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RECEDING POLAR REGIONS OF A SPHERICAL BUILDING AND THE CENTER CONJECTURE

by Bernhard MÜHLHERR & Richard M. WEISS

Abstract. — We introduce the notion of a polar region of a spherical building and use some simple observations about polar regions to give elementary proofs of various fundamental properties of root groups. We combine some of these observations with results of Timmesfeld, Balser and Lytchak to give a new proof of the center conjecture for convex chamber subcomplexes of thick spherical buildings.

Résumé. — Nous introduisons la notion de région polaire d'un immeuble sphérique et utilisons quelques observations simples sur les régions polaires pour donner des démonstrations élémentaires de diverses propriétés fondamentales des sous-groupes radiciels. Nous combinons certaines de ces observations avec des résultats de Timmesfeld, Balser et Lytchak pour donner une nouvelle preuve de la conjecture du centre pour les sous-complexes des chambres convexes des immeubles épais sphériques.

1. Introduction

In Chapter 6 of [27], we introduced certain subsets of the Cayley graph associated with a finite Coxeter group. Properties of these sets were then used in Chapter 11 of the same book to give a new, much simpler, proof of Tits' fundamental result (Theorem 11.6 in [27], first proved in [24]) that a thick irreducible spherical building of rank at least 3 satisfies the Moufang condition.

Our goal here is to develop further the ideas in Chapter 6 of [27] and to give several new applications of these ideas. This paper is organized as follows.

After making some preliminary observations about Coxeter groups in §2, we introduce in §3 the *polar regions* of a spherical building, describe

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some of their basic properties and show how these properties imply in a very simple way various fundamental properties of root groups. See, in particular, 3.1, 3.17, 3.18, 3.20 and 3.23. (Some of these properties are essential to the notion of a "root shadow space" or "long root geometry" and are well known in this context; see, for example, [6, 7, 9, 22].) In §4, we combine some of the results in §3 with a result of Timmesfeld in [21] to prove 4.8.

In §5 and §6 we combine 4.8 with a result of Balser and Lytchak in [2] and the notion of a *receding* polar region (defined in 6.5 below) to give a new proof of Tits' Center Conjecture for convex *chamber* subcomplexes of thick buildings. This new proof is free of case-by-case considerations (given 3.8 below) and seems to yield new insights into the nature of convex subcomplexes of a spherical building.

The Center Conjecture —more accurately, the Center Theorem— says that a convex subcomplex X of a spherical building Δ that is not completely reducible has a center. Here *completely reducible* means that to every simplex in X there is an opposite simplex in X and *center* means a simplex in X fixed by every element of $\mathrm{Aut}(\Delta)$ stabilizing X. (See 4.1 and 6.1.)

The Center Conjecture was proved for the buildings associated with classical groups and for buildings of rank 2 by Tits and the first author in [13] and for the remaining families of exceptional buildings by Leeb and Ramos-Cuevas in [12] and [16].

The origin of the Center Conjecture can be found in [23], where Tits sketched a proof of a result which became known later as the Borel-Tits theorem [4]. The sketch in [23] is based on a geometric argument using the spherical building associated with an isotropic simple algebraic group, whereas the proof given in [4] is based on other principles. Subsequently, Tits' original geometric arguments became the focus of great interest due to a paper by Serre [19] on complete reducibility in spherical buildings, in which Serre investigated attempts to extend the classical notion of complete reducibility for general linear groups to a notion that works for arbitrary reductive groups.

A special case of the Center Theorem has been applied in geometric invariant theory; see [11], [14, p.64] and [18]. Another, more recent, application of the Center Theorem has been obtained by K. Struyve who used it in [20] to obtain a fixed point theorem for finitely generated bounded groups of isometries acting on affine $\mathbb{R}$-buildings. This fixed point theorem validiated many results in the thesis of G. Rousseau [17], an important contribution to Bruhat-Tits theory. Other aspects of the Center Theorem are examined in [3].

In 6.8 we indicate how, at least in principle, our proof could be extended to arbitrary convex subcomplexes (rather than convex chamber subcomplexes) of thick buildings by extracting only the ‘endgame’ from the work of Leeb and Ramos-Cuevas.

Root groups play an essential role in our approach, and thickness is needed in order to apply Theorem 11.6 of [27] which assures the existence of root groups. Thus the hypothesis of thickness in our results is essential.

2. The arctic regions of a Coxeter group

We begin with a spherical —but not necessarily irreducible— Coxeter system (W, I) and let Σ denote the Cayley graph of the group W with respect to the generating set I. Thus the vertex set of Σ is W and a pair $u, v \in W$ is joined by an edge whenever $u^{-1}v \in I$. We refer to the vertices of Σ as *chambers* rather than vertices (in accordance with [27]). For each $w \in W$, we denote by Σ_w the set of chambers of Σ adjacent to w.

Let Π denote the Coxeter diagram of (W, I). Thus Π is the usual graph with vertex set I with labels on the edges from which the order of a product s_1s_2 in W for all $s_1, s_2 \in I$ can be read off and from which the pair (W, I) can be reconstructed.

We consider the set I to be a set of colors and assign colors to the edges of Σ according to the rule that an edge $\{u, v\}$ has color $s \in I$ whenever $u^{-1}v = s$. Thus every chamber of Σ is contained in exactly one edge of each color. There is a canonical isomorphism from W to the group of color-preserving automorphisms of Σ that sends $w \in W$ to the map which left-multiplies by w. We say that $u, v \in W$ are *i-adjacent* for some $i \in I$ if $\{u, v\}$ is an edge with color i.

For each subset J of I (including the empty set), a *J-residue* (or a *residue of type J*) is a maximal connected subgraph of Σ whose edges display only colors in J. Thus every chamber of Σ is contained in a unique J-residue (and this residue consists of just the one chamber if $J = \emptyset$). For each nonempty subset $J \subset I$ the pair $(\langle J\rangle, J)$ is also a Coxeter system and there is a color-preserving isomorphism from each J-residue of Σ to the edge-colored Cayley graph of (W_J, J) (by [27, 4.6]). The *rank* of a residue is the cardinality $|J|$ of its type J. In particular, the rank of Σ itself (the unique I-residue) is $|I|$.

A residue of Σ is the same thing as a simplex of the corresponding Coxeter complex.

A *reflection* of Σ is a nontrivial element of W that fixes an edge (by left multiplication) and thus interchanges the two vertices in this edge. A *root* of Σ is a set of the form

$$\{x \in W \mid \operatorname{dist}(x,u) < \operatorname{dist}(x,v)\}$$

for some ordered pair (u,v) of adjacent vertices. If $\{u,v\}$ is an edge of Σ, then the roots determined by (u,v) and by (v,u) form a partition of W and these two roots are interchanged by the unique reflection of W that interchanges u and v. The complement of a root α is called the root *opposite* α and is denoted by $-\alpha$.

We assume from now on that the reader is familiar with the basic properties of residues, roots and reflections as described in Chapters 3 and 4 of [27]. In particular, we have:

Proposition 2.1. — *Let α be a root of Σ. Then the following hold:*

(i) *There is a unique reflection $t = t_\alpha$ interchanging α with its opposite $-\alpha$.*

(ii) *For each $u \in \alpha$, either $\Sigma_u \subset \alpha$ or tu is the unique chamber in Σ_u that is contained in $-\alpha$.*

(iii) *An edge e is fixed by t if and only if e joints a chamber in α to a chamber in $-\alpha$.*

Proof. — These assertions are proved in [27, 3.11–3.14]. □

The *wall* of a root α is the set of edges fixed by the reflection t_α. We say that a root α *cuts* a residue R if both $\alpha \cap R$ and $-\alpha \cap R$ contain chambers. By [27, 4.10], this is the case precisely when $\alpha \cap R$ is a root of R.

Note that if a root α cuts a residue R, then the reflection t_α fixes edges in R and hence also fixes R (since it is color-preserving). Conversely, if t_α fixes a residue R for some root α, then R must contain chambers in both α and $-\alpha$ (by 2.1(i)).

Proposition 2.2. — *Let α and β be roots of Σ such that $\alpha \neq \pm\beta$ and suppose that $[t_\alpha, t_\beta] = 1$, where t_α and t_β are as in 2.1(i). Then $t_\alpha(\beta) = \beta$ and $t_\beta(\alpha) = \alpha$.*

Proof. — Since t_β centralizes t_α, it maps the wall of α to itself. By 2.1(iii), therefore, $t_\beta(\alpha) = \alpha$ or $t_\beta(\alpha) = -\alpha$. By [28, 29.25], there exists a residue of rank 2 cut by both α and β. This residue must contain an edge $\{x,y\}$ in the wall of β with both x and y in α. By 2.1(iii), t_β interchanges x and y. It follows that $t_\beta(\alpha) = \alpha$. By symmetry, also $t_\alpha(\beta) = \beta$. □

Definition 2.3. — *For each root α we let α' denote the set of chambers in α adjacent to a chamber in the complement $-\alpha$ of α. We call α' the border of α. (The border α' is denoted by $\partial\alpha$ in* [27]*.) Let*

$$m_\alpha = \max\bigl\{\mathrm{dist}(x, \alpha') \mid x \in \alpha\bigr\},$$

where $\mathrm{dist}(x, \alpha')$ denotes the minimal distance from the chamber x to a chamber in the set α', and let

$$R_\alpha = \bigl\{x \in \alpha \mid \mathrm{dist}(x, \alpha') = m_\alpha\bigr\}.$$

We call m_α the depth of α and we call R_α the arctic region of α (and we call the arctic region of $-\alpha$ the antarctic region of α).

Note that $m_\alpha = 0$ if and only if $\alpha = \alpha'$ if and only if $R_\alpha = \alpha$.

Definition 2.4. — *We call a root glacial if its depth is 0 (since in this case its arctic and antarctic regions cover all of Σ).*

In fact, we have the following:

Proposition 2.5. — *A root α is glacial (as defined in 2.4) if and only if there exists $s \in I$ such that $[s, I] = 1$ and every edge in the wall of α has color s.*

Proof. — Let α be a root, let $m = m_\alpha$, let $e = \{u, v\}$ be an edge in the wall of the reflection $t = t_\alpha$ with $u \in \alpha$ and let s be the color of e. By 2.1(ii), $v = tu$ and t fixes the edge e. Choose $s' \in I \setminus \{s\}$, let w be the chamber s'-adjacent to u and let T be the $\{s, s'\}$-residue containing u, v and w. Then T is a circuit of length $2|ss'|$. Furthermore, t maps T to itself and thus induces the unique reflection of T interchanging u and v. In particular, t fixes exactly two edges on T. These two edges have the same color if and only if $|ss'|$ is even. By 2.1(ii), it follows that the following are equivalent:

(i) $w \in \alpha'$.
(ii) $e = \{u, v\}$ and $\{w, tw\}$ are the two edges of T fixed by t.
(iii) T is of length 4.
(iv) $|ss'| = 2$ (i.e. $[s, s'] = 1$).

Now suppose that $m = 0$. Then by 2.1(ii), $w \in \alpha = \alpha'$. Hence $[s, s'] = 1$ and the color of the edge opposite e on T is also s. It follows that $[s, I] = 1$ and that every two edges in the wall of t that are contained in a single rank 2 residue have the same color. By [28, 29.23], therefore, every edge in the wall of α has color s.

Suppose, conversely, that $[s, I] = 1$ and every edge in the wall of α has color s. Since $[s, s'] \in [s, I] = 1$, we have $w \in \alpha'$. Since s' is arbitrary, it follows that $\Sigma_u \cap \alpha \subset \alpha'$. Since every edge in the wall of α has color s, the same argument shows that $\Sigma_w \cap \alpha \subset \alpha'$ for *every* chamber $w \in \alpha'$. Since roots are connected, it follows that $\alpha = \alpha'$. Thus $m = 0$. $\square$

THEOREM 2.6. — *Let α be a root, let R be its arctic region and let m be its depth. Then the following hold:*

(i) *R is a residue of Σ and*

$$R = \{x \in \alpha \mid \operatorname{dist}(y, \alpha') \leqslant \operatorname{dist}(x, \alpha') \text{ for all } y \in \Sigma_x\}.$$

(ii) *Let J be the type of R, let $v \in R$ and let $i \in I$. Then $i \in J$ if and only if the reflection interchanging v with the chamber i-adjacent to v commutes with t_α but is distinct from t_α.*

(iii) *A root β cuts R if and only if t_α and t_β are distinct commuting reflections.*

(iv) *If $x \in R$, then $\operatorname{dist}(x, \alpha') = m$, $\operatorname{dist}(x, t(x)) = 2m + 1$ and $x = \operatorname{proj}_R(t(x))$, where $t = t_\alpha$ and proj_R is the projection map defined in* [27, 3.23].

