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A REMARKABLE CONTRACTION OF SEMISIMPLE LIE ALGEBRAS

by Dmitri I. PANYUSHEV & Oksana S. YAKIMOVA

Abstract. — Recently, E.Feigin introduced a very interesting contraction $\mathfrak{q}$ of a semisimple Lie algebra $\mathfrak{g}$ (see arXiv:1007.0646 and arXiv:1101.1898). We prove that these non-reductive Lie algebras retain good invariant-theoretic properties of $\mathfrak{g}$. For instance, the algebras of invariants of both adjoint and coadjoint representations of $\mathfrak{q}$ are free, and also the enveloping algebra of $\mathfrak{q}$ is a free module over its centre.

Résumé. — E. Feigin a introduit la contraction $\mathfrak{q}$ d'une algèbre de Lie semi-simple $\mathfrak{g}$ dans arXiv :1007.0646 et arXiv :1101.1898. Nous démontrons que ces algèbres de Lie non-réductives conservent quelque unes des propriétés de $\mathfrak{g}$. En particulier, les algèbres des invariants des représentations adjointe et respectivement coadjointe de $\mathfrak{q}$ sont libres, et l'algèbre enveloppante de $\mathfrak{q}$ est un module libre sur son centre.

Introduction

The ground field $\mathbb{F}$ is algebraically closed and $\mathsf{char}\,\mathbb{F} = 0$. Let G be a connected semisimple algebraic group of rank l with Lie algebra $\mathfrak{g}$. Recently, E. Feigin introduced a very interesting contraction of $\mathfrak{g}$ [2]. His motivation came from some problems in Representation Theory [4], and making use of this contraction he also studied certain degenerations of flag varieties [3]. Our goal is to elaborate on invariant-theoretic properties of these contractions of semisimple Lie algebras.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$, where $\mathfrak{t}$ is a Cartan subalgebra. Then $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ is the fixed Borel subalgebra of $\mathfrak{g}$. The corresponding subgroups of G are B, U, and T. Using the vector space isomorphism $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{u}^-$, we regard $\mathfrak{u}^-$ as a B-module. If $b \in \mathfrak{b}$ and $\eta \in \mathfrak{u}^-$, then

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$(b, \eta) \mapsto b \circ \eta$ stands for the corresponding representation of $\mathfrak{b}$. That is, if $p_- \colon \mathfrak{g} \to \mathfrak{u}^-$ is the projection with kernel $\mathfrak{b}$, then $b \circ \eta = p_-([b, \eta])$.

Following [2, Sect. 2], consider the semi-direct product $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{g}/\mathfrak{b})^a = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$, where the superscript 'a' means that the $\mathfrak{b}$-module $\mathfrak{u}^-$ is regarded as an abelian ideal in $\mathfrak{q}$. We may (and will) identify the vector spaces $\mathfrak{g}$ and $\mathfrak{q}$ using the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^-$. If $(b, \eta), (b', \eta') \in \mathfrak{q}$, then the Lie bracket in $\mathfrak{q}$ is given by

$$[(b, \eta), (b', \eta')] = ([b, b'], b \circ \eta' - b' \circ \eta). \tag{0.1}$$

The corresponding connected algebraic group is $Q = B \ltimes N$, where $N = \exp((\mathfrak{u}^-)^a)$ is an abelian normal unipotent subgroup of Q. The exponential map $\exp\colon (\mathfrak{u}^-)^a \to N$ is an isomorphism of varieties, and elements of Q are written as product $s{\cdot}\exp(\eta)$, where $s \in B$ and $\eta \in \mathfrak{u}^-$. If $(s, \eta) \mapsto s{\centerdot}\eta$ is the representation of B in $\mathfrak{u}^-$, then the adjoint representation of Q is given by

$$\mathrm{Ad}_Q(s{\cdot}\exp(\eta))(b, \eta') = (\mathrm{Ad}(s)b, s{\centerdot}(\eta' - b \circ \eta)). \tag{0.2}$$

In this note, we explicitly construct certain polynomials that generate the algebras of invariants $\mathbb{F}[\mathfrak{q}]^Q$ and $\mathbb{F}[\mathfrak{q}^*]^Q$, and thereby prove that these two algebras are free. Furthermore, we also show that these polynomials generate the corresponding fields of invariants, $\mathbb{F}(\mathfrak{q})^Q$ and $\mathbb{F}(\mathfrak{q}^*)^Q$, and that $\mathbb{F}[\mathfrak{q}]$ is a free $\mathbb{F}[\mathfrak{q}]^Q$-module and $\mathbb{F}[\mathfrak{q}^*]$ is a free $\mathbb{F}[\mathfrak{q}^*]^Q$-module. The last assertion implies that the enveloping algebra of $\mathfrak{q}$, $\mathcal{U}(\mathfrak{q})$, is a free module over its centre. The Lie algebra $\mathfrak{q}$ is an *Inönü-Wigner contraction* of $\mathfrak{g}$ (see [15, Ch. 7 § 2.5]), and we also discuss the corresponding relationship between the invariants of G and Q.

Certain classes of non-reductive algebraic Lie algebras $\mathfrak{q}$ such that $\mathbb{F}[\mathfrak{q}^*]^Q$ is a polynomial ring have been studied before. They include the centralisers of nilpotent elements in $\mathfrak{sl}_{l+1}$ and $\mathfrak{sp}_{2l}$ [9], $\mathbb{Z}_2$-contractions of $\mathfrak{g}$ [10], and the truncated seaweed (biparabolic) subalgebras of $\mathfrak{sl}_{l+1}$ and $\mathfrak{sp}_{2l}$ [7]. Our result enlarges this interesting family of Lie algebras.

Let $\mathfrak{q}^*_{\rm reg}$ denote the set of regular elements of $\mathfrak{q}^*$, *i.e.*, $x \in \mathfrak{q}^*_{\rm reg}$ if and only if $\dim Q{\cdot}x$ is maximal. For many problems related to coadjoint representations, it is vital to have that $\mathrm{codim}(\mathfrak{q}^* \setminus \mathfrak{q}^*_{\rm reg}) \geqslant 2$ [10, 9]. However, we prove that if $\mathfrak{g}$ is simple and not of type $\mathbf{A}_l$, then $\mathfrak{q}^* \setminus \mathfrak{q}^*_{\rm reg}$ contains a divisor.

Notation.

- the centraliser in $\mathfrak{g}$ of $x \in \mathfrak{g}$ is denoted by $\mathfrak{g}^x$.
- $\varkappa$ is the Killing form on $\mathfrak{g}$.
- $\mathfrak{g}_{\rm reg}$ is the set of regular elements of $\mathfrak{g}$, *i.e.*, $x \in \mathfrak{g}_{\rm reg}$ if and only if $\dim \mathfrak{g}^x = l$.

- If X is an irreducible variety, then $\mathbb{F}[X]$ is the algebra of regular functions and $\mathbb{F}(X)$ is the field of rational functions on X. If X is acted upon by an algebraic group A, then $\mathbb{F}[X]^A$ and $\mathbb{F}(X)^A$ denote the subsets of respective A-invariant functions.
- If $\mathbb{F}[X]^A$ is finitely generated, then $X/\!\!/A := \operatorname{Spec}(\mathbb{F}[X]^A)$ and $\pi\colon X \to X/\!\!/A$ is determined by the inclusion $\mathbb{F}[X]^A \hookrightarrow \mathbb{F}[X]$. If $\mathbb{F}[X]^A$ is graded polynomial, then the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*.
- $\mathcal{S}^i(V)$ is the i-th symmetric power of the vector space V and $\mathcal{S}(V) = \oplus_{i\geqslant 0}\mathcal{S}^i(V)$ is the symmetric algebra of V.

