2A \xrightarrow{0} A/t^2A \xrightarrow{t}$

$A/t^2 A$ is also concentrated in degrees $-3,-2,-1$. We now show that the inflation map

$$\mathrm{Inf}_A^{A'} : \mathrm{Ext}^1_{D^-(A)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet) \longrightarrow \mathrm{Ext}^1_{D^-(A')}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$$

is not injective. This implies that $E_\infty^{1,0}$ is a proper quotient of $E_2^{1,0} = \mathrm{Ext}^1_{D^-(A)}(Z^\bullet, J\widehat{\otimes}_A Z^\bullet)$, since $A = B$ and $A' = B'$ and $E_\infty^{1,0}$ is isomorphic to the image of $\mathrm{Inf}_A^{A'}$. Consider the map of complexes $f : Z^\bullet \to J\widehat{\otimes}_A Z^\bullet[1]$ in $C^-(A)$ where $f^j = 0$ for all $j \neq -3$ and $f^{-3}\colon Z^{-3} = A \to A/t^2 A = J\widehat{\otimes}_A Z^{-2}$ sends $1 \in A$ to $t \in A/t^2 A$. Then f is not homotopic to zero which implies that f is not zero in $D^-(A)$ since the terms of $Z^\bullet$ are topologically free pseudocompact A-modules. To show that $\mathrm{Inf}_A^{A'}(f) = 0$ in $D^-(A')$, we construct a suitable bounded above complex $Q^\bullet$ of topologically free pseudocompact A'-modules together with a quasi-isomorphism $s_Q\colon Q^\bullet \to Z^\bullet$. Namely, let

$$Q^\bullet\colon \ \cdots\ (A')^2 \xrightarrow{\begin{pmatrix} t^3 & 0 \\ 0 & t^5 \end{pmatrix}} (A')^2 \xrightarrow{\begin{pmatrix} t^3 & 0 \\ 0 & 0 \\ 0 & t \end{pmatrix}} (A')^3 \xrightarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & t^5 \end{pmatrix}} (A')^2 \xrightarrow{(\,0\ t\,)} A'$$

be concentrated in degrees $\leqslant -1$, and let $s_Q = \left(s_Q^j\right)$ where $s_Q^j = 0$ for $j \notin \{-3,-2,-1\}$ and $s_Q^{-1} = \pi$, $s_Q^{-2} = (t^3\,\pi, \pi)$, $s_Q^{-3} = (t\,\pi, \pi, 0)$. It follows that $f \circ s_Q$ is homotopic to zero, and hence equal to zero in $D^-(A')$, by defining $h^j : Q^j \to J\widehat{\otimes}_A Z^j[1] = J\widehat{\otimes}_A Z^{j+1}$ by $h^j = 0$ for all $j \neq -2$ and $h^{-2} = (t\,\overline{\pi}, 0)$ where $\overline{\pi} : A' \to A/t^2$ is the natural surjection. Since s_Q is an isomorphism in $D^-(A')$, this implies that $\mathrm{Inf}_A^{A'}(f) = (f \circ s_Q) \circ (s_Q)^{-1}$ is zero in $D^-(A')$.

3.8. Proof of Lemma 3.13

As in the statement of Lemma 3.13, suppose that $(Y^\bullet, \upsilon)$ is a quasi-lift of $(Z^\bullet, \zeta)$ over A'. Using Theorem 2.10, we may assume that the terms of $Y^\bullet$ are projective pseudocompact B'-modules. Consider the triangle in $D^-(B')$

$$A\widehat{\otimes}_{A'} Y^\bullet[-1] \xrightarrow{a} J\widehat{\otimes}_{A'} Y^\bullet \xrightarrow{b} Y^\bullet \xrightarrow{c} A\widehat{\otimes}_{A'} Y^\bullet, \tag{3.67}$$

which is associated to the short exact sequence obtained by applying the functor $-\widehat{\otimes}^{\mathbf{L}}_{A'} Y^\bullet = -\widehat{\otimes}_{A'} Y^\bullet$ to the sequence $0 \to J \to A' \to A \to 0$. Applying the functor $\mathrm{Hom}_{D^-(B')}(Y^\bullet, -)$ to the triangle (3.67), one obtains

