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# LOCAL-GLOBAL PRINCIPLE FOR QUADRATIC FORMS OVER FRACTION FIELDS OF TWO-DIMENSIONAL HENSELIAN DOMAINS

by Yong HU

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ABSTRACT. — Let  $R$  be a 2-dimensional normal excellent henselian local domain in which 2 is invertible and let  $L$  and  $k$  be its fraction field and residue field respectively. Let  $\Omega_R$  be the set of rank 1 discrete valuations of  $L$  corresponding to codimension 1 points of regular proper models of  $\text{Spec } R$ . We prove that a quadratic form  $q$  over  $L$  satisfies the local-global principle with respect to  $\Omega_R$  in the following two cases: (1)  $q$  has rank 3 or 4; (2)  $q$  has rank  $\geq 5$  and  $R = A[[y]]$ , where  $A$  is a complete discrete valuation ring with a not too restrictive condition on the residue field  $k$ , which is satisfied when  $k$  is  $C_1$ .

RÉSUMÉ. — Soit  $R$  un anneau local intègre de dimension 2, normal, excellent et hensélien dans lequel 2 est inversible. Soient  $L$  son corps de fractions et  $k$  son corps résiduel. Soit  $\Omega_R$  l'ensemble des valuations discrètes de rang 1 de  $L$  correspondant aux points de codimension 1 des modèles propres réguliers de  $\text{Spec } R$ . On démontre qu'une forme quadratique  $q$  sur  $L$  satisfait le principe local-global par rapport à  $\Omega_R$  dans les deux cas suivants : (1)  $q$  est de rang 3 ou 4 ; (2)  $q$  est de rang  $\geq 5$  et  $R = A[[y]]$ , où  $A$  est un anneau de valuation discrète complet, avec une condition sur le corps résiduel  $k$  qui est satisfaite lorsque  $k$  est  $C_1$ .

## 1. Statements of results

Let  $R$  be a 2-dimensional excellent henselian local domain and let  $L$  and  $k$  be respectively its fraction field and residue field. Assume that the characteristic of  $k$  is not 2.

Colliot-Thélène, Ojanguren and Parimala [2] proved that any quadratic form of rank at least 5 over  $L$  is isotropic when  $k$  is separably closed, and that the local-global principle with respect to all discrete valuations (of

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rank 1) on  $L$  holds for quadratic forms of rank 3 or 4 when  $k$  is separably closed or finite. For the first result, the special case where  $R = \mathbb{C}[[x, y]]$  was proven earlier in [1] using the Weierstraß preparation theorem. On the other hand, Jaworski [6] proved that if  $k$  is an algebraically closed field, then quadratic forms of any rank over  $L = k((x, y))$  satisfy the local-global principle with respect to all discrete valuations on  $L$ .

In the case where  $k$  is finite, however, whether the local-global principle holds for quadratic forms of rank  $\geq 5$  is left an open question. In this paper, we give an affirmative answer to this question in the case where  $R = A[[y]]$  is the ring of formal power series in one variable over a complete discrete valuation ring  $A$ . Also, we prove that the result of Colliot-Thélène, Ojanguren and Parimala about the local-global principle for quadratic forms of rank 3 or 4 is still valid without the assumption that  $k$  is separably closed or finite.

The more precise statements are the following.

**THEOREM 1.1.** — *Let  $R$  be a 2-dimensional normal excellent henselian local domain in which 2 is invertible. Let  $L$  and  $k$  be respectively the fraction field and the residue field of  $R$ . For any regular integral scheme  $\mathcal{M}$  equipped with a proper birational morphism  $\mathcal{M} \rightarrow \text{Spec } R$ , let  $\Omega_{\mathcal{M}}$  denote the set of rank 1 discrete valuations of  $L$  that correspond to codimension 1 points of  $\mathcal{M}$ . Let  $\Omega_R$  be the union of all  $\Omega_{\mathcal{M}}$ .*

*Then the local-global principle with respect to  $\Omega_R$  holds for quadratic forms of rank 3 or 4 over  $L$ . Namely, if a quadratic form of rank 3 or 4 over  $L$  has a nontrivial zero over the  $w$ -adic completion  $L_w$  for every  $w \in \Omega_R$ , then it has a nontrivial zero over  $L$ .*

**THEOREM 1.2.** — *Let  $A$  be a complete discrete valuation ring in which 2 is invertible, and let  $K$  and  $k$  be respectively its fraction field and residue field. Let  $R = A[[y]]$  and  $L = \text{Frac}(R)$  the fraction field of  $R$ . Define  $\Omega_R$  as in Theorem 1.1.*

*Assume that the residue field  $k$  has the following property:*

- (\*) *for every finite field extension  $k'/k$ , every quadratic form of rank  $\geq 3$  over  $k'$  is isotropic.*

*Then the local-global principle with respect to discrete valuations in  $\Omega_R$  holds for quadratic forms of rank  $\geq 5$  over  $L$ .*

Recall that a field  $k$  is called a  $C_i$  field if every homogeneous polynomial of degree  $d$  in  $n > d^i$  variables has a nontrivial zero over  $k$ . A finite field extension of a  $C_i$  field is again a  $C_i$  field. Clearly, a  $C_1$  field  $k$  has property (\*). So as typical examples to which Theorem 1.2 applies, we may take

$R = \mathbb{F}[[x, y]]$  where  $\mathbb{F}$  is a finite field of characteristic  $> 2$ , or  $R = \mathcal{O}_K[[y]]$  where  $\mathcal{O}_K$  is the ring of integers of a  $p$ -adic number field  $K$  ( $p$  is an odd prime).