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1. On adjoint and coadjoint invariants of Inönü-Wigner contractions

The algebra $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$ is an Inönü-Wigner contraction of $\mathfrak{g}$. For this reason, we recall the relevant setting and then describe a general procedure for constructing adjoint and coadjoint invariants of Inönü-Wigner contractions. The $\mathbb{Z}_2$-*contractions* of $\mathfrak{g}$ (considered in [10, 11]) are special cases of Inönü-Wigner contractions, and for them such a procedure is exposed in [10, Prop. 3.1]. However, the more general situation considered here requires another proof.

For a while, we assume that G is any connected algebraic group. Let H be an arbitrary connected subgroup of G and let $\mathfrak{m}$ be a complementary subspace to $\mathfrak{h} = \operatorname{Lie} H$ in $\mathfrak{g}$. Using the vector space isomorphism $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}$, we regard $\mathfrak{m}$ as H-module. Consider the invertible linear map $\mathsf{c}_t\colon \mathfrak{g} \to \mathfrak{g}$, $t \in \mathbb{F} \setminus \{0\}$, such that $c_t(h+m) = h + tm$ $(h \in \mathfrak{h}, m \in \mathfrak{m})$ and define the Lie algebra multiplication $[\ ,\ ]_{(t)}$ on the vector space $\mathfrak{g}$ by the rule

$$[x,y]_{(t)} := \mathsf{c}_t^{-1}\big([\mathsf{c}_t(x), \mathsf{c}_t(y)]\big), \quad x, y \in \mathfrak{g}\ .$$

Write $\mathfrak{g}_{(t)}$ for the corresponding Lie algebra. The operator $(\mathsf{c}_t)^{-1} = \mathsf{c}_{t^{-1}}\colon \mathfrak{g} \to \mathfrak{g}_{(t)}$ yields an isomorphism between the Lie algebras $\mathfrak{g} = \mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(t)}$, hence all algebras $\mathfrak{g}_{(t)}$ are isomorphic. It is easily seen that $\lim_{t\to 0}\mathfrak{g}_{(t)} \simeq \mathfrak{h} \ltimes (\mathfrak{g}/\mathfrak{h})^a = \mathfrak{h} \ltimes \mathfrak{m}^a$.

The resulting Lie algebra $\mathfrak{k} := \mathfrak{h} \ltimes \mathfrak{m}^a$ is called an *Inönü-Wigner contraction* of $\mathfrak{g}$, *cf.* Example 7 in [15, Chapter 7, § 2]. The corresponding connected algebraic group is $K = H \ltimes \exp(\mathfrak{m}^a)$. We identify the vector spaces $\mathfrak{g}$ and $\mathfrak{k}$ using the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Remark. — For $\mathfrak{g}$ semisimple, the contraction $\mathfrak{g} \rightsquigarrow \mathfrak{b} \ltimes \mathfrak{u}^-$ is presented in a more lengthy way, using structure constants, in [2, Remark 2.3].

1.1. To construct invariants of the coadjoint representation of $\mathfrak{k}$, we proceed as follows. Let $f \in \mathcal{S}(\mathfrak{g}) = \mathbb{F}[\mathfrak{g}^*]$ be a homogeneous polynomial of degree n. Using the decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$, we consider the bi-homogeneous components of f:

$$f = \sum_{a\leqslant i\leqslant b} f^{(n-i,i)},$$

where $f^{(n-i,i)} \in \mathcal{S}^{n-i}(\mathfrak{h}) \otimes \mathcal{S}^{i}(\mathfrak{m}) \subset \mathcal{S}^{n}(\mathfrak{g})$, and both $f^{(n-a,a)}$ and $f^{(n-b,b)}$ are assumed to be nonzero. In particular, $f^{(n-b,b)}$ is the bi-homogeneous component having the maximal degree relative to $\mathfrak{m}$. Since $\mathfrak{g}_{(t)}$ and $\mathfrak{k}$ are just the same vector spaces, we also can regard each $f^{(n-i,i)}$ as an element of $\mathcal{S}^n(\mathfrak{g}_{(t)})$ or $\mathcal{S}^n(\mathfrak{k})$.

Theorem 1.1. — *If $f \in \mathcal{S}^n(\mathfrak{g})^G = \mathbb{F}[\mathfrak{g}^*]^G_n$, then $f^{(n-b,b)} \in \mathcal{S}^n(\mathfrak{k})^K = \mathbb{F}[\mathfrak{k}^*]^K_n$.*

Proof. — The isomorphism of Lie algebras $\mathsf{c}_{t^{-1}}\colon \mathfrak{g} \to \mathfrak{g}_{(t)}$ implies that $\sum_{a\leqslant i\leqslant b} t^{-i} f^{(n-i,i)} \in \mathcal{S}(\mathfrak{g}_{(t)})^{G_{(t)}}$ for all $t \neq 0$. It is harmless to replace the last expression with the $G_{(t)}$-invariant $f_{(t)} := \sum_{a\leqslant i\leqslant b} t^{n-i} f^{(n-i,i)}$. Since $f_{(t)}$ is killed by $\mathfrak{g}_{(t)}$ for all $t \neq 0$, its limit at 0, which is $f^{(n-b,b)}$, is killed by $\lim_{t\to 0} \mathfrak{g}_{(t)} = \mathfrak{k}$. Hence $f^{(n-b,b)}$ is K-invariant. $\square$

Let us say that $f^\bullet := f^{(n-b,b)}$ is the *highest component* of $f \in \mathbb{F}[\mathfrak{g}^*]^G_n$ (with respect to the contraction $\mathfrak{g} \rightsquigarrow \mathfrak{k}$). Denote by $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ the linear span of $\{f^\bullet \mid f \in \mathbb{F}[\mathfrak{g}^*]^G \text{ is homogeneous}\}$. Clearly, it is a graded algebra, and Theorem 1.1 implies that $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G) \subset \mathbb{F}[\mathfrak{k}^*]^K$. We say that $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ is the *algebra of highest components* for $\mathbb{F}[\mathfrak{g}^*]^G$.

Invariants of the adjoint representation of $\mathfrak{k}$ can be constructed in a similar way. Set $\mathfrak{m}^* := \mathfrak{h}^\perp$, the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Likewise, $\mathfrak{h}^* = \mathfrak{m}^\perp$. Then $\mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{h}^*$, and the adjoint operator $\mathsf{c}_t^*\colon \mathfrak{g}^* \to \mathfrak{g}^*$ is given by $\mathsf{c}_t^*(m^* + h^*) = t^{-1}m^* + h^*$ ($m^* \in \mathfrak{m}^*$, $h^* \in \mathfrak{h}^*$). Having identified $\mathfrak{q}^*$ and $\mathfrak{k}^*$, we can play the same game with homogeneous elements of $\mathcal{S}(\mathfrak{g}^*) = \mathbb{F}[\mathfrak{g}]$. If $\tilde{f} \in \mathcal{S}^n(\mathfrak{g}^*)$, then $\tilde{f}^{(i,n-i)}$ denotes its bi-homogeneous component that belongs to $\mathcal{S}^i(\mathfrak{m}^*) \otimes \mathcal{S}^{n-i}(\mathfrak{h}^*)$. The resulting assertion is the following:

Theorem 1.2. — *For $\tilde{f} \in \mathcal{S}^n(\mathfrak{g}^*)^G$, let $\tilde{f}^{(a,n-a)}$ be the bi-homogeneous component with minimal a, i.e., having the maximal degree relative to $\mathfrak{h}^* = \mathfrak{m}^\perp$. Then $\tilde{f}^{(a,n-a)} \in \mathcal{S}^n(\mathfrak{k}^*)^K$.*

Likewise, we write $\tilde{f}^\bullet := \tilde{f}^{(a,n-a)}$ and consider the algebra of highest components, $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$, which can be regarded as a graded subalgebra of $\mathbb{F}[\mathfrak{k}]^K$.

