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EXOTIC DEFORMATIONS OF CALABI-YAU MANIFOLDS

by Paolo DE BARTOLOMEIS & Adriano TOMASSINI (*)

Abstract. — We introduce *Quantum Inner State manifolds* (*QIS manifolds*) as (compact) $2n$-dimensional symplectic manifolds (M, κ) endowed with a κ-tamed almost complex structure J and with a nowhere vanishing and normalized section ϵ of the bundle $\Lambda_J^{n,0}(M)$ satisfying the condition $\overline{\partial}_J \epsilon = 0$.

We study the moduli space $\mathfrak{M}$ of QIS deformations of a given Calabi-Yau manifold, computing its tangent space and showing that $\mathfrak{M}$ is non obstructed. Finally, we present several examples of QIS manifolds.

Résumé. — On considère la classe des *variétés QIS* (*Quantum Inner State variétés*), à savoir la classe des variétés symplectiques, compactes et de dimension $2n$, munies d'une structure presque complexe J modérée par k et d'une section ϵ du fibré $\Lambda_J^{n,0}(M)$, qui ne s'annule nulle part, normalisée et satisfaisant la condition $\overline{\partial}_J \epsilon = 0$.

Le but du papier est d'étudier l'espace $\mathfrak{M}$ des modules des déformations QIS d'une variété de Calabi-Yau. À ce propos, on calcule l'espace tangent de $\mathfrak{M}$ et on montre que $\mathfrak{M}$ n'a pas d'obstructions. Plusieurs exemples de variétés QIS sont aussi exhibés.

Introduction

Nowadays, Kähler Geometry can be seen as a Holomorphic Calibrated Geometry over a Symplectic structure and thus it represents a perfect synthesis of the Symplectic and the Holomorphic worlds and a sort of chemical analysis of symplectic and holomorphic contribution can be successfully performed in order to better understand the role of the different components of the theory; *e.g.* the basic Kähler identities $[\partial, \Lambda] = i\bar{\partial}^*$, and

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$[\bar\partial, \Lambda] = -i\partial^*$ are nothing but the holomorphic splitting of the purely symplectic relation $[d, \Lambda] = d^\bigstar$, where $d^\bigstar$ denotes the symplectic codifferential of d acting on r-forms as $d^\bigstar = (-1)^{r+1}\bigstar \circ d \circ \bigstar$, and $\bigstar$ is the symplectic Hodge operator (see [18], [3]).

This suggests the possibility of various generalizations, relaxing some of the fundamental conditions.

This turns out to be especially effective in the case of Calabi-Yau manifolds, a very special class of Kähler manifolds, whose importance is well known to go far beyond Mathematics.

In [12], Hitchin introduces and studies *Generalized Calabi-Yau manifolds*. A *generalized Calabi-Yau structure* on a $2m$-dimensional manifold M is the datum of a closed complex form φ of mixed degree which is a complex pure spinor for the orthogonal vector bundle $TM \oplus T^*M$ endowed with the natural pairing $\langle , \rangle$ and such that $\langle \varphi, \overline{\varphi}\rangle \neq 0$. In particular, a generalized Calabi-Yau structure gives rise to a generalized complex structure (see [12]). The main examples of generalized Calabi-Yau manifolds are represented by *Calabi-Yau manifolds* and *symplectic manifolds*.

Among various other possible generalizations, we consider *Quantum Inner State manifolds*, (shortly *QIS manifolds*), namely (compact) symplectic $2n$-dimensional manifolds (M, κ), endowed with a κ-tamed almost complex structure J and a nowhere vanishing section ϵ of $\Lambda^{n,0}_J(M)$ satisfying

$$(-1)^{\frac{n(n+1)}{2}} i^n \epsilon \wedge \overline{\epsilon} \text{ is a volume form} \tag{0.1}$$

$$\overline{\partial}_J \epsilon = 0 \tag{0.2}$$

In particular, every Calabi-Yau manifold is a QIS manifold.

In [2] it is shown how QIS's provide preferred setting to develop symplectic topology via the theory of Lagrangian submanifolds and their Maslov classes (see also [4]); moreover, (again in [2]), it is proven that to any compact symplectic manifold (M, κ) can be attached a non compact QIS manifold whose Lagrangian submanifolds reflect the Lagrangian submanifold of M. The aim of this paper is the study of deformations of Calabi-Yau structures in the QIS directions. The celebrated Theorem of Tian-Todorov (see [16], [17]) states that, if M is a compact (non-flat) Calabi-Yau manifold of dimension $n \geqslant 3$ and $\pi : X \to S$, $0 \in S$, $\pi^{-1}(0) = M$, is the Kuranishy family of M, then S is a non-singular complex analytic space such that

$$\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} H^1(M, \Omega^{n-1}).$$

In other words, the moduli space of Calabi-Yau manifolds is unobstructed.

In the present paper, we show that QIS manifolds are somehow the perfect setting for a generalized deformation theory of Calabi-Yau manifolds, in the sense that we provide a complete description of the moduli space of QIS deformations of a given Calabi-Yau manifold and, via the implicit function theorem we are able to prove that the theory is completely unobstructed.

The paper is organized as follows. In section 1, we start by giving and fixing some preliminary notation on linear complex and symplectic structures. In section 2, we introduce QIS manifolds, finding a necessary and sufficient cohomological condition in order that, on a compact symplectic manifold (M, κ), with vanishing first Chern class, a κ-tamed almost complex structure J gives rise to a QIS structure on M. Section 3 contains the main results of the paper. We define the *moduli space* of *quantum inner state structures* on a compact Calabi-Yau manifold (M, κ, J, ϵ) as

$$\mathfrak{M} = \mathfrak{B}/\mathrm{Diff}\,(M),$$

where

$$\mathfrak{F} = \left\{(K, \eta) \in \mathfrak{T}_\kappa(M) \times \Lambda^n_*(M) \;\mid\; \eta \in \Lambda^{n,0}(M)\right\}$$

denotes the space of pairs(K, η) such that, K is a κ-tamed almost complex structure on M, η is a complex $(n, 0)$-form with respect to K, satisfying (0.1),

$$\mathfrak{B} = \left\{(K, \eta) \in \mathfrak{F} \;\mid\; \overline{\partial}_K \eta = 0\right\}$$

and the group $\mathrm{Diff}\,(M)$ acts on $\mathfrak{B}$ in a natural way.

We prove the following (see Theorem 3.5)

Theorem 0.1. — *Let (M, J, κ, ϵ) be an n-dimensional compact Calabi-Yau manifold. Denote by*

$$\mathfrak{M} := \mathfrak{B}/\mathrm{Diff}\,(M)$$

the moduli space of quantum inner state structures on M. Then the tangent space $T_{[(J,\epsilon)]}\mathfrak{M}$ to $\mathfrak{M}$ at $[(J, \epsilon)]$ satisfies

$$T_{[(J,\epsilon)]}\mathfrak{M} \subset \left(\frac{\mathrm{Ker}\,\eth_J \overline{\partial}_J}{\mathrm{Im}\,\overline{\partial}_J}\right)[1, 1]\,.$$

The operator $\eth_J$ is the Tian-Todorov operator which appears in the holomorphic deformations of Calabi-Yau manifolds. The following (see Theorem 3.6) shows that the moduli space $\mathfrak{M}$ is not obstructed.

Theorem 0.2. — *Let (M, J, κ, ϵ) be an n-dimensional compact Calabi-Yau manifold. Assume that $H^1(M, \mathbb{Z}) = 0$. Then the moduli space $\mathfrak{M}$ of quantum inner state deformations of (M, J, κ, ϵ) is unobstructed.*

Furthermore, the tangent space to $\mathfrak{M}$ at $[(J, \epsilon)]$ decomposes as

$$T_{[(J,\epsilon)]}\mathfrak{M} = H \oplus a(E) \oplus s(E)\,,$$

where H corresponds to the non-obstructed space of holomorphic deformations, $a(E) = \overline{\partial}^*_J \Lambda^{0,3}(M)$ and $s(E)$ encodes the space of calibrated deformations.

In other words, $\mathfrak{M}$ is a germ of smooth manifold with tangent space given by $T_{[(J,\epsilon)]}\mathfrak{M} = H \oplus a(E) \oplus s(E)$.