(v) *If x and y are opposite chambers of R, then $t_\alpha(y)$ and x are opposite chambers of Σ and $\operatorname{diam}(\Sigma) = \operatorname{diam}(R) + 2m + 1$.*

(vi) *The antarctic region of R is opposite R (as defined in* [27, 5.13]*) and has the same type as R.*

Proof. — Suppose first that $m = 0$ and let s be as in 2.5. Then $R = \alpha$ is a J-residue for $J = I \setminus \{s\}$, so (i) and (iv) hold, and t_α fixes *every* edge of color s and hence lies in the center of W, so (ii) and (iii) hold. It follows from [27, 5.2] that the antarctic region of α is the residue opposite R, so (vi) holds. Assertion (v) follows from (iv) and (vi) by [27, 5.14(ii)].

We can thus assume that m is positive. This means that we can use the results proved in Chapter 6 of [27]. (In this chapter it is assumed that (W, I) is irreducible, but this assumption is used only at the bottom of page 42 in [27] —where [27, 4.12] is applied— to show that $m > 0$.) Let v be a chamber in the arctic region R, let J be the set of colors on the edges $\{u, v\}$ for all chambers $u \in \Sigma_v \cap R$ and let V be the subgroup generated by all the reflections interchanging v with a chamber in $\Sigma_v \cap R$. Then v is a chamber of the set A_α defined in [27, 6.4] and by [27, 6.6 and 6.9], $A_\alpha = R$, so R is a J-residue, and the subgroup V centralizes t_α, maps α to itself and acts transitively on R. In particular, (i) holds.

Let $i \in I \setminus J$, let z be the chamber i-adjacent to v and let r be the reflection that interchanges v and z. Then $z \notin R$ by the choice of J, so

$$\operatorname{dist}(z, \alpha') \neq \operatorname{dist}(v, \alpha').$$

Therefore $r(\alpha') \neq \alpha'$ and hence $r(\alpha) \neq \alpha$. Since $z \in r(\alpha) \cap \alpha$ (since $m > 0$), we also have $r(\alpha) \neq -\alpha$. Therefore $[r, t_\alpha] \neq 1$. Thus (ii) holds.

Let β be a root. Suppose that β cuts R. Then β is V-conjugate to a root that contains the chamber v but not some chamber $u \in \Sigma_v \cap R$. It follows that $[t_\alpha, t_\beta] \in [t_\alpha, V] = 1$ (and $t_\alpha \neq t_\beta$ since $R \subset \alpha$). Suppose, conversely, that $[t_\alpha, t_\beta] = 1$ and $t_\alpha \neq t_\beta$, so that $\alpha \neq \pm\beta$. By 2.2, we have $t_\beta(\alpha) = \alpha$. It follows that $t_\beta(R) = R$. Hence β cuts R. Thus (iii) holds.

Assertions (iv) and (v) hold by [27, 6.2 and 6.6–6.8]. By (v), the residue $t(R)$ is opposite Σ. Since t is color-preserving, it has the same type as R. Thus (vi) holds. ☐

Proposition 2.7. — *Let α be a root, let $t = t_\alpha$ and let T be a residue of Σ contained in α such that the map $x \mapsto \mathrm{dist}(x, \alpha')$ is constant on T. Then $t(x) = \mathrm{proj}_{t(T)}(x)$ for all $x \in T$.*

Proof. — Choose $x \in T$ and let $n = \mathrm{dist}(x, \alpha')$. By [27, 6.1], $\mathrm{dist}(x, t(y)) \geqslant 2n + 1$ for all $y \in T$. By [27, 6.2], therefore, $\mathrm{dist}(x, t(x)) \leqslant \mathrm{dist}(x, z)$ for all $z \in t(T)$. ☐

By *gallery* we mean simply a path in Σ; see [27, 1.1].

Proposition 2.8. — *Let α, R and m be as in 2.6 and let T be a residue that is cut by α and that contains chambers of R. Then $R \cap T$ is the polar region of the root $\alpha \cap T$ of T and the depth of this root is m.*

Proof. — Let $u \in T \cap \alpha$ and let k be the distance from u to α'. By [27, 6.2], there is a minimal gallery from u to $t_\alpha(u)$ of length $2k + 1$. Since T is convex and $t_\alpha(u) \in t_\alpha(T) = T$, this gallery is contained in T. Hence the distance from u to $\alpha' \cap T$ is also k. Hence $k = m$ if $u \in R$ and $k < m$ if $u \notin R$. ☐

Proposition 2.9. — *Let α, R and J be as in 2.6. If the Coxeter diagram Π is connected and has rank at least 2, then J is as follows:*

(i) *If Π is of type $\mathsf{I}_2(n)$, then J contains exactly one of the two vertices of Π if n is even and J is the empty set if n is odd.*
(ii) *If Π is of type H_3, then J consists of all the vertices in the unique subdiagram of type $\mathsf{A}_1 \times \mathsf{A}_1$.*
(iii) *If Π is of type H_4, then J consists of all the vertices in the unique subdiagram of type H_3.*
(iv) *If Π is simply laced and thus a Dynkin diagram, then J consists of all the nodes of Π_α not adjacent to the additional node in the extended Dynkin diagram of Π.*

(v) *If* Π *can be made into a Dynkin diagram by the insertion of an arrow, then there is a unique way to do this so that* α *corresponds to a long root of the corresponding root system. In this case,* J *again consists of all the nodes of* Π *not adjacent to the additional node in the extended Dynkin diagram.*

Proof. — Assertions (i)–(iii) can be checked by hand. Suppose, therefore, that Π is connected and crystallographic and let Φ be the unique corresponding root system with respect to which α is a long root (see 2.15 below). We choose a chamber C of Φ and hence a base B of Φ with respect to which α is the highest root and then we identify the Weyl group of Φ with the Coxeter group W so that the reflections in the Weyl group corresponding to roots in the base B correspond to the elements of I. Let v be the chamber of Σ corresponding to the chamber C. We claim that $v \in R$. In any event, $v \in \alpha$ (because α is positive) and I consists of the reflections interchanging v with a chamber in Σ_v. Let

$$I^* = \big\{s \in I \mid [s, t_\alpha] = 1\big\}.$$

Thus I^* is the set of vertices of Π that are not adjacent to the additional node in the extended Dynkin diagram of Π. In particular, $t_\alpha \notin I$. Therefore $\Sigma_v \subset \alpha$ by 2.1(ii). Note, too, that if $s \in I^*$, then $s(\alpha) = \alpha$ (by 2.2) and hence $\operatorname{dist}(v, \alpha') = \operatorname{dist}(sv, \alpha')$.

Next let

$$M = \big\{x \in \Sigma_v \mid \operatorname{dist}(v, \alpha') \leqslant \operatorname{dist}(x, \alpha')\big\}$$

and

$$N = \big\{x \in \Sigma_v \mid \operatorname{dist}(v, \alpha') < \operatorname{dist}(x, \alpha')\big\}.$$

Thus $sv \in M$ for all $s \in I^*$, so $|M| \geqslant |I^*|$. Note, too, that $|M| \leqslant |I| - 1$ since Σ_v contains at least one chamber w such that $\operatorname{dist}(w, \alpha') < \operatorname{dist}(v, \alpha')$.

Suppose now that Π is not of type A_n for any $n > 1$. In this case, $|I^*| = |I| - 1$ and hence $M = \{sv \mid s \in I^*\}$ and $N = \emptyset$. By 2.6(i), therefore, M is precisely the set of neighbors of v in R and $J = I^*$. Suppose, finally, that Π is of type A_n for some $n > 1$. Then $|I^*| = |I| - 2$, so $|N| \leqslant 1$, and by [28, 29.29], there exists a (not color-preserving) automorphism σ of Σ that fixes v and the longest root α and interchanges the two elements s_1 and s_2 of I not in I^*. Hence $\sigma(N) = N$, but also $\sigma(s_1 v) = s_2 v$. It follows that $N = \emptyset$ and $\operatorname{dist}(s_i v, \alpha') < \operatorname{dist}(v, \alpha')$ for both $i = 1$ and $i = 2$. Hence $M = \{sv \mid s \in I^*\}$, so (by 2.6(i)), M is again precisely the set of neighbors of v in R and again $J = I^*$. Thus (iv) and (v) hold. □

Proposition 2.10. — *Let* α*,* R *and* J *be as in 2.6. Then the following hold:*

(i) *There is a unique connected component* Π_0 *of* Π *whose vertices are not all contained in* J.

(ii) *The intersection of* J *with the vertex set* I_0 *of* Π_0 *is as in 2.9(i)–(v) and* α *cuts every* I_0*-residue.*

(iii) α *is glacial (as defined in 2.4) if and only if* $|I_0| = 1$.

Proof. — The element t_α (like all reflections) is conjugate in W to an element of I. The elements of I conjugate in W to t_α all lie in a single connected component Π_0 of Π. If $I_0 \subset I$ is the vertex set of Π_0, then $I \setminus I_0 \subset J$ by 2.6(ii). Furthermore, $t_\alpha \in \langle I_0 \rangle$ and $\langle I_0 \rangle$ is a normal subgroup of W. Since the subgraph of Σ spanned by $\langle I_0 \rangle$ is an I_0-residue and this residue is fixed by the normal subgroup $\langle I_0 \rangle$, it follows that $\langle I_0 \rangle$ maps every I_0-residue to itself. Therefore t_α maps every I_0-residue to itself. Hence α cuts every I_0-residue. Let T be an I_0-residue containing some chamber in R. By 2.8, $\alpha \cap T$ is a root of T of depth m_α and $R \cap T$ is its polar region. We conclude that $J \cap I_0$ is as in 2.9(i)–(v) with Π_0 in place of Π and, in particular, $m_\alpha > 0$ (by [27, 4.12]) if $|I_0| > 1$. If $|I_0| = 1$, on the other hand, the root α is glacial. □

Proposition 2.11. — *Let* α *be a root, let* R *be its arctic region, let* S *be its antarctic region, let* $t = t_\alpha$*, let* v *be a chamber in* α*, let* z *be a chamber of* R*, let* $u = \mathrm{proj}_R(v)$ *and let* w *be a chamber of* α' *at minimal distance to* v*. Then* $\mathrm{dist}(u, w) = m_\alpha$*,* $tu \in S$ *and there is a minimal gallery from* z *to* tu *passing through* u*,* v *and* w *such that the subgallery from* u *to* tu *(which is of length* $2m_\alpha + 1$*) is reversed by* t*.*

Proof. — Let y be the chamber of R opposite u. By 2.6(v), y and $tu \in t(R) = S$ are opposite. By [27, 5.4], therefore,

$$\mathrm{dist}(y, tu) = \mathrm{dist}(y, v) + \mathrm{dist}(v, tu).$$

By [27, 3.22],

$$\mathrm{dist}(y, v) = \mathrm{dist}(y, u) + \mathrm{dist}(u, v)$$

and by [27, 5.4] again,

$$\mathrm{dist}(y, u) = \mathrm{dist}(y, z) + \mathrm{dist}(z, u).$$

We can thus choose a minimal gallery γ from y to tu that passes through z, u and v. Let w_1 be the unique chamber in α' on γ, let γ_0 be the subgallery of γ from u to tu, let γ_1 be the subgallery of γ_0 from u to w_1, let γ_2 be a gallery from u to w which follows γ_1 until v and then goes to w as quickly as possible and let γ_3 be the concatenation of γ_2 with $t(\gamma_2)$. Then $|\gamma_2| \leqslant |\gamma_1|$ and γ_3 is a gallery of length $2|\gamma_2| + 1$ from u to tu. Since $|\gamma_0| = 2|\gamma_1| + 1$, we conclude that $|\gamma_1| = |\gamma_2|$ and that γ_3 is minimal. Hence replacing the subgallery γ_0 of γ by γ_3 yields a gallery from y to tu with the desired properties. □

Corollary 2.12. — *Let α, R and m be as in 2.6. Then*

$$\alpha = \{x \in \Sigma \mid \operatorname{dist}(x, R) \leqslant m\}.$$

Proof. — By 2.11, every chamber in α is at distance at most m from R. The other inclusion holds by 2.3. □

Corollary 2.13. — *Let α and J be as in 2.6 and let M be the W-orbit of α. Then the map sending a root in M to its arctic region is a bijection between M and the set of J-residues of Σ.*

Proof. — Since W acts transitively on the chamber set of Σ and preserves colors, it acts transitively on the set of J-residues. Hence every J-residue is the arctic region of a root in M. Injectivity follows from 2.12. □

Notation 2.14. — Let α and β be roots of Σ such that $\beta \neq \pm\alpha$. By [28, 29.24], there exists a residue T of rank 2 cut by both α and β. The residue T is a circuit of length $2n$ for some $n \geqslant 2$. Let p denote the number of chambers in $\alpha \cap \beta \cap T$ (so $n - p$ is the number of roots of T that contain $\alpha \cap \beta \cap T$). Suppose that T_1 is a second residue of rank 2 cut by both α and β. Then $\operatorname{proj}_T(T_1)$ contains chambers in α, $-\alpha$, β and $-\beta$ (by [28, 29.16]). By [10, Prop. 3], therefore, the restriction of proj_T to T_1 is an isomorphism from T_1 to T. Hence both n and p are independent of the choice of T. We call $(n - p)\pi/n$ the *angle* between α and β and we call n the *gonality* of the pair α, β. If $\alpha = \beta$ or $-\beta$, we define the angle between them to be 0 in the first case and π in the second.

