ANNALES

DE

L'INSTITUT FOURIER

Mustafa KORKMAZ & Athanase PAPADOPOULOS

On the ideal triangulation graph of a punctured surface

Tome 62, nº 4 (2012), p. 1367-1382.

<http://aif.cedram.org/item?id=AIF_2012__62_4_1367_0>

Ann. Inst. Fourier, Grenoble
62, 4 (2012) 1367-1382

ON THE IDEAL TRIANGULATION GRAPH OF A PUNCTURED SURFACE

by Mustafa KORKMAZ & Athanase PAPADOPOULOS (*)

Abstract. — We study the ideal triangulation graph $T(S)$ of an oriented punctured surface S of finite type. We show that if S is not the sphere with at most three punctures or the torus with one puncture, then the natural map from the extended mapping class group of S into the simplicial automorphism group of $T(S)$ is an isomorphism. We also show that the graph $T(S)$ of such a surface S, equipped with its natural simplicial metric is not Gromov hyperbolic. We also show that if the triangulation graph of two oriented punctured surfaces of finite type are homeomorphic, then the surfaces themselves are homeomorphic.

Résumé. — On étudie le graphe $T(S)$ des triangulations idéales d'une surface S orientée de type fini. On montre que si S n'est pas une sphère ayant au plus quatre perforations ou un tore ayant une seule perforation, l'application naturelle du groupe modulaire étendu de S dans le groupe d'automorphismes de $T(S)$ est un isomorphisme. On montre aussi que le graphe $T(S)$ d'une telle surface n'est pas hyperbolique au sens de Gromov. On montre enfin que si les graphe des triangulations idéales de deux surfaces orientées de type fini sont homéomorphes, alors les surfaces sont elles-mêmes homéomorphes.

1. Introduction

In this paper, S is a connected orientable surface of finite type, of genus $g \geqslant 0$ without boundary and with $n \geqslant 1$ punctures. We shall assume that the Euler characteristic $\chi(S)$ of S is negative. The *mapping class group of* S, denoted by $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of S. The *extended mapping class group of* S, $\mathrm{Mod}^*(S)$, is the group of isotopy classes of all homeomorphisms of S.

Keywords: mapping class group ; surface ; arc complex ; ideal triangulation ; ideal triangulation graph ; curve complex ; Gromov hyperbolic.
Math. classification: 32G15, 20F38, 30F10.
(*) The authors' collaboration is supported by CNRS (France) and TUBITAK (Turkey).

We denote by $\overline{S}$ the surface obtained from S by filling in the punctures. Thus, $\overline{S}$ is a closed surface of genus g. The punctures can also be considered as distinguished points in $\overline{S}$, and we denote by $B \subset \overline{S}$ this set of distinguished points. An *arc* in S (or in $\overline{S}$) is the image in $\overline{S}$ of a closed interval whose interior is homeomorphically embedded in $\overline{S} \setminus B$ and whose endpoints are in B. An arc in S (or in $\overline{S}$) is said to be *essential* if it is not homotopic (relative to its endpoints) to a point in $\overline{S}$.

All homotopies of arcs considered in this paper are relative to their endpoints. An *ideal triangulation* (or, for short, a *triangulation*) of S is a maximal collection of disjoint essential arcs that are pairwise non-homotopic. An essential arc that belongs to a triangulation will also be called an *edge* of that triangulation. A triangulation will sometimes be identified with the union of its edges, and it will therefore be considered as a subset of S. A *face* of a triangulation Δ is a connected component of the complement of Δ in S (or the closure of such a component), and it will also be called a *triangle*.

An *elementary move* is the operation of obtaining a new triangulation from a given one by removing an edge and replacing it by a distinct edge. If a is the edge that is removed, then we shall say that the elementary move is *performed on* a.

An edge a in a triangulation Δ is said to be *exchangeable* if one can perform an elementary move on a, in which a is replaced by an edge distinct from a. Otherwise, the edge is said to be *non-exchangeable*. It follows from the classification of surfaces that an edge a of a triangulation Δ is non-exchangeable if and only if the following holds: a joins two distinct punctures and a is in the closure of a unique face of Δ. In this case, there is a well-defined edge a^* of Δ associated to a, which we call the edge *dual to* a, and which is the third edge of the unique triangle of Δ to which a belongs. This situation is depicted in Figure 1.1.

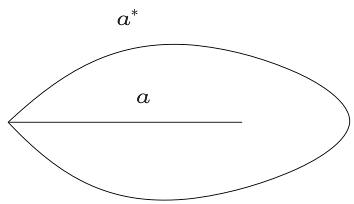


Figure 1.1. The edge a is a non-exchangeable arc in a triangulation, and a^* is its dual edge.

In this paper, we study the *ideal triangulation graph* of S. This is the simplicial graph $T(S)$ whose vertices are the isotopy classes of triangulations of S and in which an edge connects two vertices whenever these two vertices differ by an elementary move. We first prove the following rigidity result:

THEOREM 1.1. — *For any integers $g, h \geqslant 0$ and $m, n \geqslant 1$, consider the surfaces $S_{g,n}$ and $S_{h,m}$ and the associated triangulation graphs $T(S_{g,n})$ and $T(S_{h,m})$. The graphs $T(S_{g,n})$ and $T(S_{h,m})$ are homeomorphic if and only if the surfaces $S_{g,n}$ and $S_{h,m}$ are homeomorphic.*

The proof we give is a direct consequence of the fact that there are exchangeable and non-exchangeable edges in ideal triangulation graphs.