As pointed out by the referee, it is natural to ask whether the quotient topology on the moduli space $\mathfrak{M}$ is Hausdorff. Indeed, Fujiki and Schumacher in [8, Corollary 3.4] proved that the moduli spaces of Kähler and almost Kähler structures on a compact symplectic manifold with the quotient topology are Hausdorff. Similar techniques could be used to show that $\mathfrak{M}$ is Hausdorff too. We leave this problem for further investigations.

Finally, in the last section, several examples of QIS manifolds are presented and discussed.

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1. Complex structures and symplectic forms

Let V be a $2n$-dimensional real vector space. A *linear complex structure* on V is the datum of $J \in \mathrm{Aut}(V)$ satisfying $J^2 = -\mathrm{id}$. Let J be a linear complex structure on V. A positive definite scalar product g on V is said to be *J-Hermitian* if $g(Ju, Jv) = g(u, v)$, for any u, $v \in V$.

A *linear symplectic structure* on V is the datum of $\kappa \in \wedge^2 V^*$ such that $\kappa^n \neq 0$. Let κ be a symplectic structure on V. A linear complex structure J is said to be *κ-tamed* if

$$\kappa(v, Jv) > 0 \text{ for every } v \in V,\ v \neq 0$$

and J is said to be *κ-calibrated* if

$$g_J(\cdot,\cdot) = \kappa(\cdot, J\cdot).$$

defines a J-Hermitian positive definite scalar product on V. Notice that, any κ-tamed linear complex structure gives rise to a J-Hermitian metric g_J on V defined by

$$g_J(\cdot,\cdot) := \frac{1}{2}\left[\kappa(\cdot, J\cdot) - \kappa(J\cdot, \cdot)\right].$$

Denote by $\mathfrak{T}_\kappa(V)$, $\mathfrak{C}_\kappa(V)$ the set of κ-tamed, κ-calibrated linear complex structures on the symplectic vector space (V, κ) respectively. Set

$$\mathcal{T}(n) := \{X \in M_{2n,2n}(\mathbb{R}) \mid XJ_n + J_nX = 0\},$$

$$\mathcal{C}(n) := \{X \in M_{2n,2n}(\mathbb{R}) \mid X = {}^tX,\ \ XJ_n + J_nX = 0\}\,,$$

where $M_{2n,2n}(\mathbb{R})$ denotes the set of real matrices of order $2n$ and J_n is the standard complex structure on $\mathbb{R}^{2n}$. Then, $\mathfrak{T}_\kappa(V)$ is homeomorphic to a $2n^2$-dimensional cell and $\mathfrak{C}_\omega(V)$ is homeomorphic to an $(n^2{+}n)$-dimensional cell. On $\mathbb{R}^{2n}$ consider the standard symplectic structure κ_n. Then, every κ_n-tamed linear complex structure J can be uniquely represented as

$$J = (I+L)J_n(I+L)^{-1}$$

with

$$LJ_n + J_nL = 0\ ,\ ||L|| < 1$$

and every J κ_n-calibrated can be uniquely represented as

$$J = (I+L)J_n(I+L)^{-1}$$

with

$$LJ_n + J_nL = 0\ ,\ ||L|| < 1\ ,\ L = {}^tL$$

(see *e.g.* [13]).

Let (M,κ) be a symplectic manifold. We recall the following

Definition 1.1. — *An almost complex structure J is said to be tamed by κ if $\kappa_x(u,Ju) > 0$, for any $x \in M$ and any non zero tangent vector $u \in T_xM$. J is called calibrated by κ if in addition $\kappa_x(Ju,Jv) = \kappa_x(u,v)$, for any pair of tangent vectors u and v.*

Therefore, if J is a κ-tamed almost complex structure on M, then

$$g_J(\cdot,\cdot) := \frac{1}{2}\left[\kappa(\cdot,J\cdot) - \kappa(J\cdot,\cdot)\right]$$

defines a J-Hermitian metric on M. We will denote the space of κ-tamed almost complex structures on (M,κ) by $\mathfrak{T}_\kappa(M)$.

If J is an almost complex structure on M, then the space of complex r-forms $\Lambda^r(M,\mathbb{C})$ splits as $\Lambda^r(M,\mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q}_J(M)$. Then, the exterior differential decomposes as

$$d : \Lambda^{p,q}_J(M) \to \Lambda^{p+1,q-2}_J(M) \oplus \Lambda^{p+1,q}_J(M) \oplus \Lambda^{p,q+1}_J(M) \oplus \Lambda^{p-1,q+2}_J(M)$$

and

$$d = A_J + \partial_J + \overline{\partial}_J + \overline{A}_J.$$

In view of the Newlander-Nirenberg Theorem, J is integrable if and only if $A_J = 0$.

Let (M, κ) be a symplectic manifold and J be a κ-calibrated almost complex structure. In such a case, the *Chern connection* ∇^C is defined by

$$\nabla^C_X Y = \nabla^{LC}_X Y - \frac{1}{2} J((\nabla^{LC}_X J)Y),$$

where ∇^{LC} denotes the Levi-Civita connection of g_J. It turns out that

$$\nabla^C g_J = 0\,, \qquad \nabla^C J = 0\,, \qquad T^{\nabla^C} = -\frac{1}{4} N_J,$$

N_J being the *Nijenhuis tensor* of J.

2. Quantum inner state manifolds

Let (M, J, κ) be a compact Kähler manifold with $c_1(M) = 0$. Then, in view of Yau's Theorem (see [19]) on the solution of Calabi conjecture and of the structure results on Ricci-flat compact manifolds by Cheeger and Gromoll (see [5]), it follows that (M, J, κ) admits a finite covering biholomorphic to the Kähler product manifold

$$(\mathbb{T}^{2p} \times M_1 \times \cdots \times M_r, \kappa_0 \times \kappa_1 \times \cdots \times \kappa_r, J_0 \times J_1 \cdots \times J_r)\,, \tag{2.1}$$

where $(\mathbb{T}^{2p}, J_0, \kappa_0)$ is a flat torus and (M_h, J_h, κ_h), for $1 \leqslant h \leqslant r$, is a compact simply-connected irreducible manifold with $c_1(M_h) = 0$.

Therefore, for each $1 \leqslant h \leqslant r$, there exists a nowhere vanishing section $\epsilon_h \in \Lambda^{*,0}(M_h)$ such that

- $\epsilon_h \wedge \overline{\epsilon}_h$ is a volume form,
- $\nabla^{LC}_h \epsilon_h = 0$,

where ∇^{LC}_h denotes the Levi-Civita connection of $(M_h, J_h, \kappa_h, g_h)$.

We recall the well known

Definition 2.1. — *A Calabi-Yau manifold (M, J, κ, ϵ) is an n-dimensional compact Kähler manifold (M, J, κ) endowed with a nowhere vanishing $\epsilon \in \Lambda^{n,0}(M)$ satisfying*

$$\nabla^{LC}\epsilon = 0\,, \qquad \epsilon \wedge \overline{\epsilon} = (-1)^{\frac{n(n+1)}{2}}\, i^n \frac{\kappa^n}{n!}.$$

In view of the structure results about compact Ricci-flat manifolds, up to coverings, we shall confine ourselves to the case $\pi_1(M)$ is finite, namely, we assume that there is no torus factor in the decomposition (2.1), *i.e.,* M is a finite quotient of

$$(M_1 \times \cdots \times M_r, \kappa_1 \times \cdots \times \kappa_r, J_1 \times \cdots \times J_r),$$

so that

$$H^1(M,\mathbb{Z}) = 0.$$

The remaining cases will be treated elsewhere.

In order to perform a more flexible deformation theory of Calabi-Yau manifolds, we need to introduce a class of manifolds which represent a possible generalization of the Calabi-Yau ones.

We give the following

Definition 2.2. — *A quantum inner state manifold (QIS manifold), is the datum of (M,κ,J,ϵ), where (M,κ) is a (compact) symplectic $2n$-dimensional manifold, J is a κ-tamed almost complex structure on M and $\epsilon \in \Lambda^{n,0}_J(M)$ satisfies*

- $(-1)^{\frac{n(n+1)}{2}} i^n \epsilon \wedge \overline{\epsilon}$ *is a volume form*
- $\overline{\partial}_J \epsilon = 0$*;*

note that g_J can be conformally rescaled in such a way that $|\epsilon| = 1$, which gives $\nabla^C \epsilon = 0$, where ∇^C refers to the induced canonical Chern connection.

Observe that we do not require the integrability of the almost complex structure J.