Lemma 1.3. — *The graded algebras $\mathbb{F}[\mathfrak{g}^*]^G$ and $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ have the same Poincaré series, i.e., $\dim \mathbb{F}[\mathfrak{g}^*]^G_n = \dim \mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G_n)$ for all $n \in \mathbb{N}$; and likewise for $\mathbb{F}[\mathfrak{g}]^G$ and $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$.*

Proof. — Actually, the assertion concerns vector spaces. Let $\tilde{V} = \oplus_{i\in\mathbb{Z}}\tilde{V}_i$ be a finite-dimensional $\mathbb{Z}$-graded vector space and V an arbitrary subspace of $\tilde{V}$. For $v \in V$, let $v^\bullet$ denote the highest component of v with respect to the $\mathbb{Z}$-grading. Set $\mathcal{L}^\bullet(V) = \operatorname{span}\{v^\bullet \mid v \in V\}$. We claim that there is a basis for V, say $(v_1, \dots, v_m)$, such that $(v_1^\bullet, \dots, v_m^\bullet)$ is a basis for $\mathcal{L}^\bullet(V)$. (Left to the reader.) In particular, $\dim V = \dim \mathcal{L}^\bullet(V)$.

Now, apply this claim to $\tilde{V} = \mathbb{F}[\mathfrak{g}^*]_n = \bigoplus_i \mathbb{F}[\mathfrak{g}^*]_{(i,n-i)}$ and $V = \mathbb{F}[\mathfrak{g}^*]^G_n$. □

It is not always the case that $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G) = \mathbb{F}[\mathfrak{k}^*]^K$ or $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G) = \mathbb{F}[\mathfrak{k}]^K$. For instance, we will see below that, for $\mathfrak{g}$ semisimple and $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$, such an equality holds only for the invariants of the coadjoint representation. By the very construction, the algebras $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$ and $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$ are bi-graded. Moreover, it follows from [10, Theorem 2.7] that the algebras $\mathbb{F}[\mathfrak{k}^*]^K$ and $\mathbb{F}[\mathfrak{k}]^K$ are always bi-graded.

1.2. If $\mathfrak{g}$ is semisimple, then we may identify $\mathfrak{g}$ and $\mathfrak{g}^*$ (and hence $\mathcal{S}(\mathfrak{g})$ and $\mathcal{S}(\mathfrak{g}^*)$) using the Killing form $\varkappa$. If $\mathfrak{h}$ is also reductive, then $\varkappa$ is non-degenerated on $\mathfrak{h}$ and one can take $\mathfrak{m}$ to be the orthocomplement of $\mathfrak{h}$ with respect to $\varkappa$. Then $\mathfrak{h}^\perp \simeq \mathfrak{m}$ and the decompositions of $\mathfrak{g}$ and $\mathfrak{g}^*$ considered in the general setting of Inönü-Wigner contractions coincide. Moreover, we can also identify the vector spaces $\mathfrak{k}$ and $\mathfrak{k}^*$. However, to obtain invariants of the adjoint and coadjoint representations of $\mathfrak{q}$, one has to take the bi-homogeneous components of maximal degree with respect to *different* summands in the sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In this situation, Theorems 1.1 and 1.2 admit the following simultaneous formulation:

Suppose that $f \in \mathbb{F}[\mathfrak{g}]^G_n \simeq \mathcal{S}(\mathfrak{g})^G_n$ and $f = \sum_{a\leqslant i\leqslant b} f^{(n-i,i)}$ is the bi-homogeneous decomposition relative to the sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. (That is, $\deg_{\mathfrak{h}} f^{(n-i,i)} = n - i$, etc.) Then, upon identifications of vector spaces $\mathfrak{g}$,$\mathfrak{k}$, and $\mathfrak{k}^*$, we have $f^{(n-a,a)} \in \mathbb{F}[\mathfrak{k}]^K$ and $f^{(n-b,b)} \in \mathbb{F}[\mathfrak{k}^*]^K$.

Such a phenomenon was already observed in the case of $\mathbb{Z}_2$-contractions of semisimple Lie algebras, *i.e.*, if $\mathfrak{h}$ is the fixed-point subalgebra of an involution, see [10, Prop. 3.1].

2. Invariants of the adjoint representation of Q

In this section, we describe the algebra of invariants of the adjoint representation of Q.

To prove that a certain set of invariants generates the whole algebra of invariants, we use the following lemma of Igusa [6].

Lemma 2.1 (Igusa). — *Let A be an algebraic group acting regularly on an irreducible affine variety X. Suppose that S is an integrally closed finitely generated subalgebra of $\mathbb{F}[X]^A$ and the morphism $\pi\colon X \to \operatorname{Spec} S =: Y$ has the properties:*

(i) *the fibres of π over a dense open subset of Y contain a dense A-orbit;*

(ii) $\operatorname{Im}\pi$ *contains an open subset Ω of Y such that* $\operatorname{codim}(Y \smallsetminus \Omega) \geqslant 2$.

Then $S = \mathbb{F}[X]^A$. In particular, the algebra of A-invariants is finitely generated.

Remark 2.2. — A proof of the Igusa lemma is given, for example, in [11, Lemma 6.1]. This proof shows that the above condition (i) can be replaced with the condition that $S \subset \mathbb{F}[X]^A$ generates the field $\mathbb{F}(X)^A$. (In fact, it is not hard to prove that (i) holds *if and only if* S separates A-orbits in a dense open subset of X *if and only if* S generates $\mathbb{F}(X)^A$.)

Lemma 2.3. — *If $t \in \mathfrak{t}$ is regular and $u \in \mathfrak{u}$ is arbitrary, then* (i) *$t+u$ and t belong to the same* Ad *U-orbit;* (ii) $(t+u)\circ\mathfrak{u}^- = \mathfrak{u}^-$.

Proof.

(i) Clearly, $(\operatorname{Ad} U)t \subset t+\mathfrak{u}$ for all $t \in \mathfrak{t}$. If t is regular, then $\dim(\operatorname{Ad} U)t = \dim\mathfrak{u}$. It is also known that the orbits of a unipotent group acting on an affine variety are closed. Hence $(\operatorname{Ad} U)t = t+\mathfrak{u}$.

(ii) This is obvious if $u = 0$. In general, this follows from (i).

□

Theorem 2.4. — *We have $\mathbb{F}[\mathfrak{q}]^Q \simeq \mathbb{F}[\mathfrak{t}]$, and the quotient morphism $\pi_Q\colon \mathfrak{q} \to \mathfrak{t}$ is given by $(t+u,\eta) \mapsto t$.*

Proof. — Clearly, $\mathbb{F}[\mathfrak{q}]^Q = (\mathbb{F}[\mathfrak{q}]^N)^B$. We prove that 1) $\mathbb{F}[\mathfrak{q}]^N \simeq \mathbb{F}[\mathfrak{b}]$ and 2) $\mathbb{F}[\mathfrak{b}]^B \simeq \mathbb{F}[\mathfrak{t}]$.

1) Consider the projection $\pi_N : \mathfrak{q} \to \mathfrak{q}/(\mathfrak{u}^-)^a \simeq \mathfrak{b}$. Clearly, N acts trivially on $\mathfrak{q}/(\mathfrak{u}^-)^a$ and π_N is a surjective N-equivariant morphism. Hence $\mathbb{F}[\mathfrak{b}] \subset \mathbb{F}[\mathfrak{q}]^N$. By Lemma 2.1, the equality $\mathbb{F}[\mathfrak{b}] = \mathbb{F}[\mathfrak{q}]^N$ will follow from the fact that general fibres of π_N are N-orbits.