The extended mapping class group $\mathrm{Mod}^*(S)$ acts naturally on $T(S)$ by simplicial automorphisms. We prove the following.

THEOREM 1.2. — *Let S be a connected orientable surface with at least one puncture. If S is not a sphere with at most three punctures or a torus with one puncture, then the natural homomorphism $\mathrm{Mod}^*(S) \to \mathrm{Aut}(T(S))$ is an isomorphism.*

Note that the hypothesis on S made in this theorem, combined with the condition $\chi(S) \leqslant -1$, is equivalent to $\chi(S) \leqslant -2$.

The proof of Theorem 1.2 involves the consideration of the arc complex of S. This is the abstract simplicial complex, denoted by $A(S)$, whose k-simplices, for each $k \geqslant 0$, are the collections of $k+1$ distinct isotopy classes of essential arcs on S that can be represented by pairwise disjoint arcs on this surface. The ideal triangulation graph is the dual one-skeleton of the natural cell-decomposition of the decorated Teichmüller space of surfaces that is constructed in [12]. This graph has also been considered in [2] and [3]. The arc complex has been studied in [6], [8] and [5]. Results similar to Theorem 1.2, for other complexes, have been obtained, see [4] and [11].

In the proof of Theorem 1.2, we shall use the following analogous theorem for the arc complex, obtained by Irmak and McCarthy:

THEOREM 1.3. — ([5]) *Let S be a connected orientable surface with at least one puncture. If the surface S is not a sphere with at most three punctures or a torus with one puncture, then the natural homomorphism $\mathrm{Mod}^*(S) \to \mathrm{Aut}(A(S))$ is an isomorphism.*

We note that the graph $T(S)$ is also the one-skeleton of the simplicial complex dual to the arc complex $A(S)$. Thus, any automorphism of $A(S)$ induces an automorphism of the graph $T(S)$. Now since $T(S)$ is a strict subcomplex of the dual complex of $A(S)$, its automorphism group could a priori be larger than the automorphism group of $A(S)$. Theorem 1.2 shows that this is not the case.

There are two special cases of surfaces of negative characteristic that admit ideal triangulations and that are excluded by the hypothesis of Theorem 1.2: the sphere with three punctures and the torus with one puncture. Let us briefly discuss these cases.

(1) If S is a sphere with three punctures, then any ideal triangulation of S has three edges and two faces. In this case, the graph $T(S)$ is finite, and it is homeomorphic to a tripod. The center of the tripod corresponds to the unique (up to isotopy) ideal triangulation of S in which every edge is exchangeable. The three other vertices of $T(S)$ correspond to the isotopy classes of ideal triangulations in which exactly one edge is exchangeable. The automorphism group $\mathrm{Aut}(T(S))$ of $T(S)$ is isomorphic to the permutation group on three elements (permuting the three vertices of valency one of the tripod). The mapping class group of S is also isomorphic to the permutation group on three elements (the punctures of S), and the natural homomorphism $\mathrm{Mod}(S) \to \mathrm{Aut}(T(S))$ is an isomorphism. Hence, the natural homomorphism $\mathrm{Mod}^*(S) \to \mathrm{Aut}(T(S))$ is surjective and its kernel is the center of $\mathrm{Mod}^*(S)$, which is a cyclic group of order two.

(2) If S is a torus with one puncture, then the ideal triangulation graph $T(S)$ is a regular infinite tree in which every vertex has valency three. The automorphism group of such a tree is uncountable. To see this, we consider the set of one-sided infinite sequences of letters on the alphabet $\{f, n\}$ (where the letter f stands for “flip" and the letter n stands for “no flip"). This set of sequences is uncountable, and it can be injected in the simplicial automorphism group of the tree, by first choosing a base point of the tree, and then interpreting each one-sided infinite sequence as an infinite composition of flips and no flips performed on edges encountered successively, starting at the given basepoint. Thus, in the case considered, the natural homomorphism $\mathrm{Mod}^*(S) \to \mathrm{Aut}(T(S))$ is highly non-surjective since the extended mapping class group $\mathrm{Mod}^*(S)$ is countable.

By declaring that the length of each edge in $T(S)$ is one and taking the associated length metric, we may consider this graph as a metric space. With this metric, the *length* of a simplicial path between two vertices is the number of edges in that path. It is natural to ask whether $T(S)$ is Gromov-hyperbolic.