Remark 2.15. — Suppose that the Coxeter system (W, I) is crystallographic. In this case, we can choose a root system Φ (which is unique only if Π is simply laced) and an isomorphism ψ from the Weyl group W_Φ of Φ to W mapping the reflections corresponding to a base of Φ to I and, given ψ, there is a canonical correspondence between the roots of Φ and the roots of Σ (see, for example, [28, 2.8-2.13]). Let α and β be two roots of Φ. Then there exists a base of Φ containing α and a root $\beta' \in \langle \alpha, \beta \rangle$. Let C be the corresponding chamber of Φ and let s and t be the reflections of Φ corresponding to α and β'. Then the images of C under $\langle s, t \rangle$ correspond to the chambers of a rank 2 residue of Σ cut by the roots of Σ corresponding to α and β. We can now observe that the angle between the roots α and β (in the Euclidean sense) is the same as the angle between the corresponding roots of Σ as defined in 2.14 and the gonality of this pair is $|st|$.

Proposition 2.16. — *Let α and β be two roots of Σ and let R be the arctic region of α. Then α is orthogonal to β if and only if β cuts R.*

Proof. — We can assume that $\alpha \neq \pm\beta$, so $t_\alpha \neq t_\beta$. By 2.6(iii), β cuts R if and only if $[t_\alpha, t_\beta] = 1$. Let T be as in 2.14. Then $[t_\alpha, t_\beta] = 1$ if and only if the restriction of t_α commutes with the restriction of t_β to T. Hence $[t_\alpha, t_\beta] = 1$ if and only if the angle between α and β is $\pi/2$. □

Proposition 2.17. — *Let α and β be two roots of Σ and let R be the arctic region of α. Then the angle between α and β is acute if and only if $R \subset \beta$.*

Proof. — Let θ denote the angle between α and β. Suppose that $R \subset \beta$ and that θ is not acute. Then θ is, in fact, obtuse by 2.16. Let T, p and n be as in 2.14 and let $x_0, x_1, \ldots, x_{2n} = x_0$ be the unique labeling of the chambers of T such that $x_0 \in -\alpha$, $x_1 \in \alpha \cap \beta$ and x_i is adjacent to x_{i-1} for all $i \in [1, 2n]$. Thus x_1 and x_n both lie in α'. Let $z = \mathrm{proj}_R(x_n)$. Then $\mathrm{proj}_T(z) \in \alpha \cap \beta$ by [28, 29.16] (since $R \subset \alpha \cap \beta$). Thus, in particular, $\mathrm{proj}_T(z) = x_m$ for some $m \in [1, n]$. By 2.11, we have $\mathrm{dist}(z, x_n) = m_\alpha$. By [27, 3.22], there thus exists a gallery γ of length at most m_α from z to x_n passing through x_m. Let γ_0 be the first part of γ from z to x_m and let $\gamma_1 = (x_m, x_{m-1}, \ldots, x_1)$. Since $x_m \in \beta$, we have $1 \leqslant m \leqslant p$ and since θ is obtuse, we have $p < n/2$. It follows that $n - m > m - 1$. Therefore the concatenation $\gamma_0\gamma_1$ is a gallery from $z \in R$ to $x_1 \in \alpha'$ of length strictly less than m_α. By 2.3, however, $\mathrm{dist}(w, \alpha') = m_\alpha$ for every $w \in R$. We conclude that if $R \subset \beta$, then θ is acute.

Suppose, conversely, that θ is acute. By 2.16, either $R \subset \beta$ or $R \subset -\beta$. The angle between α and $-\beta$ is obtuse and hence R is not contained in $-\beta$ by the conclusion of the previous paragraph. Hence $R \subset \beta$. □

Corollary 2.18. — *Let α and β be two roots and let R be the arctic region of α. Then β contains chambers of R (some or all) if and only if the angle between α and β is at most $\pi/2$.*

Proof. — This holds by 2.16 and 2.17. □

3. The polar regions of a spherical building

We now assume that Δ is a thick (but not necessarily irreducible) spherical building of type Π. Thus the apartments of Δ are all isomorphic (as edge-colored graphs) to the graph Σ considered in the previous section.

We continue to let (W, I) denote the Coxeter system corresponding to the Coxeter diagram Π and we denote by $\mathrm{Aut}^\circ(\Delta)$ the subgroup of all color-preserving automorphisms of the building Δ.

A *root* of Δ is a root of one of its apartments. Since the arctic region of a root α is a subset of α, it is independent of the choice of the apartment containing α.

Definition 3.1. — *Let α be a root of Δ and let J be the type of the arctic region of α. The polar region of α is the unique J-residue of Δ containing the arctic region of α. Thus the polar regions of Δ are all the J-residues for all subsets J of I that are the types of arctic regions of roots.*

Let R be a polar region of Δ and let Σ be an apartment containing chambers of R. By [27, 8.13(i)], the intersection $\Sigma \cap R$ is a J-residue of Σ. Hence by 2.13, $\Sigma \cap R$ is the arctic region of a unique root α of Σ.

Remark 3.2. — If R and R' are polar regions that are opposite in Δ (as defined in [27, 9.8]), then by 2.6(vi), R and R' have the same type.

Proposition 3.3. — *Let G be a subgroup of $\mathrm{Aut}^\circ(\Delta)$, let v be a chamber of Δ and suppose that for each panel P containing v, the stabilizer of P in G acts transitively on the set of chambers in P. Then G acts transitively on the chamber set of Δ.*

Proof. — This holds because buildings are, by definition, connected. □

Proposition 3.4. — *Suppose that Δ is a residue of a thick irreducible building $\widehat{\Delta}$ and that the rank of $\widehat{\Delta}$ is at least 3. Then the following hold:*

(i) *To every root α of Δ there exists a root $\widehat{\alpha}$ of $\widehat{\Delta}$ such that $\widehat{\alpha} \cap \Delta = \alpha$.*

(ii) *The root group $U_{\widehat{\alpha}}$ of $\widehat{\Delta}$ maps Δ to itself and acts transitively and faithfully on $P \setminus \alpha$ for every panel P of Δ in the wall of α.*

(iii) *The subgroup of $\mathrm{Aut}^\circ(\Delta)$ induced by the root group $U_{\widehat{\alpha}}$ is independent of the choice of $\widehat{\alpha}$.*

Proof. — By [27, 11.6], $\widehat{\Delta}$ satisfies the Moufang condition (as defined in [27, 11.2]). Let Σ be an apartment of Δ and let α be a root of Σ. By [27, 8.13(ii)], Σ is the intersection with Δ of an apartment $\widehat{\Sigma}$ of $\widehat{\Delta}$ and α is the intersection with Δ of a root $\widehat{\alpha}$ of $\widehat{\Sigma}$. Thus (i) holds. Let $U_{\widehat{\alpha}}$ be the root group of $\widehat{\Delta}$ corresponding to $\widehat{\alpha}$ as defined in [27, 11.1]. Since $U_{\widehat{\alpha}}$ fixes chambers of Δ and is color-preserving, it maps the residue Δ to itself. By [27, 11.4], (ii) holds.

Now let $\widehat{\alpha}'$ be a second root of $\widehat{\Delta}$ such that $\widehat{\alpha}' \cap \Delta = \alpha$ and let $\widehat{\Sigma}'$ be an apartment of $\widehat{\Delta}$ containing $\widehat{\alpha}'$. By [27, 9.3], we can assume that

$\widehat{\Sigma}' \cap \Delta = \widehat{\Sigma} \cap \Delta = \Sigma$. Let P be a panel of Δ in the wall of α and let c be the unique chamber in $P \cap \alpha$. Let $\widehat{U}_c^+$ denote the subgroup of $\mathrm{Aut}^\circ(\widehat{\Delta})$ generated by the root groups $U_{\widehat{\beta}}$ of $\widehat{\Delta}$ for all roots $\widehat{\beta}$ of $\widehat{\Sigma}$ containing c. By [27, 11.11(ii)], there is a unique element g in $\widehat{U}_c^+$ mapping $\widehat{\Sigma}$ to $\widehat{\Sigma}'$. Since g is color-preserving and fixes c, it fixes P and Δ. Hence g fixes Σ. Since $\widehat{\alpha}$ is the unique root of $\widehat{\Sigma}$ containing c but not $P \cap \widehat{\Sigma}$ and $\widehat{\alpha}'$ is the unique root of $\widehat{\Sigma}'$ containing c but not $P \cap \widehat{\Sigma}'$, it follows that g maps $\widehat{\alpha}$ to $\widehat{\alpha}'$.

Let Q be an arbitrary panel of Δ containing c. If $\widehat{\beta}$ is a root of $\widehat{\Sigma}$ containing c, then (by [27, 11.1 and 11.4]) either $U_{\widehat{\beta}}$ acts trivially on Q or $\widehat{\beta}$ is the unique root of $\widehat{\Sigma}$ containing c but not the other chamber in $Q \cap \widehat{\Sigma}$ and $U_{\widehat{\beta}}$ acts regularly on $Q \setminus \{c\}$. It follows that an element of U_c^+ fixing a chamber in $Q \setminus \{c\}$ acts trivially on Q. Since g is contained in $\widehat{U}_c^+$ and fixes the two chambers in $Q \cap \Sigma$, it therefore acts trivially on Q. Since Q is arbitrary, it follows from [27, 9.7] that g acts trivially on Δ. Thus (iii) holds. □

Notation 3.5. — Let Δ and $\widehat{\Delta}$ be as in 3.4. For each root α of Δ we denote by U_α the subgroup of $\mathrm{Aut}^\circ(\Delta)$ induced by $U_{\widehat{\alpha}}$, where $\widehat{\alpha}$ is as in 3.4(i). By 3.4(iii), U_α is well defined.

HYPOTHESES 3.6. — ***For the rest of this paper*** *we assume that Δ is a residue of a thick irreducible building $\widehat{\Delta}$ of rank at least* 3 *or that Δ is a Moufang polygon (as defined in* [26, 4.2]*) other than a Moufang quadrangle of type F_4 or a Moufang octagon (as defined in* [26, 16.7 and 16.9]*). Let* Π *denote the Coxeter diagram of Δ and let (W, I) be the corresponding Coxeter system. For each root α of Δ, we let U_α be the corresponding root group (as defined in* [27, 11.1]*) if Δ is a Moufang polygon and we let U_α be as in 3.5 otherwise. (By* [27, 11.10], *these definitions coincide if Δ is a Moufang polygon and a residue of $\widehat{\Delta}$.) In every case, we let G^+ denote the subgroup of* $\mathrm{Aut}^\circ(\Delta)$ *generated by the groups U_α for all roots α of Δ.*

Notation 3.7. — Let Σ be an apartment of Δ. By [27, 11.22] applied to $\widehat{\Delta}$ (or to Δ when Δ is a Moufang polygon), for each root α of Σ and for each $g \in U_\alpha^*$, there exist unique elements $\kappa_\Sigma(g)$ and $\lambda_\Sigma(g)$ in $U_{-\alpha}^*$, where $-\alpha$ is the root of Σ opposite α, such that the product

$$\kappa_\Sigma(g) \cdot g \cdot \lambda_\Sigma(g)$$

induces the unique reflection of Σ that interchanges α and $-\alpha$. We denote this product by $\mu_\Sigma(g)$. Hence

$$U_\alpha^{\mu_\Sigma(g)} = U_{-\alpha} \quad \text{and} \quad U_{-\alpha}^{\mu_\Sigma(g)} = U_\alpha$$

for each $g \in U_\alpha$. As in [26, 6.2-6.3], we have

$$\kappa_\Sigma(g^{-1}) = \lambda_\Sigma(g)^{-1}. \tag{3.1}$$

Theorem 3.8. — *Let Δ and Π be as in 3.6 and let Π_0 be a connected component of Π. Then there exist roots α in Σ such that*

- *the type of the polar region of α contains the vertex set of every connected component of Π other than Π_0 and*
- *for each apartment Σ containing α and for each root $\beta \neq \alpha$ of Σ, either the angle θ between α and β (as defined in 2.14) is greater than $2\pi/3$ or one of the following holds:*
  - (i) *$\theta = 2\pi/3$, U_α is abelian, $[U_\alpha, U_\beta] = U_\delta$ for a unique root δ of Σ and δ is at an angle of $\pi/3$ to α and to β.*
  - (ii) *$\theta \leqslant \pi/2$, $Z(U_\alpha) \neq 1$ and $[Z(U_\alpha), U_\beta] = 1$.*

Proof. — Let Σ be an apartment of Δ. By 2.10, we can choose a root α of Σ such that the type of the polar region of α contains the vertex set of every connected component of Π other than Π_0. Let β be an arbitrary root of Σ such that $\beta \neq \pm\alpha$, let θ denote the angle between α and β, let $T \subset \Sigma$ be as in 2.14, let J be the type of T and let S be the J-residue of Δ containing T. Thus S is a generalized n-gon (by the choice of T), where $n \geqslant 2$ is the gonality of the pair α, β (as defined in 2.14). If Δ is a Moufang polygon, then $\Delta = S$ and $n \geqslant 3$. Hence if $n = 2$, then $\theta = \pi/2$ and $[U_\alpha, U_\beta] = 1$ by [27, 11.28(iii)] applied to $\widehat{\Delta}$. We suppose from now on that $n \geqslant 3$ (and hence $n = 3$, 4 or 6).