a long exact Hom sequence
(3.68)

$$\cdots \longrightarrow \mathrm{Hom}_{D^-(B')}(Y^\bullet, Y^\bullet[-1]) \to \mathrm{Hom}_{D^-(B')}(Y^\bullet, A\widehat{\otimes}_{A'} Y^\bullet[-1]) \xrightarrow{(a)_*}$$
$$\mathrm{Hom}_{D^-(B')}(Y^\bullet, J\widehat{\otimes}_{A'} Y^\bullet) \xrightarrow{(b)_*} \mathrm{Hom}_{D^-(B')}(Y^\bullet, Y^\bullet) \xrightarrow{(c)_*} \mathrm{Hom}_{D^-(B')}(Y^\bullet, A\widehat{\otimes}_{A'} Y^\bullet) \longrightarrow \cdots$$

Using that $\mathrm{Image}((b)_*)$ is a two-sided ideal with square 0 in $\mathrm{Hom}_{D^-(B')}(Y^\bullet, Y^\bullet)$, one sees that
(3.69)

$$\mathrm{Aut}^0_{D^-(B')}(Y^\bullet) \cong \mathrm{Image}((b)_*) \cong \mathrm{Hom}_{D^-(B')}(Y^\bullet, J\widehat{\otimes}_{A'} Y^\bullet)/\,\mathrm{Image}((a)_*).$$

Since $Y^\bullet$ is a bounded above complex of projective pseudocompact B'-modules, c induces an isomorphism $(c)^*\colon \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, W^\bullet) \xrightarrow{\cong} \mathrm{Hom}_{D^-(B')}(Y^\bullet, W^\bullet)$ for all complexes $W^\bullet$ in $C^-(B)$. Thus (3.69) implies

$$(3.70)\quad \mathrm{Aut}^0_{D^-(B')}(Y^\bullet) \cong \mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, J\widehat{\otimes}_{A'} Y^\bullet)/ \\ \mathrm{Image}(\mathrm{Ext}^{-1}_{D^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, A\widehat{\otimes}_{A'} Y^\bullet)),$$

where $\mathrm{Image}(\mathrm{Ext}^{-1}_{D^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, A\widehat{\otimes}_{A'} Y^\bullet))$ is the image of $\mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, A\widehat{\otimes}_{A'} Y^\bullet[-1])$ in $\mathrm{Hom}_{D^-(B)}(A\widehat{\otimes}_{A'} Y^\bullet, J\widehat{\otimes}_{A'} Y^\bullet)$ under the composition $((c)^*)^{-1} \circ (a)_* \circ (c)^*$. Since υ induces an isomorphism $J\widehat{\otimes}_A \upsilon : J\widehat{\otimes}_{A'} Y^\bullet \to J\widehat{\otimes}_A Z^\bullet$ in $D^-(B)$, Lemma 3.13 follows.

3.9. Proof of Proposition 3.14

As in the statement of Proposition 3.14, we assume the notation of §3.1 and Theorem 3.12. For simplicity, we identify $A\widehat{\otimes}_{A'} Y^j = \widetilde{Z}^j$ for all j and we identify $Z^\bullet$ with the truncation $\mathrm{Trunc}_{-p_0}(\widetilde{Z}^\bullet)$ of $\widetilde{Z}^\bullet$ at $-p_0$ which is obtained from $\widetilde{Z}^\bullet$ by replacing $\widetilde{Z}^{-p_0}$ by $\widetilde{Z}^{-p_0}/\,\mathrm{Image}(d_{\widetilde{Z}}^{-p_0-1})$ and $\widetilde{Z}^j$ by 0 for all $j < -p_0$. Let $s_Z\colon \widetilde{Z}^\bullet \to Z^\bullet$ be the resulting quasi-isomorphism where $s_Z^{-p_0}\colon \widetilde{Z}^{-p_0} \to \widetilde{Z}^{-p_0}/\,\mathrm{Image}(d_{\widetilde{Z}}^{-p_0-1}) = Z^{-p_0}$ is the natural surjection.