Theorem 1.2 implies the following:

THEOREM 1.4. — *Let S be a connected orientable surface with at least one puncture. If S is not a sphere with at most four punctures or a torus with one puncture, then the ideal triangulation graph of S is not Gromov-hyperbolic.*

Proof. — A homeomorphism of S that fixes the homotopy class of an ideal triangulation may permute the homotopy classes of the edges of this triangulation. The cardinality of the set of edges is $6g+3n-6$, where g is the genus of S and n is the number of punctures of S. A homeomorphism of S fixing the homotopy class of each edge of a given ideal triangulation is homotopic to the identity. Thus, the action of the extended mapping class group $\mathrm{Mod}^*(S)$ on the vertices of $T(S)$ has finite stabilizers, with uniformly bounded cardinality. Furthermore, the action of $\mathrm{Mod}^*(S)$ on $T(S)$ is co-compact, since up to homeomorphisms, there are only finitely many homotopy classes of ideal triangulations on any surface of finite type. To see this, we consider the surface S equipped with an arbitrary ideal triangulation as being obtained by gluing some finite set (of cardinality $4g-4+2n$) of triangles along their edges. There are only finitely many ways to glue such a finite set of triangles to get a triangulated surface. Thus, if $T(S)$ were Gromov-hyperbolic, then $\mathrm{Mod}^*(S)$ would be word-hyperbolic (see [1]). Under the hypotheses of the theorem, the extended mapping class group $\mathrm{Mod}^*(S)$ is not word-hyperbolic, since it contains free abelian groups of rank at least two. □

We note that in the cases excluded in the hypothesis of Theorem 1.4, the situation is as follows:

- If S is a sphere with one puncture, then the triangulation graph $T(S)$ is empty.
- If S is a sphere with two punctures, then $T(S)$ consists of only one vertex, hence it is hyperbolic.
- If S is a sphere with three punctures, then its triangulation graph is compact, and therefore hyperbolic.
- If S is a sphere with four punctures, then its extended mapping class group is virtually free, therefore hyperbolic. The proof of Theorem 1.4 implies then that the triangulation graph is also hyperbolic.
- If S is a torus with one puncture, then its triangulation graph is hyperbolic since, as we recalled above, it is a tree.

Theorem 1.4 implies that the large-scale geometry of the triangulation graph of S is quite different from that of the curve complex, since, by a result of Masur and Minsky, the curve complex is hyperbolic.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 uses a lemma on valency in the ideal triangulation graph.

We call the *valency* of a vertex in $T(S_{g,n})$ the number of edges abutting (locally) at that point.

LEMMA 2.1. — *For any g and n, the maximum valency of a vertex of $T(S_{g,n})$ is $6g+3n-6$, and the minimal valency of a vertex of $T(S_{g,n})$ is $6g+2n-5$. Moreover, for each integer k with $6g+2n-5 \leqslant k \leqslant 6g+3n-6$, there exists a vertex in $T(S)$ whose valency is k.*

Proof. — An easy Euler characteristic argument shows that the number of edges of an ideal triangulation Δ on $S_{g,n}$ is $6g+3n-6$. On an arbitrary punctured surface $S_{g,n}$, we can find an ideal triangulation having all of its edges on the boundary of two distinct faces, and which is therefore exchangeable. Thus, for each g and n, the maximum valency at a vertex of $T(S_{g,n})$ is $6g+3n-6$.

Now we consider the minimal valency at a vertex of $T(S_{g,n})$. This is obtained by computing the maximal number of non-exchangeable edges. If $n = 1$, then there are no non-exchangeable edges. If $n \geqslant 2$, then there is a configuration on the surface that gives the maximal number of non-exchangeable edges. This is represented in Figure 2.1. In this configuration, there are $n-1$ non-exchangeable edges. Thus, on any surface $S_{g,n}$ with $n \geqslant 1$, we can take a set of edges in the configuration of Figure 2.1 and complete it into an ideal triangulation. Notice that the statement that

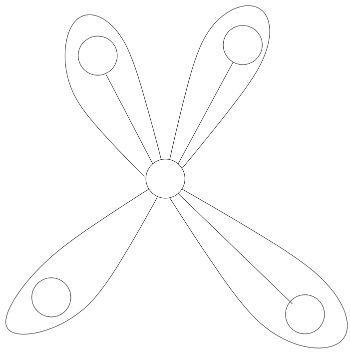


Figure 2.1. On a surface $S_{g,n}$ with $n \geqslant 2$, an ideal triangulation containing such a configuration represents a vertex of $T(S_{g,n})$ that has minimal valency. (In this picture, $n = 5$.)

on a surface $S_{g,n}$ there are at most $n-1$ non-exchangeable edges, and that this bound is attained for any surface, also holds in the case where $n=1$. Thus, in all cases, the minimal valency at a vertex of $T(S_{g,n})$ is $6g+3n-6-(n-1)=6g+2n-5$.

The second assertion of the lemma follows from similar considerations. □

Now we can prove Theorem 1.1.

From Lemma 2.1, if the two graphs $T(S_{g,n})$ and $T(S_{h,m})$ are homeomorphic, then $6g+3n-6=6h+3m-6$ and $6g+2n-5=6h+2m-5$. These two equations imply that $g=h$ and $n=m$; that is, the surfaces are homeomorphic. The converse is clear.

3. Squares and pentagons in $T(S)$

To simplify notation, we shall often identify an arc or a triangulation on S with its isotopy class.

In the proof of Theorem 1.2, we shall use two special classes of simplicial closed paths in $T(S)$. We now describe them.