Let (M,κ) be a $2n$-dimensional compact symplectic manifold with vanishing first Chern class.

We are interested in finding necessary and sufficient conditions in order that a κ-tamed almost complex structure J gives rise to a quantum inner state structure on M.

Let J be in $\mathfrak{T}_\kappa(M)$; starting from the Riemannian metric g_J, we can define the Hodge operator

$$* : \Lambda^{p,q}_J(M) \to \Lambda^{n-q,n-p}_J(M)$$

and consider the usual differential operators

$$\begin{aligned}
\overline{\partial}^*_J &= - * \partial_J * ,\\
\partial^*_J &= - * \bar{\partial}_J * ,\\
\Box_J &= \overline{\partial}^*_J \overline{\partial}_J + \overline{\partial}_J \overline{\partial}^*_J ,\\
\overline{\Box}_J &= \partial^*_J \partial_J + \partial_J \partial^*_J .
\end{aligned}$$

Since $c_1(M) = 0$, there exists a nowhere vanishing $\eta \in \Lambda^{n,0}_J(M)$. Then,

$$\overline{\partial}_J \eta = \hat{\gamma} \wedge \eta,$$

for a $(0,1)$-form $\hat{\gamma}$. Consequently, for every function $\lambda \neq 0$, we have

$$\overline{\partial}_J(\lambda\eta) = (\overline{\partial}_J \log \lambda + \hat{\gamma}) \wedge \lambda\eta.$$

Set

$$\gamma = \hat{\gamma} - \overline{\partial}_J G_J(\overline{\partial}_J^* \hat{\gamma}),$$

where G_J denotes the Green operator of $\Box_J$; the nowhere vanishing $(n,0)$-form

$$\epsilon = e^{-G_J(\overline{\partial}_J^* \hat{\gamma})}\eta$$

satisfies

$$\overline{\partial}_J \epsilon = \gamma \wedge \epsilon,$$

with $\overline{\partial}_J^* \gamma = 0$; moreover, if $\epsilon' \in \wedge_J^{n,0}(M)$ is another nowhere vanishing element such that $\bar{\partial}_J \epsilon' = \gamma' \wedge \epsilon'$, with $\bar{\partial}_J^* \gamma' = 0$, then

$$\gamma' = \gamma + \bar{\partial}_J \log \lambda,$$

for $\lambda \in C^\infty(M, \mathbb{C}^*)$, satisfying $\bar{\partial}_J^* \bar{\partial}_J \log \lambda = 0$.

Therefore, if we set

$$H_J^{0,1}(M, \mathbb{Z}) := \{\overline{\partial}_J \log \lambda \mid \lambda \in C^\infty(M, \mathbb{C}^*), \ \overline{\partial}_J^* \overline{\partial}_J \log \lambda = 0\}$$

and define

$$a(J) = [\gamma] \in \frac{\operatorname{Ker} \overline{\partial}_J^*}{H_J^{0,1}(M, \mathbb{Z})},$$

we obtain the following

Proposition 2.3. — *Let (M, κ) be a compact symplectic manifold with vanishing first Chern class. Let J be a κ-tamed almost complex structure on M. Then J determines a QIS on M if and only if $a(J) = 0$.*

Remark 2.4. — It has to be noted that $a(J)$ does not depend on the starting $(n,0)$-form η.

Remark 2.5. — For any κ-tamed almost complex structure J on a compact symplectic manifold (M, κ), it is defined

$$H_J^{1,0}(M, \mathbb{Z}) := \{\partial_J \log \lambda \mid \lambda \neq 0, \ \partial_J^* \partial_J \log \lambda = 0\}.$$

If $J \in \mathfrak{C}_\kappa$, we have that $\Box_J = \overline{\Box}_J = \Delta_{g_J}$ on functions and so:

$$H^1(M, \mathbb{Z}) = H_J^{1,0}(M, \mathbb{Z}) = H_J^{0,1}(M, \mathbb{Z}),$$

but, in the general case, relations between $H_J^{0,1}(M, \mathbb{Z})$ and $H^1(M, \mathbb{Z})$ are more complicated.

3. QIS deformations of Calabi-Yau manifolds and moduli space

In this section we are interested to deform Calabi-Yau manifolds in the Quantum Inner State directions. From now on, (M, J, κ, ϵ) will denote a compact Calabi-Yau manifold with $H^1(M, \mathbb{Z}) = 0$.

Let us start by fixing some preliminary notation.

Let (V, J) be a $2n$-dimensional real vector space endowed with a linear complex structure J. Let $0 \neq \epsilon \in \Lambda^n(V^*)^{1,0}_J$.

Definition 3.1. — *We set*

$$\circledast : \Lambda^p(V)^{1,0}_J \to \Lambda^{n-p}(V^*)^{1,0}_J$$

by the relation

$$\alpha \wedge \circledast\beta = <\alpha, \beta> \epsilon$$

where $\alpha \in \Lambda^p(V^*)^{1,0}_J$, $\beta \in \Lambda^p(V)^{1,0}_J$ *and* $<,>$ *denotes the duality pairing.*

Let

$$A^q_p(V, J) = \Lambda^q(V^*)^{0,1}_J \otimes \Lambda^p(V)^{1,0}_J.$$

Then we can extend the operator $\circledast$ to $A^q_p(V, J)$ by considering

$$1 \otimes \circledast : A^q_p(V, J) \to \Lambda^q(V^*)^{0,1}_J \otimes \Lambda^{n-p}(V^*)^{1,0}_J.$$

Let (M,J,κ, ϵ) be a compact Calabi-Yau manifold and denote by $A^q_p(M, J) = \Lambda^{0,q}_J(M) \otimes \Lambda^p_J(T^{1,0}M)$. We recall the following (see [17], [16])

Definition 3.2. — *Define*

$$\eth_J : A^q_p(M, J) \to A^q_{p-1}(M, J)$$

by

$$\eth_J = (1 \otimes \circledast)^{-1} \circ \partial_J \circ (1 \otimes \circledast).$$

Then the following holds

$$\eth^2_J = 0\,, \qquad [\eth_J, \overline{\partial}_J] = 0.$$

Since (M, J, κ) is a compact Kähler manifold, then the $\partial_J\overline{\partial}_J$-lemma holds. Hence, also the $\eth_J\overline{\partial}_J$-lemma holds, *i.e.*,

$$(\operatorname{Ker} \eth_J \cap \operatorname{Ker} \overline{\partial}_J) \cap (\operatorname{Im} \eth_J + \operatorname{Im} \overline{\partial}_J) = \operatorname{Im} \eth_J\overline{\partial}_J.$$

Remark 3.3. — The operator $\eth_J$ can be also defined as

$$\iota_{\eth_J\gamma}\epsilon = \partial_J(\iota_\gamma\epsilon).$$

The holomorphic deformation theory of Calabi-Yau manifolds is well understood via the integrable DGBV algebra $(\mathcal{A}, \overline{\partial}_J, \eth_J)$ (see [16], [17]), where

$$\mathcal{A} = \bigoplus_{p,q=0}^{\infty} A_p^q(M, J).$$

In the sequel we will set

$$\mathcal{A}[p, q] = A_p^q(M, J).$$

Denote by

$$\Lambda_*^n(M) = \{\eta \in \Lambda^n(M) \mid (-1)^{\frac{n(n+1)}{2}} (-i)^n \eta \wedge \overline{\eta} \text{ is a volume form}\};$$

set

$$\mathfrak{F} = \{(K, \eta) \in \mathfrak{T}_\kappa(M) \times \Lambda_*^n(M) \mid \eta \in \Lambda^{n,0}(M)\}$$

and

$$\mathfrak{B} = \{(K, \eta) \in \mathfrak{F} \mid \overline{\partial}_K \eta = 0\}.$$

Let (J, κ, ϵ) be a Calabi-Yau structure on the compact manifold M. We are going to study the infinitesimal structure of $\mathfrak{B}$ at the point (J, ϵ). For this purpose, let

$$\mathcal{S}^* = \{\text{Ł} \in \operatorname{End}(TM) \mid JL + LJ = 0\,, \ \| L \| < 1\}.$$

Then, as recalled in section 1, the map

$$\Phi : \mathcal{S}^* \to \mathfrak{T}_\kappa(M)\,, \qquad L \mapsto (I + L)J(I + L)^{-1}$$

is a bijection and the inverse map Φ^{-1} is given by

$$\Phi^{-1} : \mathfrak{T}_\kappa(M) \to \mathcal{S}^*\,, \qquad K \mapsto (J + K)^{-1}(J - K).$$

This means that the space $\mathfrak{T}\kappa(M)$ of κ-tamed almost complex structures on (M, κ) can be *linearized*, *i.e.*, $\mathfrak{T}\kappa(M)$ is parametrized by a dense open subset of a linear subspace of $\operatorname{End}(TM)$.