*Remark 1.3.* — Note that property  $(*)$  implies the following:

$(**)$  for every finite field extension  $K'/K$ , every quadratic form of rank  $\geq 5$  over  $K'$  is isotropic.

Indeed, the integral closure  $A'$  of  $A$  in  $K'$  is a complete discrete valuation ring and is finite over  $A$  (cf., [12, p. 28, § II.2, Prop. 3]). The residue field  $k'$  of  $A'$  is a finite extension of  $k$ . Any quadratic form  $q$  over  $K'$  is isometric to a form  $q_1 \perp t.q_2$ , where  $t$  is a uniformizer of  $A'$  and the coefficients of  $q_1, q_2$  are all units in  $A'$ . When  $q$  has rank  $\geq 5$  and  $k$  has property  $(*)$ , a standard argument using Springer's lemma (cf., Lemma 4.1) shows that  $q$  is isotropic over  $K'$ .

Let  $A, k, K$  and so on be as in Theorem 1.2. Let  $x \in A$  be a uniformizer of  $A$  and  $F = K(y)$  the function field of  $\mathbb{P}_K^1$ . For any regular integral scheme  $\mathcal{P}$  equipped with a proper flat morphism  $\mathcal{P} \rightarrow \text{Spec } A$  with generic fiber  $\mathcal{P} \times_A K \cong \mathbb{P}_K^1$ , let  $\Omega_{\mathcal{P}}$  denote the set of rank 1 discrete valuations of  $F$  that correspond to codimension 1 points of  $\mathcal{P}$ . Let  $\Omega_A$  be the union of all  $\Omega_{\mathcal{P}}$ . Then we have the following proposition.

**PROPOSITION 1.4.** — *With notation as above, let  $q/F = \langle a_1, \dots, a_r \rangle$  be a nonsingular diagonal quadratic form of rank  $r \geq 5$  with  $a_i \in A[y]$ . Let  $\Sigma \subseteq A$  be a fixed set of representatives of  $k^*$  in  $A$ . Assume that*

$$(1.1) \quad a_i = \lambda_i \cdot x^{n_i} \cdot P_i,$$

where  $\lambda_i \in \Sigma, n_i \in \{0, 1\}$  and  $P_i$  is a distinguished polynomial of degree  $m_i$  in  $A[y]$  (meaning that  $P_i$  is a monic polynomial in  $A[y]$  whose reduction mod  $x$  is  $y^{m_i} \in k[y]$ ).

If for every  $w \in \Omega_R, q$  is isotropic over the completion  $L_w$  of  $L$  with respect to  $w$ , then for every  $v \in \Omega_A, q$  is isotropic over the completed field  $F_v$ .

As we shall see at the end of the paper, Theorem 1.2 follows by combining the above proposition with a theorem of Colliot-Thélène, Parimala and Suresh [3] on quadratic forms over  $F = K(y)$ , whose proof builds upon earlier work of Harbater, Hartmann and Krashen [4].

## 2. Valuations coming from blow-ups

LEMMA 2.1. — *Let  $A$  be an excellent local domain with residue field  $k$  and  $\mathcal{X}$  an integral  $A$ -scheme of finite type. Let  $F$  be the function field of  $\mathcal{X}$  and  $v$  a rank 1 discrete valuation of  $F$  with valuation ring  $\mathcal{O}_v$ . Assume that  $v$  is centered on  $\mathcal{X}$  at a point  $x$  in the closed fiber  $X := \mathcal{X} \times_A k$  and that the residue field  $\kappa(v)$  of  $\mathcal{O}_v$  has transcendence degree  $\text{trdeg}_k \kappa(v) = \dim \mathcal{X} - 1$  over  $k$ . Let  $\mathcal{Y} = \text{Spec } \mathcal{O}_v$  and  $y \in \mathcal{Y}$  the closed point of  $\mathcal{Y}$ . Let  $f : \mathcal{Y} \rightarrow \mathcal{X}$  be the natural morphism induced by the inclusion  $\mathcal{O}_{\mathcal{X},x} \subseteq \mathcal{O}_v$ . Define schemes  $\mathcal{X}_n, n \in \mathbb{N}$  and morphisms  $f_n : \mathcal{Y} \rightarrow \mathcal{X}_n, n \in \mathbb{N}$  as follows:*

*Set  $\mathcal{X}_0 = \mathcal{X}$  and  $f_0 = f$ . When  $f_i : \mathcal{Y} \rightarrow \mathcal{X}_i$  is already defined, let  $\mathcal{X}_{i+1} \rightarrow \mathcal{X}_i$  be the blow-up of  $\mathcal{X}_i$  along the closure of  $x_i := f_i(y)$  and let  $f_{i+1} : \mathcal{Y} \rightarrow \mathcal{X}_{i+1}$  be the induced morphism.*

*Then for some large enough  $n$ , the morphism  $f_n : \mathcal{Y} \rightarrow \mathcal{X}_n$  induces an isomorphism  $\mathcal{O}_{\mathcal{X}_n, x_n} \cong \mathcal{O}_v$ .*

*Proof.* — The following proof is an easy adaptation of the proof of the geometric case, as given in [7, p. 61, Lemma 2.45].

Let  $\mathcal{O}_n := \mathcal{O}_{\mathcal{X}_n, x_n}$ . The ring theoretic construction of  $\mathcal{O}_n$  is as follows. Assume that  $\mathcal{O}_n$  (with maximal ideal  $\mathfrak{m}_n$ ) is already defined. Pick a system of generators  $z_1, \dots, z_r$  of  $\mathfrak{m}_n$  such that  $v(z_1) \leq \dots \leq v(z_r)$ . Let  $\mathcal{O}'_n = \mathcal{O}_n[z_2/z_1, \dots, z_r/z_1]$ . Then  $\mathcal{O}_{n+1}$  is the localization of  $\mathcal{O}'_n$  at  $\mathcal{O}'_n \cap \mathfrak{m}_v$ , where  $\mathfrak{m}_v$  denotes the maximal ideal of  $\mathcal{O}_v$ .