A *square* in $T(S)$ is a simple (that is, injective) closed path in this graph consisting of four edges, as represented in Figure 3.1. In this figure, elementary moves are performed on two exchangeable edges a and c, whose images under the moves are denoted, respectively, by a' and c'. Note that the existence of such an elementary move implies that the interiors of the faces containing a and c on the surface S are disjoint. The move is also

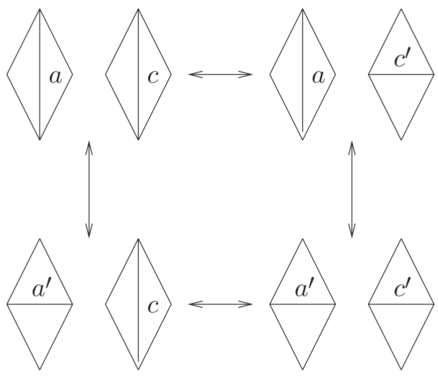


Figure 3.1. A square in the triangulation graph.

represented diagrammatically as follows:

$$\langle a, c\rangle \leftrightarrow \langle a, c'\rangle \leftrightarrow \langle a', c'\rangle \leftrightarrow \langle a', c\rangle \leftrightarrow \langle a, c\rangle.$$

In this notation, a symbol such as $\langle a, c\rangle$ represents a vertex of $T(S)$ (a triangulation of S) containing the arcs a and c. The symbol $\langle a, c\rangle \leftrightarrow \langle a, c'\rangle$ represents an elementary move between the given two triangulations, so that the arc c is changed to c' after the elementary move. The arcs not shown in the symbol remain unchanged.

A *pentagon* in $T(S)$ is a simple closed path in $T(S)$ of length five, represented diagrammatically in Figure 3.2. This pentagon is also represented diagrammatically as follows:

$$\langle a, b\rangle \leftrightarrow \langle a, e\rangle \leftrightarrow \langle e, d\rangle \leftrightarrow \langle d, c\rangle \leftrightarrow \langle c, b\rangle \leftrightarrow \langle b, a\rangle.$$

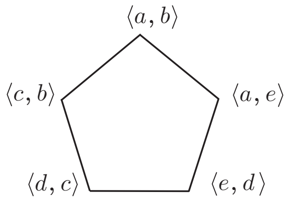


Figure 3.2. A pentagon in the triangulation graph.

Lemma 3.1. — *There are no simple closed paths of length three in $T(S)$.*

Proof. — Assume there exists a simple closed path of length three. Let us represent it by a diagram

$$\langle a, b\rangle \leftrightarrow \langle b, c\rangle \leftrightarrow \langle c, a\rangle \leftrightarrow \langle a, b\rangle$$

in which a, b and c are distinct arcs, and where the double arrows represent, as before, elementary moves. That is, the arc a is transformed into the arc c by the first move, and so on. Now the existence of the moves $\langle a, b\rangle \leftrightarrow \langle b, c\rangle$ and $\langle a, b\rangle \leftrightarrow \langle c, a\rangle$ imply that $a = b$, a contradiction. This proves the lemma. □

Lemma 3.2. — *Every simple closed path of length four in $T(S)$ is a square; that is, it is of the form described in Figure 3.1.*

Proof. — Consider a simple closed path of length four in $T(S)$, and represent it as

$$\langle a, b\rangle \leftrightarrow \langle a, c\rangle \leftrightarrow \langle c, d\rangle \leftrightarrow \langle d, e\rangle \leftrightarrow \langle a, b\rangle.$$

The existence of the last move implies that either $d = a$, or $d = b$, or $e = a$, or $e = b$. We analyze each case separately.

- The case $d = a$ is excluded, because of the existence of the move $\langle a, c\rangle \leftrightarrow \langle c, d\rangle$.
- If $d = b$ then we get two moves $\langle b, e\rangle \leftrightarrow \langle a, b\rangle$ and $\langle c, b\rangle \leftrightarrow \langle b, e\rangle$ which give $a = c$, which is excluded since we have a vertex labeled $\langle a, c\rangle$ in the path.
- If $e = a$ then we get two moves $\langle a, b\rangle \leftrightarrow \langle a, c\rangle$ and $\langle a, d\rangle \leftrightarrow \langle a, b\rangle$ which gives $d = c$, which is excluded since we have a vertex labeled $\langle c, d\rangle$ in the path.
- If $e = b$, then the path takes the form
$$\langle a, b\rangle \leftrightarrow \langle a, c\rangle \leftrightarrow \langle c, d\rangle \leftrightarrow \langle d, b\rangle \leftrightarrow \langle a, b\rangle,$$
which is of the form represented in Figure 3.1. □

Lemma 3.3. — *Every simple closed path of length five in $T(S)$ is a pentagon; that is, it is of the form described in Figure 3.2.*

Proof. — The proof is by inspection, like the proof of Lemma 3.2 (and it uses Lemma 3.1). □

In the next two lemmas, and later in the paper, if Δ and Δ' are two triangulations of S connected by an edge in $T(S)$, then the notation $\Delta - \Delta'$ will be used to denote the edge in Δ that is not in Δ'.