Let $x_0, x_1, \ldots, x_{2n} = x_0$ be the unique labeling of the chambers of the $2n$-circuit T such that x_{i-1} is adjacent to x_i for each $i \in [1, 2n]$, $x_{n+1} \in -\alpha$ and $x_n \in \alpha \cap \beta$. For each $i \in [1, n]$, let α_i be the unique root of Σ containing x_i but not x_{i-1} and let $U_i = U_{\alpha_i}$. Thus $\alpha_1 = \alpha$ and $\beta = \alpha_m$ for some $m \leqslant n$. Let $U_+ = \langle U_1, \ldots, U_n \rangle$ and let $\Omega = (U_+, U_1, U_2, \ldots, U_n)$. Then by [27, 11.27(i)], Ω is the root group sequence associated with the Moufang n-gon S (as defined in [27, 12.2]). By [26, 17.2-17.7], this root group sequence is, up to isomorphism and up to opposites (as defined in [26, 8.8–8.9]) as in [26, 16.1–16.6 or 16.8]. After replacing U_i by U_{n+1-i} for all $i \in [1, n]$ if necessary, we can thus assume that Ω is as in in [26, 16.1] with $n = 3$ or as in [26, 16.2–16.6] with $n = 4$ or as in [26, 16.8] with $n = 6$ and in each case either $U_\alpha = U_1$ and $U_\beta = U_m$ or $U_\alpha = U_n$ and $U_\beta = U_{n+1-m}$.

Suppose that $n = 3$. By [26, 16.1], U_i is abelian for each $i \in [1,3]$, $[U_1, U_3] = U_2$ and α_2 is at an angle of $\pi/3$ to α_1 and to α_3. Thus if Π_0 is simply laced, in which case n must equal 2 or 3, we are done (without having to refine our choice of α).

Suppose that $n > 3$. Then Π_0 has a unique edge with label n and J consists of the two elements of I joined by this edge. Suppose that Σ' is an apartment containing α, that β' is a root of Σ' at the same angle to α as β and that the gonality of the pair α, β' (as defined in 2.14) is 4. Let T' be a rank 2 residue of Σ' cut by both α and β'. Then T' must also be a J-residue and hence in the same G^+-orbit as T. It follows that there is an element of G^+ fixing α and mapping β' to β.

If $n = 4$, then $[U_1, U_2] = 1$, $Z(U_1) \neq 1$ and $[Z(U_1), U_3] = 1$ in each of the cases [26, 16.2–16.6]. If $n = 6$, we are in the case [26, 16.9], where U_6 is abelian and $[U_i, U_6] = 1$ for $i = 3$, 4 and 5. By the observation in the previous paragraph, we conclude that the root α_1 (which might or might not be equal to α) has the desired properties in the case $n = 4$ and the root α_6 (which might or might not be equal to α) has the desired properties in the case $n = 6$. □

It is not true that roots α as in 3.8 exist if Δ is a Moufang quadrangle of type F_4 or a Moufang octagon. (See 3.6.)

Remark 3.9. — From [26, 16.1 and 16.8] we can read off that in case (i) of 3.8 we, in fact, have

$$[u, U_\beta] = U_\delta = [U_\alpha, v]$$

for all $u \in U_\alpha^*$ and all $v \in U_\beta^*$.

Remark 3.10. — Suppose that we are in case (i) of 3.8 and choose $u \in U_\alpha^*$ and $v \in U_\beta^*$. Since α and β both cut the rank 2 residue T, both $\mu_\Sigma(u)$ and $\mu_\Sigma(v)$ (as defined in 3.7) map T to itself. From the action of these two elements on T, we see that $\mu_\Sigma(u)$ interchanges β with δ and $\mu_\Sigma(v)$ interchanges α with δ. Thus

$$U_\alpha^{\mu_\Sigma(v)} = U_\beta^{\mu_\Sigma(u)} = U_\delta, \quad U_\delta^{\mu_\Sigma(v)} = U_\alpha \quad \text{and} \quad U_\delta^{\mu_\Sigma(u)} = U_\beta.$$

Remark 3.11. — Let n and α_i for $i \in [1, n]$ be as in the proof of 3.8 By 3.10, α_1, α_2 and α_3 are all in the same G^+-orbit if $n = 3$. Suppose that $n = 4$ or 6, Then by [28, 29.52], the roots α_i and α_j are not in the same G^+-orbit if $|i - j|$ is odd. We conclude that the angle between two roots in the same G^+-orbit is 0, $\pi/3$, $\pi/2$, $2\pi/3$ or π. This observation is, of course, also a consequence of 2.15.

Remark 3.12. — It follows from [26, 16.1-16.8] and 3.4(ii) that $Z(U_\alpha)$ is nontrivial for every root α of Δ.

Definition 3.13. — *A root α is long if it satisfies the conclusions of 3.8. A root is short if it is not long.*

Remark 3.14. — Let α be a short root in an apartment Σ. It follows from the proof of 3.8 that either there exist roots in Σ at an angle of $\pi/4$ to α or Δ is a Moufang hexagon and there exists roots in Σ at an angle of $\pi/6$ to α. It also follows from the proof of 3.8 that all roots of Σ at an angle of $\pi/4$ or $\pi/6$ to α are long.

Remark 3.15. — Let α and β be two roots of Σ such that $\alpha \neq \pm\beta$, suppose that the gonality n of the pair α, β is at least 3 and let the rank 2 residue T, the labeling $x_0, x_1, \dots, x_{2n} = x_0$ of the chambers of T and the roots α_i for all $i \in [1, 2n]$ be as in the proof of 3.8, so $\alpha_1 = \alpha$ and $\alpha_m = \beta$ for some $m \in [2, n]$. The roots $\alpha_1, \alpha_2, \dots, \alpha_m$ are precisely the roots of Σ that contain $\alpha \cap \beta$ (by [28, 29.16]) and the angle between α_i and α_j is $|i-j| \cdot \pi/n$ for all $i, j \in [1, n]$. Thus the sequence $\alpha_1, \dots, \alpha_m$ is independent of the choice of T.

Proposition 3.16. — *Let α be a long root and let β be a root having the same polar region as α. Then β is in the same G^+-orbit as α (so also β is long) and $Z(U_\alpha) = Z(U_\beta)$.*

Proof. — Let R be the polar region of α (and therefore of β), let J be the type of R and let Σ be an apartment containing α. Thus $\Sigma \cap R$ is the arctic region of α in Σ. Let A be the set of roots of Σ containing chambers (some or all) of R and let B be the subgroup generated by U_β for all β in A. By 2.18, A contains precisely the roots of Σ whose angle with α is at most $\pi/2$. By 3.8, therefore, B centralizes $Z(U_\alpha)$. Since the elements of B are color-preserving and all fix some chambers in R, they all map R to itself. By 3.3 and 3.4(ii), the group B acts transitively on the set of chambers of R.

Now let Σ' be an apartment containing β. Thus $\Sigma' \cap R$ is the arctic region of β in Σ'. Replacing Σ' and β by their images under a suitable element of B, we can assume that $\Sigma \cap R \cap \Sigma'$ contains a chamber v. The group B contains U_δ for every root δ of Σ containing v. By [27, 11.11(ii)], the group B contains an element g fixing v and mapping Σ' to Σ. (In [27, 11.11(ii)] is assumed that Δ is Moufang, but with 3.4 the proof is valid verbatim.) The element g maps $\Sigma' \cap R$ to $\Sigma \cap R$ and thus (by 2.13) β to α. It also centralizes $Z(U_\alpha)$. Therefore $Z(U_\alpha) = Z(U_\beta)$. $\square$

Theorem 3.17. — *Let α and β be two long roots of Δ. Then there exists roots α' and β' contained in a single apartment such that $Z(U_{\alpha'}) = Z(U_\alpha)$, $Z(U_{\beta'}) = Z(U_\beta)$.*

Proof. — Let R and S be the polar regions of α and β. There exists an apartment Σ containing chambers of both R and S. By 2.13, there exist unique roots α' and β' of Σ whose polar regions are R and S. The claim holds, therefore, by 3.16. □

Theorem 3.18. — *Let M be a G^+-orbit of long roots, let J be the type of the polar region of a root in M and let*

$$W = \{Z(U_\alpha) \mid \alpha \in M\}.$$

Then there is a bijection ψ from the set of J-residues of Δ to W sending a J-residue R to $Z(U_\beta)$, where $\beta \in M$ and R is the polar region of β.

Proof. — By 3.3 and 3.4(ii), G^+ acts transitively on the set of chambers of Δ and hence it also acts transitively on the set of J-residues of Δ. It follows that a root lies in M if and only if its polar region is a J-residue, and every J-residue is the polar region of a root in M. The map ψ is therefore well defined (by 3.16) and surjective.

Suppose that R and R' are two J-residues of Δ. There exists an apartment Σ such that both $R \cap \Sigma$ and $R' \cap \Sigma$ are J-residues of Σ. Hence there exist unique roots α and α' of Σ such that $R \cap \Sigma$ is the arctic region of α and $R' \cap \Sigma$ is the arctic region of α'. Now suppose that $R \neq R'$. Then also $R \cap \Sigma \neq R' \cap \Sigma$ and hence $\alpha \neq \alpha'$ (by 2.13). A chamber of Σ is fixed by $Z(U_\alpha)$ if and only if it lies in α. (This holds by [27, 9.7] since for each chamber v of Σ not in α, Σ is the unique apartment of Δ that contains both v and α.) Since the analogous assertion holds for α', we conclude that $Z(U_\alpha) \neq Z(U_{\alpha'})$. Hence ψ is injective. □

Corollary 3.19. — *Let R be the polar region of a long root α and let $g \in G^+$. Then $R^g = R$ if and only if g normalizes $Z(U_\alpha)$.*

Proof. — This holds by 3.18. □

Theorem 3.20. — *Let R be the polar region of a long root α, let m be the depth of α and let x be a chamber of Δ. Then $Z(U_\alpha)$ fixes x if and only if the distance from x to R is at most m.*

Proof. — Let $y = \mathrm{proj}_R(x)$, let $k = \mathrm{dist}(x, y)$ and let Σ be an apartment containing x and y. By 2.13, there exists a unique root β of Σ whose polar region is R. By 3.16, $m = m_\beta$ and $Z(U_\alpha) = Z(U_\beta)$. By 2.12 (and [27, 8.9]), $x \in \beta$ if and only if $k \leqslant m_\beta$. As we observed in the proof of 3.18, $x \in \beta$

if and only if x is fixed by $Z(U_\beta)$. Thus x is fixed by $Z(U_\alpha)$ if and only if $k \leqslant m$. □

The next definition is taken from [22, 1.1].