The same argument as in the proof of [7, p. 61, Lemma 2.45] applies here and shows that  $\mathcal{O}_v = \bigcup_{n \geq 0} \mathcal{O}_n$ . Pick elements  $u_1, \dots, u_t \in \mathcal{O}_v \subseteq F$  such that the reductions  $\bar{u}_i$  form a transcendence basis of  $\kappa(v) = \mathcal{O}_v/\mathfrak{m}_v$  over  $k$ . Choose  $n$  big enough so that  $u_1, \dots, u_t \in \mathcal{O}_n$ . Then  $\kappa(v) = \mathcal{O}_v/\mathfrak{m}_v$  is an algebraic extension of  $\kappa(x_n) = \mathcal{O}_n/\mathfrak{m}_n$  and

$$\text{trdeg}_k \kappa(x_n) = \text{trdeg}_k \kappa(v) = \dim \mathcal{X} - 1.$$

The closure  $Z_n := \overline{\{x_n\}}$  of  $x_n$  in  $\mathcal{X}_n$  is an algebraic scheme over  $k$ . So we have

$$\dim Z_n = \text{trdeg}_k \kappa(x_n) = \dim \mathcal{X} - 1.$$

By [10, p. 334, Coro. 8.2.7], we have  $\dim \mathcal{X}_n = \dim \mathcal{X}$ . Hence,

$$\dim \mathcal{O}_n = \text{codim}(Z_n, \mathcal{X}_n) \leq \dim \mathcal{X}_n - \dim Z_n = 1.$$

But  $\mathcal{O}_n \subseteq \mathcal{O}_v$  and the discrete valuation ring  $\mathcal{O}_v$  is unequal to its fraction field  $F = \text{Frac}(\mathcal{O}_v) = \text{Frac}(\mathcal{O}_n)$ , so  $\dim \mathcal{O}_n = 1$ . Let  $R' \subseteq F$  be the normalization of  $\mathcal{O}_n$  and let  $\mathfrak{m}' = \mathfrak{m}_v \cap R'$ . Then  $R'$  is a Dedekind domain and  $R'_{\mathfrak{m}'}$  is a discrete valuation ring contained in  $\mathcal{O}_v$  with fraction field  $F$ . Therefore,  $R'_{\mathfrak{m}'} = \mathcal{O}_v$ . The ring  $\mathcal{O}_n$  is a Nagata ring (see e.g., [10, p. 340, Prop. 8.2.29

and p. 343, Thm 8.2.39]). So  $R'$  is a finitely generated  $\mathcal{O}_n$ -module. Thus we have  $R' \subseteq \mathcal{O}_N$  for some large  $N \in \mathbb{N}$ . Then it follows that  $\mathcal{O}_v = \mathcal{O}_{N+1}$ . The lemma is thus proved.  $\square$

### 3. Proof of Theorem 1.1

Theorem 1.1 is a statement generalizing [2, Thm 3.1], where the result is only established under the hypothesis that  $k$  is separably closed or finite. In our proof the observation that [2, Prop. 1.14] holds over an arbitrary field  $k$  is the key point which makes it possible to get rid of this restriction on  $k$ . In addition, Lemma 2.1 will be used in order to obtain the local-global principle for valuations in the subset  $\Omega_R$  instead of the set of all discrete valuations.

LEMMA 3.1. — *Let  $R$  be a two-dimensional normal excellent henselian local domain with fraction field  $L$ ,  $L'/L$  a finite field extension and  $R'$  the integral closure of  $R$  in  $L'$ . Let  $w'$  be a discrete valuation of  $L'$  lying over a discrete valuation  $w$  of  $L$ .*

*If  $w'$  corresponds to a codimension 1 point on a regular proper model  $\mathcal{X}'$  of  $R'$  (i.e.,  $\mathcal{X}'$  is a regular integral scheme equipped with a proper birational morphism  $\mathcal{X}' \rightarrow \text{Spec } R'$ ), then  $w$  corresponds to a codimension 1 point on a regular proper model  $\mathcal{X}$  of  $R$ .*

*Proof.* — Let  $k$  (resp.  $k'$ ) be the residue field of  $R$  (resp.  $R'$ ). Since  $R$  is excellent,  $R'$  is finite over  $R$  and hence  $k'/k$  is a finite extension. Let  $x' \in \mathcal{X}'$  be the center of  $w'$  on  $\mathcal{X}'$ ,  $p'$  the canonical image of  $x'$  in  $\text{Spec } R'$  and  $p$  the canonical image of  $p'$  in  $\text{Spec } R$ .

If  $p$  is not the closed point of  $\text{Spec } R$ , then it has codimension 1 in  $\text{Spec } R$  and the valuation ring  $\mathcal{O}_w$  of  $w$  is equal to the local ring of  $p$  in  $\text{Spec } R$ , since  $R$  is a 2-dimensional normal local domain. Let  $V$  be the complement of the closed point in  $\text{Spec } R$ . For any regular proper model  $\pi : \mathcal{X} \rightarrow \text{Spec } R$ , which exists by resolution of singularities,  $\pi^{-1}(V) \rightarrow V$  is an isomorphism since  $R$  is normal (cf., [10, p. 150, Coro. 4.4.3]). Hence, the point  $x = \pi^{-1}(p)$  has codimension 1 in  $\mathcal{X}$  and is the center of  $w$  on  $\mathcal{X}$ .