Lemma 3.4. — *Consider a square in $T(S)$, represented as $\Delta_1 \leftrightarrow \Delta_2 \leftrightarrow \Delta_3 \leftrightarrow \Delta_4 \leftrightarrow \Delta_1$. We have $\Delta_1 - \Delta_4 = \Delta_2 - \Delta_3$.*

Proof. — The proof follows from Lemma 3.2. □

Lemma 3.5. — *Consider a pentagon in $T(S)$, represented as $\Delta_1 \leftrightarrow \Delta_2 \leftrightarrow \Delta_3 \leftrightarrow \Delta_4 \leftrightarrow \Delta_5 \leftrightarrow \Delta_1$. We have $\Delta_1 - \Delta_5 = \Delta_2 - \Delta_3$.*

Proof. — The proof follows from Lemma 3.3. □

Let us note that squares and pentagons of the ideal triangulation complex are the dual two-cells of the cell decomposition of the decorated Teichmüller space of the surface constructed in [12].

4. Proof of Theorem 1.2

Let $f : T(S) \to T(S)$ be a simplicial automorphism. We associate to f a simplicial automorphism $\widetilde{f} : A(S) \to A(S)$, defined as follows using an idea in [4] and [11].

Let a be an essential arc on S. We choose a triangulation Δ in which a is an exchangeable edge. Note that such a triangulation always exists.

Since a is exchangeable in Δ, we can perform an elementary move on a, replacing it by an arc a'. Let Δ_a be the triangulation obtained from Δ by this elementary move. Since f is simplicial, the triangulations $f(\Delta)$ and $f(\Delta_a)$ (like Δ and Δ_a) are joined by an edge in $T(S)$. In other words, the two triangulations $f(\Delta)$ and $f(\Delta_a)$ differ by an elementary move. We then define $\widetilde{f}(a)$ to be the edge that is in $f(\Delta)$ but not in $f(\Delta_a)$. In short, we have

$$\widetilde{f}(a) = f(\Delta) - f(\Delta_a).$$

We now prove the following:

Proposition 4.1. — *The arc $\widetilde{f}(a)$ is independent of the choice of the triangulation Δ in which a is exchangeable.*

Proof. — Let $\Delta = \{a, c_1, \ldots, c_k\}$ and $\Delta' = \{a, c'_1, \ldots, c'_k\}$ be two triangulations in which a is exchangeable. We show that if Δ_a (resp. Δ'_a) is the triangulation obtained from Δ (resp. Δ') by the elementary move on a, then $f(\Delta) - f(\Delta_a) = f(\Delta') - f(\Delta'_a)$. This will prove the proposition.

Let R be the surface obtained by cutting S along a. There are two cases for the surface R:

(i) R has two boundary components coming from the curve a, and in this case there is one distinguished point on each of these boundary components, coming from the puncture at the endpoints of a. This occurs when the arc a joins one puncture of S to itself.

(ii) R has one boundary component, with two distinguished points on that boundary. This occurs when a joins two distinct punctures on S.

The triangulations Δ and Δ' of S naturally induce two triangulations $\{a_1, a_2, c_1, \ldots, c_k\}$ and $\{a_1, a_2, c'_1, \ldots, c'_k\}$ on R, where the edge a has been replaced by two edges a_1 and a_2. In Case (i), each of the edges a_1 and a_2 appears on a boundary component of R, and in Case (ii), the union of the edges a_1 and a_2 forms the boundary component of R, and there are two distinguished points on that boundary component.

In each case, we can join the two triangulations induced on R by Δ and Δ' by a finite sequence of elementary moves on R, in such a way that the edges a_1 and a_2 on R are left fixed by these elementary moves. This follows from Harer's result on the connectedness of the triangulation graph of a surface with boundary and with distinguished points on boundary components. (See [2]; the result is also cited in [3].)

Now gluing a_1 back to a_2 gives a simplicial path in $T(S)$ joining Δ to Δ' such that each triangulation in this path contains a as an edge. We denote the sequence of triangulations in this path by

$$\Delta = \Delta_0, \Delta_1, \dots, \Delta_l = \Delta'. \tag{4.1}$$

We may assume that this path is simple.

We shall use the following:

Lemma 4.2. — *We can assume that we can choose the path* (4.1) *in such a way that the edge a is exchangeable in each triangulation representing a vertex of this path.*

Proof. — Assume there exists a vertex in the path (4.1) that represents a triangulation in which a is not exchangeable, and let Δ_i be the triangulation represented by the first such vertex, after the vertex Δ. Then, Δ_{i+1} (respectively Δ_{i-1}) is the triangulation represented by the vertex after (respectively before) Δ_i. Note that since a is exchangeable in Δ and in Δ', we have $1 \leqslant i < l$. There are two possibilities for the vertex joining Δ_i and Δ_{i+1}, and they are represented respectively in Figure 4.1 (a) and (b).

The first possibility (top of Figure 4.1 (a)) is when the elementary move that takes Δ_i to Δ_{i+1} involves an edge that is on the boundary of a triangle having the edge a^* (the dual of a) as an edge. (Note that we exclude the case where the elementary move is performed on the edge a^*, since in that case we recover the triangulation Δ_{i-1} as a result, but we assumed the path (4.1) is simple.) In this case, we replace the subpath $\Delta_{i-1} \leftrightarrow \Delta_i \leftrightarrow \Delta_{i+1}$ by the path $\Delta_{i-1} \leftrightarrow \Delta'_i \leftrightarrow \Delta'_{i+1} \leftrightarrow \Delta_{i+1}$ represented in the bottom of Figure 4.1 (a).