To be able to compare the two lifting obstructions $\omega(Z^\bullet, A')$ and $\omega_0(Z^\bullet, Z')$, we define a particular $P^{0,\bullet}$ and a particular $\epsilon : P^{0,\bullet} \to Z^\bullet$ as in Definition 3.15 by using (Y^j, c_Y^j) from §3.1. By following Grothendieck's construction discussed in Remark 3.3, we define

$$P^{0,0} = Y^{-1}, \quad P^{0,-j} = Y^{-j-1} \oplus Y^{-j}\ (1 \leqslant j \leqslant p_0 - 1), \quad P^{0,-p_0} = Y^{-p_0}$$

and the differentials as

$$d^{-1}_{P^{0,\bullet}} = \left(-c_Y^{-2}, 1\right),$$
$$d^{-j}_{P^{0,\bullet}} = \begin{pmatrix} -c_Y^{-j-1} & 1 \\ -c_Y^{-j} \circ c_Y^{-j-1} & c_Y^{-j} \end{pmatrix} (2 \leqslant j \leqslant p_0 - 1),$$
$$d^{-p_0}_{P^{0,\bullet}} = \begin{pmatrix} 1 \\ c_Y^{-p_0} \end{pmatrix}.$$

Moreover, we define ϵ by

$$\epsilon^0 = 0, \quad \epsilon^{-j} = (0, a_Y^{-j})\ (1 \leqslant j \leqslant p_0 - 1), \quad \epsilon^{-p_0} = s_Z^{-p_0} \circ a_Y^{-p_0}$$

where $a_Y^{-j} : Y^{-j} \to A\widehat{\otimes}_{A'} \widetilde{Z}^{-j}$ is the natural surjection for $1 \leqslant j \leqslant p_0$.

Following Definition 3.15, one now computes explicitly $T^\bullet = \mathrm{Ker}(\epsilon)$ and $D^\bullet = \mathrm{Ker}(A\widehat{\otimes}_{A'}\epsilon)$ and identifies $\mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet})$ with the kernel of the surjection $\tau : A\widehat{\otimes}_{A'} T^\bullet \to D^\bullet$. This computation shows that $\mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet})$ can be identified with the truncation $\mathrm{Trunc}_{-p_0}(JY^\bullet)$ of the complex $JY^\bullet$ at $-p_0$ which is obtained from $JY^\bullet$ by replacing JY^{-p_0} by $JY^{-p_0}/\mathrm{Image}(d_{JY}^{-p_0-1})$ and JY^j by 0 for all $j < -p_0$.

We use the definition of $\omega(Z^\bullet, A') = d_2^{0,1}(\iota)$ in Theorem 3.12 which is by Lemma 3.17 equal to

$$\omega(Z^\bullet, A') = \iota[2] \circ \alpha_2[1] \circ \alpha_1$$

where α_1 and α_2 are the homomorphisms in $D^-(B)$ which occur in the triangles (3.33) and (3.34) in Definition 3.15. Using the mapping cones of the homomorphisms $\delta_D \colon D^\bullet \to A\widehat{\otimes}_{A'} T^\bullet$ and $\sigma \colon \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet}) \to A\widehat{\otimes}_{A'} T^\bullet$ in (3.33) and (3.34), respectively, one sees that one can express $\alpha_2[1] \circ \alpha_1 \in \mathrm{Hom}_{D^-(B)}(Z^\bullet, \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet})[2])$ as

$$\alpha_2[1] \circ \alpha_1 = s_J[2] \circ \tilde{\omega} \circ (s_Z)^{-1} \tag{3.71}$$

where $(s_Z)^{-1}$ is the inverse in $D^-(B)$ of the quasi-isomorphism s_Z, $\tilde{\omega}$ is as in (3.1) and

$$s_J \colon JY^\bullet \to \mathrm{Trunc}_{-p_0}(JY^\bullet) = \mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet}) \tag{3.72}$$

is the quasi-isomorphism in $C^-(B)$ resulting from truncation such that $s_J^{-p_0}$ is the natural surjection. It follows that

$$\omega(Z^\bullet, A') = \iota[2] \circ \alpha_2[1] \circ \alpha_1 = (\iota \circ s_J)[2] \circ \tilde{\omega} \circ (s_Z)^{-1}$$

in $D^-(B)$, which proves the first part of Proposition 3.14.