Now assume that  $p$  is the closed point of  $\text{Spec } R$ . Then  $x' \in \mathcal{X}'$  lies in the closed fiber of  $\mathcal{X}'/R'$  and is the generic point of an integral curve over  $k' = \kappa(p')$ . Hence, the residue field  $\kappa(w')$  of  $w'$  has transcendence degree 1 over  $k'$ . Since  $k'/k$  and  $\kappa(w')/\kappa(w)$  are finite extensions, this implies that the residue field  $\kappa(w)$  has transcendence degree 1 over  $k$ . By taking any regular proper model  $\mathcal{X} \rightarrow \text{Spec } R$  and applying Lemma 2.1 to the ring

$R$  and the  $R$ -scheme  $\mathcal{X}$ , we conclude that there is a morphism  $\mathcal{X}_n \rightarrow \mathcal{X}$  obtained by a sequence of blow-ups such that the center of  $w$  on  $\mathcal{X}_n$  is a point of codimension 1, which completes the proof.  $\square$

Given a scheme  $Y$ , we will denote by  $\mathrm{Br}(Y) = H_{\text{ét}}^2(Y, \mathbb{G}_m)$  its cohomological Brauer group.

*Proof of Theorem 1.1.* — For any  $a, b \in L^*$ , the isotropy of the rank 3 form  $\langle 1, a, b \rangle$  is equivalent to the isotropy of the rank 4 form  $\langle 1, a, b, ab \rangle$ . So we may restrict to the case of rank 4 forms. Let  $q$  be a rank 4 quadratic form over  $L$  which is isotropic over  $L_w$  for every  $w \in \Omega_R$ . After scaling we may assume without loss of generality that  $q = \langle 1, a, b, abd \rangle$  with  $a, b, d \in L^*$ .

First assume that  $d$  is a square in  $L$ . Then the quadratic form  $q$  is isomorphic to the norm form of a quaternion algebra, whose class in the Brauer group  $\mathrm{Br}(L)$  will be denoted by  $\alpha$ . The form  $q$  is isotropic if and only if  $\alpha = 0$  in the Brauer group.

Take a proper birational morphism  $\mathcal{X} \rightarrow \mathrm{Spec} R$  with  $\mathcal{X}$  a regular integral scheme such that the closed fiber  $X$  of  $\mathcal{X}/R$  is a curve over  $k$ . For each  $w \in \Omega_R$  corresponding to a codimension 1 point of  $\mathcal{X}$ , the canonical image  $\alpha_w$  of  $\alpha$  in  $\mathrm{Br}(L_w)$  is trivial since  $q$  is isotropic over  $L_w$  by assumption. In particular, the residue of  $\alpha$  at every codimension 1 point of  $\mathcal{X}$  is trivial. Since  $\mathcal{X}$  is a regular integral scheme, it follows that  $\alpha \in \mathrm{Br}(L)$  lies in the subgroup  $\mathrm{Br}(\mathcal{X})$ . By [2, Thm 1.8 (c) and Lemma 1.6], we have canonical isomorphisms  $\mathrm{Br}(\mathcal{X}) \cong \mathrm{Br}(X) \cong \mathrm{Br}(X_{\text{red}})$ . Identify  $\alpha \in \mathrm{Br}(\mathcal{X})$  with its canonical image in  $\mathrm{Br}(X_{\text{red}})$ . We will apply [2, Prop. 1.14] to show that  $\alpha = 0$ .

Let  $f : Z \rightarrow X_{\text{red}}$  be the normalization of the reduced curve  $X_{\text{red}}/k$  and let  $D \subseteq X_{\text{red}}$  be the closed subscheme defined by the conductor of  $f$ . Then [2, Prop. 1.14] says that the natural map  $\mathrm{Br}(X_{\text{red}}) \rightarrow \mathrm{Br}(Z) \times \mathrm{Br}(D)$  is injective. Let  $(\alpha_1, \alpha_2) \in \mathrm{Br}(Z) \times \mathrm{Br}(D)$  be the image of  $\alpha \in \mathrm{Br}(X_{\text{red}})$ . Each reduced irreducible component  $T$  of  $Z$  is a regular integral curve whose function field  $k(T)$  is the residue field  $\kappa(w)$  of a codimension 1 point  $w$  of the 2-dimensional regular scheme  $\mathcal{X}$ . Since  $\alpha$  vanishes in  $\mathrm{Br}(L_w)$  by hypothesis, the specialisation of  $\alpha$  in  $\mathrm{Br}(\kappa(w)) = \mathrm{Br}(k(T))$  is zero. The natural map  $\mathrm{Br}(T) \rightarrow \mathrm{Br}(k(T))$  is an injection for the regular scheme  $T$ , so the canonical image of  $\alpha$  in  $\mathrm{Br}(T)$  is zero. Since this holds for every irreducible component  $T$  of  $Z$ , we have  $\alpha_1 = 0$  in  $\mathrm{Br}(Z)$ .

To show that  $\alpha_2 = 0$  in  $\mathrm{Br}(D)$ , it suffices to prove that  $\alpha_2$  vanishes at each closed point  $x$  of  $X_{\text{red}}$ , by a 0-dimensional variant of [2, Lemma 1.6]. The point  $x$  is also a closed point of  $\mathcal{X}$ . We may choose a 1-dimensional closed integral subscheme  $C$  of  $\mathcal{X}$  which contains  $x$  as a regular point and

let  $\omega \in \mathcal{X}$  be the generic point of  $C$ . Our hypothesis implies that  $\alpha \in \text{Br}(\mathcal{X})$  vanishes at  $\omega$ , and it follows that there is a regular open subscheme  $U$  of  $C$ , containing  $x$ , such that  $\alpha|_U = 0$  in  $\text{Br}(U) \subseteq \text{Br}(\kappa(\omega))$ . Hence,  $\alpha_2(x) = \alpha(x) = 0$ . We have thus proved that  $\alpha = 0$  in  $\text{Br}(L)$ , whence the isotropy of the rank 4 quadratic form  $q = \langle 1, a, b, abd \rangle$ .