Definition 3.21. — *A rank one group is a group generated by two nontrivial nilpotent subgroups A and B such that for each $a \in A^*$ there exists $b \in B^*$ such that $bAb^{-1} = aBa^{-1}$ and vice versa.*

Proposition 3.22. — *Let Σ be an apartment, let α be a root of Σ (long or short) and let β be its opposite in Σ. Then $\langle Z(U_\alpha), Z(U_\beta)\rangle$ is a rank one group (as defined in 3.21).*

Proof. — Replacing Δ by the building $\widehat{\Delta}$ in 3.6 if necessary, we can assume that Δ is irreducible of rank at least 2. Let $A = Z(U_\alpha)$ and $B = Z(U_\beta)$. By 3.12, A and B are nontrivial and by 3.7,

$$A^{\kappa_\Sigma(u)} = B^{u^{-1}} \quad \text{and} \quad B^{\kappa_\Sigma(v)} = A^{v^{-1}}$$

for all $u \in A^*$ and all $v \in B^*$. It thus suffices to assume that U_α is nonabelian and to show that under this assumption that $\kappa_\Sigma(A) \subset B$ and $\kappa_\Sigma(B) \subset A$. By (3.1), it suffices to show that $\lambda_\Sigma(A) \subset B$. For this we choose an irreducible rank 2 residue S of Δ containing a panel in the wall of α. The residue S is a Moufang polygon but neither a Moufang quadrangle of type F_4 nor a Moufang octagon. Since U_α is nonabelian, the polygon S must be as in either 16.5 or 16.6 of [26]. By [26, 38.9-38.10], $Z(U_\alpha)$ is, in both cases, isomorphic to the group called $x_1(0, K)$ and the formulas in [26, 32.9-32.10] reveal that, in fact, $\lambda_\Sigma(A) \subset B$. □

The following result is completely standard in the special case of finite groups of Lie type. The earliest antecedent we could find is [8, 2.2]. The version we give here is due to Timmesfeld [22, II.5.20] (who proved it by different means); see also [6, 7, 9].

Theorem 3.23. — *Suppose that Δ irreducible and of rank at least 2 and let M be a G^+-orbit of long roots (as defined in 3.13). Then*

$$\{Z(U_\alpha) \mid \alpha \in M\}$$

is a set of abstract root groups of G^+ as defined in [22, II.1.1].

Proof. — Let α and β in M. By 3.17, we can assume that there is an apartment Σ containing both α and β. The claim holds, therefore, by 3.8, 3.9, 3.11 and 3.22. □

4. A theorem of Timmesfeld

The main result of this section is 4.8. The key to its proof is a result of Timmesfeld in [21]; see 4.5 below.

We continue to assume that Δ, G^+ and the groups U_α for all roots α are as in 3.6.

Definition 4.1. — *A subcomplex of a building Δ is a collection X of residues such that the following hold:*

(i) *If $R \in X$, then every residue of Δ containing R is also in X.*

(ii) *There exists m such that every residue R in X contains a residue of rank m.*

A chamber subcomplex of Δ is a subcomplex that contains chambers, equivalently, a subcomplex in which the parameter m in (ii) is 0*. A chamber subcomplex is convex if its set of chambers is convex as defined in* [27, 1.11]*. An arbitrary subcomplex is convex if it is an intersection of convex chamber subcomplexes.*

Definition 4.2. — *Let R and R' be two polar regions of Δ. Let Σ be an apartment containing chambers of both R and R' and let α and β be the unique roots of Σ whose arctic regions are $R\cap\Sigma$ and $R'\cap\Sigma$. The angle between R and R' is the angle between α and β and the gonality of the pair R, R' is the gonality of the pair α, β as defined in 2.14. By* [27, 8.20]*, these numbers are independent of the choice of Σ.*

Thus, in particular, two polar regions are opposite if and only if the angle between them is π (by [27, 9.8]).

Definition 4.3. — *Let M be a G^+-orbit of long roots and let J be the type of the polar regions of roots in M. Suppose that R and R' are J-residues, let Σ be an apartment containing chambers of both R and R', let α and β be the roots of Σ whose polar regions are R and R' and suppose that the angle between R and R' (as defined in 4.2) is $2\pi/3$. By 3.8(i) and 3.10, $[U_\alpha, U_\beta] = U_\delta$ for a unique root δ of Σ, $\delta \in M$ (so also δ is long) and $Z(U_\xi) = U_\xi$ for $\xi = \alpha$, β and δ. By 3.16, the groups U_α and U_β and hence also the group U_δ is independent of the choice of Σ. By 3.18, therefore, also the polar region of δ is independent of the choice of Σ. We call the polar region of δ the midpoint of R and R' and denote it by $R * R'$.*

Proposition 4.4. — *Let Σ, α, R and R' be as in 4.3 and choose $w \in \mu_\Sigma(U_\alpha^*)$. Then w interchanges R' and $R * R'$.*

Proof. — This holds by 3.10. □

The following key result is an immediate consequence of a theorem proved by Timmesfeld (who cites Aschbacher [1] for the underlying idea) in [21].

Proposition 4.5. — *Let J be the type of the polar region of a long root of Δ and suppose that Ω is a nonempty set of J-residues of Δ such that for all $R, R' \in \Omega$ the following hold:*

(i) *The angle between R and R' is less than π.*
(ii) *If the angle between R and R' is $2\pi/3$, then the midpoint $R * R'$, as defined in 4.3, is in Ω.*

Then there exists a residue R in Ω such that the angle between R and all the other residues in Ω is at most $\pi/2$.

Proof. — Replacing Δ by the building $\widehat{\Delta}$ if necessary, we can assume that Δ is irreducible and of rank at least 2. Let W be the set of subgroups $Z(U_\alpha)$ for all roots α whose polar region is in Ω and let H be the group generated by the subgroups in W. We know (by 3.8 and 3.11) that if $A, B \in W$, then either the angle between A and B is at most $\pi/2$ and $[A, B] = 1$ or the angle between A and B is $2\pi/3$ and $[A, B] \in W$. It thus suffices to show that $Z(H)$ contains an element of W. Let $H^{(0)} = H$ and let $H^{(m)} = [H, H^{(m-1)}]$ for all $m \geqslant 1$. By Corollaries 2.2–2.3 in [21], the group H is nilpotent, so $H^{(k)} = 1$ for some k. We can thus choose $t \geqslant 0$ to be maximal such that $H^{(t)}$ contains an element $A \in W$. If B is an arbitrary element of W, then $[A, B] \subset H^{(t+1)}$ and hence $[A, B] = 1$ by the choice of t. Therefore $A \subset Z(H)$. □

Proposition 4.6. — *Let R be the polar region of a long root, let J be the type of R, let R_2 be a J-residue at an angle of $2\pi/3$ to R, let u be a chamber of R, let v be a chamber of R_2, let Σ be an apartment containing u and v, let $R_1 = R * R_2$ as defined in 4.3 and let $z = \mathrm{proj}_{R_1}(u)$. Then the following hold:*

(i) *There is a minimal gallery from u to v that passes through z.*
(ii) *R_1 is the unique J-residue at an angle of $\pi/3$ to both R and R_2 that contains chambers of Σ.*

Proof. — Let α, δ and β be the three roots of Σ whose polar regions are, respectively, R, R_1 and R_2 and let C be the intersection of all the roots of Σ containing u and v. To prove (i), it will suffice, by [27, 3.21], to show that $z \in C$. Let ξ be a root of Σ containing u and v and let g be a nontrivial element in the group U_ξ. Since g fixes u and v and is color-preserving, it fixes both R and R_2. By 3.19, g normalizes U_α and U_β. Therefore g normalizes also $U_\delta = Z(U_\delta)$. By another application of 3.19, it follows that g fixes R_1. There is a unique apartment Σ' containing both the complement of ξ in Σ —which we call ξ_1— and the complement of ξ in Σ^g (by [28, 29.54]). We have

$$|R_1 \cap \Sigma| = |R_1 \cap \Sigma'|$$

since both intersections are residues of an apartment having the same type. We conclude that $R_1 \cap \xi \neq \emptyset$, since otherwise $R_1 \cap \Sigma'$ contains the disjoint union of $R_1 \cap \Sigma = R_1 \cap \xi_1$ and $(R_1 \cap \xi_1)^g$. By [28, 29.16], it follows that $z \in \xi$. We conclude that $z \in C$. Thus (i) holds. Let Φ be as in 2.15. If $\widehat{\alpha}$ and $\widehat{\beta}$ are roots of Φ that are at an angle of $2\pi/3$ to each other, then an arbitrary root in Φ at an angle of $\pi/3$ to both $\widehat{\alpha}$ and $\widehat{\beta}$ must lie in the plane $\langle\widehat{\alpha}, \widehat{\beta}\rangle$ and hence equal $\widehat{\alpha} + \widehat{\beta}$. Thus (ii) holds. □

Proposition 4.7. — *Let R be the polar region of a long root, let J be the type of R, let R_2 be a J-residue at an angle of $2\pi/3$ to R, let $R_1 = R * R_2$, let S_2 be a J-residue that is opposite R_1 but not opposite R. Then the angle between S_2 and R is $2\pi/3$ and R_2 is opposite $R * S_2$.*

Proof. — By 4.6(i), there is an apartment Σ containing chambers of R, R_1 and R_2. Choose $u \in R_1 \cap \Sigma$ and let $\rho = \mathrm{retr}_{\Sigma,u}$ (as defined in [27, 8.16]). Let $\bar{R} = \rho(R)$, $\bar{S}_2 = \rho(S_2)$ and $\bar{R}_i = \rho(R_i)$ for $i = 1$ and 2. Since ρ is a color-preserving homomorphism (by [27, 8.18]) whose restriction to Σ is the identity (by [27, 8.17]), it maps residues of Δ of a given type to residues of Σ of the same type and

$$\operatorname{dist}\big(\rho(x), \rho(y)\big) \leqslant \operatorname{dist}(x, y) \tag{4.1}$$

for all chambers x, y of Δ. By [27, 8.17], $\bar{R} = R \cap \Sigma$, $\bar{R}_i = R_i \cap \Sigma$ for $i = 1$ and 2 and if x is a chamber opposite u, then $\rho(x)$ is also opposite u. Since they have the same types as the opposite residues R_1 and S_2, it follows that $\bar{R}_1$ and $\bar{S}_2$ are opposite residues of Σ.

Let m be the depth of a root whose polar region is of type J. Since the angle between R and R_1 is acute, $\operatorname{dist}(x, R_1) \leqslant m$ for every $x \in R$ by 2.17. By 2.6(iv), $\operatorname{dist}(S_2, R_1) = 2m + 1$. Hence $\operatorname{dist}(x, S_2) > m$ for every $x \in R$, so (by 2.17 again) the angle between R and S_2 is obtuse. By 3.11, this angle must be $2\pi/3$. By (4.1), $\operatorname{dist}(x, \bar{R}_1) \leqslant m$ for every $x \in \bar{R}$. Since $\bar{R}_1$ and $\bar{S}_2$ are opposite, it follows by the same argument that the angle between $\bar{R}$ and $\bar{S}_2$ is also $2\pi/3$. Let $S_1 = R * S_2$ and let $\bar{S}_1 = \rho(S_1)$.

By 3.8(i), S_1 is at an angle of $\pi/3$ to S_2. By 2.17 and (4.1), it follows that $\operatorname{dist}(u, \bar{S}_2) \leqslant m$ for all $u \in \bar{S}_1$ and hence the angle between $\bar{S}_1$ and $\bar{S}_2$ is acute. By 3.11, this angle is either 0 or $\pi/3$. Similarly, the angle between

$\bar{S}_1$ and $\bar{R}$ is either 0 or $\pi/3$. Since the angle between $\bar{R}$ and $\bar{S}_2$ is $2\pi/3$, we conclude that both of these angles equal $\pi/3$.

Let $\widetilde{S}_i$ denote the J-residue of Δ that contains $\bar{S}_i$ for $i = 1$ and 2. By 4.6(ii), we have $\widetilde{S}_1 = \widetilde{S}_2 * R$. Let α be the unique root of Σ whose polar region is R and choose $w \in \mu_\Sigma(U_\alpha^*)$. By 4.4, the element w interchanges R_1 with R_2 and $\widetilde{S}_1$ with $\widetilde{S}_2$. Since R_1 and $\widetilde{S}_2$ are opposite, we conclude that R_2 and $\widetilde{S}_1$ are opposite. By (4.1), it follows that also R_2 and S_1 are opposite. □

We come now to the main result of this section.