For the second part of Proposition 3.14, let $(Y_0^\bullet, \upsilon_0)$ and $(Y'^\bullet, \upsilon')$ be two quasi-lifts of $(Z^\bullet, \zeta)$ over A'. Without loss of generality, we can assume that $Y_0^j = Y^j = Y'^j$ for all j, by using a fixed versal deformation $(U^\bullet, \phi_U)$ of $V^\bullet$

over $R = R(G, V^\bullet)$ such that $U^\bullet$ is concentrated in degrees $\leqslant -1$ and all terms of $U^\bullet$ are topologically free pseudocompact $R[[G]]$-modules. In particular, this implies that $JY_0^\bullet = JY^\bullet = JY'^\bullet$ and $\widetilde{Z}^\bullet = A\widehat{\otimes}_{A'} Y_0^\bullet = A\widehat{\otimes}_{A'} Y'^\bullet$ as complexes in $C^-(B)$. We have short exact sequences in $C^-(B')$ of the form $0 \to JY^\bullet \to Y_0^\bullet \xrightarrow{a_{Y_0}} \widetilde{Z}^\bullet \to 0$ and $0 \to JY^\bullet \to Y'^\bullet \xrightarrow{a_{Y'}} \widetilde{Z}^\bullet \to 0$. Truncating these complexes at $-p_0$ in the same way as we have done several times above and using that we have assumed that $Z^\bullet = \mathrm{Trunc}_{-p_0}(\widetilde{Z}^\bullet)$, we obtain short exact sequences in $C^-(B')$ of the form

$$\xi_0\colon 0 \to \mathrm{Trunc}_{-p_0}(JY^\bullet) \to \mathrm{Trunc}_{-p_0}(Y_0^\bullet) \xrightarrow{\mathrm{Trunc}_{-p_0}(a_{Y_0})} Z^\bullet \to 0, \tag{3.73}$$

$$\xi'\colon 0 \to \mathrm{Trunc}_{-p_0}(JY^\bullet) \to \mathrm{Trunc}_{-p_0}(Y'^\bullet) \xrightarrow{\mathrm{Trunc}_{-p_0}(a_{Y'})} Z^\bullet \to 0. \tag{3.74}$$

Since we have seen above that $\mathrm{H}_I^{-1}(A\widehat{\otimes}_{A'} P^{\bullet,\bullet})$ can be identified with $\mathrm{Trunc}_{-p_0}(JY^\bullet)$, letting $h_{\xi_0} = h_{\xi'} = \iota$ we arrive at the class η_{ξ_0} (resp. $\eta_{\xi'}$) in $\widetilde{F}_{II}^0 = F_{II}^0\, \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ represented by (ξ_0, h_{ξ_0}) (resp. $(\xi', h_{\xi'})$) as described in Definition 3.8. It follows from Lemma 3.25, parts (ii)(a) and (ii)(b), that η_{ξ_0} (resp. $\eta_{\xi'}$) is the class in $\widetilde{F}_{II}^0 = F_{II}^0\, \mathrm{H}^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ corresponding to the local isomorphism class of $(Y_0^\bullet, \upsilon_0)$ (resp. $(Y'^\bullet, \upsilon')$).