Now suppose that  $d$  is not a square in  $L$ . Let  $L' = L(\sqrt{d})$  and  $R'$  the integral closure of  $R$  in  $L'$ . Then  $R'$  and  $L'$  satisfy the same assumptions as  $R$  and  $L$ . Let  $w'$  be a discrete valuation on  $L'$  corresponding to a codimension 1 point of a regular proper model  $\mathcal{X}'/R'$ . By Lemma 3.1,  $w'$  lies over a discrete valuation  $w$  in  $\Omega_R$ . The isotropy of  $q$  over  $L_w$  implies the isotropy of  $q_{L'}$  over  $L'_{w'}$ .

Thus the quadratic form  $q_{L'}$  over  $L'$  has trivial determinant and is isotropic over  $L'_{w'}$  for every  $w' \in \Omega_{R'}$ , where the set  $\Omega_{R'}$  of discrete valuations of  $L'$  is defined in the same way as  $\Omega_R$ . By the previous case,  $q_{L'}$  is isotropic over  $L'$ . By [8, p. 197, Chap. VII, Thm 3.1], either  $q$  is isotropic over  $L$  or  $q$  contains a multiple of  $\langle 1, -d \rangle$ . In the latter case, since  $\det(q) = d \pmod{(L^*)^2}$ ,  $q$  also contains a rank 2 form of determinant  $-1$ . Hence  $q$  is isotropic over  $L$ , which completes the proof.  $\square$

#### 4. Valuations centered on the special fiber

Most of the present section and the next will be devoted to the proof of Proposition 1.4. The lemma below will be used frequently and referred to as Springer’s lemma in what follows.

LEMMA 4.1 (Springer’s lemma, [8, p. 148, Prop. VI.1.9]). — *Let  $A$  be a complete discrete valuation ring in which 2 is invertible. Let  $K$  and  $k$  be respectively its fraction field and residue field. Let  $\alpha_1, \dots, \alpha_r$  and  $\beta_1, \dots, \beta_s$  be units of  $A$  and let  $\bar{\alpha}_i \in k$  and  $\bar{\beta}_j \in k$  be their residue classes. Let  $\pi$  be a uniformizer of  $A$ .*

*Then the quadratic form  $\langle \alpha_1, \dots, \alpha_r \rangle \perp \pi \cdot \langle \beta_1, \dots, \beta_s \rangle$  over  $K$  is anisotropic if and only if the two residue forms*

$$\langle \bar{\alpha}_1, \dots, \bar{\alpha}_r \rangle \quad \text{and} \quad \langle \bar{\beta}_1, \dots, \bar{\beta}_s \rangle$$

*are both anisotropic over  $k$ .*

We shall now start the proof of Proposition 1.4. Recall that  $\Omega_A$  is the union of all  $\Omega_{\mathcal{P}}$ , where  $\mathcal{P}$  is a regular integral proper flat  $A$ -scheme with generic fiber  $\mathcal{P} \times_A K \cong \mathbb{P}_K^1$  and  $\Omega_{\mathcal{P}}$  is the set of rank 1 discrete valuations on  $F = K(y)$  that correspond to codimension 1 points of  $\mathcal{P}$ . We will fix a



discrete valuation  $v \in \Omega_A$  and let  $\mathcal{O}_v \subseteq F$  denote the valuation ring of  $v$ ,  $\pi_v \in \mathcal{O}_v$  a uniformizer of  $v$ ,  $\mathfrak{m}_v = \pi_v \mathcal{O}_v$  and  $\kappa(v)$  the residue field of  $\mathcal{O}_v$ . The  $v$ -adic completion of  $\mathcal{O}_v \subseteq F$  will be written as  $\widehat{\mathcal{O}}_v \subseteq F_v$ . If  $w$  is a discrete valuation of  $L$ , similar notations like  $\mathcal{O}_w, \mathfrak{m}_w, \kappa(w), \widehat{\mathcal{O}}_w \subseteq L_w$  and so on will be used.

Put  $\mathcal{X} = \mathbb{P}_A^1$ . Let  $\mathcal{X}_K = \mathbb{P}_K^1$  and  $\mathcal{X}_s = \mathbb{P}_k^1$  be respectively the generic and special fiber of  $\mathcal{X}$  over  $A$ . Let  $\eta \in \mathcal{X}_s = \mathbb{P}_k^1$  denote the generic point of  $\mathcal{X}_s$ . The valuation  $v \in \Omega_A$  has a unique center on the model  $\mathcal{X} = \mathbb{P}_A^1$ , which will be denoted  $P \in \mathcal{X}$ . We have the following cases:

- (1)  $P \in \mathcal{X}_s = \mathbb{P}_k^1, P \neq 0, \infty, \eta$ ;
- (2)  $P = \eta \in \mathcal{X}_s = \mathbb{P}_k^1$ ;
- (3)  $P = \infty \in \mathcal{X}_s = \mathbb{P}_k^1$  or  $P = \infty \in \mathcal{X}_K = \mathbb{P}_K^1$ ;
- (4)  $P = 0 \in \mathcal{X}_s = \mathbb{P}_k^1$ ;
- (5)  $P$  is a closed point of  $\mathbb{A}_K^1 \subseteq \mathcal{X}_K = \mathbb{P}_K^1$ .