Let V be a vector space and let P be an automorphism of V. Consider

$$\rho(P) = (P^*)^{-1} \otimes P \ \in \operatorname{Aut}(V^* \otimes V).$$

Then $\rho(P)$ acts on the space $\Lambda^r(V^*) \otimes V$. Indeed, for example, if $\alpha \in \Lambda^2(V^*) \otimes V$, then $\rho(P)(\alpha)$ is given by

$$(\rho(P)(\alpha))(v, w) = P\left(\alpha(P^{-1}(v), P^{-1}(w))\right).$$

Therefore, for

$$K = (I + L)J(I + L)^{-1},$$

we have that $\epsilon \in \Lambda_J^{n,0}(M)$ if and only if $\rho(I + L)(\epsilon) \in \Lambda_K^{n,0}(M)$.

Define now

$$\mathcal{F}^* = \{\lambda \in \mathcal{C}^\infty(M,\mathbb{C}) \mid 1+\lambda \neq 0 \text{ everywhere on } M\}.$$

Then $\mathfrak{F}$ can be linearized by using the action ρ through the following map

$$(L,\lambda) \in \mathcal{S}^* \times \mathcal{F}^* \mapsto (\Phi(L), (1+\lambda)\rho(I+L)\epsilon).$$

Let V be a vector space and $L \in \mathrm{End}(V)$; then, for every $\alpha \in \Lambda^p(V^*)$, define

$$(\tau(L)(\alpha))(v_1,\ldots,v_p) = \sum_{h=1}^{p} \alpha(v_1,\ldots,L(v_h),\ldots,v_p).$$

Hence $\tau(L) \in \mathrm{End}(\Lambda(V^*)$. Finally, for $\beta \in \Lambda(V^*)$, set

$$\tau(\beta \wedge L)(\alpha) = \beta \wedge \tau(L)(\alpha).$$

In this way, we can define $\tau(R)$ for every $R \in \Lambda(V^*) \otimes V$.

Let M be a smooth manifold and $R \in \Lambda(M) \otimes TM$. Set

$$D_R = [\tau(R), d]\,.$$

Let $L \in \mathrm{End}\,(TM) = \Lambda^1(M) \otimes TM$ and define a derivation $r(L) : \Lambda(M) \to \Lambda(M)$ in the following way:

$$\begin{aligned} &\text{for every } f \in \mathcal{C}^\infty(M) \quad r(L)(f) = 0,\\ &\text{for every } \alpha \in \Lambda^1(M) \quad r(L)(\alpha)(X,Y) = \alpha((I+L)^{-1}S(L)(X,Y)), \end{aligned}$$

where

$$S(L)(X,Y) = [L(X),L(Y)] + L^2[X,Y] - L[X,L(Y)] - L[L(X),Y].$$

Then extend $r(L)$ to every k-form α as a skew-symmetric derivation. By definition, $r(L)$ is quadratic in the endomorphism L and of order zero.

A key tool in the computation of the virtual tangent space to the moduli space of quantum inner state structures is the following lemma, whose proof will be given in the Appendix.

Lemma 3.4. — *Let $L \in \mathrm{End}\,(TM) = \Lambda^1(M) \otimes TM$ such that $(I+L) \in \mathrm{Aut}\,(TM)$. Then the following formula holds*

$$\rho(I+L)^{-1} \circ d \circ \rho(I+L) = d + D_L - r(L)\,. \tag{3.1}$$

Assume that a κ-tamed almost complex structure $K = \Phi(L)$ is given on M. Then the exterior differential d splits as

$$d = A_K + \partial_K + \overline{\partial}_K + \overline{A}_K,$$

where

$$A_K : \Lambda_K^{p,q}(M) \to \Lambda_K^{p+2,q-1}(M)$$

and K is holomorphic if and only if

$$d = \partial_K + \overline{\partial}_K.$$

Let $R \in \Lambda^{r,s+1}(M) \otimes T_J^{1,0}(M)$. For every $\alpha \in \Lambda_J^{p,q}(M)$, define

$$\urcorner_R \alpha = (D_R \alpha)^{p+r,q+s+1}.$$

Therefore, by taking the $(p, q+1)$ components of formula (3.1) of Lemma 3.4, we obtain

$$\rho(I+L)^{-1}\overline{\partial}_K \rho(I+L)\alpha = \overline{\partial}_J \alpha + \urcorner_L \alpha - (r(L)\alpha)^{p,q+1}\,. \tag{3.2}$$

Since J is holomorphic, we have

$$\overline{\partial}_J \urcorner_R := [\overline{\partial}_J, \urcorner_R] = \urcorner_{\overline{\partial}_J R}.$$

Now we are ready to prove the following

Theorem 3.5. — *Let (M, J, κ, ϵ) be an n-dimensional compact Calabi-Yau manifold. Denote by*

$$\mathfrak{M} := \mathfrak{B}/\mathrm{Diff}\,(M)$$

the moduli space of quantum inner state structures on M. Then the tangent space $T_{[(J,\epsilon)]}\mathfrak{M}$ to $\mathfrak{M}$ at $[(J,\epsilon)]$ satisfies

$$T_{[(J,\epsilon)]}\mathfrak{M} \subset \left(\frac{\mathrm{Ker}\,\eth_J \overline{\partial}_J}{\mathrm{Im}\,\overline{\partial}_J}\right)[1,1].$$

Proof. — Let

$$t \mapsto \gamma(t) = (J_t, \epsilon_t)$$

be a smooth curve in $\mathfrak{B}$, with $\gamma(0) = (J, \epsilon)$. By the above discussion,

$$J_t = (I + L_t)J(I + L_t)^{-1},$$

where $L_t = tL + o(t)$ and

$$\epsilon_t = \rho(I + L_t)(1 + \lambda_t)\,\epsilon,$$

with $\lambda_t = t\lambda + o(t)$. Then, by using formula (3.2) and the definition of $\urcorner_{L_t}$, it follows that the condition $\overline{\partial}_{J_t}\epsilon_t = 0$ holds if and only if

$$\begin{aligned} 0 &= \rho(I+L_t)^{-1}\overline{\partial}_{J_t}\left(\rho(I+L_t)(1+\lambda_t)\,\epsilon\right) = \overline{\partial}_J(1+\lambda_t)\epsilon + \urcorner_{L_t}(1+\lambda_t)\epsilon + o(t) \\ &= \overline{\partial}_J(1+\lambda_t)\epsilon - \partial_J \tau(L_t)(1+\lambda_t)\epsilon + o(t). \end{aligned}$$

Therefore,

$$0 = \frac{d}{dt}|_{t=0}\left(\overline{\partial}_J(1+\lambda_t)\,\epsilon - \partial_J\tau(L_t)\,\epsilon\right) = \overline{\partial}_J\lambda\,\epsilon - \partial_J\tau(L)\,\epsilon.$$

Hence, the tangent space to $\mathfrak{B}$ at (J,ϵ) satisfies

$$T_{[(J,\epsilon)]}\mathfrak{B} \subset \{(L,\lambda) \mid \overline{\partial}_J\lambda\,\epsilon = \partial_J\tau(L)\,\epsilon\}.$$

Observe that

$$\partial_J \tau(L)\epsilon = \eth_J L \wedge \epsilon$$

and that the $\partial_J\overline{\partial}_J$-lemma holds for (M, J), and, consequently also the $\eth_J\overline{\partial}_J$-lemma holds. Therefore, we claim that

$$T_{[(J,\epsilon)]}\mathfrak{B} = \{L \in \mathcal{A}[1,1] \mid \eth_J\overline{\partial}_J L = 0\}.$$