If $t \in \mathfrak{t}$ is regular and $u \in \mathfrak{u}$ is arbitrary, then $b = t + u$ is a regular semisimple element of $\mathfrak{g}$. By (0.2) with $s = 1$, we have

$$\mathrm{Ad}_Q(N)(b, \eta) = (b, \eta + b \circ \mathfrak{u}^-).$$

It then follows from Lemma 2.3 that $\mathrm{Ad}_Q(N)(b, \eta) = (b, \mathfrak{u}^-)$. On the other hand, $\pi_N^{-1}(b) = (b, \mathfrak{u}^-)$, *i.e.*, $\pi_N^{-1}(b)$ is a single N-orbit whenever b is regular semisimple.

2) Consider the projection $\pi_B : \mathfrak{b} \to \mathfrak{b}/\mathfrak{u} \simeq \mathfrak{t}$. Clearly, B acts trivially on $\mathfrak{b}/\mathfrak{u}$ and π_B is a surjective B-equivariant morphism. Hence $\mathbb{F}[\mathfrak{t}] \subset \mathbb{F}[\mathfrak{b}]^B$. By Lemma 2.1, the equality $\mathbb{F}[\mathfrak{t}] = \mathbb{F}[\mathfrak{b}]^B$ will follow from the fact that general fibres of π_B are B-orbits. Again, it follows from Lemma 2.3 that if $t \in \mathfrak{t}$ is regular, then $(\mathrm{Ad}\ B)t = t + \mathfrak{u} = \pi_B^{-1}(t)$. □

Remark 2.5. — Theorem 2.4 can be proved in a less informative way. Notice that $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{u} \ltimes (\mathfrak{u}^-)^a$ and therefore $\mathbb{F}[\mathfrak{t}] \subset \mathbb{F}[\mathfrak{q}]^Q$. Let $x \in \mathfrak{t}$ be regular semisimple. Then $\mathfrak{q}^x \simeq \mathfrak{g}^x = \mathfrak{t}$, since $\mathfrak{g}$ and $\mathfrak{q}$ are isomorphic as T-modules. The fibres of the morphism $\pi_Q : \mathfrak{q} \to \mathfrak{t}$, defined in Theorem 2.4, are linear spaces of dimension $\dim \mathfrak{q} - \dim \mathfrak{t} = \dim(\mathrm{Ad}\ Q)x$. Hence a general fibre contains a dense Q-orbit and Lemma 2.1 applies. We also see that the algebra $\mathbb{F}[\mathfrak{t}]$ separates Q-orbits in $\mathfrak{q}$ in general position and therefore $\mathbb{F}(\mathfrak{q})^Q = \mathbb{F}(\mathfrak{t})$.

Comparing with the adjoint representation of $\mathfrak{g}$, we see that, for $\mathfrak{q}$, the algebra of invariants remains polynomial, but the degrees of basic invariants drastically decrease! All the basic invariants in $\mathbb{F}[\mathfrak{q}]^Q$ are of degree 1. This clearly means that here $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G) \subsetneq \mathbb{F}[\mathfrak{q}]^Q$.

3. Invariants of the coadjoint representation of Q

In this section, we describe the algebra of invariants of the coadjoint representation of Q. The coadjoint representation is much more interesting since $\mathbb{F}[\mathfrak{q}^*] = \mathcal{S}(\mathfrak{q})$ is a Poisson algebra, $\mathcal{S}(\mathfrak{q})^Q$ is the centre of this Poisson algebra, and $\mathcal{S}(\mathfrak{q})$ is related to the enveloping algebra of $\mathfrak{q}$ via the Poincaré-Birkhoff-Witt theorem.

Since $\mathfrak{q}$ is isomorphic to $\mathfrak{b}\oplus\mathfrak{g}/\mathfrak{b}\simeq\mathfrak{b}\oplus\mathfrak{u}^-$ as vector space, the dual vector space $\mathfrak{q}^*$ is isomorphic to $(\mathfrak{g}/\mathfrak{b})^*\oplus\mathfrak{b}^*$. Using $\varkappa$, we identify $\mathfrak{b}^*$ with $\mathfrak{b}^-:=\mathfrak{t}\oplus\mathfrak{u}^-$ and $(\mathfrak{g}/\mathfrak{b})^*$ with $\mathfrak{u}$. To stress that $\mathfrak{q}^*$ is regarded as a Q-module and $\mathfrak{b}^-$ appears to be a Q-stable subspace, we write $\mathfrak{q}^*=\mathfrak{u}\propto\mathfrak{b}^-$. If $(b,\eta)\in\mathfrak{q}$ and $(u,\xi)\in\mathfrak{q}^*$, i.e., $u\in\mathfrak{u}$ and $\xi\in\mathfrak{b}^-$, then the coadjoint representation of $\mathfrak{q}$ is given by the formula:

$$(b,\eta)\star(u,\xi)=([b,u],\phi(u,\eta)+b\star\xi). \tag{3.1}$$

Here $(b,\xi)\mapsto b\star\xi$ is the coadjoint representation of $\mathfrak{b}$, and

$$\phi\colon \mathfrak{u}\times\mathfrak{u}^-\simeq\mathfrak{u}\times\mathfrak{u}^*\xrightarrow{\psi}\mathfrak{b}^*\simeq\mathfrak{b}^-,$$

where ψ is the *moment map* associated with the $\mathfrak{b}$-module $\mathfrak{u}$. Upon our identifications, the mapping ϕ is directly defined by

$$\varkappa(b,\phi(u,\eta)):=\varkappa([b,u],\eta)=-\varkappa(u,b\circ\eta).$$

Recall some well-known properties of the B-module $\mathfrak{u}$:

- If $\tilde e\in\mathfrak{u}$ is regular nilpotent, then $\mathfrak{g}^{\tilde e}\subset\mathfrak{u}$ [8] and hence $(\mathrm{Ad}\,B)\tilde e$ is dense in $\mathfrak{u}$.
- For any $e\in\mathfrak{u}$, the irreducible components of $(\mathrm{Ad}\,G)e\cap\mathfrak{u}$ are called *orbital varieties* and each of them has dimension $\frac{1}{2}\dim(\mathrm{Ad}\,G)e$ [14, 4.3.11].

Let $\mathrm{Mor}_G(\mathfrak{g},\mathfrak{g})$ denote the $\mathbb{F}[\mathfrak{g}]^G$-module of polynomial G-equivariant morphisms $F\colon\mathfrak{g}\to\mathfrak{g}$. By work of Kostant [8], $\mathrm{Mor}_G(\mathfrak{g},\mathfrak{g})$ is a free graded $\mathbb{F}[\mathfrak{g}]^G$-module of rank l. It was noticed by Th. Vust [16, Char. III, § 2] (see also [12]) that a homogeneous basis of this module is obtained as follows. Let $f_1,\dots,f_l$ be homogeneous algebraically independent generators of $\mathbb{F}[\mathfrak{g}]^G$. Each differential df_i determines a polynomial G-equivariant morphism (covariant) from $\mathfrak{g}$ to $\mathfrak{g}^*$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ via κ yields a homogeneous covariant (or, vector field) $F_i=\mathrm{grad} f_i\colon\mathfrak{g}\to\mathfrak{g}$. Then $F_1,\dots,F_l$ form a homogeneous basis for $\mathrm{Mor}_G(\mathfrak{g},\mathfrak{g})$. If $\deg f_i=d_i$, then $\deg F_i=d_i-1=:m_i$. It is customary to say that $\{m_1,\dots,m_l\}$ are the *exponents* of (the Weyl group of) $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is simple and $m_1\leqslant\dots\leqslant m_l$, then $m_1=1$, $m_2\geqslant 2$, and m_i+m_{l-i+1} is the Coxeter number of $\mathfrak{g}$.

The covariants F_i have the following properties:

(i) $F_i(x)\in\mathfrak{g}^x$ for all $i\in\{1,2,\dots,l\}$ and $x\in\mathfrak{g}$;

(ii) The vectors $F_1(x),\dots,F_l(x)\in\mathfrak{g}$ are linearly independent if and only if $x\in\mathfrak{g}_{\mathrm{reg}}$ [8, Theorem 9].