Our proof of Proposition 1.4 will be a case-by-case argument, which is divided into two parts with details in what follows.

*Proof of Proposition 1.4 (Part I).* — In the first part of the proof, we treat cases (1)–(4).

Case (1). The valuation  $v$  is centered at  $P \in \mathcal{X}_s \setminus \{0, \infty, \eta\}$ .

In this case, we have  $v(x) > 0$  and  $v(y) = 0$ . We may assume without loss of generality that for some  $0 \leq r_1 \leq r$ , the numbers  $n_i$  in (1.1) satisfy:

$$(4.1) \quad n_1 = \dots = n_{r_1} = 0 \quad \text{and} \quad n_{r_1+1} = \dots = n_r = 1.$$

Then  $a_1, \dots, a_{r_1}$  and  $a'_{r_1+1} = a_{r_1+1}/x, \dots, a'_r = a_r/x$  are units for  $v$ . Let

$$(4.2) \quad q_1 = \langle a_1, \dots, a_{r_1} \rangle \quad \text{and} \quad q_2 = \langle a'_{r_1+1}, \dots, a'_r \rangle.$$

Then  $q = \langle a_1, \dots, a_r \rangle = q_1 \perp x \cdot q_2$  is anisotropic only if  $q_1$  and  $q_2$  are both anisotropic. By Springer’s lemma (or Hensel’s lemma),  $q_i$  is anisotropic over  $F_v$  if and only if its residue form  $\bar{q}_i := q_i \pmod{\mathfrak{m}_v}$  is anisotropic over  $\kappa(v)$ . In the present situation, the two residue forms  $\bar{q}_i, i = 1, 2$  have coefficients in the subfield  $\kappa(P) \subseteq \kappa(v)$ . Since  $r \geq 5$ , either  $q_1$  or  $q_2$  has rank  $\geq 3$ . Assume for example  $q_1$  has rank  $\geq 3$ . The residue field  $\kappa(P)$  is a finite extension of  $k$ , so property (\*) implies that  $\bar{q}_1$  is isotropic over  $\kappa(P)$  and a fortiori over  $\kappa(v)$ . It follows that  $q$  is isotropic over  $F_v$  as desired.

Case (2). The valuation  $v$  is centered at the generic point  $\eta$  of the special fiber  $\mathcal{X}_s = \mathbb{P}_k^1$ .

In this case,  $v$  is the  $x$ -adic valuation on  $A[y]$  and  $\kappa(v) = k(y)$ . Let  $w$  be the  $x$ -adic valuation on  $A[[y]]$ , so that  $w|_{A[y]} = v|_{A[y]}$  and  $\kappa(w) = k((y))$ .

Define  $q_1$  and  $q_2$  as in (4.2). We have

$$(4.3) \quad \begin{aligned} \bar{q}_1 &:= q_1 \pmod{\mathfrak{m}_w} = \langle \lambda_1 y^{m_1}, \dots, \lambda_{r_1} y^{m_{r_1}} \rangle, \\ \bar{q}_2 &:= q_2 \pmod{\mathfrak{m}_w} = \langle \lambda_{r_1+1} y^{m_{r_1+1}}, \dots, \lambda_r y^{m_r} \rangle. \end{aligned}$$

Here we have identified each  $\lambda_i \in \Sigma \subseteq A$  with its canonical image in  $k$ . By hypothesis and Springer’s lemma, we may assume one of the two residue forms, say  $\bar{q}_1$ , is isotropic over  $k((y))$ . By (4.3),  $\bar{q}_1$  has coefficients in  $k(y)$  and is isometric to  $\mu_1 \perp y.\mu_2$  over  $k(y)$  for some nonsingular quadratic forms  $\mu_i$  over  $k$ . Indeed, if  $I$  (resp.  $J$ ) denotes the subset of  $\{1, \dots, r_1\}$  consisting of indices  $i$  such that  $m_i$  is even (resp. odd), then we may take  $\mu_1 = \langle \lambda_i \rangle_{i \in I}$  (resp.  $\mu_2 = \langle \lambda_i \rangle_{i \in J}$ ). Applying Springer’s lemma to the form  $\bar{q}_1/k((y))$  with respect to the discrete valuation ring  $k[[y]]$ , we conclude that either  $\mu_1$  or  $\mu_2$  is isotropic over  $k$ . Then it is clear that  $\bar{q}_1 \cong \mu_1 \perp y.\mu_2$  is isotropic over  $k(y) = \kappa(v)$ . Since the residue forms of  $q \pmod v$  coincide with those  $\pmod w$ , it follows from Springer’s lemma that  $q$  is isotropic over  $F_v$ .

Case (3). The valuation  $v$  is centered at  $P = \infty \in \mathcal{X}_s = \mathbb{P}_k^1$  or  $P = \infty \in \mathcal{X}_K = \mathbb{P}_K^1$ .

In this case, we have  $v(y) < 0$  and  $v(x) \geq 0$ . Put  $z = y^{-1} \in F = K(y)$ . We want to prove that  $q$  is isotropic over  $F_v$ .

Recall that the coefficients of the diagonal form  $q$  have the form  $a_i = \lambda_i.x^{n_i}.P_i$ , where  $\lambda_i \in \Sigma$ ,  $n_i \in \{0, 1\}$  and  $P_i$  is a distinguished polynomial in  $A[y]$  for each  $i$ . Let  $m_i = \deg P_i$  be the degree of  $P_i$  with respect to the variable  $y$ . Then in  $F = K(y)$  we have

$$P_i(y) = y^{m_i}(1 + z.\rho_i) \quad \text{for some } \rho_i \in A[z].$$

Set  $b_i = \lambda_i.x^{n_i}.y^{m_i} \in F$  and let  $q'/F$  be the diagonal quadratic form  $\langle b_1, \dots, b_r \rangle$ . The two forms  $q = \langle a_i \rangle$  and  $q' = \langle b_i \rangle$  are isometric over  $F_v$  since  $1 + z.\rho_i$  is a square in  $F_v$  for each  $i$ . So it suffices to prove the isotropy over  $F_v$  of the form  $q' = \langle b_i \rangle$ .