Note first that:

$$\eth_J\bar{\partial}_J L = 0 \iff \bar{\partial}_J(\eth_J L \wedge \epsilon) = 0 \iff \bar{\partial}_J\partial_J\tau(L)\epsilon = 0\,;$$

then, clearly, if (L, λ) satisfies $\overline{\partial}_J\lambda\,\epsilon = \partial_J\tau(L)\,\epsilon$, then $\eth_J\overline{\partial}_J L = 0$. Vice versa, if $L \in \mathcal{A}[1,1]$ is $\eth_J\overline{\partial}_J$-closed, then $\partial_J\overline{\partial}_J\tau(L)\epsilon = 0$ and the $\partial_J\overline{\partial}_J$-lemma implies that there exists $\beta \in \wedge_J^{n-1,0}(M)$ such that:

$$\partial_J\tau(L)\,\epsilon = \overline{\partial}_J\partial_J\beta = \overline{\partial}_J\lambda\epsilon.$$

The group $\mathrm{Diff}(M)$ naturally acts on $\mathfrak{B}$ and there is an induced action on the endomorphisms L, namely

$$\phi^\sharp(L) = \left(J + \phi_*^{-1}K\phi_*\right)^{-1}\left(J - \phi_*^{-1}K\phi_*\right),$$

where

$$K = (I + L)\,J\,(I + L)^{-1}\,.$$

Let X be a smooth vector field and $\{\phi_t^X\}\,, t\in\mathbb{R}$, be the induced 1-parameter subgroup of $\mathrm{Diff}(M)$. Then, we get

$$\frac{d}{dt}(\phi_t^X)^\sharp(O)|_{t=0} = \frac{1}{2}J\mathcal{L}_X J = -\overline{\partial}_J X.$$

On $\Lambda_J^{*,0}(M)$ we have

$$\tau(\overline{\partial}_J X) = [\overline{\partial}_J, \iota_X].$$

Therefore,

$$\tau(\overline{\partial}_J X)\,\epsilon = \overline{\partial}_J\iota_X\,\epsilon.$$

Consequently,

$$\partial_J\tau(\overline{\partial}_J X)\,\epsilon = \partial_J\overline{\partial}_J\iota_X\,\epsilon = -\overline{\partial}_J\partial_J\iota_X\,\epsilon = \overline{\partial}_J\lambda\epsilon,$$

i.e., $(\overline{\partial}_J X, \lambda) \in T_{[(J,\epsilon)]}\mathfrak{B}$. Hence, we have showed that

$$T_{[(J,\epsilon)]}\mathfrak{M} \subset \left(\frac{\mathrm{Ker}\,\eth_J\overline{\partial}_J}{\mathrm{Im}\,\overline{\partial}_J}\right)[1,1].$$

□

Let
$$H = \left(\frac{\operatorname{Ker}\eth_J \cap \operatorname{Ker}\overline{\partial}_J}{\operatorname{Im}(\eth_J\overline{\partial}_J)}\right)[1,1]$$
and
$$E = \operatorname{Ker}\eth_J \cap \operatorname{Im}\overline{\partial}^*[1,1].$$
Denote by
$$s(E) = \{\text{Ł} \in E \mid L = {}^tL\}\,, \qquad a(E) = \{\text{Ł} \in E \mid L = -{}^tL\}$$
the symmetric, respectively the skew-symmetric part of any $L \in E$. We have the following

Theorem 3.6. — *Let (M, J, κ, ϵ) be an n-dimensional compact Calabi-Yau manifold. Assume that $H^1(M,\mathbb{Z}) = 0$. Then the moduli space $\mathfrak{M}$ of quantum inner state deformations of (M, J, κ, ϵ) is unobstructed.*

Furthermore, the tangent space to $\mathfrak{M}$ at $[(J,\epsilon)]$ decomposes as
$$T_{[(J,\epsilon)]}\mathfrak{M} = H \oplus a(E) \oplus s(E)\,,$$
where H corresponds to the non-obstructed space of holomorphic deformations, $a(E) = \overline{\partial}^*_J\Lambda^{0,3}(M)$ and $s(E)$ encodes the space of calibrated deformations.

Proof. — We start by proving the first part of the statement. We will apply the implicit function theorem; recall that, setting
$$\mathfrak{F} = \{(K,\eta) \in \mathfrak{T}_\kappa(M)\times\Lambda^n_*(M) \mid \eta \in \Lambda^{n,0}(M) \text{ and } \eta\wedge\overline{\eta} \text{ is a volume form}\}$$
and
$$\mathfrak{B} = \{(K,\eta) \in \mathfrak{F} \mid \overline{\partial}_K\eta = 0\},$$
then we defined the moduli space of quantum inner state structures as
$$\mathfrak{M} := \mathfrak{B}/\operatorname{Diff}(M).$$
By Theorem 3.5, we have that
$$T_{[(J,\epsilon)]}\mathfrak{M} = \left(\frac{\operatorname{Ker}\eth_J\overline{\partial}_J}{\operatorname{Im}\overline{\partial}_J}\right)[1,1].$$
Let
$$\mathcal{S}^* = \{\text{Ł} \in \operatorname{End}(TM) \mid JL + LJ = 0\,,\ \| L \| < 1\},$$
$$\mathcal{F}^* = \{\lambda \in \mathcal{C}^\infty(M,\mathbb{C}) \mid 1 + \lambda \neq 0 \text{ everywhere on } M\},$$
and
$$\mathcal{F} = \mathcal{S}^* \times \mathcal{F}^*.$$
Then the map
$$(L,\lambda) \in \mathcal{S}^* \times \mathcal{F}^* \mapsto (\Phi(L), (1+\lambda)\rho(I+L)\epsilon).$$

linearizes $\mathfrak{F}$. Consider the following map

$$p : \mathcal{S}^* \times \mathcal{F}^* \to \Lambda^{0,1}_J(M)$$

defined as

$$p(L,\lambda) \wedge \epsilon = \rho(I+L)^{-1}\overline{\partial}_{\Phi(L)}\rho(I+L)(1+\lambda)\epsilon. \tag{3.3}$$

Then, the first order differential operator acting on functions as

$$\rho(I+L)^{-1}\overline{\partial}_{\Phi(L)}\rho(I+L)(1+\cdot)\,\epsilon$$

is elliptic.

By definition of the map p, the zero set of p is the set of quantum inner state structures on M. In view of the assumption, we have that $H^{0,1}_{\overline{\partial}_J}(M) = 0$, and, consequently, we obtain that

$$p : \mathcal{S}^* \times \mathcal{F}^* \to \operatorname{Im} \Box_J.$$

By Theorem 3.5, it follows that the differential $p_*[(0,0)]$ of p at $[(0,0)]$ is

$$p_*[(0,0)](L,\mu) = \overline{\partial}_J\mu - \eth_J L.$$

We claim that

$$\operatorname{Im} p_*[(0,0)] = \operatorname{Im} \Box_J. \tag{3.4}$$

Indeed, it is $\Lambda^{0,1}_J(M) = \operatorname{Im} \Box_J$, since $H^{0,1}_{\overline{\partial}_J}(M) = 0$. We immediately get $\operatorname{Im} p_*[(0,0)] \subset \operatorname{Im} \Box_J$.

Conversely, let $\alpha \in \operatorname{Im} \Box_J = \Lambda^{0,1}_J(M)$. We need to show that there exist $L \in \operatorname{End}(TM)\,, LJ + JL = 0$ and $\mu \in \mathcal{C}^\infty(M,\mathbb{C})$ such that

$$\alpha = \overline{\partial}_J\mu - \eth_J L.$$

By the definition of $\eth_J$-operator, it is

$$\eth_J\overline{\partial}_J\alpha = 0.$$

Therefore, by the $\eth_J\overline{\partial}$-lemma, it follows that there exists L such that

$$\overline{\partial}_J\alpha = -\eth_J\overline{\partial}L\,.$$

Hence

$$\overline{\partial}_J(\alpha + \eth_J L) = 0.$$

Consequently, the $(0,1)$-form is $\alpha + \eth_J L$ is $\overline{\partial}_J$-exact, *i.e.*, there exists $\mu \in \mathcal{C}^\infty(M,\mathbb{C})$ such that

$$\alpha = \overline{\partial}_J\mu - \eth_J L.$$

Hence, passing to a suitable Sobolev complection, using the implicit function theorem for Banach manifolds and standard regularity arguments holding for elliptic differential operators, we obtain that, for every $[(L,\, \mu)] \in T_{[(J,\,\epsilon)]}\mathfrak{M}$, there exists a curve

$$c:\ (-\delta,\, \delta) \longrightarrow \mathfrak{M},$$

such that:

$$c(0) = [(J,\, \epsilon)]\ \ ,\ \ c'(0) = [(L,\, \mu)].$$

This shows that $\mathfrak{M}$ is unobstructed.