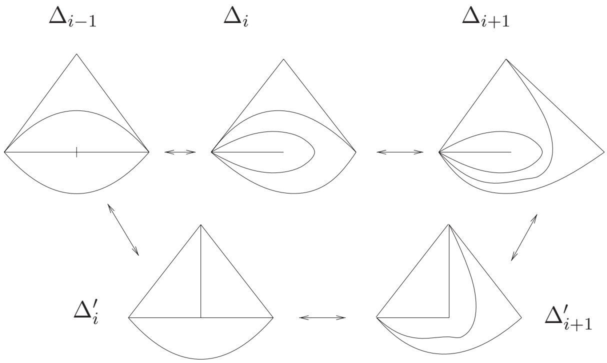
The second possibility (top of Figure 4.1 (b)) is when the elementary move that takes Δ_i to Δ_{i+1} involves an edge that is not on the boundary of a triangle whose boundary contains a^*. In Figure 4.1 (b), we have symbolically represented that elementary move on a quadrilateral that is disjoint from a and a^*. In that case, we replace the subpath $\Delta_{i-1} \leftrightarrow \Delta_i \leftrightarrow \Delta_{i+1}$ by the path $\Delta_{i-1} \leftrightarrow \Delta'_i \leftrightarrow \Delta_{i+1}$ represented in Figure 4.1 (b).

In each case, each vertex of the new simplicial path joining Δ and Δ' contains a as an edge, and the number of occurrences of vertices in this path in which a is non-exchangeable has been reduced by one. This allows us, by induction, to obtain a path joining Δ and Δ' in $T(S)$, such that every vertex is represented by a triangulation that contains a as an exchangeable edge.

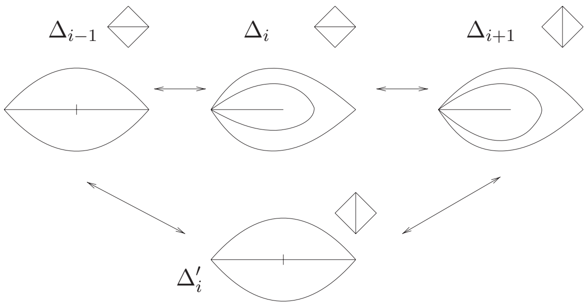
This proves Lemma 4.2 □

We now continue with the proof of Proposition 4.1.

Note that the closed paths in $T(S)$ represented in Figure 4.1 (a) and (b) are a pentagon and a square.



(a)



(b)

Figure 4.1. Replacing a path in which each vertex contains a by a path where at each vertex a is an exchangeable edge.

To prove that the arc $\widetilde{f}(a)$ is well-defined, by Lemma 4.2, it suffices to consider the case where Δ and Δ' are joined by an edge in $T(S)$.

Recall the notation $\Delta = \{a, c_1, \ldots, c_k\}$ and $\Delta' = \{a, c'_1, \ldots, c'_k\}$ at the beginning of the proof of the proposition. Since a is contained as an edge in both Δ and Δ', there is an edge distinct from a that is exchanged by the elementary move that takes Δ to Δ'. By relabeling the edges of Δ, we can assume that c_1 is the vertex that is transformed by the elementary move that takes Δ to Δ'. We use as before the notation $\Delta \leftrightarrow \Delta'$, with $\Delta = \langle a, c_1 \rangle$ and $\Delta' = \langle a, c'_1 \rangle$, to denote this elementary move, in which c'_1 is the image of c_1. Edges other than c_1 are unchanged. We distinguish two cases:

Case 1.— In the triangulation Δ, a and c_1 are two edges of a common triangle (see Figure 4.2 (a)). Since S is not a torus with one puncture or a sphere with three punctures, the edge joining Δ and Δ' in $T(S)$ belongs to a pentagon in this graph. This follows from the fact that both a and c_1 are exchangeable, and it can be seen diagrammatically in Figure 4.2 (b), where the sequence of elementary moves representing the pentagon relation is:

$$\langle a, c_1\rangle \leftrightarrow \langle a, c_1'\rangle, \leftrightarrow \langle c_1', d\rangle, \leftrightarrow \langle d, a'\rangle, \leftrightarrow \langle a', c_1\rangle, \leftrightarrow \langle c_1, a\rangle.$$

Let $\mathcal{P}$ denote this pentagon. Since $f : T(S) \to T(S)$ is simplicial, the image $f(\mathcal{P})$ of $\mathcal{P}$ by f is again a pentagon (Lemma 3.3). In the pentagon $\mathcal{P}$, the triangulation $\langle a, c_1\rangle$ is labeled Δ, the triangulation $\langle a', c_1\rangle$ is labeled Δ_a, the triangulation $\langle a, c_1'\rangle$ is labeled Δ' and the triangulation $\langle d, c_1'\rangle$ is labeled Δ_a'. We then have, by Lemma 3.5 applied to the pentagon $f(\mathcal{P})$,

$$f(\Delta) - f(\Delta_a) = f(\Delta') - f(\Delta_a').$$

This completes the proof of the fact the $\widetilde{f}(a)$ is well-defined in Case 1.