Following Definition 3.18, we now find homomorphisms $\lambda_0\colon P^{0,\bullet} \to \mathrm{Trunc}_{-p_0}(Y_0^\bullet)$ and $\lambda'\colon P^{0,\bullet} \to \mathrm{Trunc}_{-p_0}(Y'^\bullet)$ in $C^-(B')$ such that $\mathrm{Trunc}_{-p_0}(a_{Y_0}) \circ \lambda_0 = \epsilon = \mathrm{Trunc}_{-p_0}(a_{Y'}) \circ \lambda'$. Namely,

$$\lambda_0^0 = 0, \quad \lambda_0^{-j} = (d_{Y_0}^{-j} - c_Y^{-j}, 1)\ (2 \leqslant j \leqslant p_0 - 1), \quad \lambda_0^{-p_0} = e_0$$

where $e_0\colon Y^{-p_0} \to Y^{-p_0}/\,\mathrm{Image}(d_{Y_0}^{-p_0-1})$ is the natural surjection. Similarly, we define λ' by replacing d_{Y_0} by $d_{Y'}$ and e_0 by the natural surjection $e'\colon Y^{-p_0} \to Y^{-p_0}/\,\mathrm{Image}(d_{Y'}^{-p_0-1})$. Letting $\widetilde{\lambda_0}$ (resp. $\widetilde{\lambda'}$) be the restriction of λ_0 (resp. λ') to $T^\bullet$, we obtain by using triangle diagrams in $D^-(B')$ similarly to (3.39) that

$$\eta_{\xi_0} = \iota[1] \circ \widetilde{\lambda_0}[1] \circ \eta_T \quad \text{and} \quad \eta_{\xi'} = \iota[1] \circ \widetilde{\lambda'}[1] \circ \eta_T \tag{3.75}$$

in $D^-(B')$, where η_T is the connecting homomorphism in the top row of (3.39). Using the explicit computations of $T^\bullet$, $D^\bullet$ and $\tau\colon A\widehat{\otimes}_{A'} T^\bullet \to D^\bullet$ as before, one sees that there exists a quasi-isomorphism $s_D\colon \widetilde{Z}^\bullet[-1] \to D^\bullet$ in $C^-(B)$, which is independent of the local isomorphism classes of $(Y_0^\bullet, \upsilon_0)$ and $(Y'^\bullet, \upsilon')$, such that

$$\widetilde{\lambda'} - \widetilde{\lambda_0} = s_J \circ \tilde{\beta}_{Y'}[-1] \circ (s_D)^{-1} \circ (\tau \circ a_T)$$

in $D^-(B')$ where s_J is as in (3.72), $\tilde{\beta}_{Y'}$ is as in (3.2) and $a_T\colon T^\bullet \to A\widehat{\otimes}_{A'} T^\bullet$ is the natural surjection. Note that $\tau \circ a_T\colon T^\bullet \to D^\bullet$ is a quasi-isomorphism in $C^-(B')$. Hence

$$\begin{aligned}\eta_{\xi'} - \eta_{\xi_0} &= \iota[1] \circ (\widetilde{\lambda_0} - \widetilde{\lambda}')[1] \circ \eta_T \\ &= (\iota \circ s_J)[1] \circ \tilde{\beta}_{Y'} \circ \big((s_D)^{-1}[1] \circ (\tau \circ a_T)[1] \circ \eta_T\big)\end{aligned}$$

in $D^-(B')$, completing the proof of Proposition 3.14.

4. Quotients by pro-ℓ' groups

In this section, we give an application of the obstructions to lifting quasi-lifts as determined in §3. As we have assumed throughout this paper, the field k has positive characteristic ℓ, and $V^\bullet$ is a complex in $D^-(k[[G]])$ that has only finitely many non-zero cohomology groups, all of which have finite k-dimension. Without loss of generality, we may assume that $\mathrm{H}^i(V^\bullet) = 0$ unless $-p_0 \leqslant i \leqslant -1$.

Remark 4.1. — Suppose there is a short exact sequence of profinite groups

$$1 \to K \to G \to \Delta \to 1, \tag{4.1}$$

where K is a closed normal subgroup which is a pro-ℓ' group, *i.e.* the projective limit of finite groups that have order prime to ℓ. Let R be an object in $\widehat{\mathcal{C}}$, and suppose M is a projective pseudocompact $R[[\Delta]]$-module. Then the inflation $\mathrm{Inf}_\Delta^G M$ is a projective pseudocompact $R[[G]]$-module.