Now we prove the second part of the statement.

First of all, observe that

$$\frac{\operatorname{Ker} \eth_J}{\operatorname{Im} \eth_J \overline{\partial}_J} \cong \frac{\operatorname{Ker} \eth_J \overline{\partial}_J}{\operatorname{Im} \overline{\partial}_J}.$$

In view of the $\eth_J\overline{\partial}_J$-lemma, it can be easily checked that, the inclusion map

$$j : \operatorname{Ker} \eth_J \hookrightarrow \operatorname{Ker} \eth_J \overline{\partial}_J$$

induces an isomorphism

$$i_* : \frac{\operatorname{Ker} \eth_J}{\operatorname{Im} \eth_J \overline{\partial}_J} \to \frac{\operatorname{Ker} \eth_J \overline{\partial}_J}{\operatorname{Im} \overline{\partial}_J}.$$

Therefore, by Hodge Theorem, we have that

$$T_{[(J,\epsilon)]}\mathfrak{M} = \left(\frac{\operatorname{Ker} \eth_J}{\operatorname{Im} \eth_J \overline{\partial}_J} \right) [1,1] \cong H \oplus E,$$

where

$$H = \left(\frac{\operatorname{Ker} \eth_J \cap \operatorname{Ker} \overline{\partial}_J}{\operatorname{Im}(\eth_J \overline{\partial}_J)} \right) [1,1]$$

and

$$E = \operatorname{Ker} \eth_J \cap \operatorname{Im} \overline{\partial}^* [1,1].$$

Note that, for every $L \in \mathcal{A}[1,1] = \Lambda_J^{0,1}(M) \otimes T_J^{1,0}M$, we have that $\eth_J L = 0$ if and only if $\overline{\partial}^*\, {}^tL = 0$. Therefore,

$$E = s(E) \oplus a(E).$$

Consider $L \in \Lambda_J^{0,1}(M) \otimes T_J^{1,0}(M)$, with $L = -{}^tL$. Then, L induces a $(0,2)$-form on M defined as

$$\alpha_L(\overline{Z}, \overline{W}) := g_J(L(\overline{Z}), \overline{W}).$$

Hence, it follows that

$$L \in a(E) \Longleftrightarrow \alpha_L \in \overline{\partial}_J^* \Lambda_J^{0,3}(M).$$

Therefore, we obtain that

$$a(E) \cong \overline{\partial}_J^* \Lambda_J^{0,3}(M) \cong \overline{\partial}_J \Lambda_J^{0,2}(M).$$

Finally, every $L \in s(E)$ corresponds to a calibrated (virtual) deformation, L being symmetric. The theorem is proved. □

Remark 3.7. — It would be natural to characterize explicitly the locus in $\mathfrak{M}$ corresponding to the classical Calabi-Yau deformations and, possibly, to derive another proof of Tian-Todorov Lemma from Theorem 3.6.

Remark 3.8. — If (M, J, κ, ϵ) is a 3-dimensional Calabi-Yau manifold, then the set $s(E)$ can be identified with

$$\left\{ \lambda \in \mathcal{C}^\infty(M, \mathbb{C}) \ \middle|\ \int_M \lambda \frac{\kappa^3}{3!} = 0 \right\}.$$

4. Examples

We start by giving an example of an almost complex structure on a compact manifold which is calibrated by a symplectic form and with no invariant nowhere vanishing and $\overline{\partial}$-closed $(3,0)$-forms ϵ.

Example 4.1. — Let $\mathfrak{g}$ be the 6-dimensional nilpotent Lie algebras whose dual space $\mathfrak{g}^*$ is spanned by $\{e^1, \dots, e^6\}$ satisfying the following Maurer-Cartan equations:

$$\begin{cases} de^1 = 0\,, \\ de^2 = 0\,, \\ de^3 = 0\,, \\ de^4 = 0\,, \\ de^5 = e^1 \wedge e^2\,, \\ de^6 = e^3 \wedge e^4\,. \end{cases}$$

Let G be the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and $M = \Gamma \backslash G$ the associated compact 6-dimensional nilmanifold, for a uniform discrete subgroup $\Gamma \subset G$ (see [14]). Then

$$\kappa = e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6$$

defines a symplectic structure on M and

$$\varphi^1 = e^1 + ie^4\,, \qquad \varphi^2 = e^2 + ie^5\,, \qquad \varphi^3 = e^3 + ie^6,$$

are the complex $(1,0)$-forms of a κ-calibrated almost complex structure on M. Then every invariant nowhere vanishing $(3,0)$-form ϵ is a non-zero constant multiple of $\varphi^1 \wedge \varphi^2 \wedge \varphi^3$. Since $\overline{\partial}_J(\varphi^1 \wedge \varphi^2 \wedge \varphi^3) \neq 0$, we have $\overline{\partial}_J \epsilon \neq 0$.

It is not known if there exist nowhere vanishing $(3,0)$-forms ϵ on M such that $\overline{\partial}_J \epsilon = 0$.

The next example shows a curve of holomorphic structures having a nowhere vanishing holomorphic $(3,0)$-form.

Example 4.2. — Let us shortly recall the construction of the *Nakamura manifold*. On $\mathbb{C}^3$, consider the following product $*$

$$(z^1, z^2, z^3) * (w^1, w^2, w^3) = (z^1 + w^1, e^{-w^1} z^2 + w^2, e^{w^1} z^3 + w^3)\,,$$

for any $(z^1, z^2, z^3), (w^1, w^2, w^3) \in \mathbb{C}^3$.

Then $(\mathbb{C}^3, *)$ is a complex solvable Lie group admitting a uniform discrete subgroup $\Gamma \subset \mathbb{C}^3$. The compact quotient $M = \Gamma \backslash (\mathbb{C}^3, *)$ is a 3-dimensional holomorphically parallelizable manifold (see [15]).

Then

$$\varphi^1 = dz^1\,, \quad \varphi^2 = e^{z^1} dz^2\,, \quad \varphi^3 = e^{-z^1} dz^3$$

define holomorphic 1-forms on M. In [15] small deformations of the complex structure on M and on the other complex solvable manifolds are computed.

By [15, p. 98, case 1], take $t = t_{11}, t_{21} = 0, t_{22} = 0, t_{31} = 0, t_{33} = 0$. Then the corresponding curve of holomorphic structures J_t is described by

$$\zeta_t^1 = z^1 + t\overline{z}^1\,, \quad \zeta_t^2 = z^2\,, \quad \zeta_t^3 = z^3.$$

A direct computation shows that

$$\varphi_t^1 = d\zeta_t^1\,, \quad \varphi_t^2 = e^{\frac{\zeta_t^1 - t\overline{\zeta}_t^1}{1-|t|^2}} d\zeta_t^2\,, \quad \varphi_t^3 = e^{-\frac{\zeta_t^1 - t\overline{\zeta}_t^1}{1-|t|^2}} d\zeta_t^3$$

are complex invariant $(1,0)$-forms on (M, J_t). Consequently

$$d\varphi_t^1 = 0,\ d\varphi_t^2 = \frac{1}{1-|t|^2}\left[(\varphi_t^1 - t\overline{\varphi}_t^1) \wedge \varphi_t^2\right],\ d\varphi_t^3 = -\frac{1}{1-|t|^2}\left[(\varphi_t^1 - t\overline{\varphi}_t^1) \wedge \varphi_t^3\right].$$

Therefore,

$$\epsilon_t = \varphi_t^1 \wedge \varphi_t^2 \wedge \varphi_t^3$$

defines a curve of $\overline{\partial}$-closed nowhere vanishing complex (3,0)-forms on (M, J_t).