Theorem 4.8. — *Let Δ be as in 3.6, let X be a convex subcomplex of Δ, let J be the type of the polar region of some long root, let Ω be the set of all J-residues R contained in X for which there is no J-residue in X opposite R in Δ and suppose that Ω is not empty. Then there exists a J-residue R in X such that for each J-residue $R' \in \Omega$, the angle between R and R' is at most $\pi/2$.*

Proof. — By 4.5, it suffices to show that the midpoint of two J-residues in Ω at an angle of $2\pi/3$ is also in Ω. Let R and R_2 be two J-residues in Ω at an angle of $2\pi/3$ and let $R_1 = R * R_2$. By 4.6(i), every convex chamber subcomplex containing chambers of R and R_2 also contains chambers of R_1. Hence $R_1 \in X$. It remains to show there does not exist a J-residue in X opposite R_1. Suppose that S_2 is such a residue. Since $R \in \Omega$, the residues R and S_2 are not opposite. By 4.7, it follows that the angle between R and S_2 is $2\pi/3$ and that R_2 is opposite $R * S_2$. By 4.6(i) again, $R * S_2 \in X$. This is impossible, however, since $R_2 \in \Omega$. □

5. Convex subcomplexes

In this section we assemble the various additional properties of convex subcomplexes and polar regions that we will need in §6.

We continue to assume that Δ, G^+ and the groups U_α for all roots α are as in 3.6.

Proposition 5.1. — *Let R and S be two residues of Δ, let $R_1 = \mathrm{proj}_R(S)$ and let $S_1 = \mathrm{proj}_S(R)$. Then R_1 and S_1 are residues, proj_R and proj_S are homomorphisms, the restriction of proj_R to S_1 is the inverse of the restriction of proj_S to R_1 and*

$$\mathrm{dist}(x, \mathrm{proj}_S(x)) = \mathrm{dist}(\mathrm{proj}_R(y), y)$$

for all $x \in R_1$ and all $y \in S_1$.

Proof. — It follows from [27, 8.21] that the restriction of proj_R to S_1 is the inverse of the restriction of proj_S to R_1. All the remaining claims hold by [10, Prop. 3]. □

Notation 5.2. — Let R and S be residues of Δ. In light of 5.1, it is natural to call $\mathrm{dist}(x, \mathrm{proj}_S(x))$ for an arbitrary choice of $x \in \mathrm{proj}_R(S)$ the *distance between R and S*.

Proposition 5.3. — *Let R and S be two residues of Δ and let Σ be an apartment that contains chambers of both R and S. Then $R \cap \Sigma$ and $S \cap \Sigma$ are cut by the same roots of Σ if and only if the restriction of* proj_R *to S and the restriction of* proj_S *to R are inverses of each other.*

Proof. — By [28, 29.16 and 29.21], $R \cap \Sigma$ and $S \cap \Sigma$ are cut by the same roots of Σ if and only if the restriction of proj_R to $S \cap \Sigma$ and the restriction of proj_S to $R \cap \Sigma$ are both injective. Suppose that these two maps are both injective. Then $|R \cap \Sigma| = |S \cap \Sigma|$ and hence both maps are, in fact, surjective. Since proj_R and proj_S are homomorphisms, $\mathrm{proj}_R(S)$ is a residue of Δ of the same type as the residue $\mathrm{proj}_R(S \cap \Sigma)$ of Σ and $\mathrm{proj}_S(R)$ is a residue of Δ of the same type as the residue $\mathrm{proj}_S(R \cap \Sigma)$. Hence $\mathrm{proj}_R(S) = R$ and $\mathrm{proj}_S(R) = S$. By 5.1, therefore, the restriction of proj_R to S and the restriction of proj_S to R are inverses of each other. □

Proposition 5.4. — *Let R and S be two residues in a convex subcomplex X of Δ. Then* $\mathrm{proj}_R(S) \in X$.

Proof. — Let $T = \mathrm{proj}_R(S)$. By 5.1, T is a residue. If $u \in R$ and $v \in S$ are chambers, then there is a minimal gallery from u to v passing through chambers of T (by [27, 8.21]). Thus T is contained in every convex chamber subcomplex containing R and S (by 4.1(i)). □

Proposition 5.5. — *Let R and W be orthogonal polar regions, let Σ be an apartment containing chambers of both R and W, let β be the root of Σ whose arctic region is $W \cap \Sigma$ and let $T =$ $\mathrm{proj}_R(W)$. Then $\beta \cap R$ is a root of R and T is its polar region.*

Proof. — By 2.16, $\beta \cap R$ is a root of R. Let $k = \mathrm{dist}(x, \mathrm{proj}_W(x))$ for $x \in T$. By 5.1, k is independent of the choice of $x \in T$ and $k = \mathrm{dist}(\mathrm{proj}_R(y), y)$ for all $y \in \mathrm{proj}_W(R)$. Choose $x \in R \cap \beta$, let $y = \mathrm{proj}_W(x)$, let t be an element of G^+ inducing the reflection t_β on Σ and let u be a chamber of the border β' at minimal distance from x. By 2.11, there exists a minimal gallery in Σ from $t(y)$ to y passing through x and u which is reversed by t and $\mathrm{dist}(u, y) = m_\beta$. Since β cuts R, the element t maps R to itself. Thus

$t(x) \in t(R) = R$. Since R is convex, it follows that $u \in R$. Thus

$$\mathrm{dist}(\beta', x) = \mathrm{dist}(\beta' \cap R, x) = \mathrm{dist}(u, x) = m_\beta - \mathrm{dist}(x, W).$$

We have $\mathrm{dist}(x, W) \geqslant k$ with equality if and only if $x \in T$. Therefore $T \cap \Sigma$ is precisely the set of chambers in $R \cap \Sigma$ at maximal distance from $\beta' \cap R$. By 2.3, $T \cap \Sigma$ is therefore the arctic region of $\beta \cap R$. □

Proposition 5.6. — *Let R be the polar region of a root, let S be a residue opposite R, let Σ be an apartment containing chambers of both R and S and let W be a polar region such that $W \cap \Sigma$ is the arctic region of a root of Σ that cuts both R and S. Then the following hold:*

(i) *For each chamber $x \in \mathrm{proj}_S(W)$, there exists a minimal gallery from x to $\mathrm{proj}_R(x)$ that passes through $\mathrm{proj}_W(x)$ and $\mathrm{proj}_W(\mathrm{proj}_R(x))$.*

(ii) *If X is a convex subcomplex that contains R and $\mathrm{proj}_S(W)$, then X also contains $\mathrm{proj}_W(R)$ and $\mathrm{proj}_W(S)$.*

Proof. — Let $T = \mathrm{proj}_R(W)$, let $V = \mathrm{proj}_S(W)$, let $T_1 = \mathrm{proj}_W(R)$, let $V_1 = \mathrm{proj}_W(S)$, let α be the root of Σ whose arctic region is $R \cap \Sigma$ and let t be an element of G^+ that induces the reflection t_α on Σ. By 2.16, α cuts W and hence t maps W to itself. Since t interchanges R with S, it therefore interchanges T with V as well as T_1 with V_1. By [28, 29.16], we have $T_1 \cap \Sigma \subset \alpha$.

Choose $x \in V \cap \Sigma$. We have $t(x) = \mathrm{proj}_R(x)$ by 2.6(iv). Let $v = \mathrm{proj}_W(t(x))$. Thus $v \in T_1 \cap \Sigma \subset \alpha$. Choose $w \in \alpha'$ at minimal distance to v. Since $t(x) \in t(V) = T$, we have $t(x) = \mathrm{proj}_R(v)$ (by 5.1). By 2.11, there is a minimal gallery from $t(x)$ to x that passes through $v \in T_1$, w and $t(v) \in V_1$. Since every chamber of $R \cap \Sigma$ is at distance m_α from α' (by 2.3), it follows that

$$\mathrm{dist}(v, \alpha') = m_\alpha - \mathrm{dist}(T, T_1)$$

(where $\mathrm{dist}(T, T_1)$ is as in 5.2). Since x is an arbitrary chamber of $V \cap \Sigma$, we conclude that every chamber of $T_1 \cap \Sigma$ is at the same distance from α'. Hence $z = \mathrm{proj}_{T_1}(t(z))$ for all $z \in T_1$ by 2.7. It follows that

$$\mathrm{dist}(x, \mathrm{proj}_R(x)) = \mathrm{dist}(V, V_1) + \mathrm{dist}(V_1, T_1) + \mathrm{dist}(T_1, T).$$

We conclude that for each $x \in V$ (whether or not $x \in \Sigma$), there exists a minimal gallery from x to $\mathrm{proj}_R(x)$ that passes through both $\mathrm{proj}_W(x) \in V_1$ and

$$\mathrm{proj}_W(\mathrm{proj}_R(x)) \in T_1.$$

Thus (i) holds. For all $x \in V$ and $y \in R$, there exists a minimal gallery from x to y that passes through $\mathrm{proj}_R(x)$ (by [27, 8.21]). Thus every convex chamber subcomplex containing chambers of V and R contains chambers of V_1 and T_1. Hence (ii) holds. □

Proposition 5.7. — *Let R and S be opposite residues of Δ, let R_1 and R_2 be opposite residues of R and let $S_1 = \mathrm{proj}_S(R_2)$. Then S_1 is opposite R_1 in Δ.*

Proof. — Let u and v be chambers of R_1 and R_2, respectively, that are opposite in R and let $w = \mathrm{proj}_S(v)$. By [27, 9.11(i)], $v = \mathrm{proj}_R(w)$. By [27, 9.11(ii)], therefore, there is a minimal gallery from w to u passing through v that has the same type as a minimal gallery connecting opposite chambers of Δ. Hence w is opposite u in Δ, v is contained in the unique apartment Σ containing u and w and

$$S_1 \cap \Sigma = \mathrm{proj}_{S\cap\Sigma}\big(\mathrm{op}_{R\cap\Sigma}(R_1 \cap \Sigma)\big).$$

By [27, 5.14(ii)], $S_1 \cap \Sigma$ is opposite $R_1 \cap \Sigma$ in Σ. Hence R_1 is opposite S_1 in Δ (by [27, 9.8]). □

Proposition 5.8. — *Let R and S be polar regions that are opposite in Δ and let T be a polar region of R. Then there exists a polar region W of Δ that is orthogonal to both R and S and an apartment Σ containing chambers of S, T and W such that $\mathrm{proj}_W(R)$ and $\mathrm{proj}_W(S)$ are opposite polar regions of W and $\mathrm{proj}_R(W) = T$.*

Proof. — Let Σ be an apartment containing chambers of S and T, let α be the root of Σ whose polar region is R and let $-\alpha$ be its opposite in Σ (so S is the polar region of $-\alpha$). By [27, 8.13], there is a root β of Σ such that $\beta \cap R$ is a root of R whose polar region is T. Let W be the polar region of β. By 2.16, W is orthogonal to R, α cuts W and hence W is also orthogonal to S. By three applications of 5.5, $T = \mathrm{proj}_R(W)$, $\mathrm{proj}_W(R)$ is the polar region of $\alpha \cap W$ and $\mathrm{proj}_W(S)$ is the polar region of $-\alpha \cap W$. Since $\alpha \cap W$ and $-\alpha \cap W$ are opposite roots of W, $\mathrm{proj}_W(R)$ and $\mathrm{proj}_W(S)$ are opposite polar regions of W. □

Proposition 5.9. — *Let R, B and W be polar regions such that W is orthogonal to both R and B and $\mathrm{proj}_W(R)$ is opposite $\mathrm{proj}_W(B)$ in W. Then R is opposite B in Δ and*

$$\mathrm{proj}_R\big(\mathrm{proj}_B(W)\big) = \mathrm{proj}_R(W).$$

Proof. — Let $R_1 = \mathrm{proj}_W(R)$, let $B_1 = \mathrm{proj}_W(B)$, let $u \in R_1$, let $v = \mathrm{proj}_R(u)$, let z be a chamber of B_1 opposite u in W, let $w = \mathrm{proj}_B(z)$, let Σ be an apartment containing v and z and let Σ' be an apartment containing w and u. We have $\mathrm{proj}_W(v) = u$ and $z = \mathrm{proj}_W(w)$ by 5.1. There thus exist minimal galleries from v to z passing through u and from u to w passing through z. Hence Σ and Σ' both contain z and u. Since z and u are opposite in W, it follows that $W \cap \Sigma = W \cap \Sigma'$ (by [27, 9.2]). Let α be the root of Σ whose polar region is R and let β be the root of Σ' whose polar region is B. By two applications of 5.5, $\alpha \cap W$ is a root of W whose polar region is R_1 and $\beta \cap W$ is a root of W whose polar region is B_1. Since R_1 and B_1 are opposite in W, the roots $\alpha \cap W$ and $\beta \cap W$ are opposite roots of $W \cap \Sigma$. Let P be a panel in the wall of $\alpha \cap W$. Then P is also in the wall of $\beta \cap W$. Hence there exists an element g in G^+ mapping Σ to Σ', P to itself and α to β. It follows that $R^g = B$ and that α and β have the same depth.