Case 2.— In the triangulation Δ, a and c_1 are not edges of the same triangle. In this case, we can perform elementary moves on a and c_1 independently from each other, and we obtain the following square in $T(S)$:

$$\langle a, c_1\rangle \leftrightarrow \langle a, c_1'\rangle, \leftrightarrow \langle a', c_1'\rangle, \leftrightarrow \langle a', c_1\rangle, \leftrightarrow \langle a, c\rangle.$$

Similarly to the preceding case, the triangulation $\langle a, c_1\rangle$ is labeled Δ, the triangulation $\langle a', c_1\rangle$ is labeled Δ_a, the triangulation $\langle a, c_1'\rangle$ is labeled Δ' and the triangulation $\langle a', c_1'\rangle$ is labeled Δ_a', and we get again, this time by Lemma 3.4,

$$f(\Delta) - f(\Delta_a) = f(\Delta') - f(\Delta_a').$$

This shows that $\widetilde{f}(a)$ is well-defined in each case.

This completes the proof of Proposition 4.1. $\square$

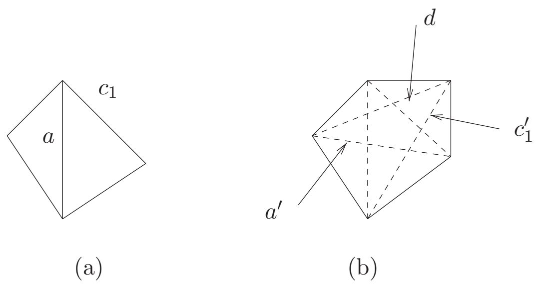


Figure 4.2. The star in the right hand side figure corresponds to a pentagon relation.

The following naturality formula will be useful.

Proposition 4.3. — *For any ideal triangulation* $\Delta = \{c_0, c_1, \ldots, c_k\}$, *we have*

$$f(\Delta) = \{\widetilde{f}(c_0), \widetilde{f}(c_1), \ldots, \widetilde{f}(c_k)\}.$$

Proof. — It suffices to prove that $f(\Delta) \subset \{\widetilde{f}(c_0), \widetilde{f}(c_1), \ldots, \widetilde{f}(c_k)\}$. Let a be any edge in Δ. If a is exchangeable, then it follows from the definition that $\widetilde{f}(a)$ is in the simplex $f(\Delta)$. If a is not exchangeable, then its dual edge a^* is exchangeable in Δ. Let $\Delta' = \Delta_{a^*}$ be the ideal triangulation obtained from Δ by exchanging the edge a^*. Finally, let Δ'_a be the ideal triangulation obtained from Δ' by exchanging the edge a. Since there is an edge in $T(S)$ joining Δ to Δ', and an edge in $T(S)$ joining Δ' to Δ'_a, and since f is simplicial, there is an edge in $T(S)$ joining $f(\Delta)$ to $f(\Delta')$, and an edge in $T(S)$ joining $f(\Delta')$ to $f(\Delta'_a)$. Since, by definition, $\widetilde{f}(a)$ belongs to the triangulation $f(\Delta')$, the edge $\widetilde{f}(a)$ is transformed by the move $f(\Delta') \leftrightarrow f(\Delta'_a)$. Therefore, $\widetilde{f}(a)$ is not an edge of $f(\Delta'_a)$. Now since $\widetilde{f}(a)$ is not transformed by the move $f(\Delta) \leftrightarrow f(\Delta')$, we conclude that $\widetilde{f}(a)$ is an edge of $f(\Delta)$. □

Corollary 4.4. — *The map* $\widetilde{f} : A(S) \to A(S)$ *is simplicial.*

□

Proposition 4.5. — *If* f *and* h *are two automorphisms of* $T(S)$, *then* $\widetilde{fh} = \widetilde{f}\,\widetilde{h}$.

Proof. — Let a be (the isotopy class of) an arc on S. We show that $\widetilde{fh}(a) = \widetilde{f}(\widetilde{h}(a))$.

If $\Delta = \{a, c_1, c_2, \ldots, c_k\}$ be a triangulation in which a is exchangeable and if $\Delta' = \{a', c_1, c_2, \ldots, c_k\}$ is the triangulation obtained from Δ by an elementary move on a, then $\widetilde{h}(a) = h(\Delta) - h(\Delta')$, and $\widetilde{fh}(a) = fh(\Delta) - fh(\Delta')$.