The next example gives a formal non-Kähler QIS manifold

Example 4.3. — Let $\mathfrak{g}$ be the Lie algebra whose dual vector space is spanned by $\{e^1, \dots, e^6\}$ satisfying the following structure equations

$$\begin{cases} de^1 = 0\,, \\ de^2 = 0\,, \\ de^3 = e^1 \wedge e^3\,, \\ de^4 = -e^1 \wedge e^4\,, \\ de^5 = e^1 \wedge e^5\,, \\ de^6 = -e^1 \wedge e^6\,. \end{cases}$$

Then $\mathfrak{g}$ is a 6-dimensional (real) *completely solvable* Lie algebra, *i.e.*, ad_X has real eigenvalues, for any $X \in \mathfrak{g}$. It turns out that the simply-connected Lie group G having $\mathfrak{g}$ as Lie algebra, admits a uniform discrete subgroup $\Gamma \subset G$ (see [9]). Hence $M = \Gamma\backslash G$ is a compact 6-dimensional solvmanifold. Let J be the almost complex structure on M arising by the complex $(1,0)$-forms defined by

$$\varphi^1 = e^1 + ie^2\,, \quad \varphi^2 = e^3 + ie^4\,, \quad \varphi^3 = e^5 + ie^6$$

and the compatible symplectic form

$$\kappa = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6.$$

Then we obtain:

$$\overline{\partial}_J \varphi^1 = 0\,, \quad \overline{\partial}_J \varphi^2 = \frac{1}{2}\varphi^1 \wedge \overline{\varphi}^2\,, \quad \overline{\partial}_J \varphi^3 = \frac{1}{2}\varphi^1 \wedge \overline{\varphi}^3.$$

Therefore,

$$\epsilon = \varphi^1 \wedge \varphi^2 \wedge \varphi^3$$

defines a nowhere vanishing $(3,0)$-form on M satisfying $\overline{\partial}_J \epsilon = 0$.

Hence $(\Gamma\backslash G, \kappa, J, \epsilon)$ is a QIS manifold.

It can be checked (see also [9]) that the Riemannian manifold (M, g_J) is *geometrically formal*, *i.e.*, the product of any two harmonic forms is still harmonic. Hence, it follows that the de Rham complex of M is formal in the sense of Deligne, Griffiths, Morgan and Sullivan (see [6]). Furthermore, (M, κ) satisfies the Hard Lefschetz condition. On the other hand, by a result of Hasegawa (see [9] and [10]), M has no Kähler structure, since the Lie algebra $\mathfrak{g}$ is completely solvable.

We will construct now a QIS structure on a compact quotient of a 6-dimensional nilpotent Lie group.

Example 4.4. — Let $\mathfrak{g}$ be the 6-dimensional real Lie algebra with dual space $\mathfrak{g}^*$ spanned by $\{e^1, \ldots, e^6\}$ satisfying the following

$$\begin{cases} de^1 = 0\,, \\ de^2 = 0\,, \\ de^3 = 0\,, \\ de^4 = e^1 \wedge e^2\,, \\ de^5 = e^1 \wedge e^3 + e^1 \wedge e^4\,, \\ de^6 = e^2 \wedge e^4\,. \end{cases}$$

Then, there exists a compact quotient $M = \Gamma\backslash G$, where G is the simply-connected Lie group with Lie algebra $\mathfrak{g}$. A straightforward computation shows that the following data

$$\kappa = e^1 \wedge e^6 + e^2 \wedge e^5 + e^3 \wedge e^4,$$

$$\varphi^1 = e^1 + ie^6\,, \qquad \varphi^2 = e^2 + ie^5\,, \qquad \varphi^3 = e^3 + ie^4,$$

and

$$\epsilon = \varphi^1 \wedge \varphi^2 \wedge \varphi^3,$$

give rise to a quantum inner state structure on M.

Finally, we will present a QIS structure on a family of cohomologically Kähler compact 4-dimensional manifolds.

Example 4.5. — Let

$$G(c) = \left\{ A = \begin{pmatrix} e^{-cx^1} & 0 & 0 & x^2 \\ 0 & e^{cx^1} & 0 & x^3 \\ 0 & 0 & 1 & x^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x^1, x^2, x^3 \in \mathbb{R} \right\}$$

and $c \in \mathbb{R}$ is such that $e^c + e^{-c}$ is a integer different from 2. Then there exists a uniform discrete subgroup $\Gamma(c)$ such that $\Gamma(c)\backslash G(c)$ is a family of 3-dimensional compact solvmanifolds (see [1]). Consider

$$M(c) := \Gamma(c)\backslash G(c) \times S^1;$$

then in [7] it is proved that the compact solvmanifolds $M(c)$ are cohomologically Kähler. The following

$$\alpha^1 = dx^1\,, \qquad \alpha^2 = e^{cx^1} dx^2\,, \qquad \alpha^3 = e^{-cx^1} dx^2\,, \qquad \alpha^4 = dt$$

define 1-forms on $M(c)$. Then

$$d\alpha^1 = 0\,, \quad d\alpha^2 = c\,\alpha^1 \wedge \alpha^2\,, \quad d\alpha^3 = -c\,\alpha^1 \wedge \alpha^3\,, \quad d\alpha^4 = 0.$$

Set

$$\begin{aligned}\kappa &= \alpha^1\wedge\alpha^4+\alpha^2\wedge\alpha^3\,,\\ \psi^1 &= \alpha^1+i\alpha^4\,,\\ \psi^2 &= \alpha^2+i\alpha^3\,,\\ \epsilon &= \psi^1\wedge\psi^2\end{aligned}$$

Then, it is immediate to check that $(M(c),\kappa,J,\epsilon)$ is a family of 4-dimensional quantum inner state solvmanifolds. It has to be remarked that $M(c)$ has no holomorphic structures. Indeed, a direct computation shows that the Lie algebras $\mathfrak{g}(c)$ are completely solvable. Therefore, in view of the Hattori Theorem (see [11]), $H^*(M(c))\cong H^*(\mathfrak{g}(c))$. In particular, $b_1(M(c))=2$ and consequently, if the completely solvable solvmanifold $M(c)$ had a holomorphic structure, then it should carry a Kähler metric; but this is impossible by [9].

Example 4.6. — Let $\mathbb{R}^{2n}$ with coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ and let $\mathbb{T}^{2n}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Define

$$K=\begin{pmatrix}0 & -e^{-\Lambda}\\ e^{\Lambda} & 0\end{pmatrix}\quad\text{with}\quad \Lambda=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

and $\lambda_1,\,\dots,\,\lambda_n\in\mathcal{C}^\infty(\mathbb{R}^{2n},\,\mathbb{R})$; it is immediate to check that

$$J=\begin{pmatrix}0 & -e^{-\Lambda}\\ e^{\Lambda} & 0\end{pmatrix}=(I+L)\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix}(I+L)^{-1}$$

where

$$L=\begin{pmatrix}\Omega & 0\\ 0 & -\Omega\end{pmatrix}$$

with

$$\Omega=\begin{pmatrix}\frac{1-e^{\lambda_1}}{1+e^{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1-e^{\lambda_n}}{1+e^{\lambda_n}}\end{pmatrix}.$$

Therefore, J gives rise to an almost complex structure on the torus $\mathbb{T}^{2n}$ and it can be verified that J is κ_0-calibrated, where $\kappa_0=\sum_{j=1}^n dx_j\wedge dy_j$ is the standard symplectic structure on $\mathbb{T}^{2n}$. A global $(1,\,0)$-frame is given by:

$$\left\{Z_j:=\frac{1}{2}\left(\frac{\partial}{\partial x_j}-ie^{\lambda_j}\frac{\partial}{\partial y_j}\right)\,,\ 1\leqslant j\leqslant n\right\},$$

with (1, 0)-coframe:

$$\left\{\zeta_j := dx_j + ie^{-\lambda_j} dy_j\,,\ \ 1 \leqslant j \leqslant n\right\}.$$

A direct computation yields

$$\begin{aligned}
[\bar Z_j,\, \bar Z_k]^{1,0} &= -\frac{1}{2} N_J(\bar Z_j,\, \bar Z_k) = \bar Z_k(\lambda_j) Z_j - \bar Z_j(\lambda_k) Z_k \\
[Z_j,\, \bar Z_k]^{1,0} &= -\left(\bar Z_k(\lambda_j) Z_j + Z_j(\lambda_k) Z_k\right).
\end{aligned} \tag{4.1}$$

Let

$$\epsilon := \zeta_1 \wedge \ \ldots\ \wedge \zeta_n\,;$$

then, ϵ defines a complex volume form on $(\mathbb{T}^{2n}, K)$ satisfying:

$$\bar\partial_J \epsilon = \gamma \wedge \epsilon$$

with

$$\gamma = -\sum_{j=1}^{n} \sum_{h=1}^{n} \zeta_h([Z_h, \bar Z_j])\bar\zeta_j.$$

Hence, by (4.1), we obtain

$$\gamma = \bar\partial_J \left(\sum_{h=1}^{n} \lambda_h\right) + \sum_{j=1}^{n} Z_j(\lambda_j)\bar\zeta_j.$$

We are going to modify ϵ to obtain a $\overline{\partial}_J$-closed complex volume form. Assume that:

$$\sum_{j=1}^{n} Z_j(\lambda_j)\bar\zeta_j = \bar\partial_J \mu;$$

(this is certainly true if *e.g.* λ_j does not depend on x_j, y_j). Then

$$\bar\partial_J e^{-\mu - \sum_{j=1}^{n} \lambda_j} \epsilon = 0$$

and consequently $(\kappa_0, J, e^{-\mu - \sum_{j=1}^{n} \lambda_j}\epsilon)$ determines a QIS structure on $\mathbb{T}^{2n}$.