It follows that $(F_1(x),\dots,F_l(x))$ is a basis for $\mathfrak{g}^x$ if and only if $x\in\mathfrak{g}_{\mathrm{reg}}$.

Lemma 3.1. — *If $x\in\mathfrak{b}$, then $F_i(x)\in\mathfrak{b}$. If $y\in\mathfrak{u}$, then $F_i(y)\in\mathfrak{u}$.*

Proof. — If $x \in \mathfrak{b} \cap \mathfrak{g}_{\rm reg}$, then $\mathfrak{g}^x \subset \mathfrak{b}$. (Indeed, $[\mathfrak{b}, x] \subset \mathfrak{u}$, hence $\dim \mathfrak{b}^x \geqslant \operatorname{rk} \mathfrak{g}$. On the other hand, $\mathfrak{b}^x \subset \mathfrak{g}^x$ and $\dim \mathfrak{g}^x = \operatorname{rk} \mathfrak{g}$.) Hence $F_i(x) \in \mathfrak{g}^x \subset \mathfrak{b}$. Since $\mathfrak{b} \cap \mathfrak{g}_{\rm reg}$ is open and dense in $\mathfrak{b}$, the assertion follows.

If $y \in \mathfrak{u} \cap \mathfrak{g}_{\rm reg}$, *i.e.*, y is regular nilpotent, then $\mathfrak{g}^y \subset \mathfrak{u}$ [8]. The rest is the same. □

Consequently, letting $P_i := F_i|_{\mathfrak{u}}$, we obtain the covariants $P_1, \dots, P_l \in \operatorname{Mor}_B(\mathfrak{u}, \mathfrak{u})$. Actually, we consider the P_i's as B-equivariant morphisms $P_i \colon \mathfrak{u} \to \mathfrak{u} \subset \mathfrak{b}$. Using these covariants, we define polynomials $\widehat{P}_i \in \mathbb{F}[\mathfrak{q}^*] = \mathbb{F}[\mathfrak{u} \propto \mathfrak{b}^-]$ by the formula

$$\widehat{P}_i(u, \xi) = \varkappa(P_i(u), \xi), \quad i = 1, \dots, l, \tag{3.2}$$

where $u \in \mathfrak{u}$ and $\xi \in \mathfrak{b}^-$.

LEMMA 3.2. — *We have $\widehat{P}_i \in \mathbb{F}[\mathfrak{q}^*]^Q$.*

Proof. — Since $Q = B \ltimes N$, it suffices to verify that $\widehat{P}_i$ is both B- and N-invariant.

1) $\widehat{P}_i$ is B-invariant, since P_i is B-equivariant.
2) For polynomials obtained from covariants P_i as in (3.2), the invariance with respect to the commutative unipotent group N is equivalent to that $[P_i(u), u] = 0$, $u \in \mathfrak{u}$. Indeed, for $\eta \in \mathfrak{u}^-$, the coadjoint action of $\exp(\eta) \in N$ is given by $\exp(\eta)\star(u, \xi) = (u, \xi + \phi(u, \eta))$. Then

$$\begin{aligned} \widehat{P}_i(\exp(\eta){\cdot}(u, \xi)) &= \varkappa(P_i(u), \xi + \phi(u, \eta)) \\ &= \varkappa(P_i(u), \xi)) + \varkappa(P_i(u), \phi(u, \eta)) \\ &= \widehat{P}_i(u, \xi) + \varkappa([P_i(u), u], \eta). \end{aligned}$$

Hence $\widehat{P}_i(\exp(\eta){\cdot}(u, \xi)) = \widehat{P}_i(u, \xi)$ for all η if and only if $[P_i(u), u] = 0$. The latter follows from the corresponding property (i) for F_i.

□

Remark. — We prove below that $\widehat{P}_i$ is the highest component of $f_i \in \mathbb{F}[\mathfrak{g}^*]^G$. In view of Theorem 1.1, this also implies that $\widehat{P}_i$ is Q-invariant.

THEOREM 3.3. — *The algebra $\mathbb{F}[\mathfrak{q}^*]^Q$ is freely generated by $\widehat{P}_1, \dots, \widehat{P}_l$, and $\mathbb{F}(\mathfrak{q}^*)^Q$ is the fraction field of $\mathbb{F}[\mathfrak{q}^*]^Q$.*

Proof. — Consider the morphism

$$\pi \colon \mathfrak{q}^* = \mathfrak{u} \propto \mathfrak{b}^- \to \mathbb{A}^l,$$

given by $\pi(u, \xi) = (\widehat{P}_1(u, \xi), \dots, \widehat{P}_l(u, \xi))$. As in Section 2, to prove that π is the quotient by Q, we are going to apply Lemma 2.1 to π.

If $e \in \mathfrak{u}$ is regular, then $P_1(e), \dots, P_l(e)$ are linearly independent and form a basis for $\mathfrak{g}^e = \mathfrak{u}^e$. Therefore, (3.2) implies that π is onto, and condition (ii) in Lemma 2.1 is satisfied.

Let us prove that $\mathbb{F}(\mathfrak{q}^*)^Q = \mathbb{F}(\widehat{P}_1, \dots, \widehat{P}_l)$. Consider the morphism

$$\tilde{\pi}\colon \mathfrak{q}^* \to (\mathfrak{q}^*/\mathfrak{b}^-) \times \mathbb{A}^l = \mathfrak{u} \times \mathbb{A}^l$$

defined by $\tilde{\pi}(u, \xi) = (u, \widehat{P}_1(u, \xi), \dots, \widehat{P}_l(u, \xi))$. If $e \in \mathfrak{u} \cap \mathfrak{g}_{\mathrm{reg}}$, then Eq. (3.2) shows that $\tilde{\pi}^{-1}(e, a)$ is an affine subspace of $\mathfrak{q}^*$ for any $a \in \mathbb{A}^l$, and $\dim \tilde{\pi}^{-1}(e, a) = \dim \mathfrak{b} - l = \dim \mathfrak{u}$. As in the proof of Theorem 2.4, this implies that $\tilde{\pi}^{-1}(e, a)$ is a sole N-orbit. Thus, the coordinate functions on $\mathfrak{u}$ and $\widehat{P}_1, \dots, \widehat{P}_l$ separate generic N-orbits of maximal dimension. By the Rosenlicht theorem [1, 1.6], this implies that all these functions generate the field of N-invariants on $\mathfrak{q}^*$, *i.e.*, $\mathbb{F}(\mathfrak{q}^*)^N = \mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l)$. Since B has an open orbit in $\mathfrak{u}$, we have $\mathbb{F}(\mathfrak{u})^B = \mathbb{F}$. Hence

$$\mathbb{F}(\mathfrak{q}^*)^Q = (\mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l))^B = \mathbb{F}(\widehat{P}_1, \dots, \widehat{P}_l).$$

In view of Remark 2.2, this is sufficient for using Lemma 2.1, and we conclude that $\widehat{P}_1, \dots, \widehat{P}_l$ generate the algebra of Q-invariants on $\mathfrak{q}^*$. $\square$

Remark 3.4. — Although we have proved that $\mathbb{F}(\mathfrak{q}^*)^N = \mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l)$, it is not true that $\mathbb{F}[\mathfrak{q}^*]^N = \mathbb{F}[\mathfrak{u}][\widehat{P}_1, \dots, \widehat{P}_l]$. The reason is that the morphism $\tilde{\pi}$ defined in the previous proof does not satisfy condition (ii) of Lemma 2.1. That is, the closure of the complement of $\operatorname{Im} \tilde{\pi}$ contains a divisor. One can prove that this divisor is equal to $D \times \mathbb{A}^l$, where $D = \mathfrak{u} \smallsetminus (\mathrm{Ad}\ B)\tilde{e} = \mathfrak{u} \smallsetminus (\mathfrak{u} \cap \mathfrak{g}_{\mathrm{reg}})$. Actually, we can explicitly point out a function in $\mathbb{F}[\mathfrak{q}^*]^N \smallsetminus \mathbb{F}[\mathfrak{u}][\widehat{P}_1, \dots, \widehat{P}_l]$. Let v be a non-zero vector in the one-dimensional space $\mathfrak{b}^U$. We can regard v as a linear function on $\mathfrak{b}^-$ and hence on $\mathfrak{q}^*$. Making use of Eq. (0.1), it is not hard to check that the subalgebra $(\mathfrak{u}^-)^a \subset \mathfrak{q}$ commutes with v, *i.e.*, v is a required N-invariant in the symmetric algebra $\mathcal{S}(\mathfrak{q})$.