Since the map $h : T(S) \to T(S)$ is simplicial and since the triangulation Δ is related to Δ' in $T(S)$ by an edge, the triangulation $h(\Delta)$ is also related to the triangulation $h(\Delta')$ by an edge. Since $h(\Delta)$ contains the arc $\widetilde{h}(a)$ as an exchangeable arc, we can use $h(\Delta)$ and $h(\Delta')$ to define $\widetilde{f}(\widetilde{h}(a))$, and we obtain:

$$\begin{aligned}\widetilde{f}(\widetilde{h}(a)) &= f\left(h(\Delta)\right) - f\left(h(\Delta')\right) \\ &= (fh)(\Delta) - (fh)(\Delta') \\ &= \widetilde{fh}(a).\end{aligned}$$

This completes the proof. □

Proposition 4.6. — *For any automorphism $f \in \mathrm{Aut}(T(S))$, the associated map $\widetilde{f} : A(S) \to A(S)$ is an automorphism.*

Proof. — Let $h = f^{-1} \in \mathrm{Aut}(T(S))$ be the inverse of f. By Proposition 4.5, we have $\widetilde{f}\widetilde{h} = \widetilde{\mathrm{I}} = \widetilde{h}\widetilde{f}$, where I is the identity map of $T(S)$. Now from the definitions, the map $\widetilde{\mathrm{I}} : A(S) \to A(S)$ associated to I is the identity map of $A(S)$. Thus, $\widetilde{f}$ has an inverse, which shows that $\widetilde{f} : A(S) \to A(S)$ is a bijection.

Since $\widetilde{f}$ is also simplicial by Corollary 4.4, it is an automorphism of $A(S)$. $\square$

Finally, we prove the following theorem which, together with Theorem 1.3, implies Theorem 1.2.

Theorem 4.7. — *If the surface S is not a sphere with at most three punctures or a torus with one puncture, then the map $\phi : \mathrm{Aut}(T(S)) \to \mathrm{Aut}(A(S))$ defined by $f \mapsto \widetilde{f}$ is an isomorphism.*

Proof. — Let $\Theta : \mathrm{Mod}^*(S) \to \mathrm{Aut}(A(S))$ and $\Phi : \mathrm{Mod}^*(S) \to \mathrm{Aut}(T(S))$ denote the natural maps. Let $\Psi : \mathrm{Aut}(T(S)) \to \mathrm{Aut}(A(S))$ be defined as $\Psi(f) = \widetilde{f}$. Proposition 4.5 implies that Ψ is a homomorphism. It is clear that $\Psi\Phi = \Theta$. Since Θ is an isomorphism by Theorem 1.3, the homomorphism Ψ is onto.

$$\begin{tikzcd} \mathrm{Mod}^*(S) \arrow[d, "\Phi"] \arrow[dr, "\Theta"] & \\ \mathrm{Aut}(T(S)) \arrow[r, "\Psi"] & \mathrm{Aut}(A(S)) \end{tikzcd}$$

We now show that Ψ is one-to-one. Let f be an element of $\mathrm{Aut}(T(S))$ such that $\Psi(f) = \widetilde{f}$ is the identity automorphism of $A(S)$. By Proposition 4.3, the automorphism f acts trivially on $T(S)$. Hence, Ψ is one-to-one.

This completes the proof of the theorem. $\square$

BIBLIOGRAPHY

[1] M. Gromov, "Hyperbolic groups", in *Essays in Group Theory, edited by S.M. Gersten*, MSRI Publications 8, Springer-Verlag, 1987, p. 75-263.

[2] J. L. Harer, "Stability of the homology of the mapping class groups of orientable surfaces", *Annals of Math.* **121** (1985), p. 215-249.

[3] A. Hatcher, "On triangulations of surfaces", *Top. and its Appl.* **41** (1991), p. 189-194, A new version is available in the author's webpage.

[4] E. Irmak & M. Korkmaz, "Automorphisms of the Hatcher-Thurston complex", *Isr. J. Math.* **162** (2007), p. 183-196.

[5] E. Irmak & J. D. McCarthy, “Injective simplicial maps of the arc complex”, *Turkish Journal of Mathematics* **33** (2009), p. 1-16.

[6] N. V. Ivanov, “Automorphisms of Teichmüller modular groups”, Lecture Notes in Math., no. 1346, Springer-Verlag, Berlin and New York, 1988, p. 199-270.

[7] N. V. Ivanov & J. D. McCarthy, “On injective homomorphisms between Teichmüller modular groups”, *I. Invent. Math.* **135** (1999), no. 2, p. 425-486.

[8] M. Korkmaz, “Automorphisms of complexes of curves on punctured spheres and on punctured tori”, *Topology and its Applications* **95** (1999), no. 2, p. 85-111.

[9] M. Korkmaz & A. Papadopoulos, “On the arc and curve complex of a surface”, Math. Proc. Cambridge Philos. Soc., to appear.

[10] F. Luo, “Automorphisms of the complex of curves”, *Topology* **39** (2000), no. 2, p. 283-298.

[11] D. Margalit, “Automorphisms of the pants complex”, *Duke Math. J.* **121** (2004), no. 3, p. 457-479.

[12] R. C. Penner, “The decorated Teichmüller space of punctured surfaces”, *Communications in Mathematical Physics* **113** (1987), p. 299-339.

Manuscrit reçu le 23 août 2010,
accepté le 12 janvier 2011.

Mustafa KORKMAZ
Department of Mathematics,
Middle East Technical University,
06531 Ankara,
Turkey.
korkmaz@metu.edu.tr

Athanase PAPADOPOULOS
Max-Planck-Institut für Mathematik,
Vivatsgasse 7,
53111 Bonn, Germany

Institut de Recherche Mathématique Avancée,
Université de Strasbourg and CNRS,
7 rue René Descartes,
67084 Strasbourg Cedex, France.
athanase.papadopoulos@math.unistra.fr