Assume now that the panel P is chosen to be of minimal distance to u among all the panels in the wall of $\alpha \cap W$ and let $x = \mathrm{proj}_P(v)$. By 2.11, we can choose a minimal gallery γ of length m_α from v to x passing through u. Let $\rho = \mathrm{retr}_{\Sigma,v}$ and let $S = \rho(B)$. Thus S is a residue of Σ of the same type as R. The concatenation γ_0 of γ with the reverse of γ^g is a gallery from v to $v^g \in B$ of length $2m_\alpha + 1$. By 2.11, γ_0 has the same type as a minimal gallery from v to its projection to the residue of Σ opposite $R \cap \Sigma$. By [27, 8.17], $\rho(\gamma_0)$ is a gallery of the same type from v to $\rho(v^g)$. Furthermore, $\rho(v^g) \in \rho(R^g) = \rho(B) = S$, and S has the same type as a residue opposite $R \cap \Sigma$ by 3.2. We conclude that S is the arctic region of Σ opposite $R \cap \Sigma$ and $\rho(v^g) = \mathrm{proj}_S(v)$. Let y be the unique chamber opposite v in Σ. Then $y \in S$ and by [27, 8.17] again, the chambers in $\rho^{-1}(y) \in B$ are all opposite v in Δ. Since B has the same type as S, it follows that B is opposite R in Δ.

We can now replace Σ by an apartment containing γ_0, so that Σ contains chambers of R, B and W and a unique root whose polar region is R, which we continue to call α. Let t be an element of G^+ that induces the reflection t_α on Σ. Then t maps W to itself and interchanges $R \cap \Sigma$ and $B \cap \Sigma$. Hence t interchanges $\mathrm{proj}_R(W)$ with $\mathrm{proj}_B(W)$. By 2.6(iv), $\mathrm{proj}_R(y) = t(y)$ for every chamber $y \in B \cap \Sigma$. Thus proj_R maps $\mathrm{proj}_{B\cap\Sigma}(W)$ to $\mathrm{proj}_{R\cap\Sigma}(W)$. By 5.1, it follows that proj_R maps $\mathrm{proj}_B(W)$ to $\mathrm{proj}_R(W)$. □

Proposition 5.10. — *Let α and β be distinct roots of the same apartment Σ whose gonality (as defined in 2.14) is 4 and let R and S be their polar regions. Then $R \cap S$ contains chambers if and only if the angle between R and S is $\pi/4$.*

Proof. — Suppose that $R \cap S$ contains chambers. Then R and S have different types. Since the gonality of the pair α, β is 4, the angle between R and S must be either $\pi/4$ or $3\pi/4$. By 2.17, the angle is $\pi/4$.

Suppose, conversely, that the angle between R and S is $\pi/4$. Let T be a rank 2 residue cut by α and β (which exists by 2.14). By 3.6, there is a unique two-element subset J of I such that the elements of J are joined by an edge with label 4 in the Coxeter diagram Π. This set J must be the type of T. By [28, 29.52], α and β are not in the same G^+ orbit. Thus R is not of the same type as S. For each chamber x in $R \cap \Sigma$, let S_x be the unique polar region containing x that has the same type as S and let β_x be the root of Σ whose polar region is S_x.

Suppose that the gonality of the pair α, β_x is 2 —and thus α is orthogonal to β_x— for each $x \in R \cap \Sigma$. In particular, $\beta_x \neq \beta$ and hence $S_x \neq S$ for each $x \in R \cap \Sigma$. Let t be an element of G^+ mapping Σ to itself that induces the reflection t_α on Σ. Thus t maps S_x to itself for each $x \in R \cap \Sigma$ (by 2.16). Hence for all $x \in R \cap \Sigma$, every minimal gallery from x to $t(x)$ is contained in S_x. By 2.11, there exists $x \in R \cap \Sigma$ such that a minimal gallery from x to $t(x)$ passes through chambers of S. This implies, however, that $S = S_x$ since S and S_x have the same type.

We conclude that for some $x \in R \cap \Sigma$, the gonality of the pair α, β_x is 4. As we have seen above, this implies that the angle between α and β_x is $\pi/4$. Let T_x be a rank 2 residue cut by α and β_x. Then T_x is also a J-residue. It follows that we can choose an element $g \in G^+$ stabilizing Σ, mapping T to T_x, mapping a panel of T in the wall of α to a panel in T_x in the wall of α and a panel of T in the wall of β to a panel of T_x in the wall of β_x. Hence g maps α to itself and β to β_x. Hence $R^g = R$ and $S^g = S_x$. Since $R \cap S_x$ contains chambers, we conclude that $R \cap S$ does too. □

Proposition 5.11. — *Let R and S be polar regions at an angle of $3\pi/4$, let Σ be an apartment containing chambers of both R and S, let α and β be the roots of Σ whose polar regions are R and S, let the roots α_i for $i \in [1,4]$ be as in 3.15 and let R_i be the polar region of α_i for all $i \in [1,4]$, so $R_1 = R$, $R_4 = S$ and the angle between R_i and R_j is $|i-j| \cdot \pi/4$ for all $i,j \in [1,4]$. Then every convex subcomplex containing R and S also contains R_2 and R_3.*

Proof. — Let X be a convex subcomplex containing R and S, let u be an arbitrary chamber of $R \cap \Sigma$, let v be an arbitrary chamber of $S \cap \Sigma$, let t be an element of G^+ mapping Σ to itself that induces the reflection t_α on Σ, let $z = \mathrm{proj}_R(t(v))$ and let T be a rank 2 residue cut by α and β (so, in particular, $T^t = T$). By 5.10, $R \cap R_2$ contains chambers. By considering the action of t on T, we see that t interchanges R_2 and $R_4 = S$. Hence $t(v) \in R_2$, so $z \in R \cap R_2$ by [27, 3.25]. By 2.11, there exists a minimal gallery from u to v that passes through $t(v)$ and z. In particular, every chamber of $R \cap \Sigma$ adjacent to z is farther from v than z is. Therefore $z = \mathrm{proj}_R(v)$. We conclude that $\mathrm{proj}_R(S \cap \Sigma)$ is contained in the residue $R \cap R_2 \cap \Sigma$ of Σ. Since proj_R is a homomorphism, it follows that $\mathrm{proj}_R(S)$ is contained $R \cap R_2$. By 4.1(i) and 5.4, therefore, $R_2 \in X$. By symmetry, we have $R_3 \in X$ as well. □

Proposition 5.12. — *Let R, S and T be polar regions such that R is orthogonal to T and the angle between R and S is $\pi/4$ as is the angle between S and T. Then the following hold:*

(i) *$S \cap T = \mathrm{proj}_T(R)$ and $R \cap S = \mathrm{proj}_R(T)$.*

(ii) *$R \cap S$ is opposite $S \cap T$ in S.*

Proof. — Let Σ be an apartment that contains chambers in both $R \cap S$ and $S \cap T$. It suffices to prove the claims under the assumption that $\Delta = \Sigma$ (and $G^+ = W$). Let Φ be as in 2.15. If $\widehat{\alpha}$ and $\widehat{\beta}$ are roots of Φ at an angle of $\pi/2$ to each other, then an arbitrary root in Φ at an angle of $\pi/4$ to both $\widehat{\alpha}$ and $\widehat{\beta}$ must lie in the plane $\langle \widehat{\alpha}, \widehat{\beta} \rangle$ and hence be proportional to $\widehat{\alpha} + \widehat{\beta}$. It follows that S is the only arctic region that is at an angle of $\pi/4$ to both R and T. Thus if some element of G^+ stabilizes R and T, then it stabilizes S as well. This implies that if a root that cuts R and T, then it also cuts S.

By 5.1 and 5.3, the residues $\mathrm{proj}_R(T)$ and $\mathrm{proj}_T(R)$ are cut by the same roots. Therefore if a root cuts $\mathrm{proj}_R(T)$, it must also cut S. By [27, 3.25], we have $\mathrm{proj}_R(T \cap S) \subset R \cap S$. Suppose that $\mathrm{proj}_R(T)$ is not contained in $R \cap S$. Since $\mathrm{proj}_R(T)$ is connected, it follows that we can choose adjacent chambers u and v in $\mathrm{proj}_R(T)$ such that $u \in S$ but $v \notin S$, but then the unique root containing v but not u cuts $\mathrm{proj}_R(T)$ but it does not (by [28, 29.28]) cut S. We conclude that $\mathrm{proj}_R(T) \subset R \cap S$.

Now let α be the root whose arctic region is R, let $t = t_\alpha$, let $W = S^t$, let β be the root whose arctic region is S and let T_1 be a rank 2 residue cut by both α and β. Since the angle between α and β is $\pi/4$, the gonality of the pair α, β is 4. Considering the action of t on T_1, we see that β^t is a root orthogonal to β, so W is orthogonal to S. Since $T^t = T$ (by 2.16), the polar region W is at an angle of $\pi/4$ to T. Therefore every root that cuts W and S also cuts T, as we showed above for R, S and T in place of S, T and W.

Let ξ be a root that cuts $R \cap S$. Then the reflection t_ξ stabilizes R and hence $[t, t_\xi] = 1$. Since t_ξ also stabilizes S, it thus stabilizes W as well. Hence ξ cuts W. Since ξ also cuts S, it follows that ξ cuts T (as observed in the previous paragraph). By [27, 3.25] and [28, 29.16], it follows that ξ cuts $S \cap T$. By symmetry, we conclude that a root cuts $R \cap S$ if and only if it cuts $S \cap T$. By 5.3, therefore, $\mathrm{proj}_{R \cap S}(S \cap T) = R \cap S$. Therefore

$R \cap S \subset \mathrm{proj}_R(T)$. Thus $R \cap S = \mathrm{proj}_R(T)$ (since we showed the other inclusion above). By symmetry, $S \cap T = \mathrm{proj}_T(R)$. Hence (i) holds.

Now let β' be the root whose arctic region is W, let $s = t_{\beta'}$ (so $s = t_\beta^t$) and let $V = W^s$. Thus V is opposite W. Considering the action of s on the roots cutting the rank 2 residue T_1, we see that s stabilizes S (so V is orthogonal to S) and interchanges R and T. By 5.5, $\mathrm{proj}_S(V)$ and $\mathrm{proj}_S(W)$ are the polar regions of the opposite roots $\beta' \cap S$ and $-\beta' \cap S$ of S and hence are opposite residues of S. By (i), we have $\mathrm{proj}_S(V) = R \cap S$ and $\mathrm{proj}_S(W) = S \cap T$. Hence (ii) holds. □

Proposition 5.13. — *Let R, S and W be polar regions such that R and S are opposite and W is at an angle of $\pi/4$ to S. Then W is at an angle of $3\pi/4$ to R.*

Proof. — By 5.10, $S \cap W$ contains chambers. Hence we can choose an apartment Σ containing chambers of $S \cap W$ and R. Let Φ be as in 2.15. The elements of Φ that correspond to the roots of Σ whose polar regions are R, S and W lie in a single plane and hence the sum of the angle between W and R and the angle between W and S is π. □

Proposition 5.14. — *Let R_i for $i \in [1, 4]$ be polar regions such that the angle between R_1 and R_4 is $3\pi/4$ and the angle between R_{i-1} and R_i is $\pi/4$ for all $i \in [2, 4]$. Then R_1 is orthogonal to R_3.*

Proof. — Let Σ be an apartment containing chambers in $R_1 \cap R_2$ and $R_2 \cap R_3$ and let Φ be as 2.15. As in the proof of 5.13, it suffices to observe that the elements of Φ that correspond to the roots of Σ whose polar regions are R_1, R_2, R_3 and R_4 must all lie in a single plane. □

Proposition 5.15. — *Let X be a convex chamber subcomplex of Δ. If to every polar region of a long root in X there exists an opposite residue in X, then also to every polar region of a short root in X there exists an opposite residue in X.*

Proof. — Suppose first that Δ is a generalized hexagon. In this case, the polar regions are the vertices, so we can talk about "long" and "short" vertices. Let y be a short vertex in X. Since X is a chamber subcomplex, there exists a vertex x adjacent to y (and hence long) such that the edge $\{x, y\}$ is in X. Since x is long, we can choose a vertex $x' \in X$ opposite x. Again because X is a chamber subcomplex, there exists y' adjacent to x' such that the edge $\{x', y'\}$ is in X (and hence $y' \in X$). Suppose that y' is not opposite y. Then there is a unique path $(x_0, \ldots, x_6)$ of length 6 such that $x_0 = x$, $x_1 = y$, $x_5 = y'$ and $x_6 = x'$. Since X is convex, we have

$\{x_{i-1}, x_i\} \in X$ for all $i \in [1,6]$. By the same argument applied to the pair y, x_2 in place of y, x, we can assume that there exists a path $(u_0, \ldots, u_6)$ of length 6 such that $u_0 = x_2$, $u_1 = x_1 = y$ and $\{u_{i-1}, u_i\} \in X$ for all $i \in [1,6]$. Let Σ be the unique apartment that contains the path

$$(u_3, u_2, y, x_2, x_3, \ldots, x_6)$$

of length 7. Since X is convex, every edge on Σ is in X. Thus the unique vertex of Σ opposite y is in X.