5. Appendix

We restate and prove Lemma 3.4.

Lemma 3.4 *Let* $L \in \mathrm{End}\,(TM) = \Lambda^1(M) \otimes TM$ *such that* $(I + L) \in \mathrm{Aut}\,(TM)$*. Then the following formula holds*

$$\rho(I + L)^{-1} \circ d \circ \rho(I + L) = d + D_L - r(L)\,.$$

Proof. — It is enough to prove that the formula above is true for $f \in \Lambda^0(M)$ and $\alpha \in \Lambda^1(M)$ respectively. First of all, let $f \in \Lambda^0(M)$. Indeed, for every vector field X, we have:

$$\big(\rho(I+L)^{-1} \circ d \circ \rho(I+L)\big)\, f(X) = df(X+LX)\,;$$

the second right hand of (3.1) applied to f reads as

$$(d+D_L+r(L))f(X) = df(X)+(\tau(L)\circ df)(X) = df(X)+df(LX) = df(X+LX)$$

so that the formula holds in this case.

Let now $\alpha \in \Lambda^1(M)$. Then, for every pair of vector fields X, Y on M, we have

$$\begin{aligned}
&\big(\rho(I+L)^{-1} \circ d \circ \rho(I+L)\big)\, \alpha(X,Y) = \\
&= \quad X\alpha(Y) + LX\alpha(Y) - Y\alpha(X) - LY\alpha(X) + \\
&\qquad -\alpha\left((I+L)^{-1}[(I+L)X,(I+L)Y]\right) \\
&= \quad d\alpha(X,Y) + LX\alpha(Y) - LY\alpha(X) + \alpha([X,Y]) - \alpha\left((I+L)^{-1}[X,Y]\right) \\
&\qquad -\alpha\left((I+L)^{-1}[X,LY]\right) - \alpha\left((I+L)^{-1}[LX,Y]\right) - \alpha\left((I+L)^{-1}[LX,LY]\right).
\end{aligned}$$

By using that $I-(I+L)^{-1} = (I+L)^{-1}L$, we may write the above expression as

$$\begin{aligned}
&\big(\rho(I+L)^{-1} \circ d \circ \rho(I+L)\big)\, \alpha(X,Y) = \\
&\quad = \quad d\alpha(X,Y) + LX\alpha(Y) - LY\alpha(X) + \\
&\qquad +\alpha\left((I+L)^{-1}L[X,Y]\right) - \alpha\left((I+L)^{-1}[X,LY]\right) \\
&\qquad -\alpha\left((I+L)^{-1}[LX,Y]\right) - \alpha\left((I+L)^{-1}[LX,LY]\right).
\end{aligned} \tag{5.1}$$

We expand now the right hand side of the formula: we need to compute $r(\alpha)$,

$D_L\alpha$ and $d\alpha$. We get

$$\begin{aligned}
r(L)(\alpha)(X,Y) &= \alpha((I+L)^{-1}[LX,LY]) + \alpha(L[X,Y]) - \alpha((I+L)^{-1}L[X,Y]) \\
&\quad - \alpha([X,LY]) + \alpha((I+L)^{-1}[X,LY]) - \alpha([LX,Y]) \\
&\quad + \alpha((I+L)^{-1}[LX,Y]),
\end{aligned} \tag{5.2}$$

where we used again that $(I+L)^{-1}L = I-(I+L)^{-1}$.

For $D_L\alpha$ we obtain

$$\begin{aligned}
D_L\alpha(X,Y) &= (\tau(L)\circ d)(\alpha)(X,Y) - (d\circ\tau(L))(\alpha)(X,Y) = LX\alpha(Y) + \\
&\quad - Y\alpha(LX) - \alpha([LX,Y]) + X\alpha(LY) - LY\alpha(X) + \\
&\quad - \alpha([X,LY]) - X\alpha(LY) + Y\alpha(LX) + \alpha(L[X,Y])\,,
\end{aligned}$$

i.e.,

$$D_L\alpha(X,Y) = LX\alpha(Y) - \alpha([LX,Y]) - LY\alpha(X) - \alpha([X,LY]) + \alpha(L[X,Y]). \tag{5.3}$$

Then, taking into account (5.1), (5.2) and (5.3), we get that

$$(\rho(I+L)^{-1} \circ d \circ \rho(I+L))(\alpha)(X,Y) = (d + D_L - r(L))(\alpha)(X,Y).$$

□

BIBLIOGRAPHY

[1] L. Auslander, L. Green & F. Hahn, *Flows on homogeneous spaces*, With the assistance of L. Markus and W. Massey, and an appendix by L. Greenberg. Annals of Mathematics Studies, No. 53, Princeton University Press, Princeton, N.J., 1963, vii+107 pages.

[2] P. de Bartolomeis, "Some constructions with Symplectic Manifolds and Lagrangian Submanifolds", preprint.

[3] P. de Bartolomeis & A. Tomassini, "On formality of some symplectic manifolds", *Internat. Math. Res. Notices* (2001), no. 24, p. 1287-1314.

[4] ———, "On the Maslov index of Lagrangian submanifolds of generalized Calabi-Yau manifolds", *Internat. J. Math.* **17** (2006), no. 8, p. 921-947.

[5] J. Cheeger & D. Gromoll, "On the structure of complete manifolds of nonnegative curvature", *Ann. of Math. (2)* **96** (1972), p. 413-443.

[6] P. Deligne, P. Griffiths, J. Morgan & D. Sullivan, "Real homotopy theory of Kähler manifolds", *Invent. Math.* **29** (1975), no. 3, p. 245-274.

[7] M. Fernández & A. Gray, "Compact symplectic solvmanifolds not admitting complex structures", *Geom. Dedicata* **34** (1990), no. 3, p. 295-299.

[8] A. Fujiki & G. Schumacher, "The moduli space of Kähler structures on a real compact symplectic manifold", *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 1, p. 141-168.

[9] K. Hasegawa, "Complex and Kähler structures on compact solvmanifolds", *J. Symplectic Geom.* **3** (2005), no. 4, p. 749-767, Conference on Symplectic Topology.

[10] ———, "A note on compact solvmanifolds with Kähler structures", *Osaka J. Math.* **43** (2006), no. 1, p. 131-135.

[11] A. Hattori, "Spectral sequence in the de Rham cohomology of fibre bundles", *J. Fac. Sci. Univ. Tokyo Sect. I* **8** (1960), p. 289-331 (1960).

[12] N. Hitchin, "Generalized Calabi-Yau manifolds", *Q. J. Math.* **54** (2003), no. 3, p. 281-308.

[13] J. Lafontaine & M. Audin, "Introduction: applications of pseudo-holomorphic curves to symplectic topology", in *Holomorphic curves in symplectic geometry*, Progr. Math., vol. 117, Birkhäuser, Basel, 1994, p. 1-14.

[14] A. I. Malcev, "On a class of homogeneous spaces", *Amer. Math. Soc. Translation* **1951** (1951), no. 39, p. 33.

[15] I. Nakamura, "Complex parallelisable manifolds and their small deformations", *J. Differential Geometry* **10** (1975), p. 85-112.

[16] G. Tian, "Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric", in *Mathematical aspects of string theory (San Diego, Calif., 1986)*, Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, p. 629-646.

[17] A. N. Todorov, “The Weil-Petersson geometry of the moduli space of SU($n \geqslant 3$) (Calabi-Yau) manifolds. I”, *Comm. Math. Phys.* **126** (1989), no. 2, p. 325-346.

[18] D. Yan, “Hodge structure on symplectic manifolds”, *Adv. Math.* **120** (1996), no. 1, p. 143-154.

[19] S. T. Yau, “On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I”, *Comm. Pure Appl. Math.* **31** (1978), no. 3, p. 339-411.

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