Recall that, for an algebraic group A with Lie algebra $\mathfrak{a}$, the *index* of $\mathfrak{a}$, $\operatorname{ind} \mathfrak{a}$, is defined as the minimal codimension of an A-orbit in the coadjoint representation. By the Rosenlicht theorem, one has $\operatorname{ind} \mathfrak{a} = \operatorname{trdeg} \mathbb{F}(\mathfrak{a}^*)^A$. It is easily seen that the index cannot decrease under contractions, hence $\operatorname{ind} \mathfrak{q} \geqslant \operatorname{ind} \mathfrak{g} = l$. The above description of the field of Q-invariants implies that

Corollary 3.5. — $\operatorname{ind} \mathfrak{q} = l$.

Theorem 3.6. — *The polynomial ring $\mathbb{F}[\mathfrak{q}^*]$ is a free $\mathbb{F}[\mathfrak{q}^*]^Q$-module.*

Proof. — Since it is already known that $\mathbb{F}[\mathfrak{q}^*]^Q$ is a polynomial algebra (of Krull dimension l), it suffices to prove that the quotient morphism $\pi\colon \mathfrak{q}^* \to \mathfrak{q}^*/\!\!/Q \simeq \mathbb{A}^l$ is equidimensional [13, Prop. 17.29]. This, in turn, will follow from the fact that the null-cone $\mathcal{N} = \pi^{-1}(\pi(0))$ is of dimension $\dim\mathfrak{q}-l$. To estimate the dimension of $\mathcal{N}$, consider the projection $p\colon \mathcal{N} \to \mathfrak{u}$ and partition $\mathfrak{u}$ into finitely many orbital varieties (the irreducible components of $(\mathrm{Ad}\ G)e_i\cap\mathfrak{u}$), where $\{e_i\}$ runs over a finite set of representatives of all nilpotent G-orbits. Let Z_i be an irreducible component of $(\mathrm{Ad}\ G)e_i\cap\mathfrak{u}$. Since $\pi = (\widehat{P}_1,\dots,\widehat{P}_l)$, Eq. (3.2) shows that

$$\dim p^{-1}(Z_i) = \dim Z_i + \dim\mathfrak{b} - \dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\}.$$

As $\dim Z_i = \frac{1}{2}\dim(\mathrm{Ad}\ G)e_i$, the condition that $\dim p^{-1}(Z_i) \leqslant \dim\mathfrak{q} - l$ can easily be transformed into

$$\dim\mathfrak{g}^{e_i} + 2\dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\} \geqslant 3l. \tag{3.3}$$

Recall that $P_1,\dots,P_l$ are just the restrictions to $\mathfrak{u}$ of basic covariants $F_1,\dots,F_l$, and $F_j = \operatorname{grad} f_j$. Consequently, $\dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\}$ equals the rank of the differential at e of the quotient morphism $\pi_{\mathfrak{g},G}\colon \mathfrak{g} \to \mathfrak{g}/\!\!/G$. Therefore, (3.3) is precisely the inequality proved in [11, Theorem 10.6]. □

Corollary 3.7. — *The enveloping algebra $\mathcal{U}(\mathfrak{q})$ is a free module over its centre $\mathcal{Z}(\mathfrak{q})$.*

Proof. — This is a standard consequence of the fact that $\mathbb{F}[\mathfrak{q}^*] = \mathcal{S}(\mathfrak{q})$ is a free module over $\mathcal{S}(\mathfrak{q})^Q$, $\mathcal{S}(\mathfrak{q})^Q$ is the centre of the Poisson algebra $\mathcal{S}(\mathfrak{q})$, and $\mathrm{gr}\ \mathcal{Z}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^Q$, *cf.* [8, Theorem 21], [5, Theorem 3.3]. □

Remark 3.8. — By Theorem 3.6, the irreducible components of all fibres of $\pi\colon \mathfrak{q}^* \to \mathfrak{q}^*/\!\!/Q \simeq \mathbb{A}^l$ are of dimension $\dim\mathfrak{q} - l$. However, unlike the case of the (co)adjoint representation of $\mathfrak{g}$, the zero fibre of π is highly reducible. For, if $\dim\mathfrak{g}^{e_i} + 2\dim\operatorname{span}\{P_1(e_i),\dots,P_l(e_i)\} = 3l$, then every irreducible component of $(\mathrm{Ad}\ G)e_i\cap\mathfrak{u}$ gives rise to an irreducible component of $\pi^{-1}(\pi(0))$. A complete classification of nilpotent elements of $\mathfrak{g}$ satisfying this equality is contained in [11, § 10].

Theorem 3.9. — *We have $\mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G) = \mathcal{S}(\mathfrak{q})^Q$. The polynomials $\widehat{P}_1,\dots,\widehat{P}_l \in \mathbb{F}[\mathfrak{q}^*]^Q = \mathcal{S}(\mathfrak{q})^Q$ are the highest components of $f_1,\dots,f_l \in \mathcal{S}(\mathfrak{g})^G$ in the sense of Subsection 1.1.*

Proof.

1) Since $\deg \widehat{P}_i = \deg f_i$ for all i, it follows from Lemma 1.3 and Theorem 3.3 that $\mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G)$ and $\mathcal{S}(\mathfrak{q})^Q$ have the same Poincaré series. Hence these algebras coincide.

2) Recall that $\deg f_i = d_i = m_i + 1$. According to Theorem 1.1, we have to take the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^-$ and pick the bi-homogeneous component of f_i of maximal degree with respect to $\mathfrak{u}^-$.

If the component $f_i^{(0,d_i)} \in \mathcal{S}^{d_i}(\mathfrak{u}^-)$ were non-trivial, then it would be a Q-invariant in $\mathcal{S}(\mathfrak{q})$ and in particular a B-invariant (Theorem 1.1). Recall that if we work in $\mathfrak{q}$, then $\mathfrak{u}^- \simeq \mathfrak{g}/\mathfrak{b}$ as B-module. Since $\mathcal{S}(\mathfrak{g}/\mathfrak{b}) \simeq \mathbb{F}[\mathfrak{u}]$ and $\mathbb{F}[\mathfrak{u}]^B = \mathbb{F}$, we get a contradiction. Hence $f_i^{(0,d_i)} = 0$.