Now suppose that Δ is not a generalized hexagon. Let $R \in X$ be the polar region of a short root, let Σ be an apartment containing chambers of R and let α be the root of Σ whose polar region is R. Since α is short and Δ is not a generalized hexagon, there exists (by 3.14) a long root β of Σ at an angle of $\pi/4$ to α. Let W be the polar region of β. Since β is long, there exists a polar region $S \in X$ opposite W. By 5.13, the angle between R and S is $3\pi/4$. Let R_2 and R_3 be the polar regions in X obtained by applying 5.11 to R and S. By 3.14, R_2 is the polar region of a long root, so there exists a polar region $B \in X$ opposite R_2. The angle between R_3 and B is $3\pi/4$ (again by 5.13). Now let R_2' and R_3' be the polar regions in X obtained by applying 5.11 with R_3 and B in place of R and S. Then R and R_3' are both orthogonal to R_3. Since the angle between R_2 and R_3' is $3\pi/4$ (by 5.13 again), R_2' is orthogonal R_2 by 5.14. By 5.12(i) applied to R, R_2 and R_3, we have $\mathrm{proj}_{R_3}(R) = R_2 \cap R_3$. By 5.12(ii) applied to R_2, R_3 and R_2', the residues $R_2 \cap R_3$ and $R_2' \cap R_3$ are opposite in R_3. By 5.12(i) applied to R_3, R_2' and R_3', we have $\mathrm{proj}_{R_3}(R_3') = R_2' \cap R_3$. We conclude that $\mathrm{proj}_{R_3}(R_3')$ is opposite $\mathrm{proj}_{R_3}(R)$ in R_3. By 5.9, it follows that R_3' is opposite R in Δ. $\square$

6. The center conjecture

All of the following observations are standard:

Notation 6.1. — Let Δ be a thick spherical building. By [5, II.10A.5], there is a canonical geometrical realization of Δ as a complete metric space $(|\Delta|, d)$ with respect to which the subspaces corresponding to apartments (which we also call apartments) are convex and isometric to unit n-spheres for n one less than the rank of Δ, and roots correspond to hemispheres. Furthermore, the group $\mathrm{Aut}(\Delta)$ acts on $(|\Delta|, d)$ and $(|\Delta|, d)$ is CAT(1). The subspace $|u|$ of $|\Delta|$ corresponding to a chamber u of Δ is isometric to an n-dimensional simplex whose dimension is the topological dimension of $(|\Delta|, d)$, and the residues of Δ containing the chamber v correspond the faces of this simplex. Every point of $|\Delta|$ is contained in a unique face (of one of these simplices) of minimal dimension. By [5, II.1.4(i)], two points of $|\Delta|$ at distance less that π are joined by a unique geodesic segment. A subset W of $|\Delta|$ is called convex if for every two points in W at distance less than π from one another, the unique geodesic joining them lies in W. The subspace of $|\Delta|$ corresponding to a convex subcomplex of Δ is convex. This is because roots (i.e. hemispheres) of $|\Delta|$ are convex, every two points of $|\Delta|$ are contained in an apartment (because every two chambers of Δ are contained in an apartment) and every convex set of chambers in an apartment of Δ is an intersection of roots (by [28, 29.20]).

Definition 6.2. — *Let Δ and $(|\Delta|, d)$ be as in 6.1, let $G = \mathrm{Aut}(\Delta)$, let R be a polar region, let Σ be an apartment containing chambers of R and let α be the root of Σ whose polar region is R. Suppose that Δ is irreducible and let F be the intersection of $|u|$ for all chambers $u \in R$. By 2.9, either F is a point or $\Pi = \mathsf{A}_{n+1}$ and F is a 1-simplex. Let $p = F$ in the first case and let p be the midpoint of F in the second. This point p is the unique fixed point of $\mathrm{Aut}(\Sigma)$ in the hemisphere $|\alpha|$ (even in the case that $\Pi = \mathsf{A}_{n+1}$, when the building Δ itself might have only color-preserving automorphisms). It follows that p is the unique point of $|\alpha|$ such that $|\alpha|$ is the intersection of the sphere $|\Sigma|$ with the ball $B_{\pi/2}(p)$ of radius $\pi/2$ in $|\Delta|$ centered at p. Thus $u \in |\Sigma|$ is fixed by U_α if and only if $u \in B_{\pi/2}(p)$. We call p the pole of the polar region R. Now suppose that α is a long root and let $A = Z(U_\alpha)$. By 3.16, A is independent of the choice of Σ. Since every point of $|\Delta|$ is contained in some apartment containing p (which necessarily contains chambers of R), we conclude that the fixed point set of A in $|\Delta|$ is precisely $B_{\pi/2}(p)$.*

Remark 6.3. — Let Δ be irreducible and let p and q be the poles of two polar regions R and S as defined in 6.2. By 2.15, $d(p, q)$ is precisely the angle between R and S as defined in 4.2. If, in addition, R and S have the same type and the roots α and β in 4.2 are long, then by 3.8 and 3.11, $[Z(U_\alpha), Z(U_\beta)] = 1$ if and only if $d(p, q) \leqslant \pi/2$.

We apply [2] in the proof of the next result.

Theorem 6.4. — *Let Δ be a thick spherical building and let X be a convex subcomplex of Δ. Then one of the following holds:*

(i) *To every polar region R of a long root contained in X there exists a polar region in X that is opposite R in Δ.*

(ii) *There is a residue C in X such that every automorphism of Δ (color-preserving or not) that maps X to itself also maps C to itself.*

Proof. — By [13, 1.2 and 3.4], we can assume that Δ is irreducible and of rank at least 3. Let $G = \mathrm{Aut}(\Delta)$ and let $(|\Delta|, d)$ be as in 6.1. For each point $x \in |\Delta|$, let R_x be the residue of Δ corresponding to the unique minimal face of $|\Delta|$ containing x. Thus $G_x = G_{R_x}$ for each $x \in |\Delta|$. Let Q be the set of subsets J of I that are the types of polar regions of Δ. Since Δ is irreducible, we have $|Q| \leqslant 2$. For each $J \in Q$, let Ω_J denote the set of poles (as defined in 6.2) of all J-residues in X for which there is no opposite polar region in X. Suppose that (i) does not hold. Thus there exists $J \in Q$ such that Ω_J is non-empty. By 4.8 and 6.3, there exists $p \in \Omega_J$ such that Ω_J is contained in the ball $B_{\pi/2}(p)$. By 6.2, this ball is the fixed point set of a nontrivial subgroup of G. Thus also the convex closure of Ω_J is contained in $B_{\pi/2}(p)$. By [2, Prop. 1.4] applied to this closure, there is a point x in $|\Delta|$ such that $G_{\Omega_J} \subset G_x$. It remains only to consider the case that G_X is not contained in G_{Ω_J}. Then $|Q| = 2$ and G_X contains elements interchanging Ω_J and $\Omega_{J'}$, where J' is the other element of Q. Hence also $\Omega_{J'}$ is non-empty, so by 4.8 and [2, Prop. 1.4] again, there is a point x' in $|\Delta|$ such that $G_{\Omega_{J'}} \subset G_{x'}$. The subgroup G_X fixes the subset $\{x, x'\}$ and the elements of G_X that are not color-preserving interchange x and x'. By 2.9, the Coxeter diagram Π of Δ must be F_4 (since in every other case with $|I| \geqslant 3$, the group G acts trivially on Q). By [25, 2.39], therefore, opposite residues of Δ have the same type. Suppose that x and x' are opposite. Then the type of R_x must be fixed by the nontrivial automorphism τ of Π. Hence we can choose $u \in |\Delta|$ such that R_u contains R_x properly and the type of R_u is not fixed by τ. Since $G_x \subset G_u$, there is a unique point u' such that $\{u, u'\}$ is a G_X-orbit. Since R_u and $R_{u'}$ have different types, u and u' are not opposite, so we can let w be the midpoint of the unique geodesic segment joining u and u'. If x and x' are not opposite, simply let w be the midpoint of the unique geodesic segment joining x and x'. Then (ii) holds with $C = R_w$ (whether or not x and x' are opposite). □

The Center Conjecture for convex *chamber* subcomplexes in thick irreducible buildings follows immediately from 6.4 and [15, Thm. 2]. We conclude this paper with a different proof based on the notion of a receding polar region (in place of [15]):

Definition 6.5. — *A residue T of Δ will be called a receding polar region if there exist residues*

$$R = R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_k$$

such that R is a polar region of Δ, $T = R_k$ and R_i is a polar region of R_{i-1} for all $i \in [1, k]$.

Proposition 6.6. — *Suppose that Δ is as in 3.6 and that X is a convex subcomplex of Δ such that to every polar region in X, there exists an opposite residue in X. Then to every receding polar region in X, there exists an opposite residue in X.*

Proof. — Let T and $R = R_0, \dots, R_k$ be as in 6.5 and suppose that $T \in X$. Our goal is to show that there is a residue in X opposite T. By 5.4, 5.7 and induction, it suffices to show that there is a residue in X that is opposite T in R under the assumption that $k = 1$.

Let $S \in X$ be a residue opposite R in Δ, let W and Σ be as in 5.8 and let $V_1 = \mathrm{proj}_W(S)$ and $T_1 = \mathrm{proj}_W(R)$, so that V_1 and T_1 are opposite polar regions of W and $\mathrm{proj}_R(W) = T$. By 5.6(ii), $V_1 \in X$, and thus also $W \in X$. Since $W \in X$, there exists a polar region $A \in X$ that is opposite W in Δ. We now apply 5.8 a second time, with W in place of R, A in place of S and V_1 in place of T to conclude that there exist an apartment Σ' and a polar region B that is orthogonal to both W and A such that Σ' contains chambers of A, B and V_1, $\mathrm{proj}_B(A)$ and $\mathrm{proj}_B(W)$ —which we denote by C, respectively, D— are opposite polar regions of B and $\mathrm{proj}_W(B) = V_1$. By 5.6(ii) again, $C \in X$. By 5.9, $\mathrm{proj}_R(D) = T$ and B is opposite R in Δ. Since B and R are opposite, the restriction of proj_R to B is an isomorphism from B to R (by [27, 9.11(i)]). We conclude that $\mathrm{proj}_R(C)$ is a residue of R that is contained in X (because R and C are in X) and opposite T in R (because C and D are opposite in B). □

The Center Conjecture for chamber subcomplexes in thick irreducible buildings follows now directly from 5.15, 6.4 and 6.6. Here is the proof:

Theorem 6.7. — *Let Δ be a thick building and let X be a convex chamber subcomplex of Δ. Then one of the following holds:*

(i) *To every residue R in X there exists a residue in X that is opposite R in Δ.*
(ii) *There is a residue R in X such that every automorphism of Δ (color-preserving or not) that maps X to itself also maps R to itself.*

Proof. — By [13, 1.2 and 3.4], we can assume that Δ is irreducible and of rank at least 3. Hence Δ satisfies 3.6. Suppose that (ii) does not hold. By 6.4, to every polar region of a long root in X there is an opposite residue in X. By 5.15, it follows that to *every* polar region in X there is an opposite residue in X. By 6.6, therefore, to every receding polar region in X there is an opposite residue in X. Since chambers are receding polar regions, we conclude that to every chamber in X there is an opposite chamber in X. Hence (i) holds by 4.1(i). □

Remark 6.8. — The proof of the Center Conjecture in [16] for buildings of type E_8 begins on page 18 with a convex subcomplex X that does not satisfy 6.4(ii) and arrives at the conclusion 6.4(i) in [16, 5.17] on page 41; it takes only two more pages to arrive at the conclusion (in [16, 5.24]) that residues of *arbitrary* type in X have opposites in X. This indicates that at least in principle, it should be possible to give a relatively short proof of the Center Conjecture for arbitrary convex subcomplexes (not only convex *chamber* subcomplexes) of thick buildings based on 6.4 by adapting just a small part of the arguments in [12] and [16].

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