We may assume the numbers  $n_i$  are given as in (4.1), so that  $q' = q'_1 \perp x.q'_2$  with

$$q'_1 = \langle \lambda_1 y^{m_1}, \dots, \lambda_{r_1} y^{m_{r_1}} \rangle, \quad q'_2 = \langle \lambda_{r_1+1} y^{m_{r_1+1}}, \dots, \lambda_r y^{m_r} \rangle.$$

There are diagonal quadratic forms  $\mu_j, j = 1, \dots, 4$ , where  $\mu_1, \mu_2$  have coefficients in  $\{\lambda_1, \dots, \lambda_{r_1}\} \subseteq \Sigma$  and  $\mu_3, \mu_4$  have coefficients in  $\{\lambda_{r_1+1}, \dots, \lambda_r\} \subseteq \Sigma$ , such that  $q'_1 \cong \mu_1 \perp y.\mu_2$  and  $q'_2 \cong \mu_3 \perp y.\mu_4$  over  $F = K(y)$ . Observe that the two residue forms of  $q$  with respect to the  $x$ -adic valuation on  $F$  are isometric to the forms  $\mu_1 \perp y.\mu_2$  and  $\mu_3 \perp y.\mu_4$ . A close inspection of the above proof for case (2) shows that not all of the four forms  $\mu_j$  are

anisotropic over  $k$ . Since

$$q' \cong \mu_1 \perp y \cdot \mu_2 \perp x \cdot (\mu_3 \perp y \cdot \mu_4) \quad \text{over } F = K(y),$$

it follows easily that  $q'$  is isotropic over  $F_v$ , whence the isotropy of  $q$  over  $F_v$ .

Case (4). The valuation  $v$  is centered at the origin  $P = 0 \in \mathbb{P}_k^1$  of the special fiber.

By the definition of the set  $\Omega_A$ , the valuation  $v \in \Omega_A$  corresponds to a codimension 1 point  $p$  of a regular proper model  $\mathcal{P}/A$  of  $\mathbb{P}_K^1$ . Since the center of  $v$  on  $\mathcal{X}$  lies in the special fiber,  $v(x) > 0$ . The point  $p \in \mathcal{P}$  lies in the special fiber of  $\mathcal{P}/A$  since otherwise the valuation  $v$  must be trivial on  $K = \text{Frac}(A)$ . The residue field  $\kappa(v)$  is then the function field of a curve over  $k$ . So we have

$$\text{trdeg}_k \kappa(v) = 1 = \dim \mathbb{P}_A^1 - 1.$$

By Lemma 2.1, there is a scheme  $\mathcal{X}_n \rightarrow \mathcal{X} = \mathbb{P}_A^1$  obtained by a sequence of blow-ups at closed points lying over  $0 \in \mathcal{X}_s = \mathbb{P}_k^1$  such that  $\mathcal{O}_v = \mathcal{O}_{\mathcal{X}_n, x_n} \subseteq F$  for some codimension 1 point  $x_n \in \mathcal{X}_n$ . If we consider the same sequence of blow-ups which is carried out on  $\text{Spec } A[[y]]$  this time, then we get a discrete valuation  $w \in \Omega_R$  of  $L$  which extends  $v$ . Now we have inclusions  $A[y] \subseteq \mathcal{O}_v \subseteq \mathcal{O}_w$  and  $\kappa(v) = \kappa(w)$ . Let  $q_1, q_2$  be diagonal quadratic forms with coefficients in  $\widehat{\mathcal{O}}_v^*$  such that

$$q \cong q_1 \perp \pi_v \cdot q_2 \quad \text{over } F_v.$$

Since  $q$  is isotropic over  $L_w$  by assumption, applying Springer's lemma to  $w$  shows that  $\bar{q}_1 = q_1 \pmod{\mathfrak{m}_v}$  or  $\bar{q}_2 = q_2 \pmod{\mathfrak{m}_v}$  has a nontrivial zero in  $\kappa(w) = \kappa(v)$ . One more application of Springer's lemma, with respect to  $v$  this time, proves that  $q$  is isotropic over  $F_v$ . □

### 5. End of the proof

To prove Proposition 1.4 in case (5), we need the following form of the Weierstraß preparation theorem.

LEMMA 5.1 (Weierstraß). — *Let  $A$  be a complete discrete valuation ring and  $A[[y]]$  the ring of formal power series in one variable over  $A$ . Let  $P \in A[y]$  be a distinguished polynomial and  $f \in A[[y]]$ .*

(i) *For any  $g \in A[[y]]$ , there is a unique expression*

$$g = Q \cdot P + R$$

where  $Q \in A[[y]]$  and  $R \in A[y]$  is a polynomial of degree  $\leq \deg P - 1$ . In particular,

$$A[y]/(P) \cong A[[y]]/(P).$$

(ii) If  $f$  divides  $P$  in  $A[y]$ , then there is a unit  $u$  in  $A$  such that  $uf$  is a distinguished polynomial.

*Proof.* — (i) See e.g., [13, p. 114, Prop. 7.4]. Note that the isomorphism  $A[y]/(P) \cong A[[y]]/(P)$  implies that  $P$  is irreducible in  $A[y]$  if and only if  $P$  is irreducible in  $A[[y]]$  and that  $P$  divides a polynomial  $f$  in  $A[y]$  if and only if  $P$  divides  $f$  in  $A[[y]]$ .