Then next possible component is $f_i^{(1,m_i)} \in \mathfrak{b} \otimes \mathcal{S}^{m_i}(\mathfrak{u}^-)$. Using the identifications $\mathfrak{b}^* \simeq \mathfrak{b}^-$ and $\mathfrak{u}^* \simeq \mathfrak{u}^-$, we have $f_i^{(1,m_i)} \in \mathbb{F}[\mathfrak{b}^-]_1 \otimes \mathbb{F}[\mathfrak{u}]_{m_i}$. That is, if considered as a function on $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{u}$, it can be written as $f_i^{(1,m_i)}(\xi, u) = \kappa(\bar{P}_i(u), \xi)$ for some morphism $\bar{P}_i\colon \mathfrak{u} \to \mathfrak{b}$ of degree m_i. As we have already proved that $f_i^{(0,d_i)} = 0$, $\bar{P}_i(u)$ is nothing but the value of $\operatorname{grad} f_i$ at u. Hence $\bar{P}_i = P_i$, and we are done. □

4. Further properties of the coadjoint representation

4.1. For the classical Lie algebras, the basic covariants $F_i\colon \mathfrak{g} \to \mathfrak{g}$ (and hence P_i) have a simple description:

- if $x \in \mathfrak{sl}_{l+1}$, then $F_i(x) = x^i$, $i = 1, 2, \ldots, l$;
- if $x \in \mathfrak{sp}_{2l}$ or $\mathfrak{so}_{2l+1}$, then $F_i(x) = x^{2i-1}$, $i = 1, 2, \ldots, l$;
- if $x \in \mathfrak{so}_{2l}$, then $F_i(x) = x^{2i-1}$, $i = 1, 2, \ldots, l-1$. The covariant F_l that is related to the pfaffian is described as follows. Let x be a skew-symmetric matrix of order $2l$. For $i \ne j$, let $x_{[ij]}$ be the skew-symmetric sub-matrix of order $2l-2$ obtained by deleting ith and jth row and column. Set $a_{ij} = \mathsf{Pf}(x_{[ij]})$ if $i \ne j$, and $a_{ii} = 0$. Then $F_l(x) = (a_{ij})_{i,j=1}^{2l}$. Clearly, $\deg F_l = l - 1$, as required.

Results of Sections 2 and 3 explicitly yield the bi-degrees of basic invariants for $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$. For $\mathbb{F}[\mathfrak{q}]^Q$, all the basic invariants have bi-degrees $(1, 0)$. For $\mathbb{F}[\mathfrak{q}^*]^Q$, the basic invariants have bi-degrees $(m_i, 1)$, *i.e.*, $\widehat{P}_i \in \mathcal{S}^{m_i}(\mathfrak{u}^-) \otimes \mathfrak{b}$. In particular, for the coadjoint representation, the total degrees of the basic Q-invariants remain the same as for G.

4.2. Hereafter we assume that $\mathfrak{g}$ is simple and the basic invariants $f_1, \ldots, f_l \in \mathbb{F}[\mathfrak{g}]^G$ are numbered such that $d_i \leqslant d_{i+1}$. Then $d_l = \mathsf{h}$ is the Coxeter number of $\mathfrak{g}$. We show that the corresponding Q-invariant $\widehat{P}_l$ has a rather simple form. In fact, it turns out to be a product of linear forms.

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$ and Δ^+ the subset of positive roots corresponding to $\mathfrak{u}$. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ (resp. θ) be the set of simple roots (resp. the highest root) in Δ^+. Then $\theta = \sum_{i=1}^l a_i\alpha_i$ and $\sum_{i=1}^l a_i = \mathsf{h} - 1$. For any $\gamma \in \Delta$, $\mathfrak{g}_\gamma$ denotes the corresponding root subspace, and we fix a nonzero vector $e_\gamma \in \mathfrak{g}_\gamma$.

Lemma 4.1. — *Up to a scalar multiple, we have* $\widehat{P}_l = e_{-\alpha_1}^{a_1} \cdots e_{-\alpha_l}^{a_l} e_\theta \in \mathcal{S}(\mathfrak{q})^Q$.

Proof. — Recall that $\mathfrak{q} = \mathfrak{b} \oplus \mathfrak{u}^-$ as vector space, and here $e_\theta \in \mathfrak{b}$ and $e_{-\alpha_i} \in \mathfrak{u}^-$. By the very construction, $\widehat{P} := e_{-\alpha_1}^{a_1} \cdots e_{-\alpha_l}^{a_l} e_\theta$ is a T-invariant in $\mathcal{S}(\mathfrak{q})$. Then, using Eq. (0.1), one readily verifies that $\widehat{P}$ is both U-invariant and N-invariant. Hence $\widehat{P}$ is a polynomial in $\widehat{P}_1, \dots, \widehat{P}_l$. Since $\textsf{bi-deg}\, \widehat{P} = (\mathsf{h}-1, 1)$ and $m_i < m_l$ for $i < l$, the subspace of bi-degree $(m_l, 1) = (\mathsf{h}-1, 1)$ in $\mathcal{S}(\mathfrak{q})^Q$ is one-dimensional and spanned by $\widehat{P}_l$. Hence the assertion. □

Since $\dim \mathfrak{q} = \dim \mathfrak{g}$, $\textsf{ind}\, \mathfrak{q} = \textsf{ind}\, \mathfrak{g}$, and the (total) degrees of the basic invariants of the coadjoint representations for G and Q coincide, we have the equality

$$\sum_{i=1}^{l} \deg \widehat{P}_i = \frac{\dim \mathfrak{q} + \textsf{ind}\, \mathfrak{q}}{2}, \tag{4.1}$$

which is very useful in the study of the coadjoint representation, see *e.g.* [10, Theorem 1.2]. Unfortunately, $\mathfrak{q}$ does not always possess another important ingredient, the so-called *codim-2* property. Recall that $x \in \mathfrak{q}^*$ is said to be *regular* if $\dim Q{\cdot}x$ is maximal. The set of all regular elements is denoted by $\mathfrak{q}^*_{\mathrm{reg}}$. It is an open subset of $\mathfrak{q}^*$, and we say that $\mathfrak{q}$ has the *codim-2* property if $\operatorname{codim}(\mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathrm{reg}}) \geqslant 2$.

Theorem 4.2. — *The algebra* $\mathfrak{q}$ *does not have the codim-2 property if* $\mathfrak{g}$ *is not of type* $\boldsymbol{A}_l$.

Proof. — Suppose that $\mathfrak{q}$ has the *codim-2* property. Since (4.1) is satisfied, it follows from [10, Theorem 1.2] that the differentials $(d\widehat{P}_i)_x$, $i = 1, \dots, l$, are linearly independent if and only if $x \in \mathfrak{q}^*_{\mathrm{reg}}$. In particular, any divisor $\tilde{D} \subset \mathfrak{q}^*$ contains a point where the differentials of $\widehat{P}_1, \dots, \widehat{P}_l$ are linearly independent.

On the other hand, Lemma 4.1 shows that if $a_i \geqslant 2$ for some i, then $d\widehat{P}_l$ vanishes at the hyperplane $\{e_{-\alpha_i} = 0\}$, where $e_{-\alpha_i}$ is regarded as a linear function on $\mathfrak{u}$ and hence on $\mathfrak{q}^*$. Thus, $\mathfrak{q}$ cannot have the *codim-2* property unless $a_i = 1$ for all i, *i.e.*, $\mathfrak{g}$ is of type $\mathbf{A}_l$. □

To prove the converse of this theorem, we need some preparations. For $\alpha_i \in \Pi$, let $\mathfrak{u}_i \subset \mathfrak{u}$ denote the kernel of the linear form $u \mapsto \kappa(e_{-\alpha_i}, u)$. By [8], $\mathfrak{u} \setminus \mathfrak{u} \cap \mathfrak{g}_{\rm reg} = \cup_i \mathfrak{u}_i$. Set

(4.2)
$$\mathcal{Y} = \mathcal{Y}(\mathfrak{q}^*) = \Big\{ x \in \mathfrak{q}^* \mid (d\widehat{P}_1)_x, \dots, (d\widehat{P}_l)_x \ \text{ are linearly independent} \Big\}.$$

Proposition 4.3. — *If* $\mathfrak{g} = \mathfrak{sl}_{l+1}$, *then* $\operatorname{codim}(\mathfrak{q}^* \setminus \mathcal{Y}) \geqslant 2$.