Proposition 4.2. — *Suppose G and Δ are as in Remark 4.1, G has finite pseudocompact cohomology, and $V^\bullet$ is isomorphic to the inflation $\mathrm{Inf}_\Delta^G V_\Delta^\bullet$ of a bounded above complex $V_\Delta^\bullet$ of pseudocompact $k[[\Delta]]$-modules. Then the two deformation functors $\widehat{F}^G = \widehat{F}^G_{V^\bullet}$ and $\widehat{F}^\Delta = \widehat{F}^\Delta_{V_\Delta^\bullet}$ which are defined according to Definition 2.11 are naturally isomorphic. In consequence, $R(G,V^\bullet) \cong R(\Delta, V_\Delta^\bullet)$ and $(U(G,V^\bullet),\phi_U) \cong (\mathrm{Inf}_\Delta^G U(\Delta,V_\Delta^\bullet), \mathrm{Inf}_\Delta^G \phi_U)$.*

Proof. — It follows from the definition of finite pseudocompact cohomology (see Definition 2.12) and from Remark 4.1 that Δ also has finite pseudocompact cohomology. It will be enough to show that the two deformation functors $\widehat{F}^G = \widehat{F}^G_{V^\bullet}$ and $\widehat{F}^\Delta = \widehat{F}^\Delta_{V_\Delta^\bullet}$ are naturally isomorphic.

Let $0 \to J \to A' \to A \to 0$ be an extension of objects A', A in $\widehat{\mathcal{C}}$ with $J^2 = 0$, and let $(Z_\Delta^\bullet, \zeta_\Delta)$ be a quasi-lift of $V_\Delta^\bullet$ over A. By Theorem 2.10, we may assume that the terms of $Z_\Delta^\bullet$ are projective pseudocompact $A[[\Delta]]$-modules. Hence $(Z^\bullet, \zeta) = (\mathrm{Inf}_\Delta^G Z_\Delta^\bullet, \mathrm{Inf}_\Delta^G \zeta_\Delta)$ is a quasi-lift of $V^\bullet$ over A, and by Remark 4.1 the terms of $Z^\bullet$ are projective pseudocompact $A[[G]]$-modules. By Remark 2.7, we can truncate $Z^\bullet_\Delta$, and hence $Z^\bullet = \mathrm{Inf}_\Delta^G\, Z^\bullet_\Delta$, so as to be able to assume Hypothesis 3.1 for both $Z^\bullet_\Delta$ and $Z^\bullet$. Moreover, in view of Remark 4.1, we can choose the projective resolutions $P^{\bullet,\bullet} \to Z^\bullet \to 0$ and $P^{\bullet,\bullet}_\Delta \to Z^\bullet_\Delta \to 0$ in Definition 3.4 such that $P^{\bullet,\bullet} = \mathrm{Inf}_\Delta^G\, P^{\bullet,\bullet}_\Delta$ and such that $P^{0,\bullet}_\Delta$, and hence $P^{0,\bullet}$, is acyclic. We can also arrange that the projective Cartan-Eilenberg resolutions $M^{\bullet,\bullet,\bullet}$ and $M^{\bullet,\bullet,\bullet}_\Delta$ in Definition 3.6 satisfy $M^{\bullet,\bullet,\bullet} = \mathrm{Inf}_\Delta^G\, M^{\bullet,\bullet,\bullet}_\Delta$. Following the definition of (3.23) and (3.24), we see that the natural inflation homomorphisms from Δ to G identify the sequences of low degree terms for G and for Δ. Using Theorem 3.12(i), it follows that the obstruction to lifting $(Z^\bullet_\Delta, \zeta_\Delta)$ over A' vanishes if and only if the obstruction to lifting $(Z^\bullet, \zeta)$ over A' vanishes. Using Theorem 3.12(ii), we see that if these obstructions vanish, then the set of all local isomorphism classes of quasi-lifts of $(Z^\bullet, \zeta)$ over A' is in bijection with the set of all local isomorphism classes of quasi-lifts of $(Z^\bullet_\Delta, \zeta_\Delta)$ over A'. This implies Proposition 4.2. □

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