(ii) Assume  $P = fg$  with  $g \in A[y]$ . The hypothesis implies that the coefficient  $a_0$  of  $y^{\deg f}$  in  $f$  is a unit in  $A$  since  $P$  is a monic polynomial. Let  $k$  be the residue field of  $A$  and let  $A[y] \rightarrow k[y]$ ,  $F \mapsto \bar{F}$  denote the canonical reduction map. By considering the factorization  $y^{\deg P} = \bar{P} = \bar{f} \cdot \bar{g}$  in  $k[y]$ , we see that  $u := a_0^{-1} \in A^*$  has the required property.  $\square$

*Proof of Proposition 1.4 (Part II).* — We now consider the only remaining case, case (5). This is the case where the center  $P$  of the valuation  $v$  lies in  $\mathbb{A}_K^1 \subseteq \mathcal{X}_K = \mathbb{P}_K^1$ .

We have  $\mathcal{O}_{\mathcal{X},P} = \mathcal{O}_v$  since the two rings are both discrete valuation rings with fraction field  $F$ . So  $v$  is defined by an irreducible polynomial  $f \in A[y]$  with  $x \nmid f$ .

If none of the polynomials  $P_i, i = 1, \dots, r$  is divisible by  $f$ , then  $q$  has coefficients in  $\mathcal{O}_v^* = \mathcal{O}_{\mathcal{X},P}^*$ . Now the residue field  $\kappa(v) = \kappa(P)$  is a finite extension of  $K$  and the residue form  $\bar{q} = q \pmod{\mathfrak{m}_v}$  has rank  $r \geq 5$ . By property (\*\*) (cf., Remark 1.3),  $\bar{q}$  is isotropic over  $\kappa(v)$ . It follows from Springer’s lemma (or Hensel’s lemma) that  $q$  is isotropic over  $F_v$ .

Assume next  $f$  divides some  $P_i$ , say  $f|P_1$ . By Lemma 5.1, multiplying  $f$  by a unit in  $A$  if necessary, we may assume that  $f$  is an irreducible distinguished polynomial. In  $A[[y]]$ ,  $f$  is still an irreducible element. The  $f$ -adic valuation on  $R = A[[y]]$  determines a discrete valuation  $w \in \Omega_R$  which extends  $v \in \Omega_A$ . We have

$$\kappa(v) = \text{Frac}(A[y]/(f)) = \text{Frac}(A[[y]]/(f)) = \kappa(w)$$

and  $F_v \subseteq L_w$ . Using the argument with the first and second residue forms and Springer’s lemma, we conclude as in case (4) that  $q$  is isotropic over  $F_v$ .  $\square$

We are now ready to give the proof of Theorem 1.2.

*Proof of Theorem 1.2.* — Let  $q$  be any quadratic form of rank  $r \geq 5$  over  $L = \text{Frac}(R)$  and assume that  $q$  is isotropic over  $L_w$  for every  $w \in \Omega_R$ . Without loss of generality, we may assume  $q = \langle a_1, \dots, a_r \rangle$  for some nonzero

elements  $a_i \in R = A[[y]]$ . By the usual form of the Weierstraß preparation theorem (see e.g., [13, p. 115, Thm 7.3]), each  $a_i$  may be written as

$$a_i = x^{n_i} \cdot P_i \cdot U_i \quad \text{with } n_i \in \mathbb{N}, U_i \in R^* \\ \text{and } P_i \text{ a distinguished polynomial in } A[y].$$

For any power series  $f = \sum_{i=0}^{\infty} a_i y^i \in R = A[[y]]$  which is invertible in  $R$ , letting  $\lambda \in \Sigma$  be the unique element such that  $\lambda^{-1} a_0 \equiv 1 \pmod{x A}$ , we have

$$\lambda^{-1} f \equiv 1 \pmod{\mathfrak{m}_R}.$$

Since  $R$  is complete, it follows that  $\lambda^{-1} f$  is a square in  $R$ . So after scaling out squares we may assume that the coefficients  $a_i$  have the form described in Proposition 1.4. Now the quadratic form  $q$  is defined over  $F = K(y)$  and by Proposition 1.4, it is isotropic over  $F_v$  for every  $v \in \Omega_A$ . The local-global principle with respect to discrete valuations in  $\Omega_A$  is proved for quadratic forms of rank  $\geq 3$  in [3, Thm 3.1 and Remark 3.2]. Hence,  $q$  is isotropic over  $F$  and a fortiori over  $L$ .  $\square$

*Remark 5.2.* — In Theorem 1.2, assume that  $A = k[[x]]$  with  $k$  a  $C_1$  field of characteristic  $\neq 2$  or  $A = \mathcal{O}_K$  with  $K$  a  $p$ -adic number field ( $p$  an odd prime). Then every quadratic form of rank  $\geq 9$  is isotropic over  $F = K(y)$ . In the former case, it is well-known that  $F = k((x))(y)$  is a  $C_3$  field. For the case  $A = \mathcal{O}_K$ , this statement is firstly proved by Parimala and Suresh [11], and then two more recent proofs using different methods are given in [4, Coro. 4.15] and [3, Coro. 3.4] as consequences of their main theorems. Still another proof (including the case  $p = 2$ ), which builds upon the work of Heath-Brown [5], has been announced by Leep [9].

An easy argument using the Weierstraß preparation theorem shows that every quadratic form of rank  $\geq 9$  is isotropic over  $L = \text{Frac}(A[[y]])$ . So in these cases, the local-global principle in Theorem 1.2 is only interesting for quadratic forms of rank  $5 \leq r \leq 8$ .

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