Proof. — Let $a = (e, \xi)$ and $a' = (e', \xi')$ be typical elements of $\mathfrak{q}^*$, where $e, e' \in \mathfrak{u}$ and $\xi, \xi' \in \mathfrak{b}^-$. According to formulae of Subsection 4.1, $\widehat{P}_i(e, \xi) = \kappa(e^i, \xi)$. Recall that $(d\widehat{P}_i)_a \in \mathfrak{q}$ and $\langle (d\widehat{P}_i)_a, a'\rangle$ is the coefficient of t in the expansion of $\widehat{P}_i(a + ta')$. Consequently,

$$\big\langle (d\widehat{P}_i)_a, a' \big\rangle = \kappa(e^i, \xi') + \kappa\bigg(\sum_{k+m=i-1} e^k e' e^m, \xi \bigg).$$

The vector $(d\widehat{P}_i)_a$ has the $\mathfrak{b}$- and $\mathfrak{u}^-$-components, and this equality shows that:

- the $\mathfrak{b}$-component of $(d\widehat{P}_i)_a$ equals e^i;
- the $\mathfrak{u}^-$-component of $(d\widehat{P}_i)_a$, say $(d\widehat{P}_i)_a\{\mathfrak{u}^-\}$, is determined by the equation $\kappa((d\widehat{P}_i)_a\{\mathfrak{u}^-\}, e') = \kappa(\sum_{k+m=i-1} e^k e' e^m, \xi)$.

Let $\mathcal{O}^{\rm reg}$ and $\mathcal{O}^{\rm sub}$ denote the regular and subregular nilpotent orbits in $\mathfrak{sl}_{l+1}$, respectively. Then $\overline{\mathcal{O}^{\rm sub} \cap \mathfrak{u}} = \cup_j \mathfrak{u}_j$. If $e \in \mathcal{O}^{\rm reg} \cap \mathfrak{u}$, then the $\mathfrak{b}$-components of $(d\widehat{P}_i)_{(e,\xi)}$, $i = 1, \dots, l$, are linearly independent, regardless of ξ. Hence $(\mathcal{O}^{\rm reg} \cap \mathfrak{u}) \times \mathfrak{b}^- \subset \mathcal{Y}$.

If $e \in \mathcal{O}^{\rm sub} \cap \mathfrak{u}$, then the $\mathfrak{b}$-components of $(d\widehat{P}_i)_{(e,\xi)}$, $i = 1, \dots, l-1$, are still linearly independent for any ξ, but $e^l = 0$. However, if e is sufficiently general, then the $\mathfrak{u}^-$-component of $(d\widehat{P}_l)_{(e,\xi)}$ appears to be nonzero for all ξ that belong to a dense open subset of $\mathfrak{b}^-$. More precisely, suppose that $e \in \mathfrak{u}_j$ and $\kappa(e, e_{-\alpha_i}) \neq 0$ for $i \neq j$. Taking $e' = e_{\alpha_j}$, one readily computes that $\sum_{k+m=l-1} e^k e' e^m = e^{j-1} e_{\alpha_i} e^{l-j}$ is a nonzero multiple of e_θ. Hence, one can take any ξ such that $\kappa(\xi, e_\theta) \neq 0$.

Thus, there is a dense open subset $\Omega \subset \cup_i \mathfrak{u}_i \times \mathfrak{b}^-$ such that $\Omega \subset \mathcal{Y}$, and the assertion follows. □

It turns out that Proposition 4.3 together with (4.1) is sufficient to conclude that for $\mathfrak{g} = \mathfrak{sl}_{l+1}$, $\mathfrak{q}$ has the *codim-2* property. This follows from the following general assertion:

THEOREM 4.4. — *Let R be a connected algebraic group with Lie algebra $\mathfrak{r}$. Suppose that* (i) $\mathbb{F}[\mathfrak{r}^*]^R = \mathbb{F}[p_1, \ldots, p_m]$ *is a graded polynomial algebra,* (ii) $\mathsf{ind}\,\mathfrak{r} = m$*, and* (iii) $\sum_{i=1}^m \deg p_i = (\dim \mathfrak{r} + \mathsf{ind}\,\mathfrak{r})/2$*. Then the following conditions are equivalent:*

(1) $\operatorname{codim}(\mathfrak{r}^* \smallsetminus \mathfrak{r}^*_{\rm reg}) \geqslant 2$*;*

(2) $\operatorname{codim}(\mathfrak{r}^* \smallsetminus \mathcal{Y}(\mathfrak{r}^*)) \geqslant 2$*, where $\mathcal{Y}(\mathfrak{r}^*)$ is defined as in* (4.2) *via the p_i's.*

If these conditions are satisfied, then actually $\mathfrak{r}^*_{\rm reg} = \mathcal{Y}(\mathfrak{r}^*)$.

Proof. — The implication (1) $\Rightarrow$ (2) is already proved in [10, Theorem 1.2].

To prove the converse, one can slightly adjust the proof given in [10], see also the proof of Theorem 1.2 in [9]. Set $n = \dim \mathfrak{r}$. Let $T(\mathfrak{r}^*)$ denote the tangent bundle of $\mathfrak{r}^*$. The main part of that proof consists in a construction of two homogeneous polynomial sections of $\wedge^{n-m} T(\mathfrak{r}^*)$, denoted $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Write $(\mathfrak{V}_i)_x$ for the value of $\mathfrak{V}_i$ at $x \in \mathfrak{r}^*$. These sections have the following properties:

(a) There exist nonzero polynomials $F_1, F_2 \in \mathbb{F}[\mathfrak{r}^*]$ such that $F_1 \mathfrak{V}_1 = F_2 \mathfrak{V}_2$;

(b) $(\mathfrak{V}_1)_x \neq 0$ if and only if $x \in \mathfrak{r}^*_{\rm reg}$

(c) $(\mathfrak{V}_2)_x \neq 0$ if and only if $x \in \mathcal{Y}(\mathfrak{r}^*)$;

(d) $\deg \mathfrak{V}_1 = (n-m)/2$ and $\deg \mathfrak{V}_2 = \sum_{i=1}^m (\deg p_i - 1)$.

This only requires assumptions (i) and (ii). If (iii) is also satisfied, then $\deg \mathfrak{V}_1 = \deg \mathfrak{V}_2$. Therefore either of conditions (1),(2) implies the other. Moreover, properties **(a)** and **(b)** imply that if (1) is satisfied, then $\deg F_2 = 0$, *i.e.*, $F_2 \in \mathbb{F}^\times$. Likewise, **(a)** and **(c)** imply that if (2) is satisfied, then $\deg F_1 = 0$. This yields the last assertion. $\square$

Since $\mathfrak{q} = \mathfrak{b} \ltimes \mathfrak{u}^-$ does not have the *codim-2* property if $\mathfrak{g}$ is not of type $\mathbf{A}_l$, we cannot immediately conclude that in all cases $x \in \mathfrak{q}^*_{\rm reg}$ if and only if $(d\widehat{P}_1)_x, \ldots, (d\widehat{P}_l)_x$ are linearly independent. Nevertheless, the fact that $\widehat{P}_1, \ldots, \widehat{P}_l$ are the highest components of the basic G-invariants $f_1, \ldots, f_l$ allows to circumvent this difficulty. It can be shown in general (see [17]) that the coadjoint representation $(Q \colon \mathfrak{q}^*)$ has the following property:

CLAIM 4.5. — *For $x \in \mathfrak{q}^*$ the following conditions are equivalent:*

- *The orbit $Q{\cdot}x$ is of maximal dimension, which is* $\dim \mathfrak{q} - l$ *in this situation;*
- *The differentials $(d\widehat{P}_i)_x$, $i = 1, \ldots, l$, are linearly independent.*

This generalise a result of Kostant obtained for semisimple Lie algebras [8, Theorem 9].

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